Cantor families of periodic solutions for completely resonant nonlinear wave equations

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Abstract: We prove existence of small amplitude, $2\pi/\omega$-periodic in time solutions of completely resonant nonlinear wave equations with Dirichlet boundary conditions, for any frequency $\omega$ belonging to a Cantor-like set of asymptotically full measure and for a new set of nonlinearities. The proof relies on a suitable Lyapunov-Schmidt decomposition and a variant of the Nash-Moser Implicit Function Theorem. In spite of the complete resonance of the equation we show that we can still reduce the problem to a finite dimensional bifurcation equation. Moreover, a new simple approach for the inversion of the linearized operators required by the Nash-Moser scheme is developed. It allows to deal also with nonlinearities which are not odd and with finite spatial regularity.

Keywords: Nonlinear Wave Equation, Infinite Dimensional Hamiltonian Systems, Periodic Solutions, Variational Methods, Lyapunov-Schmidt reduction, small divisors, Nash-Moser Theorem.

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Contents

1 Introduction 2

1.1 Main result ................................................. 4
1.2 The Lyapunov-Schmidt reduction ......................... 6
1.2.1 The 0th order bifurcation equation .................... 6
1.2.2 About the proof of Theorem 1.2 ..................... 7
1.2.3 About the proof of Theorem 1.1 ..................... 9

2 Solution of the ($Q_2$)-equation 9

3 Solution of the ($P$)-equation 12

3.1 The Nash-Moser scheme ................................. 13
3.2 $C^\infty$ extension ........................................... 17
3.3 Measure estimate ................................. 21

4 Analysis of the linearized problem: proof of (P3) 24

4.1 Decomposition of $L_n(\delta,v_1,w)$ .................... 24
4.2 Step 1: Inversion of $D$ .................................. 25
4.3 Step 2: Inversion of $L_n$ ............................... 28

5 Solution of the ($Q_1$)-equation 32

5.1 The ($Q_1$)-equation for $\delta = 0$ .............. 32
5.2 Proof of Theorem 1.3 ................................. 33

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1 Introduction

We consider the completely resonant nonlinear wave equation

\[
\begin{cases}
  u_{tt} - u_{xx} + f(x, u) = 0 \\
  u(t, 0) = u(t, \pi) = 0
\end{cases}
\]

where the nonlinearity

\[ f(x, u) = a_p(x) u^p + O(u^{p+1}), \quad p \geq 2 \]

is analytic in \( u \) but is only \( H^1 \) with respect to \( x \).

We look for small amplitude, \( 2\pi/\omega \)-periodic in time solutions of equation (1) for all frequencies \( \omega \) in some Cantor set of positive measure, actually of full density at \( \omega = 1 \).

Equation (1) is an infinite dimensional Hamiltonian system possessing an elliptic equilibrium at \( u = 0 \). The frequencies of the linear oscillations at 0 are \( \omega_j = j \), \( \forall j = 1, 2, \ldots \), and therefore satisfy infinitely many resonance relations. Any solution \( v = \sum_{j \geq 1} a_j \cos(jt + \theta_j) \sin(jx) \) of the linearized equation at \( u = 0 \),

\[
\begin{cases}
  u_{tt} - u_{xx} = 0 \\
  u(t, 0) = u(t, \pi) = 0
\end{cases}
\]

is \( 2\pi \)-periodic in time. For this reason equation (1) is called a completely resonant Hamiltonian PDE.

Existence of periodic solutions close to a completely resonant elliptic equilibrium for finite dimensional Hamiltonian systems has been proved in the celebrated theorems of Weinstein [27], Moser [22] and Fadell-Rabinowitz [14]. The proofs are based on the classical Lyapunov-Schmidt decomposition which splits the problem into two equations: the range equation, solved through the standard Implicit Function Theorem, and the bifurcation equation, solved via variational arguments.

For proving existence of small amplitude periodic solutions of completely resonant Hamiltonian PDEs like (1) two main difficulties must be overcome:

(i) a “small denominators” problem which arises when solving the range equation;

(ii) the presence of an infinite dimensional bifurcation equation: which solutions \( v \) of the linearized equation (2) can be continued to solutions of the nonlinear equation (1)?

The “small denominators” problem (i) is easily explained: the eigenvalues of the operator \( \partial_{tt} - \partial_{xx} \) in the spaces of functions \( u(t, x) \), \( 2\pi/\omega \)-periodic in time and such that, say, \( u(t, \cdot) \in H^1_0(0, \pi) \) for all \( t \), are \( -\omega^2 l^2 + j^2 \), \( l \in \mathbb{Z}, j \geq 1 \). Therefore, for almost every \( \omega \in \mathbb{R} \), the eigenvalues accumulate to 0. As a consequence, for most \( \omega \), the inverse operator of \( \partial_{tt} - \partial_{xx} \) is unbounded and the standard Implicit Function Theorem is not applicable.

The appearance of small denominators is a common feature of Hamiltonian PDEs. This problem was first solved by Kuksin [19] and Wayne [20] using KAM theory (other existence results of quasi-periodic solutions with KAM theory were obtained e.g. in [21], [23], [10], see also [20] and references therein).

In [12] Craig-Wayne introduced for Hamiltonian PDEs the Lyapunov-Schmidt reduction method and solved the range equation via a Nash-Moser Implicit function technique. The major difficulty concerns the inversion of the linearized operators obtained at any step of the Nash-Moser iteration because the eigenvalues may be arbitrarily small (this is the small divisor problem (i)). The Craig-Wayne method to control such inverses is based on the Fröhlich-Spencer technique [15] and (in the wave equation with Dirichlet boundary conditions) works for nonlinearities \( f(x, u) \) which can be extended to analytic, odd, periodic functions so that the Dirichlet problem on \([0, \pi]\) is equivalent to the \( 2\pi \)-periodic problem within
the space of all odd functions. A key property exploited in this case is that the “off-diagonal” terms of the linearized operator (seen as an infinite dimensional matrix in Fourier basis) decay exponentially fast away from the diagonal. At the end of the Nash-Moser iteration, due to the “small divisor problem” (i), the range equation is solved only for a Cantor set of parameters.

We mention that the Craig-Wayne approach has been extended by Su [25] to some case where the nonlinearity has only low Sobolev regularity (for periodic conditions) and by Bourgain to find also quasi-periodic solutions [7]-[8].

The previous results apply for example to non-resonant or partially resonant Hamiltonian PDEs like $u_{tt} - u_{xx} + a_1(x)u = f(x,u)$ where the bifurcation equation is finite dimensional (2 dimensional in [12] and $2m$ dimensional in [13]). With a non-degeneracy assumption (“twist condition”) the bifurcation equation is solved in [12]-[13] by the Implicit function Theorem finding a smooth path of solutions which intersects “transversally”, for a positive measure set of frequencies, the Cantor set where also the range equation has been solved.

On the other hand, for completely resonant PDE like (1) where $a_1(x) = 0$, both small divisor difficulties and infinite dimensional bifurcation phenomena occur. It was quoted in [11] as an important problem.

The first existence results for small amplitude periodic solutions of (1) have been obtained in [18] for the nonlinearity $f(x,u) = u^3$, and in [2] for $f(x,u) = u^3 + O(u^5)$, imposing on the frequency $\omega$ the “strongly non-resonance” condition $|\omega - j| \geq \gamma/l, \forall l \neq j$. For $0 < \gamma < 1/6$, the frequencies $\omega$ satisfying such condition accumulate to $\omega = 1$ but form a set $\mathcal{W}_\gamma$ of zero measure. For such $\omega$’s the spectrum of $\partial_{tt} - \partial_{xx}$ does not accumulate to 0 and so the small divisor problem (i) is by-passed. Next, problem (ii) is solved by means of the Implicit Function Theorem, observing that the $0th$-order bifurcation equation (which is an approximation of the exact bifurcation equation) possesses, for $f(x,u) = u^3$, non-degenerate periodic solutions, see [3].

In [5]-[6], for the same set $\mathcal{W}_\gamma$ of strongly non-resonant frequencies, existence and multiplicity of periodic solutions has been proved for any nonlinearity $f(u)$. The novelty of [5]-[6] was to solve the bifurcation equation via a variational principle at fixed frequency which, jointly with min-max arguments, enables to find solutions of (1) as critical points of the Lagrangian action functional. More precisely, the bifurcation equation is, for any fixed $\omega \in \mathcal{W}_\gamma$, the Euler Lagrange equation of a “reduced Lagrangian action functional” which possesses non trivial critical points of Mountain-pass type [1], see also remark 1.4.

Unlike [2]-[5]-[6], a new feature of the results of this paper is that the set of frequencies $\omega$ for which we prove existence of $2\pi/\omega$-periodic in time solutions of (1) has positive measure, actually has full density at $\omega = 1$.

The existence of periodic solutions for a set of frequencies of positive measure has been proved in [9] in the case of periodic boundary conditions in $x$ and for the nonlinearity $f(x,u) = u^3 + \sum_{4 \leq j \leq d} a_j(x)u^j$ where the $a_j(x)$ are trigonometric cosine polynomials in $x$. The nonlinear equation $u_{tt} - u_{xx} + u^3 = 0$ possesses a continuum of small amplitude, analytic and non-degenerate periodic solutions in the form of traveling waves $u(t,x) = \delta p_0(\omega t + x)$ where $\omega^2 = 1 + \delta^2$ and $p_0$ is a non-trivial $2\pi$-periodic solution of the ordinary differential equation $p''_0 = -p_0^3$. With these properties at hand the small divisors problem (i) is solved via a Nash-Moser Implicit function Theorem adapting the estimates of Craig-Wayne [12] for non-resonant PDEs.

Recently, the existence of periodic solutions of (1) for frequencies $\omega$ in a set of positive measure has been proved in [16] using the Lindstedt series method to solve the small divisor problem. [16] applies to odd analytic nonlinearities like $f(u) = au^3 + O(u^5)$ with $a \neq 0$ (the term $u^3$ guarantees a non-degeneracy property). The reason for which $f(u)$ is odd is that the solutions are obtained as analytic sine-series in $x$, see remark 1.1.

We also quote the recent paper [17] on the standing wave problem for a perfect fluid under gravity and with infinite depth which leads to a nonlinear and completely resonant second order equation.

In this paper we prove the existence of $2\pi/\omega$-periodic solutions of the completely resonant wave equation (1) with Dirichlet boundary conditions for a set of frequencies $\omega$’s with full density at $\omega = 1$.

\footnote{Actually [18] deals with the case of periodic boundary conditions in $x$, i.e. $u(t,x + 2\pi) = u(t,x)$.}
and for a new set of nonlinearities $f(x,u)$, including for example $f(x,u) = u^2$.

We do not require that $f(x,u)$ can be extended on $(-\pi,\pi) \times \mathbb{R}$ to a function $g(x,u)$, smooth w.r.t. $u$, satisfying the oddness assumption $g(-x,-u) = -g(x,u)$ and we assume only $H^1$-regularity in the spatial variable $x$, see assumption (H).

To deal with these cases we develop a new approach for the inversion of the linearized operators which is different from the one of Craig-Wayne-Bourgain \cite{12,7,8}. Our method -presented in section 1.1- is quite elementary especially requiring that the frequencies $\omega$ satisfy the Diophantine first order Melnikov non-resonance condition of Definition 3.3 with $1 < \tau < 2$, see comments regarding the $(P)$-equation in subsection 1.2.2.

To handle the presence of an infinite dimensional bifurcation equation (and the connected problems which arise in a direct application of the Craig-Wayne method, see subsection 1.2.2) we perform a further finite dimensional Lyapunov-Schmidt reduction. Under the condition that the $0$th-order bifurcation equation possesses a non-degenerate solution we find periodic solutions of (1) for asymptotically full measure sets of frequencies.

We postpone to subsection 1.2 a detailed description of our method of proof.

1.1 Main result

Normalizing the period to $2\pi$, we look for solutions of

$$\begin{align*}
\omega^2 u_{t\ell} - u_{xx} + f(x,u) &= 0 \\
u(t,0) &= u(t,\pi) = 0
\end{align*}$$

(3)

in the Hilbert space

$$X_{\sigma,s} := \left\{ u(t,x) = \sum_{l \in \mathbb{Z}} \exp(ilt) u_l(x) \mid u_l \in H^1_0((0,\pi),\mathbb{R}), \quad u_l(x) = u_{-l}(x) \quad \forall l \in \mathbb{Z}, \right. $$

$$\left. \quad \text{and} \quad \|u\|_{\sigma,s}^2 := \sum_{l \in \mathbb{Z}} \exp(2l\sigma l)(l^2s + 1)\|u_l\|_{L^2}^2 < +\infty \right\}.$$

For $\sigma > 0,s \geq 0$, the space $X_{\sigma,s}$ is the space of all even, $2\pi$-periodic in time functions with values in $H^1_0((0,\pi),\mathbb{R})$, which have a bounded analytic extension in the complex strip $|\text{Im } t| < \sigma$ with trace function on $|\text{Im } t| = \sigma$ belonging to $H^1(T,H^1_0((0,\pi),\mathbb{R}))$.

For $2s > 1$, $X_{\sigma,s}$ is a Banach algebra with respect to multiplication of functions, namely

$$u_1,u_2 \in X_{\sigma,s} \implies u_1u_2 \in X_{\sigma,s} \quad \text{and} \quad \|u_1u_2\|_{\sigma,s} \leq C\|u_1\|_{\sigma,s}\|u_2\|_{\sigma,s}.$$ 

It is natural to look for solutions of (3) which are even in time because equation (1) is reversible.

A weak solution $u \in X_{\sigma,s}$ of (3) is a classical solution because the map $x \mapsto u_{xx}(t,x) = \omega^2 u_{t\ell}(t,x) - f(x,u(t,x))$ belongs to $H^1_0(0,\pi)$ for all $t \in T$ and hence $u(t,\cdot) \in H^1((0,\pi))$. Hence $u(t,\cdot) \in H^1((0,\pi))$.

**Remark 1.1** Let us explain why we have chosen $H^1_0((0,\pi),\mathbb{R})$ as configuration space instead of $Y := \{u(x) = \sum_{j \geq 1} u_j \sin(jx) \mid \sum_j \exp(2aj j^2)|u_j|^2 < +\infty \}$ as in \cite{12,10} which is natural if the nonlinearity $f(x,u)$ can be extended to an analytic function in both variables odd function. For non odd nonlinearities $f$ (even analytic) it is not possible to find a non trivial, smooth solution of (1) with $u(t,\cdot) \in Y$ for all $t$. For example assume that $f(x,u) = u^2$. Deriving twice the equation w.r.t. $x$ and using that $u(t,0) = 0$, $u_{xx}(t,0) = 0$, $u_{ttxx}(t,0) = 0$, we deduce $-u_{xxxx}(t,0) + 2u_x(t,0) = 0$. Now $u_{xxxx}(t,0) = 0$, $\forall t$, because all the even derivatives of any function in $Y$ vanish at $x = 0$. Hence $u_x(t,0) = 0, \forall t$. But this implies, using again the equation, that $\partial_x^4 u(t,0) = 0$, $\forall k, \forall t$. Hence, by the analyticity of $u(t,\cdot) \in Y$, $u \equiv 0$.

The space of the solutions of the linear equation $v_{t\ell} - v_{xx} = 0$ that belong to $H^1_0(T \times (0,\pi),\mathbb{R})$ and are even in time is

$$V := \left\{ v(t,x) = \sum_{l \geq 1} 2\cos(lt)u_l \sin(lx) \mid u_l \in \mathbb{R}, \quad \sum_{l \geq 1} l^2|u_l|^2 < +\infty \right\}.$$ 

\footnote{The proof is as in \cite{24} recalling that $H^1_0((0,\pi),\mathbb{R})$ is a Banach algebra with respect to multiplication of functions.}
\[ V \text{ can also be written as } \ V := \left\{ v(t, x) = \eta(t + x) - \eta(t - x) \mid \eta \in H^1(T, \mathbb{R}) \text{ with } \eta \text{ odd} \right\}. \]

We assume that the nonlinearity \( f \) satisfies

\[ (H) \quad f(x, u) = \sum_{k \geq p} a_k(x)u^k, \quad p \geq 2, \quad \text{and } a_k(x) \in H^1((0, \pi), \mathbb{R}), \quad \text{verify } \sum_{k \geq p} \|a_k\|_{H^1} \rho^k < +\infty \text{ for some } \rho > 0. \]

**Theorem 1.1** Assume that \( f(x, u) \) satisfies assumption \((H)\) and

\[ f(x, u) = \begin{cases} a_2u^2 + \sum_{k \geq 4} a_k(x)u^k & a_2 \neq 0 \\ a_3(x)u^3 + \sum_{k \geq 4} a_k(x)u^k & \langle a_3 \rangle := (1/\pi) \int_0^{\pi} a_3(x)dx \neq 0. \end{cases} \]

Then, \( s > 1/2 \) being given, there exist \( \delta_0 > 0, \sigma > 0 \) and a \( C^\infty \)-curve \([0, \delta_0) \ni \delta \to u(\delta) \in X_{\pi/2,s} \) with the following properties:

- (i) \( \|u(\delta) - \delta v\|_{\pi/2,s} = O(\delta^2) \) for some \( v \in V \cap X_{\pi,s}, \ v \neq \{0\} \);
- (ii) There exists a Cantor set \( C \subset [0, \delta_0) \) of asymptotically full measure, i.e. satisfying
  \[ \lim_{\eta \to 0^+} \frac{\operatorname{meas}(C \cap (0, \eta))}{\eta} = 1, \quad (4) \]
  such that, \( \forall \delta \in C, \ u(\delta) \) is a \( 2\pi \)-periodic, even in time, classical solution of \([3]\) with respectively
  \[ \omega = \omega(\delta) = \begin{cases} \sqrt{1 - 2\delta^2} & \text{or} \\ \sqrt{1 + 2\delta^2 \text{sign}(a_3)} \end{cases}. \]

As a consequence, \( \forall \delta \in C, \ \tilde{u}(\delta)(t, x) := u(\delta)(\omega(\delta)t, x) \) is a \( 2\pi/\omega(\delta) \)-periodic, even in time, classical solution of equation \([4]\).

By \([4]\) also the Cantor-like set \( \{\omega(\delta) \mid \delta \in C\} \) has asymptotically full measure at \( \omega = 1 \).

**Remark 1.2** The same conclusions of Theorem 1.1 hold true also for \( f(x, u) = a_4u^4 + O(u^8) \) with \( \omega^2 = 1 - 2\delta^6 \). This was recently proved in \([3]\) as a further application of the techniques of the present paper, see remark 1.5.

Theorem 1.1 is related to Theorem 1.2 stated in the next subsection.

**Remark 1.3** Under the hypotheses of Theorem 1.1 we could also get multiplicity of periodic solutions as a consequence of Theorem 1.2 and Lemmas 6.1 and 6.3. More precisely, there exist \( n_0 \in \mathbb{N} \) and a Cantor-like set \( C \) of asymptotically full measure, such that \( \forall \delta \in C, \text{ equation } \|[4]\text{ has a } 2\pi/(n\omega(\delta))\text{-periodic solution } u_n \text{ for any } n_0 \leq n \leq N(\delta) \text{ with } \lim_{\delta \to 0} N(\delta) = \infty (u_n \text{ is in particular } 2\pi/\omega(\delta)\text{-periodic}). \] This can be seen as an analogue for \([4]\) of the well known multiplicity results of Weinstein-Moser \([27],[22]\) and Fadell-Rabinowitz \([14]\) which hold in finite dimension. Multiplicity of solutions of \([4]\) was also obtained in \([6]\), but only for the zero measure set of “strongly non-resonant” frequencies \( W_\gamma \).
1.2 The Lyapunov-Schmidt reduction

Instead of looking for solutions of (3) in a shrinking neighborhood of 0 it is a convenient devise to perform the rescaling

\[ u \rightarrow \delta u, \quad \delta > 0 \]

obtaining

\[
\begin{align*}
\omega^2 u_{tt} - u_{xx} + \delta^{p-1} g_\delta(x,u) &= 0 \\
u(t,0) = u(t,\pi) &= 0
\end{align*}
\]

(5)

where

\[ g_\delta(x,u) := \frac{f(x,\delta u)}{\delta^p} = a_p(x)u^p + \delta a_{p+1}(x)u^{p+1} + \ldots. \]

To find solutions of (5) we try to implement the Lyapunov-Schmidt reduction according to the orthogonal decomposition

\[ X_{\sigma,s} = (V \cap X_{\sigma,s}) \oplus (W \cap X_{\sigma,s}) \]

(6)

(6)

The simplest situation occurs when \( \Pi_V (a_p(x)v^p) \neq 0 \).

Looking for solutions \( u = v + w \) with \( v \in V, w \in W \), we are led to solve the bifurcation equation (called the \((Q)\)-equation) and the range equation (called the \((P)\)-equation)

\[
\begin{align*}
-\frac{\omega^2 - 1}{2} \Delta v &= \delta^{p-1} \Pi_V g_\delta(x,v + w) \\
L_\omega w &= \delta^{p-1} \Pi_W g_\delta(x,v + w)
\end{align*}
\]

(7)

where

\[
\Delta v := v_{xx} + v_{tt}, \quad L_\omega := -\omega^2 \partial_{tt} + \partial_{xx}
\]

and \( \Pi_V : X_{\sigma,s} \rightarrow V, \Pi_W : X_{\sigma,s} \rightarrow W \) denote the projectors respectively on \( V \) and \( W \).

1.2.1 The 0th order bifurcation equation

In order to find non-trivial solutions of (7) we impose a suitable relation between the frequency \( \omega \) and the amplitude \( \delta \) (\( \omega \) must tend to 1 as \( \delta \to 0 \)).

The simplest situation occurs when

\[ \Pi_V (a_p(x)v^p) \neq 0 . \]

(8)

Assumption (8) amounts to require that

\[ \exists v \in V \text{ such that } \int_\Omega a_p(x)\partial^{p+1}(t,x) \, dt \, dx \neq 0, \quad \Omega := T \times (0,\pi) \]

(9)

which is verified iff \( a_p(\pi - x) \neq (-1)^p a_p(x) \), see Lemma 7.1 in the Appendix.

When condition (8) (equivalently (9)) holds, we set the “frequency-amplitude” relation

\[ \frac{\omega^2 - 1}{2} = \varepsilon, \quad |\varepsilon| := \delta^{p-1}, \]

so that system (7) becomes

\[
\begin{align*}
-\Delta v &= \Pi_V g(\delta,x,v + w) \\
L_\omega w &= \varepsilon \Pi_W g(\delta,x,v + w)
\end{align*}
\]

(10)
where
\[ g(\delta, x, u) := s^* g_0(x, u) = s^* \left( a_p(x)u^p + \delta a_{p+1}(x)u^{p+1} + \ldots \right) \quad \text{and} \quad s^* := \text{sign}(\varepsilon). \]

When \( \delta = 0 \) (and hence \( \varepsilon = 0 \)), system (10) reduces to \( w = 0 \) and the “0th-order bifurcation equation”
\[ -\Delta v = s^* \Pi_V(a_p(x)v^p) \tag{11} \]
which is the Euler-Lagrange equation of the functional \( \Phi_0 : V \to \mathbb{R} \)
\[ \Phi_0(v) = \frac{||v||^2_{H^1}}{2} - s^* \int_\Omega a_p(x)\left( \frac{v^{p+1}}{p+1} \right) dx \tag{12} \]
where \( ||v||^2_{H^1} := \int_\Omega v^2 + v^2 dx dt \).

By the Mountain-pass theorem [1], taking
\[ s^* := \begin{cases} 1 & \text{i.e. } \varepsilon > 0, \omega > 1, \text{ if } \exists v \in V \text{ such that } \int_\Omega a_p(x)v^{p+1} > 0 \\ -1 & \text{i.e. } \varepsilon < 0, \omega < 1, \text{ if } \exists v \in V \text{ such that } \int_\Omega a_p(x)v^{p+1} < 0 \end{cases} \tag{13} \]
there exists at least one nontrivial critical point of \( \Phi_0 \), i.e. a solution of (11).

We shall say that a solution \( \overline{\psi} \in V \) of equation (11) is non degenerate if 0 is the only solution of the linearized equation at \( \overline{\psi} \), i.e. \( \ker \Phi_0'(\overline{\psi}) = \{0\} \).

If condition (8) is violated (as for \( f(x, u) = a_2u^2 \)) the right hand side of equation (11) vanishes. In this case the correct 0th-order non-trivial bifurcation equation will involve higher order nonlinear terms and another “frequency-amplitude” relation is required, see subsection 1.2.3.

For the sake of clarity we shall develop all the details when the 0th order bifurcation equation is (11). In subsection 6.2 we shall describe the changes for dealing with other cases.

We can also look for \( 2\pi/n \)-time-periodic solutions of the 0th order bifurcation equation (11) (they are particular \( 2\pi \)-periodic solutions). Let
\[ V_n := \{ v \in V \mid v \text{ is } 2\pi/n \text{ periodic in time} \} \]
\[ = \{ v(t, x) = \eta(nt + nx) - \eta(nt - nx) \mid \eta \in H^1(T, \mathbb{R}) \text{ with } \eta \text{ odd} \}. \tag{14} \]

If \( v \in V_n \) then \( \Pi_V(a_p(x)v^p) \in V_n \), and the critical points of \( \Phi_0|_{V_n} \) are the solutions of equation (11) which are \( 2\pi/n \) periodic. Also \( \Phi_0|_{V_n} \) possesses a Mountain pass critical point for any \( n \), see [6].

We shall say that a solution \( \overline{\psi} \in V_n \) of (11) is non degenerate in \( V_n \) if 0 is the only solution in \( V_n \) of the linearized equation at \( \overline{\psi} \), i.e. \( \ker \Phi_0'(\overline{\psi}) = \{0\} \).

**Theorem 1.2** Let \( f \) satisfy (8) and (H). Assume that \( \overline{\psi} \in V_n \) is a non trivial solution of the 0th order bifurcation equation (11) which is non degenerate in \( V_n \).

Then the conclusions of Theorem 1.1 hold with \( \omega = \omega(\delta) = \sqrt{1 + 2s^*\delta^{p-1}} \).

**1.2.2 About the proof of Theorem 1.2**

Sections 2–4 are devoted to the proof of Theorem 1.2. Without genuine loss of generality, the proof is carried out for \( n = 1 \), and we shall explain why it works for \( n > 1 \) as well at the end of section 5.

The natural way to deal with [10] is to solve first the \((P)\)-equation (for example through a Nash-Moser procedure) and then to insert the solution \( w(\delta, v) \) in the \((Q)\)-equation. However, since here \( V \) is infinite dimensional a serious difficulty arises: if \( v \in V \cap X_{\sigma_0} \), then the solution \( w(\delta, v) \) of the range equation, obtained with any Nash-Moser iteration scheme will have a lower regularity, e.g. \( w(\delta, v) \in X_{\sigma_0/2,s} \). Therefore in solving next the bifurcation equation for \( v \in V \), the best estimate we can obtain is \( v \in V \cap X_{\sigma_0/2,s+2} \), which makes the scheme incoherent. Moreover we have to ensure that the \( 0th \)-order bifurcation equation (11) has solutions \( v \in V \) which are analytic, a necessary property to initiate an
analytic Nash-Moser scheme (in [12]-[13] these problems do not arise since the bifurcation equation is finite dimensional).

We overcome these difficulties thanks to a reduction to a finite dimensional bifurcation equation on a subspace of \( V \) of dimension \( N \) independent of \( \omega \). This reduction can be implemented, in spite of the complete resonance of equation (\ref{system}), thanks to the compactness of the operator \((-\Delta)^{-1}\).

We introduce the decomposition \( V = V_1 \oplus V_2 \) where

\[
V_1 := \left\{ v \in V \mid v(t, x) = \sum_{i=1}^{N} 2 \cos(lt) u_i \sin(lx), \ u_i \in \mathbb{R} \right\},
\]

\[
V_2 := \left\{ v \in V \mid v(t, x) = \sum_{l \geq N+1} 2 \cos(lt) u_i \sin(lx), \ u_i \in \mathbb{R} \right\}.
\]

Setting \( v := v_1 + v_2 \), with \( v_1 \in V_1, v_2 \in V_2 \), system (\ref{system}) is equivalent to

\[
\begin{align*}
-\Delta v_1 &= \Pi_{V_1} g(\delta, x, v_1 + v_2 + w) \quad \text{(Q1)} \\
-\Delta v_2 &= \Pi_{V_2} g(\delta, x, v_1 + v_2 + w) \quad \text{(Q2)} \\
L_{\omega} w &= \varepsilon \Pi_{V} g(\delta, x, v_1 + v_2 + w) \quad \text{(P)}
\end{align*}
\]

where \( \Pi_{V_i} : X_{\tau,s} \rightarrow V_i \) (\( i = 1, 2 \)), denote the orthogonal projectors on \( V_i \) (\( i = 1, 2 \)).

Our strategy to find solutions of system (\ref{system}) - and hence to prove Theorem 1.2 - is the following.

**Solution of the (Q2) equation.** We solve first the (Q2)-equation obtaining \( v_2 = v_2(\delta, v_1, w) \in V_2 \cap X_{\tau,s+2} \) when \( w \in W \cap X_{\tau,s} \), by the Contraction Mapping Theorem provided we have chosen \( N \) large enough and \( 0 < \sigma \leq \tau \) small enough, depending on the nonlinearity \( f \) but independent of \( \delta \), see section 2.

**Solution of the (P) equation.** Next we solve the (P)-equation, obtaining \( w = w(\delta, v_1) \in W \cap \mathcal{X}_{7/2,s} \) by means of a Nash-Moser type Implicit Function Theorem for \((\delta, v_1)\) belonging to some Cantor-like set \( B_{\infty} \) of parameters, see Theorem 3.1 in section 3.

Our approach for the inversion of the *linearized operators* at any step of the Nash-Moser iteration is different from the Craig-Wayne-Bourgain method. We develop \( u(t, \cdot) \in H_{\delta}^{1}((0, \pi), \mathbb{R}) \) in time-Fourier expansion only and we distinguish the “diagonal part” \( D = \text{diag}\{D_{k}\}_{k \in \mathbb{Z}} \) of the operator that we want to invert. Next, using Sturm-Liouville theory (see Lemma 4.1), we diagonalize each \( D_{k} \) in a suitable basis of \( H_{\delta}^{1}((0, \pi), \mathbb{R}) \) (close to, but different from \( \sin(jx)_{j \geq 1} \)). Assuming a “first order Melnikov non resonance condition” (Definition 3.3) we prove that its eigenvalues are polynomially bounded away from 0 and so we invert \( D \) with sufficiently good estimates (Corollary 1.4). The presence of the “off-diagonal” Toeplitz operators requires to analyze the “small divisors”: for our method it is sufficient to prove that the product of two “small divisors” is larger than a constant if the corresponding “singular sites” are close enough, see Lemma 4.5. This holds true if the Diophantine exponent \( \tau \in (1, 2) \) by the lower bound of Lemma 4.3.

Moreover, for \( \tau \in (1, 2) \) the non-resonance Diophantine conditions are particularly simple, see Definition 3.3 and the Cantor set \( B_{\infty} \) in Theorem 3.1. This restriction for the values of the exponent \( \tau \) simplifies also the proof of Lemma 4.9 where the loss of derivatives due to the small divisors is compensated by the regularizing property of the map \( v_2 \).

**Solution of the (Q1)-equation.** Finally, in section 3 we consider the finite dimensional (Q1)-equation.

We could define a smooth functional \( \Psi : [0, \delta_0] \times V_1 \rightarrow \mathbb{R} \) such that any critical point \( v_1 \in V_1 \) of \( \Psi(\delta, \cdot) \) with \( (\delta, v_1) \in B_{\infty} \) (the Cantor-like set of parameters for which the (P) equation is solved exactly) gives rise to an exact solution of (\ref{system}), see [5]. Moreover it would be possible to prove the existence of a critical point \( v_1(\delta) \) of \( \Psi(\delta, \cdot) \), \( \forall \delta > 0 \) small enough, using the Mountain pass theorem [11].

However, since the section \( E_{\delta} := \{ v_1 \mid (\delta, v_1) \in B_{\infty} \} \) has “gaps” (except for \( \delta \) in a zero measure set, see Remark 1.4), the difficulty is to prove that \( (\delta, v_1(\delta)) \in B_{\infty} \) for a large set of \( \delta \)’s. Although \( B_{\infty} \) is in some sense a “large” set, this property is not obvious. In this paper, we prove that it holds at least if the path \( (\delta \rightarrow v_1(\delta)) \) is \( C^{1} \) (Proposition 3.2) and so intersects “transversally” the Cantor set \( B_{\infty} \).

This is why we require in Theorem 1.2 non-degenerate solutions of the \( 0 \)-th order bifurcation equation [11]. This condition enables to use the Implicit function theorem, yielding a smooth path \( (\delta \rightarrow v_1(\delta)) \) of solutions of the (Q1)-equation.
Remark 1.4 The section $E_3$ has “no gaps” iff the frequency $\omega(\delta) = \sqrt{1 + 2s^* \delta^{p-1}}$ belongs to the uncountable zero-measure set $W_\gamma := \{ |\omega - j| \geq \gamma/l, \forall j \neq l, l \geq 0, j \geq 1 \}$ of $[3]$. This explains why in $[2]-[6]$ we had been able to prove the existence of periodic solutions for ANY nonlinearity $f$, solving the bifurcation equation with variational methods.

We lay the stress on the fact that the parts on the (Q2) and (P)-equations do not use the non-degeneracy condition. We hope that we will be able to improve our results relaxing the non-degeneracy condition in a subsequent work, using the variational formulation of the (Q1) equation and results on properties of critical sets for parameter depending functionals.

1.2.3 About the proof of Theorem 1.1

To deduce Theorem 1.1 when $f(x,u) = a_3(x)u^3 + O(u^4)$ and $\langle a_3 \rangle \neq 0$ we just have to prove that the 0th order bifurcation equation

$$-\Delta v = s^* \Pi_V(a_3(x)v^3)$$

possesses, at least for $n$ large, a non-degenerate solution in $V_n$. Choosing $s^* \in \{-1, 1\}$ so that $s^* \langle a_3 \rangle > 0$ this is proved in Lemma 6.1.

In the case $f(x,u) = a_2u^2 + O(u^4)$, condition $[\delta]$ is violated because $\Pi_Vv^2 \equiv 0$, and we have to use a development in $\delta$ of higher order, as in $[5]$. Imposing in $[7]$ the frequency-amplitude relation

$$\frac{\omega^2 - 1}{2} = -\delta^2$$

the correct 0th-order bifurcation equation turns out to be (see subsection 6.2)

$$-\Delta v + 2a_2^2 \Pi_V(vL^{-1}(v^2)) = 0$$

(18)

where $L^{-1} : W \rightarrow W$ is the inverse operator of $-\partial_{tt} + \partial_{xx}$. (18) is the Euler-Lagrange equation of

$$\Phi_0(v) = \frac{\|v\|_{H_1}^2}{2} + \frac{a_2^2}{2} \int_\Omega v^2L^{-1}v^2$$

(19)

which again possesses Mountain pass critical points because $\int_\Omega v^2L^{-1}v^2 < 0$, $\forall v \in V$, see $[5]$.

The existence of a non-degenerate critical point of $(\Phi_0)|_{V_n}$ for $n$ large enough is proved in Lemma 6.3. This implies, as in Theorem 1.2, the conclusions of Theorem 1.1.

Remark 1.5 Also when $f(x,u) = a_3u^3 + O(u^4)$ condition $[\delta]$ is violated because $\Pi_Vv^4 \equiv 0$. Imposing the frequency-amplitude relation $\omega^2 - 1 = -25\delta^2$, the correct 0th-order bifurcation equation turns out to be

$$-\Delta v + 4a_3^2 \Pi_V(v^3L^{-1}(v^4)) = 0.$$  

(20)

The existence of a solution of $[20]$ which is non-degenerate in $V_n$ for $n$ large enough is proved in $[3]$. This implies the conclusions of Theorem 1.1.

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2 Solution of the (Q2)-equation

The main assumption of Theorem 1.2 says that at least one of the critical points of $\Phi_0$ defined in $[12]$ or of the restriction of $\Phi_0$ to some $V_n$, called $\mathcal{V}$, is non-degenerate. For definiteness, we shall assume that $\mathcal{V}$ is non-degenerate in the whole space $V$.

\footnote{Note that $\langle a_3 \rangle \neq 0$ implies condition $[\delta]$ because $a_3(\pi - x) \neq -a_3(x)$ and so $\Pi_V(a_3(x)v^3) \neq 0$.}
By the regularizing property
\[(−Δ)^{−1} : V ∩ H^k(Ω) → V ∩ H^{k+2}(Ω), \quad ∀ k ≥ 0,
\]
and a direct bootstrap argument, \( \pi ∈ H^k(Ω) \) \( ∀ k ≥ 0 \). Therefore\( \pi ∈ V ∩ C^∞(Ω) \).

In the sequel of the paper \( s > 1/2 \) is fixed once for all. We also fix some \( R > 0 \) such that
\[
||\pi||_{0,s} < R.
\] (21)

By the analyticity assumption \((H)\) on the nonlinearity \( f \) and the Banach algebra property of \( X_{σ,s} \), there is a constant \( K_0 > 0 \) such that
\[
\left\| g(δ, x, u) \right\|_{σ,s} = \left\| \sum_{k ≥ p} a_k(x)δ^{k−p}u^k \right\|_{σ,s} \leq \sum_{k ≥ p} \left\| a_k \right\|_{H^1}δ^{k−p}K_0^{k−1}\left\| u \right\|_{σ,s}^k \leq C\left\| u \right\|_{σ,s}^p \sum_{k ≥ p} \left\| a_k \right\|_{H^1}\left( δK_0 \left\| u \right\|_{σ,s} \right)^{k−p} \leq C'\left\| u \right\|_{σ,s}^p
\] (22)
in the open domain \( U_δ := \{ u ∈ X_{σ,s} \mid δK_0 \left\| u \right\|_{σ,s} < ρ \} \) because the power series \( \sum_{k ≥ p} \left\| a_k \right\|_{H^1}\rho^{k−p} < +∞ \) by \((H)\). The Nemitsky operator
\[X_{σ,s} \ni u → g(δ, x, u) ∈ X_{σ,s} \] is in \( C^∞(U_δ, X_{σ,s}) \).

We specify that all the norms \( \left\| \right\|_{σ,s} \) are equivalent on \( V_1 \). In the sequel
\[B(ρ, V_1) := \{ v_1 ∈ V_1 \mid \left\| v_1 \right\|_{0,s} ≤ ρ \} .\]

The fact that \( \pi ∈ V ∩ X_{σ,s} \) for some \( σ > 0 \) is a consequence of the following Lemma.

**Lemma 2.1 (Solution of the (Q2)-equation)** There exists \( N ∈ \mathbb{N}_+ \), \( σ := \ln 2/N > 0 \), \( δ_0 > 0 \) such that:

a) \( ∀ 0 ≤ σ ≤ σ, \forall ||v_1||_{0,s} ≤ 2R, \forall ||w||_{σ,s} ≤ 1, \forall δ ∈ (0, δ_0), \) there exists a unique \( v_2 = v_2(δ, v_1, w) ∈ V_2 ∩ X_{σ,s} \) with \( ||v_2(δ, v_1, w)||_{σ,s} ≤ 1 \) which solves the \((Q2)\)-equation.

b) \( v_2(0, Π_1, \pi, 0) = Π_2\pi. \)

c) \( v_2(δ, v_1, w) ∈ X_{σ,s+2} \), the function \( v_2(δ, v_1, w) ∈ C^∞ [0, δ_0) \times (B(2R; V_1) × B(1; W ∩ X_{σ,s}) ; V_2 ∩ X_{σ,s+2}) \)
and \( D^s v_2 \) is bounded on \( [0, δ_0) \times B(2R; V_1) × B(1; W ∩ X_{σ,s}) \) for any \( k ∈ \mathbb{N} \).

d) If in addition \( ||w||_{σ,s'} = +∞ \) for some \( s' ≥ s \), then (provided \( δ_0 \) has been chosen small enough) \( \left\| v_2(δ, v_1, w) \right\|_{σ,s'+2} ≤ K(s', ||w||_{σ,s'}) \).

**Proof.** Fixed points of the nonlinear operator \( N(δ, v_1, w, \cdot) : V_2 → V_2 \) defined by
\[N(δ, v_1, w, v_2) := (−Δ)^{−1}\Pi_2 g(δ, x, v_1 + w + v_2) \]
ar solutions of equation \((Q2)\). For \( w ∈ W ∩ X_{σ,s}, v_2 ∈ V_2 ∩ X_{σ,s} \) we have \( N(δ, v_1, w, v_2) ∈ V_2 ∩ X_{σ,s+2} \)
since \( g(δ, x, v_1 + w + v_2) ∈ X_{σ,s} \) and because of the regularizing property of the operator \((−Δ)^{−1}\Pi_2 : X_{σ,s} → V_2 ∩ X_{σ,s+2} \).

**Proof of a).** Let \( B := \{ v_2 ∈ V_2 ∩ X_{σ,s} \mid ||v_2||_{σ,s} ≤ 1 \} \). We claim that there exists \( N ∈ \mathbb{N}, σ > 0 \) and \( δ_0 > 0 \), such that \( ∀ 0 ≤ σ ≤ σ, ||v_1||_{0,s} ≤ 2R, ||w||_{σ,s} ≤ 1, δ ∈ (0, δ_0) \) the operator \( v_2 → N(δ, v_1, w, v_2) \) is a contraction in \( B \), more precisely
\[
(i) \quad ||v_2||_{σ,s} ≤ 1 \Rightarrow ||N(δ, v_1, w, v_2)||_{σ,s} ≤ 1;
(ii) \quad v_2, \tilde{v}_2 ∈ B ⇒ ||N(δ, v_1, w, v_2) − N(δ, v_1, w, \tilde{v}_2)||_{σ,s} ≤ (1/2)||v_2 − \tilde{v}_2||_{σ,s}.
\]

5 Even if \( a_k(x) ∈ H^1(0, π; \mathbb{R}) \) only, because the projection \( Π_2 \) has a regularizing effect in the variable \( x \).

6 If \( f ∈ C^∞(A, Y) \) means, if \( A \) is not open, that there is an open neighborhood \( U \) of \( A \) and an extension \( \tilde{f} ∈ C^∞(U, Y) \) of \( f \).
Let us prove \((i)\). \(\forall u \in X_{\sigma, s}, \|(-\Delta)^{-1}\Pi v_2 u\|_{\sigma, s} \leq (C/(N + 1)^2)\|u\|_{\sigma, s}\) and so, \(\forall \|w\|_{\sigma, s} \leq 1, \|v_1\|_{0, s} \leq 2R\), \(\delta \in [0, \delta_0]\), using \([22]\).

\[
\|N(\delta, v_1, w, v_2)\|_{\sigma, s} \leq \frac{C}{(N + 1)^2} \left\|g(\delta, x, v_1 + v_2 + w)\right\|_{\sigma, s} \leq \frac{C'}{(N + 1)^2} \left(\|v_1\|_{\sigma, s}^p s + \|v_2\|_{\sigma, s}^p s + \|w\|_{\sigma, s}^p\right)
\]

\[
\leq \frac{C'}{(N + 1)^2} \left(\exp(\sigma p N)\|v_1\|_{\sigma, s}^p + \|v_2\|_{\sigma, s}^p + 1\right) \leq \frac{C'}{(N + 1)^2} \left((4R)^p + \|v_2\|_{\sigma, s}^p + 1\right)
\]

for \(\exp(\sigma N) \leq 2\), where we have used that \(\|v_1\|_{\sigma, s} \leq \exp(\sigma N)\|v_1\|_{0, s} \leq 4R\).

For \(N\) large enough (depending on \(R\)) we get

\[
\|v_2\|_{\sigma, s} \leq 1 \Rightarrow \|N(\delta, v_1, w, v_2)\|_{\sigma, s} \leq \frac{C'}{(N + 1)^2} \left((4R)^p + 1 + 1\right) \leq 1
\]

and \((ii)\) follows, taking \(\bar{\sigma} := \ln 2/N\). Property \((ii)\) can be proved similarly and the existence of a unique solution \(v_1(\delta, v_1, w) \in B\) follows by the Contraction Mapping Theorem.

**Proof of \(b\).** We may assume that \(N\) has been chosen so large that \(\Pi v_2 \|_{0, s} \leq 1/2\). Since \(\varpi\) solves equation \([13]\), \(\Pi v_2 \varpi\) solves the \((Q2)\)-equation associated with \((\delta, v_1, w) = (0, \Pi v_2 \varpi, 0)\). Since \(\Pi v_2 \varpi = \mathcal{N}(0, \Pi v_2 \varpi, 0, \Pi v_2 \varpi)\) \(\in B\), we deduce \(\Pi v_2 \varpi = v_2(0, \Pi v_2 \varpi, 0)\).

**Proof of \(c\).** As a consequence of \((ii)\) the linear operator \(I - Dv_2 \mathcal{N}\) is invertible at the fixed point of \(\mathcal{N}(\delta, v_1, w, \cdot)\). Since the map \((\delta, v_1, w, v_2) \mapsto \mathcal{N}(\delta, v_1, w, v_2)\) is \(C^{\infty}\), by the Implicit function Theorem \(v_2 : \mathcal{N}(\delta, v_1, w, \cdot)\) \(\|\varpi\|_{\sigma, s} \leq 2R, \|w\|_{\sigma, s} \leq 1\) \(\Rightarrow V_2 \cap X_{\sigma, s}\) is a \(C^{\infty}\) map. Hence, since \((-\Delta)^{-1}\Pi v_2\)

\[
v_2(\delta, v_1, w) = (-\Delta)^{-1}\Pi v_2 \left(g(\delta, x, v_1 + v_2(\delta, v_1, w))\right),
\]

\[
(23)
\]

by the regularity of the Nemitsky operator induced by \(g, v_2(\cdot, \cdot) \in C^\infty((0, \delta_0) \times B(2R; V_1) \times B(1; W \cap X_{\sigma, s}), V_2 \cap X_{\sigma, s+2})\). The estimates for the derivatives can be obtained similarly.

**Proof of \(d\).** Let us firstly prove the following : if \(\|\varpi\|_{\sigma, s}\) is small enough then

\[
u \in X_{\sigma, r} \Rightarrow g(\delta, x, u) \in X_{\sigma, r}, \quad \forall r \geq s.
\]

We first observe that, since \(r \geq s > 1/2\), for \(u, v \in H^r(\mathbb{R}/2\pi \mathbb{Z})\), we have \(\|uv\|_{H^r} \leq C_r(\|u\|_{\infty}\|v\|_{H^r} + \|v\|_{\infty}\|u\|_{H^r})\). This is a consequence of the Gagliardo-Nirenberg inequalities. Hence there is a positive constant \(K_r\) such that

\[
\|u^l\|_{H^r} \leq K_r -1 \|u|^{l-1}\|u\|_{H^r} \leq K_r -1 \|u\|_{H^r}^{-1} \|u\|_{H^r}^1, \quad \forall u \in H^r(\mathbb{R}/2\pi \mathbb{Z}), \forall l \geq 1 .
\]

Considering the extension of a function \(u \in X_{\sigma, r}\) to the complex strip of width \(\sigma\) and using that \(H^r(0, \pi)\) is a Banach algebra, we can derive that \(\forall r \geq s, \quad \|u^l\|_{\sigma, r} \leq K_l -1 \|u\|_{\sigma, r} \leq K_l -1 \|u\|_{\sigma, r}^{-1} \|u\|_{\sigma, r}^1\).

\[
\|g(\delta, x, u)\|_{\sigma, r} = \|\sum_{k \geq p} a_k(x)\delta^{k-p} u^k\|_{\sigma, r} \leq \|u^p\|_{\sigma, r} \sum_{k \geq p} \|a_k\|_{H^1} \|(\delta^{k-p})\|_{\sigma, r}
\]

\[
\leq \|u^p\|_{\sigma, r} \|a_p\|_{H^1} + \sum_{k \geq p} \|a_k\|_{H^1} (1) \|\delta\|_{\sigma, s}^1 \|u\|_{\sigma, r}^{-1} \|u\|_{\sigma, r}^1 < +\infty
\]

for \(\|\varpi\|_{\sigma, s}\) small enough.

Now, assume that \(\|\varpi\|_{\sigma, s'} < +\infty\) for some \(s' \geq s\). Since \(v_2(\delta, v_1, w) \in X_{\sigma, s}\) solves equation \([23]\), by a direct bootstrap argument using the regularizing properties of \((-\Delta)^{-1}\Pi v_2 : X_{\sigma, r} \rightarrow V_2 \cap X_{\sigma, s+2}\) and that \(\|v_1\|_{\sigma, r} < +\infty, \forall r \geq s\), we derive that \(v_2(\delta, v_1, w) \in X_{\sigma, s'+2}\) and \(\|v_2(\delta, v_1, w)\|_{\sigma, s'+2} \leq K(s', \|\varpi\|_{\sigma, s'})\).

**Remark 2.1** Lemma \([2, 7]\) implies, in particular, that the solution \(\varpi\) of the 0th-order bifurcation equation \([17]\) is not only in \(V \cap C^\infty(\Omega)\) but actually belongs to \(V \cap X_{\varpi, s+2}\) and therefore is analytic in \(t\) and hence in \(x\).

We stress that we shall consider as fixed the constants \(N\) and \(\varpi\) obtained in Lemma \([2, 1]\) which depend only on the nonlinearity \(f\) and on \(\bar{\varpi}\). On the contrary, we shall allow \(\delta_0\) to decrease in the next sections.
3 Solution of the \((P)\)-equation

By the previous section we are reduced to solve the \((P)\)-equation with \(v_2 = v_2(\delta, v_1, w)\), namely

\[
L_n w = \varepsilon \Pi_W \Gamma(\delta, v_1, w)
\]

where

\[
\Gamma(\delta, v_1, w)(t, x) := g\left(\delta, x, v_1(t, x) + w(t, x) + v_2(\delta, v_1, w)(t, x)\right).
\]

The solution \(w = w(\delta, v_1)\) of the \((P)\)-equation \((25)\) will be obtained by means of a Nash-Moser Implicit Function Theorem for \((\delta, v_1)\) belonging to a Cantor-like set of parameters.

We consider the orthogonal splitting \(W = W^{(n)} \oplus W^{(n)\perp}\) where

\[
W^{(n)} = \left\{ w \in W \mid w = \sum_{|l| \leq L_n} \exp(ilt) w_l(x) \right\}, \quad W^{(n)\perp} = \left\{ w \in W \mid w = \sum_{|l| > L_n} \exp(ilt) w_l(x) \right\}
\]

and \(L_n\) are integer numbers (we will choose \(L_n = L_0 2^n\) with \(L_0 \in \mathbb{N}\) large enough). We denote by

\[
P_n : W \to W^{(n)} \quad \text{and} \quad P_n^\perp : W \to W^{(n)\perp}
\]

the orthogonal projectors onto \(W^{(n)}\) and \(W^{(n)\perp}\).

The convergence of the recursive scheme is based on properties \((P1)-(P2)-(P3)\) below.

- (P1) (Regularity) \(\Gamma(\cdot, \cdot, \cdot) \in C^\infty\left([0, \delta_0) \times B(2R; V_1) \times B(1; W \cap X_{\sigma, s}, X_{\sigma, s})\right)\). Moreover, \(D^k \Gamma\) is bounded on \([0, \delta_0) \times B(2R; V_1) \times B(1; W \cap X_{\sigma, s})\) for any \(k \in \mathbb{N}\).

(P1) is a consequence of the \(C^\infty\)-regularity of the Nemitsky operator induced by \(g(\delta, x, u)\) on \(X_{\sigma, s}\) and of the \(C^\infty\)-regularity of the map \(v_2(\cdot, \cdot, \cdot)\) proved in Lemma \(\text{[2.1]}\).

- (P2) (Smoothing estimate) \(\forall w \in W^{(n)\perp} \cap X_{\sigma, s}\) and \(\forall 0 \leq \sigma' \leq \sigma\), \(\|w\|_{\sigma', s} \leq \exp\left(-L_n(w - \sigma\cdot)\right) \|w\|_{\sigma, s}\).

The standard property \((P2)\) follows from

\[
\|w\|_{\sigma', s}^2 = \sum_{|l| > L_n} \exp(2\sigma'|l|)(l^2 s + 1)\|w_l\|_{H^1}^2 = \sum_{|l| > L_n} \exp(-2(\sigma - \sigma')|l|)\exp(2\sigma|l|)(l^2 s + 1)\|w_l\|_{H^1}^2 \leq \exp(-2(\sigma - \sigma')L_n)\|w\|_{\sigma, s}^2
\]

The next property \((P3)\) is an invertibility property of the linearized operator \(L_n(\delta, v_1, w) : D(L_n) \subset W^{(n)} \to W^{(n)}\) defined by

\[
L_n(\delta, v_1, w)[h] := L_n h - \varepsilon P_n \Pi_W D w \Gamma(\delta, v_1, w)[h].
\]

Throughout the proof, \(w\) will be the approximate solution obtained at a given step of the Nash-Moser iteration.

The invertibility of \(L_n(\delta, v_1, w)\) is obtained excising the set of parameters \((\delta, v_1)\) for which 0 is an eigenvalue of \(L_n(\delta, v_1, w)\). Moreover, in order to have bounds for the norm of the inverse operator \(L_n^{-1}(\delta, v_1, w)\) which are sufficiently good for the recursive scheme, we also excise the parameters \((\delta, v_1)\) for which the eigenvalues of \(L_n(\delta, v_1, w)\) are too small.

We prefix some definitions.

**Definition 3.1 (Mean value)** For \(\Omega := T \times (0, \pi)\) we define

\[
M(\delta, v_1, w) := \frac{1}{|\Omega|} \int_{\Omega} \partial_u g\left(\delta, x, v_1(t, x) + w(t, x) + v_2(\delta, v_1, w)(t, x)\right) dxdt.
\]

Note that \(M(\cdot, \cdot, \cdot) : [0, \delta_0) \times B(2R; V_1) \times B(1; W \cap X_{\sigma, s}) \to \mathbb{R}\) is a \(C^\infty\)-function.
The recurrence, the set of parameters $A(25)$ for $(\sigma,s)$ sufficiently large.

**Definition 3.3 (First order Melnikov non-resonance condition)** Let $\Delta_{n}(v_1,w)$ be the set of parameters $\Delta_{n}(v_1,w)$ for which we have the solution $w_{\delta,v_1}(\delta,v_1)$. We define for some large integer $L_0$ small enough (note that $\Delta_{n}(v_1,w)$ is defined by a finite set of inequalities).

**Remark 3.1** The intersections of the sets $\Delta_{n}(v_1,w)$ over all possible $(v_1,w)$ in a neighborhood of $0$ and over all $n$ contains, for $|\epsilon|\gamma^{-1}$ small, the zero measure, uncountable set $W$ defined as $\Delta_{n}(\delta,v_1)$ for some $0 < \gamma < 1/6$ introduced in [2]. See Remark 1.4 for consequences on the existence of periodic solutions.

We claim that:

- **(P3) (Invertibility of $L_n$)** There exist positive constants $\mu$, $\delta_0$ such that, if $[w]|_{\gamma,s} \leq \mu$, $\|v_1\|_{0,s} \leq 2R$ and $\delta \in \Delta_{n}(v_1,w) \cap [0,\delta_0)$ for some $0 < \gamma < 1$, $1 < \tau < 2$, then $L_n(\delta,v_1)$ is invertible and the inverse operator $L_n^{-1}(\delta,v_1): W(\gamma) \to W(\gamma)$ satisfies

$$\left\| L_n^{-1}(\delta,v_1)[h] \right\|_{\gamma,s} \leq \frac{C}{\gamma} \left( L_n \right)^{-1} \|h\|_{\gamma,s}$$

for some positive constant $C > 0$.

Property (P3) is the real core of the convergence proof and where the analysis of the small divisors enters into play. Property (P3) is proved in section [4].

### 3.1 The Nash-Moser scheme

We are going to define recursively a sequence $\{w_n\}_{n \geq 0}$ with $w_n = w_n(\delta,v_1) \in W(\gamma)$, defined on smaller and smaller sets of “non-resonant” parameters $A_n \subseteq \cdots \subseteq A_1 \subseteq A_0 := \{(\delta,v_1) \mid \delta \in [0,\delta_0), \|v_1\|_{0,s} \leq 2R\}$. The sequence $(w_n(\delta,v_1))$ will converge to a solution $w(\delta,v_1)$ of the $(P)$-equation for $(\delta,v_1) \in A_\infty := \cap_{n \geq 1} A_n$. The main goal of the construction is to show that, at the end of the recurrence, the set of parameters $A_\infty := \cap_{n \geq 1} A_n$ for which we have the solution $w(\delta,v_1)$ remains sufficiently large.

We define inductively the sequence $\{w_n\}_{n \geq 0}$. Define the “loss of analyticity” $\gamma_n$ by

$$\gamma_n := \frac{\gamma_0}{n^2 + 1}, \quad \sigma_0 = \sigma, \quad \sigma_{n+1} = \sigma_n - \gamma_n, \quad \forall \ n \geq 0,$$

where we choose $\gamma_0 > 0$ small such that the “total loss of analyticity”

$$\sum_{n \geq 0} \gamma_n = \sum_{n \geq 0} \frac{\gamma_0}{(n^2 + 1)} \leq \frac{\sigma}{2}, \quad \text{i.e.} \quad \sigma_n \geq \frac{\sigma}{2} > 0, \ \forall n.$$

We also assume

$$L_n := L_0 2^n, \quad \forall \ n \geq 0,$$

for some large integer $L_0$ specified in the next Proposition.
Proposition 3.1 (Induction) Let $A_0 := \{ (\delta, v_1) \mid \delta \in [0, \delta_0), \|v_1\|_{0,s} \leq 2R \}$. Assume $L_0 := L_0(\gamma, \tau) > 0$, $\varepsilon_0 := \varepsilon_0(\gamma, \tau) > 0$, such that for $\delta_0^{-1}\gamma^{-1} < \varepsilon_0$, there exists a sequence $(w_n)_{n \geq 0}$, $w_n = w_n(\delta, v_1) \in W^{(n)}$, of solutions of the equation

$$(P_n) \quad L_\omega w_n - \varepsilon P_{n+1} \Pi W \Gamma(\delta, v_1, w_n) = 0,$$ 

defined inductively for $(\delta, v_1) \in A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_1 \subseteq A_0$ where

$$A_n := \{ (\delta, v_1) \in A_{n-1} \mid \delta \in \Delta^\gamma_{n-1}(v_1, w_{n-1}) \} \subseteq A_{n-1},$$

$$w_n(\delta, v_1) = \sum_{i=0}^n w_n(\delta, v_1),$$

and $h_i = h_i(\delta, v_1) \in W(i^\circ)$ satisfy $\|h_0\|_{0,s} \leq \|\varepsilon K_0, \|h_i\|_{s,i} \leq \|\varepsilon\|_{\gamma}^{-1} \exp(-\chi^i)$ $\forall 1 \leq i \leq n$ for some $1 < \chi < 2$ and some constant $K_0 > 0$.

We define

$$A_\infty := \cap_{n \geq 0} A_n.$$ 

Remark 3.2 For a given $(\delta, v_1)$, the sequence $(w_n)$ may be finite because the iterative process stops after $w_{k-1}$ if $\delta \notin \Delta^\gamma_k(v_1, w_{k-1})$, i.e., if $(\delta, v_1) \notin A_k$. However, from this possibly finite sequences, we shall define a $C^\infty$ map $\tilde{w}(\delta, v_1)$ on the whole set $A_0$, and Cantor-like set $B_\infty$, such that $B_\infty \subseteq A_\infty$ and $\tilde{w}(\delta, v_1) \in B_\infty$ is an exact solution of the $(P)$-equation. It will be justified in Proposition 3.3 that $B_\infty$ is a “large” set. As a consequence also $A_\infty$ is “large”.

Proof. The proof proceeds by induction.

First step: initialization. Let $L_0$ be given. If $|\omega - 1|L_0 \leq 1/2$ then $L_{\omega |W^{(0)}}$ is invertible and $\|L^{-1}_\omega \|_{0,s} \leq 2\|h\|_{0,s}, \forall h \in W^{(0)}$. Indeed the eigenvalues of $L_{\omega |W^{(0)}}$ are $-\omega j^2 + j^2, \forall 0 \leq l \leq L_0, j \geq 1$. Let $l \neq l$, and

$$| - \omega j^2 + j^2 | = | - \omega l + j| (\omega l + j) \geq (|j - l| - |\omega - 1| L_0) (\omega l + j) \geq (1 - \frac{1}{2}).$$

By the Implicit function theorem, using Property (P1), there exist $K_0 > 0$, $\varepsilon_1 := \varepsilon_1(\gamma, L_0) > 0$ such that if $|\varepsilon| \gamma^{-1} < \varepsilon_1$ and $\forall v_1 \in B(2R, V_1)$, equation $(P_0)$ has a unique solution $w_0(\delta, v_1)$ satisfying

$$\|w_0(\delta, v_1)\|_{0,s} \leq K_0|\varepsilon|.$$ 

Moreover, for $\delta_0^{-1}\gamma^{-1} < \varepsilon_1$, the map $h_1^l(\delta, v_1) \mapsto w_0(\delta, v_1)$ is in $C^\infty(A_0, W^{(0)})$ and $\|D^k w_0(\delta, v_1)\|_{0,s} \leq C(k)$.

Second step: iteration. Fix some $\chi \in (1, 2)$. Let $\varepsilon_2 := \varepsilon_2(L_0, \gamma, \tau) \in (0, \varepsilon_1(\gamma, L_0))$ be small enough such that

$$\varepsilon_2 \max(1, eK_0\gamma) \sum_{i \geq 0} \exp(-\chi^i) \left(\frac{1 + \frac{1}{\gamma}}{\gamma_0} \right)^{2(\tau+1)} < \mu$$

where $\mu$ is defined in property (P3) and $\beta := (2 - \tau)/\tau$.

Suppose we have already defined a solution $w_n = w_n(\delta, v_1) \in W^{(n)}$ of equation $(P_n)$ satisfying the properties stated in the Proposition. We want to define

$$w_{n+1} = w_{n+1}(\delta, v_1) := w_n(\delta, v_1) + h_{n+1}(\delta, v_1),$$

as an exact solution of the equation

$$(P_{n+1}) \quad L_\omega w_{n+1} - \varepsilon P_{n+1} \Pi W \Gamma(\delta, v_1, w_{n+1}) = 0.$$ 

In order to find a solution $w_{n+1} = w_n + h_{n+1}$ of equation $(P_{n+1})$ we write, for $h \in W^{(n+1)},$

$$L_\omega (w_n + h) - \varepsilon P_{n+1} \Pi W \Gamma(\delta, v_1, w_n + h) = L_\omega w_n - \varepsilon P_{n+1} \Pi W \Gamma(\delta, v_1, w_n)$$

$$+ L_\omega h - \varepsilon P_{n+1} \Pi W D_\omega \Gamma(\delta, v_1, w_n)[h] + R(h)$$

$$= r_n + L_{n+1}(\delta, v_1, w_n)[h] + R(h)$$

(32)
where, since \( w_n \) solves equation \((P_n)\),
\[
\begin{aligned}
\begin{cases}
  r_n := L_\omega w_n - \varepsilon P_{n+1} \Pi W \Gamma(\delta, v_1, w_n) = -\varepsilon P_n^0 P_{n+1} \Pi W \Gamma(\delta, v_1, w_n) \in W^{(n+1)} \\
  R(h) := -\varepsilon P_{n+1} \Pi W \left( \Gamma(\delta, v_1, w_n + h) - \Gamma(\delta, v_1, w_n) - D_v \Gamma(\delta, v_1, w_n) [h] \right).
\end{cases}
\end{aligned}
\]

The term \( r_n \) is “super-exponentially” small because, using properties \((P2)\) and \((P1)\),
\[
\| r_n \|_{\sigma_{n+1, s}} \leq \varepsilon |\varepsilon| C \exp \left( -L_n \gamma_n \right) \left\| P_{n+1} \Pi W \Gamma(\delta, v_1, w_n) \right\|_{\sigma_{n, s}} \leq \varepsilon |\varepsilon| C' \exp \left( -L_n \gamma_n \right) \left\| \Gamma(\delta, v_1, w_n) \right\|_{\sigma_{n, s}},
\]
(33)

being \( \|w_n\|_{\sigma_{n, s}} \) bounded independently of \( n \) since, by the induction hypothesis,
\[
\|w_n\|_{\sigma_{n, s}} \leq \sum_{i=0}^{n} \|h_i\|_{\sigma_{i, s}} \leq \max(1, eK_0 \gamma) \varepsilon |\gamma|^{-1} \sum_{i=0}^{\infty} \exp(\varepsilon i),
\]
(34)

with \( h_0 := w_0 \). The term \( R(h) \) is “quadratic” in \( h \), since, by property \((P1)\) and \((34)\),
\[
\left\| R(h) \right\|_{\sigma_{n+1, s}} \leq \left\| C |\varepsilon| \|h\|^2 \right\|_{\sigma_{n+1, s}} \leq C |\varepsilon| \left( \|\|_{\sigma_{n+1, s}} + \|h\|_{\sigma_{n+1, s}} \right) \|h - h'\|_{\sigma_{n+1, s}}
\]
\]
(35)

for all \( h, h' \in W^{(n+1)} \) with \( \|h\|_{\sigma_{n+1, s}}, \|h'\|_{\sigma_{n+1, s}} \) small enough.

Since \( w_{n+1} = \sum_{i=0}^{n} h_i \) with \( \|h_i\|_{\sigma_{i, s}} \leq \max(1, eK_0 \gamma) \varepsilon |\gamma|^{-1} \exp(-\varepsilon i) \), and \( \sigma_i - \sigma_{i+1} \geq \gamma_i := \gamma_0/(1+i^2) \), \( \forall i = 0, \ldots, n \),
\[
[w_{n+1}]_{\sigma_{n+1, s}} \leq \sum_{i=0}^{n} \|h_i\|_{\sigma_{i, s}} \leq \max(1, eK_0 \gamma) \varepsilon |\gamma|^{-1} \sum_{i=0}^{\infty} \exp(-\varepsilon i) \left( \frac{1+i^2}{\gamma_0} \right)^{\frac{2(\gamma-1)}{\gamma}} < \mu
\]

for \( |\varepsilon|^{-1} \leq \varepsilon_2 \) and by \((31)\).

Hence, by property \((P3)\), the linear operator \( L_{n+1}(\delta, v_1, w_n) : D(L_{n+1}) \subset W^{(n+1)} \rightarrow W^{(n+1)} \) is invertible for \( (\delta, v_1) \) restricted to the set of parameters
\[
A_{n+1} := \left\{ (\delta, v_1) \in A_n \mid \delta \in \Delta_{n+1}^\tau(v_1, w_n) \right\} \subseteq A_n,
\]
(36)

and the inverse operator satisfies
\[
\left\| L_{n+1}(\delta, v_1, w_n)^{-1} \right\|_{\sigma_{n+1, s}} \leq C' (L_{n+1})^{-1}, \quad \forall (\delta, v_1) \in A_{n+1}.
\]
(37)

By \((32)\), equation \((P_{n+1})\) for \( w_{n+1} = w_n + h \) is equivalent to find \( h \in W^{(n+1)} \) solving
\[
h = -L_{n+1}(\delta, v_1, w_n)^{-1} \left( r_n + R(h) \right),
\]
namely to look for a fixed point
\[
h = G(\delta, v_1, w_n, h), \quad h \in W^{(n+1)},
\]
(38)

of the nonlinear operator
\[
G(\delta, v_1, w_n, \cdot) : W^{(n+1)} \rightarrow W^{(n+1)}, \quad G(\delta, v_1, w_n, h) := -L_{n+1}(\delta, v_1, w_n)^{-1} \left( r_n + R(h) \right).
\]

To complete the proof of the Proposition we need the following Lemma.

**Lemma 3.1 (Contraction)** There exist \( L_0(\gamma, \tau) > 0, \varepsilon_0(\gamma, \tau) \), such that, \( \forall |\varepsilon|^{-1} < \varepsilon_0 \), the operator \( G(\delta, v_1, w_n, \cdot) \) is, for any \( n \geq 0 \), a contraction in the ball
\[
B(\rho_{n+1}; W^{(n+1)}) := \left\{ h \in W^{(n+1)} \mid \|h\|_{\sigma_{n+1, s}} \leq \rho_{n+1} := \frac{|\varepsilon|}{\gamma} \exp(-\chi^{n+1}) \right\}.
\]
PROOF. We first prove that $G(\delta, v_1, w_n, \cdot)$ maps the ball $B(\rho_{n+1}; W^{(n+1)})$ into itself.

By (37), (33) and (35),
\[
\|G(\delta, v_1, w_n, h)\|_{\sigma_{n+1}; s} = \left\|L_{n+1}(\delta, v_1, w_n)^{-1}(r_n + R(h))\right\|_{\sigma_{n+1}; s} \\
\leq \frac{C}{\gamma}(L_{n+1})^{r-1}\left(\|r_n\|_{\sigma_{n+1}; s} + \|R(h)\|_{\sigma_{n+1}; s}\right) \\
\leq \frac{C'}{\gamma}(L_{n+1})^{r-1}\left(\|\varepsilon\exp(-L_n\gamma_n) + \|h\|^2\|h\|_{\sigma_{n+1}; s}\right).
\] (39)

By (39), if $\|h\|_{\sigma_{n+1}; s} \leq \rho_{n+1}$ then
\[
\|G(\delta, v_1, w_n, h)\|_{\sigma_{n+1}; s} \leq \frac{C'}{\gamma}(L_{n+1})^{r-1}|\varepsilon|\left(\exp(-L_n\gamma_n) + \rho_{n+1}^2\right) \leq \rho_{n+1}
\]
provided that
\[
C'\frac{|\varepsilon|}{\gamma}(L_{n+1})^{r-1}\exp(-L_n\gamma_n) \leq \frac{\rho_{n+1}}{2} \quad \text{and} \quad C'\frac{|\varepsilon|}{\gamma}(L_{n+1})^{r-1}\rho_{n+1} \leq \frac{1}{2} \quad \text{(40)}
\]
The first inequality in (40) becomes, for $\rho_{n+1} := |\varepsilon|\gamma^{-1}\exp(-\chi^{n+1})$,
\[
C'(L_{n+1})^{r-1}\exp(-L_n\gamma_n) \leq \frac{1}{2} \exp(-\chi^{n+1})
\]
which, for $L_n := L_02^n$, $\gamma_n := \gamma_0/(1 + n^2)$ and $L_0 := L_0(\gamma, \tau) > 0$ large enough, is satisfied $\forall n \geq 0$.

Next, the second inequality in (40) becomes
\[
C'\frac{|\varepsilon|^2}{\gamma^2}(L_0(\gamma, \tau)2^{n+1})^{r-1}\exp(-\chi^{n+1}) \leq \frac{1}{2}
\]
which is satisfied for $|\varepsilon|\gamma^{-1} \leq \varepsilon_0(L_0(\gamma, \tau)) \leq \varepsilon_2$ small enough, $\forall n \geq 0$.

With similar estimates, using (35), we can prove that $\forall h, h' \in B(\rho_{n+1}; W^{(n+1)})$,
\[
\|G(\delta, v_1, w_n, h') - G(\delta, v_1, w_n, h)\|_{\sigma_{n+1}; s} \leq \frac{1}{2}\|h - h'\|_{\sigma_{n+1}; s}
\]
again for $L_0$ large enough and $|\varepsilon|\gamma^{-1} \leq \varepsilon_0(L_0(\gamma, \tau))$ small enough, uniformly in $n$, and we conclude that $G(\delta, v_1, w_n, \cdot)$ is a contraction on $B(\rho_{n+1}; W^{(n+1)})$.

By the standard Contraction Mapping Theorem we deduce the existence, for $L_0(\gamma, \tau)$ large enough and $|\varepsilon|\gamma^{-1} \leq \varepsilon_0(L_0(\gamma, \tau))$, of a unique $h_{n+1} \in W^{(n+1)}$ solving (38) and satisfying
\[
\|h_{n+1}\|_{\sigma_{n+1}; s} \leq \rho_{n+1} = \frac{|\varepsilon|}{\gamma}\exp(-\chi^{n+1}).
\]

Summarizing, $w_{n+1}(\delta, v_1) = w_n(\delta, v_1) + h_{n+1}(\delta, v_1)$ is a solution in $W^{(n+1)}$ of equation $(P_{n+1})$, defined for $(\delta, v_1) \in A_{n+1} \subseteq A_n \subseteq \ldots \subseteq A_1 \subseteq A_0$, and $w_{n+1}(\delta, v_1) = \sum_{i=0}^{n+1} h_i(\delta, v_1)$ where $h_i = h_i(\delta, v_1) \in W^{(i)}$ satisfy $\|h_i\|_{\sigma_i; s} \leq |\varepsilon|\gamma^{-1}\exp(-\chi^i)$ for some $\chi \in (1, 2)$, $\forall i = 1, \ldots, n + 1$, $\|h_0\|_{\sigma_0; s} \leq K_0|\varepsilon|$.

**Remark 3.3** A difference with respect to the usual “quadratic” Nash-Moser scheme, is that $h_n(\delta, v_1)$ is found as an exact solution of equation $(P_n)$, and not just a solution of the linearized equation $r_n + L_{n+1}(\delta, v_1, w_n)[h] = 0$. It appears to be more convenient to prove the regularity of $h_n(\delta, v_1)$ with respect to the parameters $(\delta, v_1)$, see Lemma 3.2.

**Corollary 3.1** (Solution of the $(P)$-equation) For $(\delta, v_1) \in A_\infty := \cap_{n \geq 0} A_n$, $\sum_{i \geq 0} h_i(\delta, v_1)$ converges in $X_{\gamma/2, s}$ to a solution $w(\delta, v_1) \in W \cap X_{\gamma/2, s}$ of the $(P)$-equation (25) and $\|w(\delta, v_1)\|_{\gamma/2, s} \leq C|\varepsilon|\gamma^{-1}$. The convergence is uniform in $A_\infty$. 
Proof. By Proposition 3.1, for $(\delta, v_1) \in A_\infty := \cap_{n \geq 0} A_n$,\
\[ \sum_{i=0}^{\infty} \|h_i(\delta, v_1)\|_{\sigma, \gamma} \leq \sum_{i=0}^{\infty} \|h_i(\delta, v_1)\|_{\sigma, \gamma} \leq \max(1, \epsilon K_0) \sum_{i=0}^{\infty} \frac{|\epsilon|}{\gamma} \exp(-\gamma^i) < +\infty. \] (41)

Hence the series of functions $w = \sum_{i=0}^{\infty} h_i$ converges normally and, by (41), $\|w(\delta, v_1)\|_{\sigma, \gamma} \leq C|\epsilon|\gamma^{-1}$ with $C := \max(1, \epsilon K_0) \sum_{i=0}^{\infty} \exp(-\gamma^i)$.

Let us justify that $L_\omega w = \epsilon \Pi_W \Gamma(\delta, v_1, w)$. Since $w_n$ solves equation $(P_n)$,
\[ L_\omega w_n = \epsilon P_n \Pi_W \Gamma(\delta, v_1, w_n) = \epsilon \Pi_W \Gamma(\delta, v_1, w_n) - \epsilon P_n^+ \Pi_W \Gamma(\delta, v_1, w_n). \] (42)

We have, by (P2), (P1) and since $\sigma_n - (\sigma/2) \geq \gamma_n := \gamma_0/(n^2 + 1)$,
\[ \|P_n^+ \Pi_W \Gamma(\delta, v_1, w_n)\|_{\sigma, \gamma} \leq C \exp \left( - L_n(\sigma_n - (\sigma/2)) \right) \leq C \exp \left( - \gamma_0 \left( \frac{L_0^2 n}{(n^2 + 1)} \right) \right). \]

Hence, by (P1), the right hand side in (42) converges in $X_{\sigma/2, \gamma}$ to $\Gamma(\delta, v_1, w)$. Moreover, since $(w_n) \to w$ in $X_{\sigma/2, \gamma}$, $(L_\omega w_n) \to L_\omega w$ in the sense of distributions. Hence $L_\omega w = \epsilon \Pi_W \Gamma(\delta, v_1, w)$.

3.2 $C^\infty$ extension

Before proving the key property (P3) on the linearized operator we prove a “Whitney-differentiability” property for $w(\delta, v_1)$ extending $w(\cdot, \cdot)$ in a $C^\infty$-way on the whole $A_0$.

For this, some bound on the derivatives of $h_n = w_n - w_{n-1}$ is required.

Lemma 3.2 (Estimates for the derivatives of $h_n$ and $w_n$) For $\epsilon_0 \gamma^{-1} = \delta_0^{-1} \gamma^{-1}$ small enough, the function $(\delta, v_1) \mapsto h_n(\delta, v_1)$ is in $C^\infty(A_n, W^{(n)})$, $\forall n \geq 0$, and the $k^{th}$-derivative $D^k h_n(\delta, v_1)$ satisfies
\[ \left\| D^k h_n(\delta, v_1) \right\|_{\sigma_n, \gamma} \leq K_1(k) \exp(-\gamma^n) \] (43)
for $\gamma \in (1, \chi)$ and a suitable positive constant $K_1(k, \chi)$, $\forall n \geq 0$.

As a consequence, the function $(\delta, v_1) \mapsto w_n(\delta, v_1) = \sum_{i=0}^{n} h_i(\delta, v_1)$ is in $C^\infty(A_n, W^{(n)})$ and the $k^{th}$-derivative $D^k w_n(\delta, v_1)$ satisfies
\[ \left\| D^k w_n(\delta, v_1) \right\|_{\sigma_n, \gamma} \leq K_2(k) \] (44)
for a suitable positive constant $K_2(k)$.

Proof. By the first step in the proof of Proposition 3.1, $h_0 = w_0$ depends smoothly on $(\delta, v_1)$, and $\left\| D^k w_0(\delta, v_1) \right\|_{\sigma_0, \gamma} \leq C(k)$.

Next, assume by induction that $h_n = h_n(\delta, v_1)$ is a $C^\infty$ map defined in $A_n$. We shall prove that $h_{n+1} = h_{n+1}(\delta, v_1)$ is $C^\infty$ too.

First recall that $h_{n+1} = h_{n+1}(\delta, v_1)$ is defined, in Proposition 3.1, for $(\delta, v_1) \in A_{n+1}$ as a solution in $W^{(n+1)}$ of equation $(P_{n+1})$, namely
\[ (P_{n+1}) \quad U_{n+1}(\delta, v_1, h_{n+1}(\delta, v_1)) = 0 \]
where
\[ U_{n+1}(\delta, v_1, h) := L_\omega(w_n + h) - \epsilon P_{n+1} \Pi_W \Gamma(\delta, v_1, w_n + h). \]

We claim that $D_h U_{n+1}(\delta, v_1, h_{n+1}) = L_{n+1}(\delta, v_1, w_{n+1})$ is invertible and
\[ \left\| \left( D_h U_{n+1}(\delta, v_1, h_{n+1}) \right)^{-1} \right\|_{\sigma_{n+1}, \gamma} \leq C'(L_{n+1})^{-1}. \] (45)
Now Equation \((P_{n+1})\) can be written as
\[ h + q_{n+1}(\delta, v_1, h) = 0, \]
where
\[ q_{n+1}(\delta, v_1, h) = (\Delta)^{-1}[L_\omega w_n - (\omega^2 + 1)h_{tt} - \varepsilon P_{n+1}\Pi_W \Gamma(\delta, v_1, w_n + h)]. \]

The map \(q_{n+1} : [0, \delta_0] \times V_1 \times W^{(n+1)} \to W^{(n+1)}\) implies the injectivity and hence (noting that \(D_hq_{n+1}(\delta, v_1, h_{n+1})\) is compact) the invertibility of \(I + D_hq_{n+1}(\delta, v_1, h_{n+1})\). As a consequence, by the Implicit Function Theorem, the map \((\delta, v_1) \mapsto h_{n+1}(\delta, v_1)\) is in \(C^\infty(A_{n+1}, W^{(n+1)})\).

Let us prove \((45)\). Using \((P1)\) and \(\|w_{n+1} - w_n\|_{\sigma_{n+1, s}} = \|h_{n+1}\|_{\sigma_{n+1, s}} \leq |\varepsilon| \gamma^{-1} \exp(-\chi^{n+1})\), we get
\[
\left\|\mathcal{L}_{n+1}(\delta, v_1, w_{n+1}) - \mathcal{L}_{n+1}(\delta, v_1, w_n)\right\|_{\sigma_{n+1, s}} = \left\|\varepsilon P_{n+1}\Pi_W \left(D_u \Gamma(\delta, v_1, w_{n+1}) - D_u \Gamma(\delta, v_1, w_n)\right)\right\|_{\sigma_{n+1, s}} \\
\leq C|\varepsilon| \|h_{n+1}\|_{\sigma_{n+1, s}} \leq \frac{C\varepsilon^2}{\gamma} \exp(-\chi^{n+1}).
\]

Therefore
\[
\mathcal{L}_{n+1}(\delta, v_1, w_{n+1}) = \mathcal{L}_{n+1}(\delta, v_1, w_n) \left[\text{Id} + \mathcal{L}_{n+1}(\delta, v_1, w_n)^{-1}\left(\mathcal{L}_{n+1}(\delta, v_1, w_{n+1}) - \mathcal{L}_{n+1}(\delta, v_1, w_n)\right)\right]
\]
is invertible whenever (recall \((37)\) and \((46)\))
\[
\left\|\mathcal{L}_{n+1}(\delta, v_1, w_n)^{-1}\left(\mathcal{L}_{n+1}(\delta, v_1, w_{n+1}) - \mathcal{L}_{n+1}(\delta, v_1, w_n)\right)\right\|_{\sigma_{n+1, s}} \leq \frac{C}{\gamma} (L_{n+1})^{-1} \frac{\varepsilon^2}{\gamma} \exp(-\chi^{n+1}) < \frac{1}{2}
\]
which is true, provided that \(|\varepsilon| \gamma^{-1}\) is small enough, for all \(n \geq 0\) (note that \((L_{n+1})^{-1} = (L_0 2^{n+1})^{-1} << \exp(\chi^{n+1})\) for \(n\) large). Furthermore, by \((47), (37), (48)\), estimate \((45)\) holds.

We now prove in detail estimate \((43)\) for \(k = 1\). Differentiating equation \((P_{n+1})\) with respect to some coordinate \(\lambda\) of \((\delta, v_1) \in A_{n+1}\), we obtain
\[
(P'_{n+1}) \quad \mathcal{L}_{n+1}(\delta, v_1, w_{n+1}) \left[\partial_\lambda h_{n+1}(\delta, v_1)\right] = -\left(\partial_\lambda U_{n+1}\right)(\delta, v_1, h_{n+1}(\delta, v_1))
\]
and therefore, by \((45)\),
\[
\left\|\partial_\lambda h_{n+1}\right\|_{\sigma_{n+1, s}} \leq \frac{C}{\gamma} (L_{n+1})^{-1} \left\|\left(\partial_\lambda U_{n+1}\right)(\delta, v_1, h_{n+1})\right\|_{\sigma_{n+1, s}}.
\]
To estimate the right hand side of \((49)\), first notice that since \(w_n = w_n(\delta, v_1)\) solves
\[ L_\omega w_n = \varepsilon P_n\Pi_W \Gamma(\delta, v_1, w_n), \quad \forall(\delta, v_1) \in A_n, \]
we have
\[ U_{n+1}(\delta, v_1, h) = L_\omega h + \varepsilon(P_n\Pi_W \Gamma(\delta, v_1, w_n) - P_{n+1}\Pi_W \Gamma(\delta, v_1, w_n + h)). \]
Let us write
\[
\left(\partial_\lambda U_{n+1}\right)(\delta, v_1, h) = \left(\partial_\lambda U_{n+1}\right)(\delta, v_1, 0) + r(\delta, v_1, h)
\]
where
\[
\left(\partial_\lambda U_{n+1}\right)(\delta, v_1, 0) = \left(P_n - P_{n+1}\right)\Pi_W \partial_\lambda[\varepsilon(\delta) \Gamma(\delta, v_1, w_n(\delta, v_1))]
\]
\[
= -\varepsilon P_n^1\Pi_W \left[\left(\partial_\delta \Gamma(\delta, v_1, w_n) + (\partial_\delta \Gamma)(\delta, v_1, w_n)\right)[\partial_\lambda w_n]\right] - \partial_\lambda(\varepsilon(\delta)) P_n^1\Pi_W \Gamma(\delta, v_1, w_n)
\]
\]
(51)
and
\[
 r(\delta, v_1, h) := -\varepsilon P_{n+1} \Pi W \left[ (\partial_{\lambda} \Gamma)(\delta, v_1, w_n + h) - (\partial_{\lambda} \Gamma)(\delta, v_1, w_n) \right] \\
- \varepsilon P_{n+1} \Pi W \left[ (\partial_{\mu} \Gamma)(\delta, v_1, w_n + h) - (\partial_{\mu} \Gamma)(\delta, v_1, w_n) \right] [\partial_{\lambda} w_n] \\
+ \partial_{\lambda} (L_{\omega}(\delta) h) + \partial_{\lambda} (\varepsilon(\delta)) P_{n+1} \Pi W (\Gamma(\delta, v_1, w_n) - \Gamma(\delta, v_1, w_n + h)),
\]
with \( \partial_{\lambda} (L_{\omega}(\delta) h) = 0, \partial_{\lambda} (\varepsilon(\delta)) = 0 \) if \( \lambda \neq \delta \) and
\[
\partial_{\delta} (L_{\omega}(\delta) h) = -2(p-1)\delta^{p-2} h, \quad \partial_{\delta} (\varepsilon(\delta)) = (p-1)\delta^{p-2}.
\]
By (P1), (51), properties (P2), (P1)
\[
\| (\partial_{\lambda} U_{n+1})(\delta, v_1, 0) \|_{\sigma_{n+1, s}} \leq \exp (-L_n \gamma n) \| (\partial_{\lambda} \Gamma)(\delta, v_1, w_n) + (\partial_{\mu} \Gamma)(\delta, v_1, w_n) [\partial_{\lambda} w_n] \|_{\sigma_{n, s}}
\]
\[
+ \| \Gamma(\delta, v_1, w_n) \|_{\sigma_{n, s}}
\]
\[
\leq C \exp (-L_n \gamma n) \left( 1 + \| \partial_{\lambda} w_n \|_{\sigma_{n, s}} \right).
\]
Combining (49), (50), (51), (55) and the bound \( \| h_{n+1} \|_{\sigma_{n+1, s}} \leq |\varepsilon|^{-1} \exp (-\chi^{n+1}) \), we get
\[
\| \partial_{\lambda} h_{n+1} \|_{\sigma_{n+1, s}} \leq C \gamma \gamma (L_n \gamma n)^{\tau+1} \left( |\varepsilon|^{\gamma} \exp (-\chi^{n+1}) + \exp (-L_n \gamma n) \right) \left( 1 + \| \partial_{\lambda} w_n \|_{\sigma_{n, s}} \right)
\]
\[
\leq C(\chi) \exp (-\chi^{n+1}) \left( 1 + \sum_{i=0}^{n} \| \partial_{\lambda} h_i \|_{\sigma_{i, s}} \right)
\]
for any \( \chi \in (1, \chi) \). By (56), the sequence \( a_n := \| \partial_{\lambda} h_n \|_{\sigma_{n, s}} \) satisfies
\[
a_0 \leq C \text{ and } a_{n+1} \leq C(\chi) \exp (-\chi^{n+1}) \left( 1 + a_0 + \ldots + a_n \right)
\]
which implies (by induction)
\[
\| \partial_{\lambda} h_n \|_{\sigma_{n, s}} = a_n \leq K(\chi)^n \exp (-\chi^n), \quad \forall n \geq 0,
\]
provided that \( K(\chi) \) has been chosen large enough. We can prove in the same way the general estimate (43) from which (44) follows.

Since, by (43), \( h_n(\delta, v_1) = O(\varepsilon^{-1} \exp (-\chi^n)) \) and the “non-resonant” set \( A_n \) is obtained at each step deleting strips of size \( O(\gamma / L_n^{\alpha}) \), we can define (by interpolation, say) a \( C^\infty \)-extension \( \tilde{w}(\delta, v_1) \) of \( w(\delta, v_1) \) for all \( (\delta, v_1) \in A_0 \).

Let
\[
\tilde{A}_n := \left\{ (\delta, v_1) \in A_n \mid \text{dist}((\delta, v_1), \partial A_n) \geq \frac{2}{L_n^{\gamma}} \right\} \subset A_n
\]
where \( \nu^{-1} > 0 \) is some small constant to be specified later, see Lemma 3.3.

**Lemma 3.3** (Whitney \( C^\infty \) Extension \( \tilde{w} \) of \( w \) on \( A_0 \)) There exists a function \( \tilde{w}(\delta, v_1) \in C^\infty(A_0, W \cap X_{\gamma/2, s}) \) satisfying
\[
\| \tilde{w}(\delta, v_1) \|_{\gamma/2, s} \leq \frac{|\varepsilon|}{\gamma} C, \quad \| D^k \tilde{w}(\delta, v_1) \|_{\gamma/2, s} \leq \frac{C(k)}{\nu^k}, \quad \forall (\delta, v_1) \in A_0, \forall k \geq 1,
\]
for some $C(k) > 0$, such that,
\[
\forall (\delta, v_1) \in \tilde{A}_n := \cap_{n \geq 0} \tilde{A}_n, \quad \tilde{w}(\delta, v_1) \text{ solves the } (P) - \text{equation (25)}.
\]
Moreover there exists a sequence $\tilde{w}_n \in C^\infty(A_0, W^{(n)})$ such that
\[
\tilde{w}_n(\delta, v_1) = w_n(\delta, v_1), \quad \forall (\delta, v_1) \in \tilde{A}_n
\]
and
\[
\left\| \tilde{w}(\delta, v_1) - \tilde{w}_n(\delta, v_1) \right\|_{\pi/2, s} \leq \frac{|\varepsilon C|}{\gamma} \exp(-\bar{\chi}^n),
\]
\[
\left\| D^k \tilde{w}(\delta, v_1) - D^k \tilde{w}_n(\delta, v_1) \right\|_{\pi/2, s} \leq \frac{C(k)}{\nu^k} \exp(-\bar{\chi}^n), \quad \forall (\delta, v_1) \in A_0,
\]
for some $\bar{\chi} \in (1, \infty)$.

**Proof.** Let $\varphi : \mathbb{R} \times V_1 \to \mathbb{R}_+$ be a $C^\infty$ function supported in the open ball $B(0, 1)$ of center 0 and radius 1 with $\int_{\mathbb{R} \times V_1} \varphi \ d\mu = 1$. Here $\mu$ is the Borelian positive measure of $\mathbb{R} \times V_1$ defined by $\mu(E) = m(L^{-1}(E))$ where $L$ is some automorphism from $\mathbb{R}^{N+1}$ to $\mathbb{R} \times V_1$ and $m$ is the Lebesgue measure in $\mathbb{R}^{N+1}$.

Let $\varphi_n : \mathbb{R} \times V_1 \to \mathbb{R}_+$ be the "mollifier"
\[
\varphi_n(\lambda) := \left( \frac{L_0^3}{\nu} \right)^{N+1} \varphi\left( \frac{L_0^3}{\nu} \lambda \right)
\]
(here $\lambda := (\delta, v_1)$) which is a $C^\infty$ function satisfying
\[
supp \varphi_n \subset B\left(0, \frac{\nu}{L_0^3}\right) \quad \text{and} \quad \int_{\mathbb{R} \times V_1} \varphi_n \ d\mu = 1.
\]

Next we define $\psi_n : \mathbb{R} \times V_1 \to \mathbb{R}_+$ as
\[
\psi_n(\lambda) := \left( \varphi_n * \chi_{A_n^*} \right)(\lambda) = \int_{\mathbb{R} \times V_1} \varphi_n(\lambda - \eta) \chi_{A_n^*}(\eta) \ d\mu(\eta)
\]
where $\chi_{A_n^*}$ is the characteristic function of the set
\[
A_n^* := \left\{ \lambda = (\delta, v_1) \in A_n \mid \text{dist}(\lambda, \partial A_n) \geq \frac{\nu}{L_0^3} \right\} \subset A_n,
\]

namely $\chi_{A_n^*}(\lambda) := 1$ if $\lambda \in A_n^*$, and $\chi_{A_n^*}(\lambda) := 0$ if $\lambda \notin A_n^*$.

The function $\psi_n$ is $C^\infty$ and, $\forall k \in \mathbb{N}, \forall \lambda \in \mathbb{R} \times V_1$,
\[
|D^k \psi_n(\lambda)| = \left| \int_{\mathbb{R} \times V_1} D^k \varphi_n(\lambda - \eta) \chi_{A_n^*}(\eta) \ d\mu(\eta) \right|
\]
\[
\leq \int_{\mathbb{R} \times V_1} \left| \left( \frac{L_0^3}{\nu} \right)^k \left( \frac{L_0^3}{\nu} \right)^{N+1} \left( D^k \varphi \right) \left( \frac{L_0^3}{\nu} (\lambda - \eta) \right) \right| \ d\mu(\eta)
\]
\[
= \left( \frac{L_0^3}{\nu} \right)^k \int_{\mathbb{R} \times V_1} |D^k \varphi| \ d\mu = \left( \frac{L_0^3}{\nu} \right)^k C(k)
\]
where $C(k) := \int_{\mathbb{R} \times V_1} |D^k \varphi| \ d\mu$. Furthermore, by (61) and the definition of $A_n^*$ and $\tilde{A}_n$,
\[
0 \leq \psi_n(\lambda) \leq 1, \quad supp \psi_n \subset int A_n \quad \text{and} \quad \psi_n(\lambda) = 1 \quad \text{if} \quad \lambda \in \tilde{A}_n.
\]
Finally we can define $\tilde{w}_n : A_0 \to W^{(n)}$ by
\[
\tilde{w}_0(\lambda) := w_0(\lambda), \quad \tilde{w}_{n+1}(\lambda) := \tilde{w}_n(\lambda) + \tilde{h}_{n+1}(\lambda) \in W^{(n+1)},
\]
where
\[\hat{h}_{n+1}(\lambda) := \begin{cases} \psi_{n+1}(\lambda)h_{n+1}(\lambda) & \text{if } \lambda \in A_{n+1} \\ 0 & \text{if } \lambda \notin A_{n+1} \end{cases}\]
is in \(C^\infty(A_0, W^{(n+1)})\) because \(\text{supp} \psi_{n+1} \subset \text{int } A_{n+1}\) and, by Lemma 3.2, \(h_{n+1} \in C^\infty(A_{n+1}, W^{(n+1)})\).

Therefore we have
\[\hat{w}_n(\lambda) = \sum_{i=0}^n \hat{h}_i(\lambda), \quad \hat{w}_n \in C^\infty(A_0, W^{(n)})\]
and (58) holds.

By the bounds (62) and (43) we obtain, \(\forall k \in \mathbb{N}, \forall \lambda \in A_0, \forall n \geq 0,\)
\[\left\|\hat{h}_{n+1}(\lambda)\right\|_{\sigma_{n+1, s}} \leq \frac{\left|\varepsilon\right|K}{\gamma} \exp(-\overline{\chi}^n), \quad \left\|D^k\hat{h}_{n+1}(\lambda)\right\|_{\sigma_{n+1, s}} \leq C(k, \overline{\chi})^n \left(\frac{L_{n+1}^3}{\nu}\right)^k \exp(-\overline{\chi}^n) \leq \frac{K(k)}{\nu^k} \exp(-\overline{\chi}^n)\]
for some \(1 < \overline{\chi} < \overline{\bar{x}}\) and some positive constant \(K(k)\) large enough. As a consequence, the sequence \((\hat{w}_n)\) (and all its derivatives) converges uniformly in \(A_0\) for the norm \(\left\| \cdot \right\|_{\sigma/2, s}\) on \(W\), to some function \(\bar{w}(\delta, v_1) \in C^\infty(A_0, W \cap X_{\pi/2, s})\) which satisfies (57), (59) and (60).

Finally, note that if \(\lambda \notin A_\infty := \cap_{n \geq 0} A_n\) then the series \(\hat{w}(\lambda) = \sum_{n \geq 1} \hat{h}_n(\lambda)\) is a finite sum. On the other hand, if \(\lambda \in A_\infty := \cap_{n \geq 0} A_n\) then \(\hat{w}(\lambda) = \overline{w}(\lambda)\) solves the \((P)\)-equation (25).

**Remark 3.4** If \((\delta, v_1) \notin \bar{A}_\infty\) we claim that \(\bar{w}(\delta, v_1)\) solves the \((P)\)-equation up to exponentially small remainders: there exists \(\alpha > 0, \delta_0(\gamma, \tau) > 0\) such that, \(\forall 0 < \delta \leq \delta_0(\gamma, \tau)\)
\[\left\|L_\omega \bar{w}(\delta, v_1) - \varepsilon \Pi_W \Gamma(\delta, v_1, \bar{w}(\delta, v_1))\right\|_{\sigma/4, s} \leq \frac{\left|\varepsilon\right|}{\gamma} \exp\left(-\frac{1}{\delta^s}\right).\]

Since we shall not use this property in the present paper we do not give here the proof.

### 3.3 Measure estimate

We now replace the set \(\bar{A}_\infty\) with a smaller Cantor-like set \(B_\infty\) which has the advantage of being independent of the iteration steps. This is more convenient for the measure estimates required in section 5 (this issue is discussed differently in [12]).

Define
\[B_n := \left\{ (\delta, v_1) \in \bar{A}_0 \mid \delta \in \Delta^{2_{\gamma, \tau}}_n(v_1, \bar{w}(\delta, v_1)) \right\}\]
where we have replaced \(\gamma\) with \(2\gamma\) in the definition of \(\Delta^{\gamma, \tau}_n\), see Definition 3.3. Note that \(B_n\) does not depend on the approximate solution \(w_n\) but only on the fixed function \(\bar{w}\).

**Lemma 3.4** If \(\nu \gamma^{-1} > 0\) and \(|\varepsilon| \gamma^{-1}\) are small enough, then
\[B_n \subset \bar{A}_n, \quad \forall n \geq 0.\]

Hence \(B_\infty := \cap_{n \geq 1} B_n \subset \bar{A}_\infty \subset A_\infty\) and so, if \((\delta, v_1) \in B_\infty\) then \(\bar{w}(\delta, v_1)\) solves the \((P)\)-equation (25).

**Proof.** We shall prove the Lemma by induction. First \(B_0 \subset \bar{A}_0\). Suppose next that \(B_n \subset \bar{A}_n\) holds. In order to prove that \(B_{n+1} \subset \bar{A}_{n+1}\), take any \((\delta, v_1) \in B_{n+1}\). We have to justify that the ball \(B((\delta, v_1), 2\nu/L_{n+1}^3) \subset A_{n+1}\).

First, since \(B_{n+1} \subset B_n \subset \bar{A}_n, (\delta, v_1) \in \bar{A}_n,\) hence, since \(L_{n+1} > L_n, B((\delta, v_1), 2\nu/L_{n+1}^3) \subset A_n,\)

Let \((\delta', v_1') \in B((\delta, v_1), 2\nu/L_{n+1}^3).\) Since \((\delta, v_1) \in \bar{A}_n\) we have \(\bar{w}_n(\delta, v_1) = w_n(\delta, v_1)\). Moreover by (44)
\[\left\|Dw_n\right\|_{\sigma/2, s} \leq C.\]
By (59) we can derive
\[\left\|w_n(\delta', v_1') - \bar{w}(\delta, v_1)\right\|_{\sigma/2, s} \leq \left\|w_n(\delta', v_1') - w_n(\delta, v_1)\right\|_{\sigma/2, s} + \left\|w_n(\delta, v_1) - \bar{w}(\delta, v_1)\right\|_{\sigma/2, s}\]
\[\leq \frac{2\nu C}{L_{n+1}^3} + \frac{C|\varepsilon|}{\gamma} \exp(-\overline{\chi}^n)\]
Hence, by (63), setting \( \omega' := \sqrt{1 + 2(\delta')^{p-1}} \) and \( \varepsilon' := (\delta')^{p-1} \) (for simplicity of notation suppose \( s^* = 1 \))

\[
|\omega' l - j - \varepsilon M(\omega', v_j') w_n(\delta', v'_j)| \geq |\omega l - j - \varepsilon M(\delta, v_1, \omega(\delta, v_1))| - l \frac{C\nu}{L_{n+1}^3} - C \frac{|\varepsilon|^2}{\gamma} \exp(-\tilde{\chi}^n)
\]

\[
\geq \frac{2\gamma}{(l+j)^2} - C \frac{\nu}{L_{n+1}^2} - C \frac{|\varepsilon|^2}{\gamma} \exp(-\tilde{\chi}^n) \geq \frac{\gamma}{(l+j)^2}
\]

for all \( 1/3|\varepsilon| < l < L_{n+1}, l \neq j, j \leq 2L_{n+1} \), whenever

\[
\frac{\gamma}{(3L_{n+1})^2} \geq C \left( \frac{\nu}{L_{n+1}^2} + \frac{|\varepsilon|^2}{\gamma} \exp(-\tilde{\chi}^n) \right).
\] (64)

(64) holds true, for \(|\varepsilon|\gamma^{-1} \) and \( \nu\gamma^{-1} \) small, for all \( n \geq 0 \), because \( \tau < 2 \) and \( \lim_{n \to \infty} L_{n+1}^2 \exp(-\tilde{\chi}^n) = 0 \). It results that \( B((\delta, v_1), 2\nu/L_{n+1}) \subset A_{n+1}. \)

Up to now, we have not justified that

\[
B_\infty \subset \tilde{A}_\infty \subset A_\infty
\] (65)

are not reduced to \( \{ \delta = 0 \} \times B(2R, V_1) \). It is a consequence of the following result which shall be applied in section 5.

**Proposition 3.2 (Measure estimate of \( B_\infty \))** Let \( V_1 : [0, \delta_0) \to V_1 \) be a \( C^1 \) function. Then

\[
\lim_{\eta \to 0^+} \frac{\text{meas}\{ \delta \in [0, \eta] \mid (\delta, V_1(\delta)) \in B_\infty \}}{\eta} = 1.
\] (66)

**Proof.** Let \( 0 < \eta < \delta_0 \). Define

\[
C_{V_1, \eta} := \{ \delta \in (0, \eta) \mid (\delta, V_1(\delta)) \in B_\infty \} \quad \text{and} \quad D_{V_1, \eta} := (0, \eta) \setminus C_{V_1, \eta}.
\]

By the definition \( B_\infty := \cap_{n \geq 1} B_n \) (see also the expression of \( B_\infty \) in the statement of Theorem 3.1 where for simplicity of notation we suppose \( s^* = 1 \))

\[
D_{V_1, \eta} = \{ \delta \in (0, \eta) \mid \left| \omega(\delta)l - j - \frac{\delta^{p-1}m(\delta)}{2j} \right| < \frac{2\gamma}{(l+j)^2} \quad \text{or} \quad \left| \omega(\delta)l - j \right| < \frac{2\gamma}{(l+j)^2}
\]

for some \( l > \frac{1}{3\delta^{p-1}}, l \neq j \}

where \( m(\delta) := M(\delta, V_1(\delta), \omega(\delta, V_1(\delta))) \) is a function in \( C^1([0, \delta_0), \mathbb{R}) \) since \( \omega(\cdot, \cdot) \) is in \( C^\infty(A_0, W \cap \mathcal{X}_{\pi/2, s}) \) and \( V_1 \) is \( C^1 \). This implies, in particular,

\[
|m(\delta)| + |m'(\delta)| \leq C, \quad \forall \delta \in [0, \delta_0/2]
\] (67)

for some positive constant \( C \).

We claim that, for any interval \( [\delta_1/2, \delta_1) \subset [0, \eta] \subset [0, \delta_0/2] \) the following measure estimate holds:

\[
\text{meas}(D_{V_1, \eta} \cap \left[ \frac{\delta_1}{2}, \delta_1 \right]) \leq K_1(\tau) \gamma \eta^{(p-1)(\tau-1)} \text{meas}(\left[ \frac{\delta_1}{2}, \delta_1 \right])
\] (68)

for some constant \( K_1(\tau) > 0 \).

Before proving (68) we show how to conclude the proof of the Lemma. Writing \( (0, \eta) = \cup_{n \geq 0} [\eta/2^{n+1}, \eta/2^n] \) and applying the measure estimate (68) to any interval \( [\delta_1/2, \delta_1] = [\eta/2^{n+1}, \eta/2^n] \), we get

\[
\text{meas}(D_{V_1, \eta} \cap [0, \eta]) \leq K_1(\tau) \gamma \eta^{(p-1)(\tau-1)} \eta,
\]

whence \( \lim_{\eta \to 0^+} \text{meas}(C_{V_1, \eta} \cap (0, \eta))/\eta = 1 \), proving Proposition 3.2.
We now prove (68). We have
\[ D_{\mathcal{V}_1, \eta} \cap \left[ \frac{\delta_1}{2}, \delta_1 \right] \subset \bigcup_{(l,j) \in I_R} R_{l,j}(\delta_1) \]  
where
\[ R_{l,j}(\delta_1) := \left\{ \delta \in \left[ \frac{\delta_1}{2}, \delta_1 \right] \mid \left| \omega(\delta)l - j - \frac{\delta^{p-1}m(\delta)}{2j} \right| < \frac{2\gamma}{(l+j)^{1/2}} \text{ or } \left| \omega(\delta)l - j \right| < \frac{2\gamma}{(l+j)^{1/2}} \right\} \]
and
\[ I_R := \left\{ (l,j) \mid l > \frac{1}{3\delta_1^{p-1}}, l \neq j, \frac{j}{l} \in [1 - c_0\delta_1^{p-1}, 1 + c_0\delta_1^{p-1}] \right\} \]
(note indeed that \( R_{l,j}(\delta_1) = \emptyset \) unless \( j/l \in [1 - c_0\delta_1^{p-1}, 1 + c_0\delta_1^{p-1}] \) for some constant \( c_0 > 0 \) large enough).

Next, let us prove that
\[ \text{meas}(R_{l,j}(\delta_1)) = O\left( \frac{\gamma}{l^{1/2} \delta_1^{p-2}} \right). \]  
(70)

Define \( f_{l,j}(\delta) := \omega(\delta)l - j - (\delta^{p-1}m(\delta)/2j) \) and \( S_{l,j}(\delta_1) := \{ \delta \in [\delta_1/2, \delta_1] : \left| f_{l,j}(\delta) \right| < 2\gamma/(l+j)^{1/2} \} \). Provided \( \delta_0 \) has been chosen small enough (recall that \( j, l \geq 1/3\delta_0^{p-1} \)),
\[ \left| \partial_\delta f_{l,j}(\delta) \right| = \frac{(p-1)\delta^{p-2} - (p-1)\delta^{p-2}m(\delta)}{2j} \geq \frac{(p-1)\delta^{p-2}(l-C)}{2j} \]
and therefore \( \left| \partial_\delta f_{l,j}(\delta) \right| \geq (p-1)\delta_1^{p-2}/2 \) for any \( \delta \in [\delta_1/2, \delta_1] \). This implies
\[ \text{meas}(S_{l,j}(\delta_1)) \leq \frac{4\gamma}{(l+j)^{1/2}} \leq \frac{4\gamma}{(l+j)^{1/2}} \leq \frac{2\gamma}{(l+j)^{1/2}} = O\left( \frac{\gamma}{l^{1/2} \delta_1^{p-2}} \right). \]

Similarly we can prove
\[ \text{meas}\left( \left\{ \delta \in \left[ \frac{\delta_1}{2}, \delta_1 \right] : \left| \omega(\delta)l - j \right| < \frac{2\gamma}{(l+j)^{1/2}} \right\} \right) = O\left( \frac{\gamma}{l^{1/2} \delta_1^{p-2}} \right) \]
and the measure estimate (70) follows.

Now, by (69), (70) and since, for a given \( l \), the number of \( j \) for which \((l,j) \in I_R \) is \( O(\delta_1^{p-1}) \),
\[ \text{meas}(D_{\mathcal{V}_1, \eta} \cap \left[ \frac{\delta_1}{2}, \delta_1 \right]) \leq \sum_{(l,j) \in I_R} \text{meas}(R_{l,j}(\delta_1)) \leq C \sum_{l \geq 1/3\delta_0^{p-1}} \delta_1^{p-1}l \leq K_2(\tau)\gamma \delta_1^{1+(p-1)(\tau-1)} \]
whence we obtain (68) since \( 0 < \delta_1 < \eta \).  

We summarize the main result of this section as follows:

**Theorem 3.1 (Solution of the \( (P) \)-equation)** For \( \delta_0 := \delta_0(\gamma, \tau) > 0 \) small enough, there exist a \( C^\infty \)-function \( \tilde{w} : A_0 := \{ (\delta, v_1) \mid \delta \in [0, \delta_0), \| v_1 \|_{0,s} \leq 2R \} \) \(\to W \cap X_{\pi/2,s} \) satisfying (57), and a “large” -see (69). Cantor set
\[ B_\infty := \left\{ (\delta, v_1) \in A_0 : \left| \omega(\delta)l - j - s^*\delta^{p-1} \frac{M(\delta, v_1, \tilde{w}(\delta, v_1))}{2j} \right| \geq \frac{2\gamma}{(l+j)^{1/2}}, \right\} \]
\[ \left| \omega(\delta)l - j \right| \geq \frac{2\gamma}{(l+j)^{1/2}}, \forall l \geq 1/3\delta_0^{p-1}, l \neq j \right\} \subset A_0, \]
where \( \omega(\delta) = \sqrt{1 + 2s^*\delta^{p-1}} \) and \( M(\delta, v_1, w) \) is defined in Definition 3.1, such that
\[ \forall (\delta, v_1) \in B_\infty, \quad \tilde{w}(\delta, v_1) \text{ solves the } (P) - \text{equation } (25). \]
4 Analysis of the linearized problem: proof of (P3)

We prove in this section the key property (P3) on the inversion of the linear operator \( \mathcal{L}_n(\delta, v_1, w) \) defined in (28).

Through this section we shall use the notations
\[
F_k := \{ f \in H^1_0((0, \pi); \mathbb{R}) \mid \int_0^\pi f(x) \sin(kx) \, dx = 0 \} = (\sin(kx))^{-1}
\]
whence the space \( W \), defined in (6), writes
\[
W = \{ h = \sum_{k \in \mathbb{Z}} \exp(ikt)h_k \in X_{0, \delta} \mid h_k = h_{-k}, \ h_k \in F_k, \ \forall k \in \mathbb{Z} \}
\]
and the corresponding projector \( \Pi_W : X_{\pi, \delta} \to W \) is
\[
(\Pi_W h)(t, x) = \sum_{k \in \mathbb{Z}} \exp(ikt)(\pi_k h_k)(x)
\]
where \( \pi_k : H^1_0((0, \pi); \mathbb{R}) \to F_k := (\sin(kx))^{-1} \) is the \( L^2 \)-orthogonal projector onto \( F_k \)
\[
(\pi_k f)(x) := f(x) - \left( \frac{2}{\pi} \int_0^\pi f(x) \sin(kx) \, dx \right) \sin(kx).
\]
Note that \( \pi_{-k} = \pi_k \). Hence, since \( h_k = h_{-k} \), \( \pi_k h_k = \pi_{-k} h_{-k} \).

4.1 Decomposition of \( \mathcal{L}_n(\delta, v_1, w) \)

Recalling (26), the operator \( \mathcal{L}_n(\delta, v_1, w) : D(\mathcal{L}_n) \subset W^{(n)} \to W^{(n)} \) writes
\[
\mathcal{L}_n(\delta, v_1, w)[h] := L_\omega h - \varepsilon P_n \Pi_W \frac{\partial}{\partial_x} \Gamma(\delta, v_1, w)[h]
\]
\[
= L_\omega h - \varepsilon P_n \Pi_W \left( \partial_u g(\delta, x, v_1 + w + v_2(\delta, v_1, w)) \left( h + \partial_w v_2(\delta, v_1, w)[h] \right) \right)
\]
\[
= L_\omega h - \varepsilon P_n \Pi_W \left( a(t, x) h \right) - \varepsilon P_n \Pi_W \left( a(t, x) \partial_w v_2(\delta, v_1, w)[h] \right)
\]
where, for brevity, we have set
\[
a(t, x) := \partial_u g(\delta, x, v_1(t, x) + w(t, x) + v_2(\delta, v_1, w)(t, x)).
\]
In order to invert \( \mathcal{L}_n \) it is convenient to perform a Fourier expansion and represent the operator \( \mathcal{L}_n \) as a matrix, distinguishing a “diagonal” matrix \( D \) and a “off-diagonal Toepliz” matrix. The main difference with respect to the analogue procedure of Craig-Wayne-Bourgain [12]-[8] is that we shall develop \( \mathcal{L}_n \) only in time-Fourier basis and not also in the spatial fixed basis formed by the eigenvectors \( \sin(jx) \) of the linear operator \(-\partial_{xx}\). The reason is that this is more convenient to deal with nonlinearities \( f(x, u) \) with finite regularity in \( x \) and without oddness assumptions. Each diagonal element \( D_k \) is a differential operator acting on functions of \( x \). Next, using Sturm-Liouville theory, we shall diagonalize each \( D_k \) in a suitable basis of eigenfunctions close, but different, from \( \sin jx \), see Lemma 4.1 and Corollary 4.1.

Performing a time-Fourier expansion, the operator \( L_\omega := -\omega^2 \partial_t + \partial_{xx} \) is diagonal since
\[
L_\omega \left( \sum_{|k| \leq L_n} \exp(ikt)h_k \right) = \sum_{|k| \leq L_n} \exp(ikt)(\omega^2 k^2 + \partial_{xx})h_k.
\]
The operator \( h \to P_n \Pi_W(a(t, x) h) \) is the composition of the multiplication operator for the function \( a(t, x) = \sum_{l \in \mathbb{Z}} \exp(ilt) a_l(x) \) with the projectors \( \Pi_W \) and \( P_n \). As usual, in Fourier expansion, the multiplication operator is described by a “Toepliz matrix”
\[
a(t, x) h(t, x) = \sum_{|k| \leq L_n, l \in \mathbb{Z}} \exp(ilt) a_{l-k}(x) h_k(x).
\]
and, recalling (71) and (27),

\[ P_n \Pi_W(a(t, x) h) = \sum_{|k|, |l| \leq L_n} \exp(i k t) \pi_l(a_{l-k}(x) h_k) \]

\[ = \sum_{|k| \leq L_n} \exp(i k t) \pi_k(a_0(x) h_k) + \sum_{|k|, |l| \leq L_n, k \neq l} \exp(i l t) \pi_l(a_{l-k} h_k) \]  

(75)

where we have distinguished the “diagonal” term

\[ \sum_{|k| \leq L_n} \exp(i k t) \pi_k(a_0(x) h_k) = P_n \Pi_W(a_0(x) h) \]  

(76)

with \( a_0(x) := \frac{1}{2\pi} \int_0^{2\pi} a(t, x) \, dt \), from the “off-diagonal Toeplitz” term

\[ \sum_{|k|, |l| \leq L_n, k \neq l} \exp(i l t) \pi_l(a_{l-k} h_k) = P_n \Pi_W(\pi(t, x) h) \]  

(77)

where

\[ \pi(t, x) := a(t, x) - a_0(x) \]

has zero time-average.

By (72), (75), (76) and (77), we can decompose

\[ \mathcal{L}_n(\delta, v_1, w) = D - \mathcal{M}_1 - \mathcal{M}_2 \]

where \( D, \mathcal{M}_1, \mathcal{M}_2 \) are the linear operators

\[
\begin{align*}
Dh := L_n h - \varepsilon P_n \Pi_W(a_0(x) h) \\
\mathcal{M}_1 h := \varepsilon P_n \Pi_W(\pi(t, x) h) \\
\mathcal{M}_2 h := \varepsilon P_n \Pi_W(a(t, x) \partial_u v_2[h]) .
\end{align*}
\]  

(78)

To invert \( \mathcal{L}_n \) we first (Step 1) prove that, assuming the “first order Melnikov non-resonance condition \( \delta \in \Delta_n^{-\gamma}(v_1, w) \) [see Definition 3.3] the diagonal (in time) linear operator \( D \) is invertible, see Corollary 4.2. Next (Step 2) we prove that the “off-diagonal Toeplitz” operator \( \mathcal{M}_1 \) (Lemma 4.8) and \( \mathcal{M}_2 \) (Lemma 4.9) are small enough with respect to \( D \), yielding the invertibility of the whole \( \mathcal{L}_n \) (note that we do not decompose the term \( \mathcal{M}_2 \) in a diagonal and “off-diagonal term”). More precisely, the crucial bounds of Lemma 4.5 enable us to prove via Lemma 4.6 that the operator \(|D|^{-1/2} \mathcal{M}_1|D|^{-1/2}\) has small norm, whereas the norm of \(|D|^{-1/2} \mathcal{M}_2|D|^{-1/2}\) is controlled thanks to the regularizing properties of the map \( v_2 \).

4.2 Step 1: Inversion of \( D \)

The first aim is to diagonalize (both in time and space) the linear operator \( D \), see Corollary 4.1. By (74) and (76), the operator \( D \) is yet diagonal in time-Fourier basis, and, \( \forall \ h \in W^{(n)} \), the \( k^{th} \) time Fourier coefficient of \( Dh \) is

\[ (Dh)_k = (\omega^2 k^2 + \partial_x x) h_k - \varepsilon \pi_k(a_0(x) h_k) \equiv D_k h_k \]

where \( D_k : \mathcal{D}(D_k) \subset F_k \to F_k \) is the operator

\[ D_k u = \omega^2 k^2 u - S_k u \quad \text{and} \quad S_k u := -\partial_x x u + \varepsilon \pi_k(a_0(x) u) . \]

Note that \( S_k = S_{-k} \).

We now have to diagonalize (in space) each Sturm-Liouville type operator \( S_k \) and to study its spectral properties.
In the next Lemma 4.1 we shall find a basis of eigenfunctions \( v_{k,j} \) of \( S_k : D(S_k) \subset F_k \to F_k \) which are orthonormal for the scalar product of \( F_k \),

\[
(u, v)_\varepsilon := \int_0^\pi u_xv_x + \varepsilon a_0(x)uv \, dx.
\]

For \( |\varepsilon| a_0|_{\infty} < 1 \), \( (\cdot, \cdot)_\varepsilon \) actually defines a scalar product on \( F_k \subset H_0^1((0, \pi); \mathbb{R}) \) and its associated norm is equivalent to the \( H^1 \)-norm defined by \( ||u||_{H^1}^2 := \int_0^\pi u_x^2(x) \, dx \), since

\[
||u||_{H^1}^2 \left( 1 - |\varepsilon| a_0|_{\infty} \right) \leq ||u||_\varepsilon^2 \leq ||u||_{H^1}^2 \left( 1 + |\varepsilon| a_0|_{\infty} \right) \quad \forall u \in F_k. \tag{79}
\]

(79) follows from \( \int_0^\pi u(x)^2 \, dx \leq \int_0^\pi u_x(x)^2 \, dx, \forall u \in H_0^1(0, \pi), \) and

\[
\left| \int_0^\pi \varepsilon a_0(x)u^2 \, dx \right| \leq |\varepsilon| a_0|_{\infty} \int_0^\pi u^2 \, dx.
\]

**Lemma 4.1 (Sturm-Liouville)** The operator \( S_k : D(S_k) \subset F_k \to F_k \) possesses a \( (\cdot, \cdot)_\varepsilon \)-orthonormal basis \( (v_{k,j})_{j \geq 1,j \neq |k|} \) of eigenvectors with positive, simple eigenvalues

\[
0 < \lambda_{k,1} < \ldots < \lambda_{k,|k|-1} < \lambda_{k,|k|+1} < \ldots < \lambda_{k,j} < \ldots \quad \text{with} \quad \lim_{j \to \infty} \lambda_{k,j} = +\infty
\]

and \( \lambda_{k,j} = \lambda_{-k,j}, v_{-k,j} = v_{k,j} \). Moreover, \( (v_{k,j})_{j \geq 1,j \neq |k|} \) is an orthogonal basis also for the \( L^2 \)-scalar product in \( F_k \).

The asymptotic expansion as \( j \to +\infty \) of the eigenfunctions \( \varphi_{k,j} := v_{k,j}/\|v_{k,j}\|_{L^2} \) of \( S_k \) and its eigenvalues \( \lambda_{k,j} \) is

\[
\varphi_{k,j} - \sqrt{\frac{2}{\pi}} \sin(jx) = O\left( \frac{|\varepsilon| a_0|_{\infty}}{j} \right)
\]

and

\[
\lambda_{k,j} = \lambda_{k,j}(\delta, v_1, w) = j^2 + \varepsilon M(\delta, v_1, w) + O\left( \frac{|\varepsilon| a_0|_{H^1}}{j} \right) \tag{80}
\]

where \( M(\delta, v_1, w) \), introduced in Definition 3.1, is the mean value of \( a_0(x) \) on \( (0, \pi) \).

**Proof.** In the Appendix. We remark that we do not directly apply some known result for Sturm-Liouville operators because of the projection \( \pi_k \). ■

By Lemma 4.1 each linear operator \( D_k : D(D_k) \subset F_k \to F_k \) possesses a \( (\cdot, \cdot)_\varepsilon \)-orthonormal basis \( (v_{k,j})_{j \geq 1,j \neq |k|} \) of real eigenvectors with real eigenvalues \( (\omega^2 k^2 - \lambda_{k,j})_{j \geq 1,j \neq |k|} \).

As a consequence

**Corollary 4.1 (Diagonalization of \( D \))** The operator \( D \) (acting in \( W^n \)) is the diagonal operator \( \text{diag}\{\omega^2 k^2 - \lambda_{k,j}\} \) in the basis \( \{\cos(kt)\varphi_{k,j} : k \geq 0, j \geq 1, j \neq |k|\} \) of \( W^n \).

By Lemma 4.1

\[
\min_{|k| \leq L_n} |\omega^2 k^2 - \lambda_{k,j}| \to +\infty \quad \text{as} \quad j \to +\infty,
\]

and so, by Corollary 4.1 the linear operator \( D \) is invertible iff all its eigenvalues \( \{\omega^2 k^2 - \lambda_{k,j}\}_{|k| \leq L_n, j \geq 1, j \neq |k|} \) are different from zero.

In this case, we can define \( D^{-1} \) as well as \( |D|^{-1/2} : W^n \to W^n \) by

\[
|D|^{-1/2}h := \sum_{|k| \leq L_n} \exp(ikt)|D_k|^{-1/2}h_k, \quad \forall h = \sum_{|k| \leq L_n} \exp(ikt)h_k
\]

\(^2\)Because the least eigenvalue of \(-\partial_{xx}\) with Dirichlet B.C. on \((0, \pi)\) is 1.
where $|D_k|^{-1/2} : F_k \to F_k$ is the diagonal operator defined by

$$|D_k|^{-1/2} v_{k,j} := \frac{v_{k,j}}{\sqrt{\omega^2 k^2 - \lambda_{k,j}}}, \quad \forall j \geq 1, \ j \neq |k|.$$  

(81)

The “small divisor problem” (i) is that some of the eigenvalues of $D_k \omega^2 k^2 - \lambda_{k,j}$, can become arbitrarily small for $(k, j) \in \mathbb{Z}^2$ sufficiently large and therefore the norm of $|D|^{-1/2}$ can become arbitrarily large as $L_n \to \infty$.

In order to quantify this phenomenon, we define for all $|k| \leq L_n$

$$\alpha_k := \min_{j \neq |k|} |\omega^2 k^2 - \lambda_{k,j}|.$$  

(82)

Note that $\alpha_{-k} = \alpha_k$.

**Lemma 4.2** Suppose $\alpha_k \neq 0$. Then $D_k$ is invertible and, for $\varepsilon$ small enough,

$$\left\| |D_k|^{-1/2} u \right\|_{H^1} \leq \frac{2}{\sqrt{\alpha_k}} \|u\|_{H^1}.$$  

(83)

**Proof.** For any $u = \sum_{j \neq |k|} u_j v_{k,j} \in F_k$, by (81), and using that $(v_{k,j})_{j \neq |k|}$ is an orthonormal basis for the $(\ , \ )_\varepsilon$ scalar product on $F_k$,

$$\left\| |D_k|^{-1/2} u \right\|_{\varepsilon}^2 = \left\| \sum_{j \neq |k|} \frac{u_j v_{k,j}}{\sqrt{\omega^2 k^2 - \lambda_{k,j}}} \right\|_{\varepsilon}^2 = \sum_{j \neq |k|} \frac{|u_j|^2}{\omega^2 k^2 - \lambda_{k,j}} \leq \frac{1}{\alpha_k} \sum_{j \neq |k|} |u_j|^2 = \frac{\|u\|^2}{\alpha_k}.$$  

Hence, since, by (80), the norms $\| \cdot \|_{\varepsilon}$ and $\| \cdot \|_{H^1}$ are equivalent, (83) follows for $\varepsilon$ small enough. □

The condition “$\alpha_k \neq 0$, $\forall |k| \leq L_n$” depends very sensitively on the parameters $(\delta, v_1)$. Assuming the “first order Melnikov non-resonance condition” $\delta \in \Delta^{\varepsilon, \tau}_n(v_1, w)$ (see Definition 3.3) with $\tau \in (1, 2)$, we obtain, in Lemma 4.3, a lower bound of the form $c^\gamma/|k|^{\tau - 1}$ for the moduli of the eigenvalues of $D_k$ (namely $\alpha_k \geq c^\gamma/|k|^{\tau - 1}$) and, therefore, in Corollary 4.2 sufficiently good estimates for the inverse of $D$.

**Lemma 4.3** (Lower bound for the eigenvalues of $D$) There is $c > 0$ such that if $\delta \in \Delta^{\varepsilon, \tau}_n(v_1, w) \cap [0, \delta_0)$ and $\delta_0$ is small enough (depending on $\gamma$), then

$$\alpha_k := \min_{j \geq 1, j \neq |k|} |\omega^2 k^2 - \lambda_{k,j}| \geq \frac{c^\gamma}{|k|^{\tau - 1}} > 0, \quad \forall 0 < |k| \leq L_n.$$  

(84)

Moreover $\alpha_0 \geq 1/2$.

**Proof.** Since $\alpha_{-k} = \alpha_k$ it is sufficient to consider $k \geq 0$. By the asymptotic expansion (80) for the eigenvalues $\lambda_{k,j}$, using that $\|a_0\|_{H^1}, \ |M(\delta, v_1, w)| \leq C$,

$$|\omega^2 k^2 - \lambda_{k,j}| = \left| \omega^2 k^2 - j^2 - \varepsilon M(\delta, v_1, w) + O\left(\frac{\varepsilon \|a_0\|_{H^1}}{j}\right) \right|$$

$$= \left| \left( \omega k - \sqrt{j^2 + \varepsilon M(\delta, v_1, w)} \right) \left( \omega k + \sqrt{j^2 + \varepsilon M(\delta, v_1, w)} \right) + O\left(\frac{\varepsilon}{j}\right) \right|$$

$$\geq \left| \omega k - j - \varepsilon \frac{M(\delta, v_1, w)}{2j} \right| + O\left(\frac{\varepsilon^2}{j^2} \frac{|\varepsilon|}{j} \right) \omega k - C \frac{|\varepsilon|}{j}$$

$$\geq \left| \omega k - j - \varepsilon \frac{M(\delta, v_1, w)}{2j} \right| \omega k - C \left(\frac{\varepsilon^2 k^2}{j^2} + \frac{|\varepsilon|}{j} \right) \geq \frac{\gamma \omega k}{(k + \varepsilon)^\tau} - C \frac{\varepsilon^2 k^2}{j^2} - \frac{|\varepsilon|}{j},$$

(85)

since $\delta \in \Delta^{\varepsilon, \tau}_n(v_1, w)$. If $\alpha_k := \min_{j \geq 1, j \neq |k|} |\omega^2 k^2 - \lambda_{k,j}|$ is attained at $j = j(k)$, i.e. $\alpha_k = |\omega^2 k^2 - \lambda_{k,j}|$, then $|\omega k - j| \leq 1$ (provided $|\varepsilon|$ is small enough). Therefore, using that $1 < \tau < 2$ and $|\omega - 1| \leq 2|\varepsilon|$, we can derive (84) from (85), for $|\varepsilon|$ small enough. □
Corollary 4.2 (Estimate of $|D|^{-1/2}$) If $\delta \in \Delta_{n}^{\gamma, \tau}(v_{1}, w) \cap [0, \delta_{0})$ and $\delta_{0}$ is small enough, then $D : D(D) \subset W^{(n)} \rightarrow W^{(n)}$ is invertible and $\forall s' \geq 0$

$$\left\| |D|^{-1/2}h \right\|_{\sigma, s'} \leq \frac{C}{\sqrt{\gamma}} \| h \|_{\sigma, s' + \frac{\gamma}{\gamma - 1}} \quad \forall h \in W^{(n)}. \quad (86)$$

PROOF. Since $|D|^{-1/2}h := \sum_{|k| \leq L_{n}} \exp(ikt)|Dk|^{-1/2}h_{k}$, we get, using (83) and (84),

$$\left\| |D|^{-1/2}h \right\|_{\sigma, s'}^{2} = \sum_{|k| \leq L_{n}} \exp(2\sigma|k|)(1 + k^{2s'}) \left\| |D|^{-1/2}h_{k} \right\|_{H^{1}}^{2} \leq \sum_{|k| \leq L_{n}} \exp(2\sigma|k|)(1 + k^{2s'}) \frac{4}{\alpha_{k}} \| h_{k} \|_{H^{1}}^{2} \leq 8\| h_{0} \|_{H^{1}}^{2} + C \sum_{0 < |k| \leq L_{n}} \exp(2\sigma|k|)(1 + k^{2s'}) \frac{|k|^{\tau-1}}{\gamma} \| h_{k} \|_{H^{1}}^{2} \leq \frac{C}{\gamma} \| h \|_{\sigma, s'}^{2} \leq \frac{C}{\gamma} \| h \|_{\sigma, s' + \frac{\gamma}{\gamma - 1}}^{2}$$

proving (86). ■

4.3 Step 2: Inversion of $L_{n}$

To show the invertibility of $L_{n} : D(L_{n}) \subset W^{(n)} \rightarrow W^{(n)}$ it is a convenient devise to write

$$L_{n} = D - M_{1} - M_{2} = |D|^{1/2}(U - R_{1} - R_{2})|D|^{1/2}$$

where

$$U := |D|^{-1/2}D|D|^{-1/2} = |D|^{-1}D \quad \text{and} \quad R_{i} := |D|^{-1/2}M_{i}|D|^{-1/2}, \quad i = 1, 2.$$ 

We shall prove the invertibility of $U - R_{1} - R_{2}$ showing that, for $\varepsilon$ small enough, $R_{1}$ and $R_{2}$ are small perturbations of $U$.

Lemma 4.4 (Estimate of $\|U^{-1}\|$) $U : W^{(n)} \rightarrow W^{(n)}$ is an invertible operator and its inverse $U^{-1}$ satisfies, $\forall s' \geq 0$,

$$\left\| U^{-1}h \right\|_{\sigma, s'} = \| h \|_{\sigma, s'} \left(1 + O(\varepsilon \| a_{0} \|_{H^{1}})\right) \quad \forall h \in W^{(n)}. \quad (87)$$

PROOF. $U_{k} := |Dk|^{-1}D_{k} : F_{k} \rightarrow F_{k}$ being orthogonal for the $(\cdot, \cdot)_{\varepsilon}$ scalar product, it is invertible and $\forall u \in F_{k}, \| U_{k}^{-1}u \|_{\varepsilon} = \| u \|_{\varepsilon}$. Hence, by (79),

$$\forall u \in F_{k}, \quad \| U_{k}^{-1}u \|_{H^{1}} = \| u \|_{\varepsilon}(1 + O(\varepsilon \| a_{0} \|_{H^{1}}))$$.

Therefore, $U = |D|^{-1}D$, being defined by $(Uh)_{k} = U_{k}h_{k}, \forall |k| \leq L_{n}, U$ is invertible, $(U^{-1}h)_{k} = U_{k}^{-1}h_{k}$ and (87) holds. ■

The estimate of the “off-diagonal” operator $R_{1} : W^{(n)} \rightarrow W^{(n)}$ requires a careful analysis of the “small divisors” and the use of the “first order Mehnikov non-resonance condition” $\delta \in \Delta_{n}^{\gamma, \tau}(v_{1}, w)$, see Definition 3.3. For clarity, we enounce such property separately.

Lemma 4.5 (Analysis of the Small Divisors) Let $\delta \in \Delta_{n}^{\gamma, \tau}(v_{1}, w) \cap [0, \delta_{0}), \text{ with } \delta_{0} \text{ small. There exists } C > 0 \text{ such that, } \forall l \neq k,$

$$\frac{1}{\alpha_{k}\alpha_{l}} \leq C \frac{|k - l|^{2\gamma} \gamma}{\gamma^{2}|\varepsilon|^{|\gamma| - 1}} \quad \text{where} \quad \beta := \frac{2 - \tau}{\tau}. \quad (88)$$
PROOF. To obtain (88) we distinguish different cases.

- **FIRST CASE:** \(|k - l| \geq (1/2)[\max(|k|, |l|)]^{\beta}\). Then \((\alpha_k\alpha_l)^{-1} \leq C|k - l|^{2\tau - \gamma}/\gamma^2\).

Indeed we can estimate both \(\alpha_k\), \(\alpha_l\) with the lower bound \([84]\), \(\alpha_k \geq c_\gamma/|k|^{\tau - 1}\), \(\alpha_l \geq c_\gamma/|l|^{\tau - 1}\). Using that \(0 < \beta < 1\), we obtain

\[
\frac{1}{\alpha_k\alpha_l} \leq C|k|^{\tau - 1}l|^{\tau - 1}/\gamma^2 \leq C\left[\max(|k|, |l|)\right]^{2(\tau - 1)}/\gamma^2 \leq C'|k - l|^{2\tau - \gamma}/\gamma^2.
\]

In the other cases we have \(0 < |k - l| < (1/2)[\max(|k|, |l|)]^{\beta}\). We observe that in this situation, \(\text{sign}(l) = \text{sign}(k)\) and, to fix the ideas, we assume in the sequel that \(l, k \geq 0\) (the estimate for \(k, l < 0\) is the same, since \(\alpha_k\alpha_l = \alpha_{-k}\alpha_{-l}\)). Moreover, since \(\beta \leq 1\), we have \(\max(k, l) = k\) or \(l - k \leq (1/2)|l|^{\beta} \leq (1/2)|l|\). Hence \(l \leq 2k\); similarly \(k \leq 2l\).

- **SECOND CASE:** \(0 < |k - l| < (1/2)[\max(|k|, |l|)]^{\beta}\) and \((|k| \leq 1/3|\varepsilon|\text{ or }|l| \leq 1/3|\varepsilon|)\). Then \((\alpha_k\alpha_l)^{-1} \leq C/\gamma\).

Suppose, for example, that \(0 \leq k \leq 1/3|\varepsilon|\). We claim that if \(\varepsilon\) is small enough, then \(\alpha_k \geq (k + 1)/8\).

Indeed, \(\forall j \neq k\),

\[
|\omega_k - j| = |\omega_k - k - j| \geq |k - j| - |\omega - 1| \geq 1 - 2|\varepsilon| k \geq \frac{1}{3}.
\]

Therefore \(\forall j \leq k \leq 1/3|\varepsilon|\), \(\forall j \neq k, j \geq 1, |\omega^2k^2 - j^2| = |\omega_k - j| |\omega_k + j| \geq (\omega k + 1)/3 \geq (k + 1)/6\) and so

\[
\alpha_k := \min_{j \geq 1, k \neq j} |\omega^2k^2 - \lambda_{k,j}| = \min_{j \geq 1, k \neq j} |\omega^2k^2 - j^2 - \varepsilon M(\delta, \nu_1, w) + O\left(\frac{|\varepsilon|\|a_0\|_{L^1}}{j}\right)| \geq \frac{k + 1}{6} - |\varepsilon| C \geq \frac{k + 1}{8}.
\]

Next, we estimate \(\alpha_l\). If \(0 \leq l \leq 1/3|\varepsilon|\) then \(\alpha_l \geq 1/8\) and therefore \((\alpha_k\alpha_l)^{-1} \leq 64\). If \(l > 1/3|\varepsilon|\), we estimate \(\alpha_l\) with the lower bound \([84]\) and so, since \(l \leq 2k\) and \(1 < \tau < 2\)

\[
\frac{1}{\alpha_k\alpha_l} \leq C\left|\frac{\tau - 1}{\gamma}\right| \leq C'\left|\frac{\tau - 1}{k^2 - \gamma}\right| \leq C'/\gamma.
\]

In the remaining cases we consider \(|k - l| < (1/2)[\max(|k|, |l|)]^{\beta}\) and both \(|k|, |l| > 1/3|\varepsilon|\). We have to distinguish two sub-cases. For this, \(\forall k \in \mathbb{Z}\), let \(j = \bar{j}(k) \geq 1\) be an integer such that \(\alpha_k := \min_{l \neq k} |\omega^2k^2 - \lambda_{k,n}| = |\omega^2k^2 - \lambda_{j,k}|\). Analogously let \(i = \bar{i}(k) \geq 1\) be an integer such that \(\alpha_l := |\omega^2l^2 - \lambda_{i,l}|\).

- **THIRD CASE:** \(0 < |k - l| < (1/2)[\max(|k|, |l|)]^{\beta}\), \(|k|, |l| > 1/3|\varepsilon|\text{ and }|k - l| = |j - i|\). Then \((\alpha_k\alpha_l)^{-1} \leq C/\gamma|\varepsilon|^{\tau - 1}\).

Indeed \(|\omega_k - j| - (\omega_l - i)| = |\omega(k - l) - (j - i)| = |\omega - 1||k - l| \geq |\varepsilon|/2\) and therefore \(|\omega_k - j| \geq |\varepsilon|/4\) or \(|\omega_l - i| \geq |\varepsilon|/4\). Assume for instance that \(|\omega_k - j| \geq |\varepsilon|/4\). Then \(|\omega^2k^2 - j^2| = |\omega(k - j)| |\omega_k + j| \geq |\varepsilon|\omega_k/2 \geq |\varepsilon|(1/2\varepsilon)(k/2)\) and so, for \(\varepsilon\) small enough, \(\alpha_k \geq |\varepsilon| k/4\). Hence, since \(l \leq 2k\) and \(k > 1/3|\varepsilon|\),

\[
\frac{1}{\alpha_k\alpha_l} \leq C\left|\frac{\tau - 1}{\gamma|\varepsilon|k}\right| \leq \frac{C}{\gamma|\varepsilon|^{\tau - 1}} \leq \frac{C}{\gamma|\varepsilon|^{\tau - 1}}.
\]

- **FOURTH CASE:** \(0 < |k - l| < (1/2)[\max(k, l)]^{\beta}, k, l > 1/3|\varepsilon|\text{ and }|k - l| \neq |j - i|\). Then \((\alpha_k\alpha_l)^{-1} \leq C/\gamma^2\).

Using that \(\omega\) is \(\gamma\tau\)-Diophantine, we get

\[
\left|\omega(k - l) - (j - i)\right| \geq \frac{\gamma}{|k - l|^{\tau - 1}} \geq \frac{C\gamma}{\left|\max(k, l)\right|^{\beta\tau}} \geq \frac{C}{2} \left(\frac{\gamma}{k^{\beta\tau}} + \frac{\gamma}{l^{\beta\tau}}\right).
\]
so that $|wk - j| \geq C\gamma/2k^{3+\beta}$ or $|wl - i| \geq C\gamma/2l^{3+\beta}$. Therefore $|\omega^2k^2 - j^2| \geq C'\gamma k^{1-\beta} = C'\gamma k^{\tau-1}$, since $\beta := (2 - \tau)/\tau$. Hence, for $\varepsilon$ small enough, $\alpha_l \geq C'\gamma k^{\tau-1}/2$. We estimate $\alpha_l$ with the worst possible lower bound and so, using also $l \leq 2k$, we obtain

$$\frac{1}{\alpha_k\alpha_l} \leq \frac{CL^{\tau-1}}{C'\gamma^2} \leq \frac{C}{\gamma^2}.$$  

Collecting the estimates of all the previous cases, [88] follows.

**Remark 4.1** The analysis of the small divisors in the Cases II)-III)-IV) of the previous Lemma corresponds, in the language of [12], to the property of "separation of the singular sites".

**Lemma 4.6 (Bound of an off-diagonal operator)** Assume that $\delta \in \Delta_n^{-\tau}(v_1, w) \cap [0, \delta_0)$ and let, for some $s' \geq s$, $b(t, x) \in \mathcal{X}_{s, s'} + \frac{1}{2}$ satisfy $b_0(x) = 0$, i.e. $\int_0^{2\pi} b(t, x) dt \equiv 0, \forall x \in (0, \pi)$. Define the operator $T_n : W^{(n)} \to W^{(n)}$ by

$$T_n h := |D|^{-1/2} P_n \Pi_W \left(b(t, x) |D|^{-1/2} h\right).$$

There is a constant $\tilde{C}$, independent of $b(t, x)$ and of $n$, such that

$$\left\|T_n h\right\|_{\sigma, s'} \leq \frac{\tilde{C}}{|\varepsilon|^2} \left\|b\right\|_{\sigma, s'} \frac{\tilde{C}}{|\varepsilon|^2} \left\|h\right\|_{\sigma, s'} \quad \forall h \in W^{(n)}.$$  

**Proof.** For $h \in W^{(n)}$, we have $(T_n h)(t, x) = \sum_{|k| \leq L_n} (T_n h)_k(x) \exp(ikt)$, with

$$(T_n h)_k = |D|^{-1/2}\pi_k \left(b \left|D\right|^{-1/2} h\right)_k = |D|^{-1/2}\pi_k \left[\sum_{|l| \leq L_n} b_{k-l} |D|^{-1/2} h_l\right].$$  

(89)

Set $B_m := \|b_m(x)\|_{H^1}$. From [88] and [88], using that $B_0 := \|b_0(x)\|_{H^1} = 0$,

$$\left\|T_n h\right\|_{H^1} \leq C \sum_{|l| \leq L_n, l \neq k} \frac{B_{k-l} \sqrt{\alpha_k \alpha_l}}{\sqrt{\alpha_k \alpha_l}} \|h_l\|_{H^1}. \quad (90)$$

Hence, by [88]

$$\left\|T_n h\right\|_{H^1} \leq \frac{C}{\gamma |\varepsilon|^{2+\tau}} s_k \quad \text{where} \quad s_k := \sum_{|l| \leq L_n} B_{k-l} |k-l|^{-\tau/2} \|h_l\|_{H^1}. \quad (91)$$

By [91], setting $\tilde{s}(t) := \sum_{|k| \leq L_n} s_k \exp(ikt)$ (with $s_{-k} = s_k$),

$$\left\|T_n h\right\|_{\sigma, s'}^2 = \sum_{|k| \leq L_n} \exp(2\sigma |k|)(k^{2s'} + 1) \left\|T_n h\right\|_{H^1}^2 \leq \frac{C^2}{\gamma^2 |\varepsilon|^{2+\tau}} \sum_{|k| \leq L_n} \exp(2\sigma |k|)(k^{2s'} + 1) s_k^2 = \frac{C^2}{\gamma^2 |\varepsilon|^{2+\tau}} \left\|\tilde{s}\right\|_{\sigma, s'}^2. \quad (92)$$

It turns out that $\tilde{s} = P_n(\tilde{b}c)$ where $\tilde{b}(t) := \sum_{l \leq L_n} |l|^{-\tau/2} B_l \exp(ilt)$ and $\tilde{c}(t) := \sum_{|l| \leq L_n} \|h_l\|_{H^1} \exp(ilt)$. Therefore, by [92] and since $s' > 1/2$,

$$\left\|T_n h\right\|_{\sigma, s'} \leq \frac{C}{\gamma |\varepsilon|^{2+\tau}} \|\tilde{b}\|_{\sigma, s'} \leq \frac{C}{\gamma |\varepsilon|^{2+\tau}} \|\tilde{b}\|_{\sigma, s'} \|\tilde{c}\|_{\sigma, s'} \leq \frac{C}{\gamma |\varepsilon|^{2+\tau}} \|b\|_{\sigma, s'} \|\tilde{c}\|_{\sigma, s'} \quad \text{since} \ |\tilde{b}|_{\sigma, s'} \leq \|b\|_{\sigma, s'} \leq \|b\|_{\sigma, s'+1/2}, \quad \text{and} \ |\tilde{c}|_{\sigma, s'} = \|h\|_{\sigma, s'}.$$  

Before proving the smallness of the "off-diagonal" operator $R_1$ and of $R_2$ we need the following preliminary Lemma which gives a suitable estimate of the multiplicative function $a(t, x)$.
Lemma 4.7 There are \( \mu > 0, \delta_0 > 0 \) and \( C > 0 \) with the following property: if \( \|w\|_{\sigma,s} \leq 2R, \|w\|_{\sigma,s} \leq \mu \) and \( \delta \in [0, \delta_0) \), then \( \|a\|_{\sigma,s+2(\tau-1)} \leq C \).

Proof. By the Definition 3.2 of \( [w]_{\sigma,s} \) there are \( h_i \in W^i \), \( 0 \leq i \leq q \), and a sequence \( \sigma_i > \sigma \), such that \( w = h_0 + h_1 + \ldots + h_q \) and

\[
\sum_{i=0}^{q} \frac{\|h_i\|_{\sigma,s+2(\tau-1)}}{(\sigma_i - \sigma)^{2(\tau-1)}} \leq 2\|w\|_{\sigma,s} \leq 2\mu . \tag{93}
\]

An elementary calculus, using that \( \max_k \{ (\sigma_i - \sigma)k \} \leq C(\alpha)/(\sigma_i - \sigma)^{\alpha} \), gives

\[
\|h_i\|_{\sigma,s+2(\tau-1)} \leq C(\tau) \frac{\|h_i\|_{\sigma,s+2(\tau-1)}}{(\sigma_i - \sigma)^{2(\tau-1)}} . \tag{94}
\]

Hence, by (93)-(94)

\[
\|w\|_{\sigma,s+2(\tau-1)} \leq \sum_{i=0}^{q} \|h_i\|_{\sigma,s+2(\tau-1)} \leq \sum_{i=0}^{q} C(\tau) \frac{\|h_i\|_{\sigma,s+2(\tau-1)}}{(\sigma_i - \sigma)^{2(\tau-1)}} \leq C(\tau)2\mu .
\]

By Lemma 2.1, provided \( \delta_0 \) is small enough, also \( \|v_2(\delta, v_1, w)\|_{\sigma,s+2(\tau-1)} \leq C' \) and therefore

\[
\|a\|_{\sigma,s+2(\tau-1)} = \|\partial_t g(\delta, x, v_1 + w + v_2(\delta, v_1, w))\|_{\sigma,s+2(\tau-1)} \leq C .
\]

This bound is a consequence of the analyticity assumption \((H)\) on the nonlinearity \( f \), the Banach algebra property of \( X_{\sigma,s+2(\tau-1)/\beta} \), and can be obtained as in (22).

Lemma 4.8 (Estimate of \( R_1 \)) Under the hypotheses of (P3), there exists a constant \( C > 0 \) depending on \( \mu \) such that

\[
\|R_1 h\|_{\sigma,s+\frac{\tau-1}{\gamma}} \leq \|\partial_t\|_{\sigma,s+\frac{\tau-1}{\gamma}} \|h\|_{\sigma,s+\frac{\tau-1}{\gamma}} \quad \forall h \in W^\gamma .
\]

Proof. Recalling the definition of \( R_1 := |D|^{-1/2}M_1|D|^{-1/2} \) and \( M_1 \), and using Lemma 4.6 since \( \sigma(t, x) \) has zero time-average,

\[
\|R_1 h\|_{\sigma,s+\frac{\tau-1}{\gamma}} = \||D|^{-1/2}M_1|D|^{-1/2}h\|_{\sigma,s+\frac{\tau-1}{\gamma}} = \|\partial_t\|_{\sigma,s+\frac{\tau-1}{\gamma}} \|D|^{-1/2}P_{\delta,\pi}W(\pi)|D|^{-1/2}h\|_{\sigma,s+\frac{\tau-1}{\gamma}} \leq \|\partial_t\|_{\sigma,s+\frac{\tau-1}{\gamma}} \|\pi\|_{\sigma,s+\frac{\tau-1}{\gamma}} \|h\|_{\sigma,s+\frac{\tau-1}{\gamma}} \leq \|\partial_t\|_{\sigma,s+\frac{\tau-1}{\gamma}} \|h\|_{\sigma,s+\frac{\tau-1}{\gamma}}
\]

since \( 0 < \beta < 1 \) and, by Lemma 4.7, \( \|\pi\|_{\sigma,s+\frac{\tau-1}{\gamma}} \leq \|a\|_{\sigma,s+\frac{\tau-1}{\gamma}} \leq C . \)

The “smallness” of \( R_2 := |D|^{-1/2}M_2|D|^{-1/2} \) w.r.t. \( U \), is just a consequence of Lemma 4.7 and of the regularizing property of \( \partial_w v_2 : X_{\sigma,s} \to X_{\sigma,s+2} \) proved in Lemma 2.1, by (80) the “loss of \( \tau-1 \) derivatives” due to \( |D|^{-1/2} \) applied twice, is compensated by the gain of 2 derivatives due to \( \partial_w v_2 : X_{\sigma,s} \to X_{\sigma,s+2} \).

Lemma 4.9 (Estimate of \( R_2 \)) Under the hypotheses of (P3), there exists a constant \( C > 0 \) depending on \( \mu \) such that

\[
\|R_2 h\|_{\sigma,s+\frac{\tau-1}{\gamma}} \leq C\|\partial_t\|_{\sigma,s+\frac{\tau-1}{\gamma}} \|h\|_{\sigma,s+\frac{\tau-1}{\gamma}} \quad \forall h \in W^\gamma .
\]
Once the \(|Q5\) solution of the \(|Q\) finite dimensional \(|Q\) where
\[\begin{align*}
\text{Hence also the linear operator } \delta &\text{ invertible by Lemma 4.4 and, by Lemmas 4.9 and 4.8, provided } \tau < 3 \text{ and by Lemma 4.7},
\|a\|_{\sigma,s+\tau-1} \leq \|a\|_{\sigma,s+2(\tau-1)/\gamma} \leq C. \quad \blacksquare
\end{align*}\]

Proof of property \((P3)\) completed. Under the hypothesis of \((P3)\), the linear operator \(U\) is invertible by Lemma 4.4 and, by Lemmas 4.9 and 4.8 provided \(\delta\) is small enough,
\[\left\|U^{-1}R_1\right\|_{\sigma,s+\tau-1}, \left\|U^{-1}R_2\right\|_{\sigma,s+\tau-1} < \frac{1}{4}.\]
Therefore also the linear operator \(U - R_1 - R_2\) is invertible and its inverse satisfies
\[\left\|(U - R_1 - R_2)^{-1}h\right\|_{\sigma,s+\tau-1} = \left\|(I - U^{-1}R_1 - U^{-1}R_2)^{-1}U^{-1}h\right\|_{\sigma,s+\tau-1} \leq 2\|U^{-1}h\|_{\sigma,s+\tau-1} \leq C\|h\|_{\sigma,s+\tau-1} \quad \forall h \in W^{(n)}.\]
Hence \(L_n\) is invertible, \(L_n^{-1} = |D|^{-1/2}(U - R_1 - R_2)^{-1}|D|^{-1/2} : W^{(n)} \to W^{(n)}\), and by \((86)\), \((95)\),
\[\left\|L_n^{-1}h\right\|_{\sigma,s} = \left\| |D|^{-1/2}(U - R_1 - R_2)^{-1}|D|^{-1/2}h\right\|_{\sigma,s} \leq \frac{C}{\sqrt{\gamma}} \left\|(U - R_1 - R_2)^{-1}|D|^{-1/2}h\right\|_{\sigma,s+\tau-1} \leq \frac{C'}{\gamma} \left\|h\right\|_{\sigma,s+\tau-1} \leq \frac{C''}{\gamma} (L_n)^{-1}\left\|h\right\|_{\sigma,s} \]
because \(h \in W^{(n)}\). This completes the proof of property \((P3)\).

5 Solution of the \((Q1)\)-equation

Once the \((Q2)\) and \((P)\)-equations are solved (with “gaps” for the latter), the last step is to find solutions of the finite dimensional \((Q1)\)-equation
\[\Delta v_1 = \Pi_{V_1} G(\delta,v_1) \quad (97)\]
where
\[G(\delta,v_1)(t,x) := g \left( \delta,x,v_1(t,x) + \bar{w}(\delta,v_1)(t,x) + v_2(\delta,v_1,\bar{w}(\delta,v_1))(t,x) \right).\]
We are interested in solutions \((\delta,v_1)\) which belong to the Cantor set \(B_\infty\).

5.1 The \((Q1)\)-equation for \(\delta = 0\)

For \(\delta = 0\) the \((Q1)\)-equation \((97)\) reduces to
\[- \Delta v_1 = \Pi_{V_1} G(0,v_1) = s^* \Pi_{V_1} \left( a_p(x)(v_1 + v_2(0,v_1,0))^p \right) \quad (98)\]
which is the Euler-Lagrange equation of \(\Psi_0 : B(2R,V_1) \to \mathbb{R},\)
\[\Psi_0(v_1) := \Phi_0(v_1 + v_2(0,v_1,0)), \quad (99)\]
where \( \Phi_0 : V \to \mathbb{R} \) is defined in \([12]\).

In fact, since \( v_2(0, v_1, 0) \) solves the \((Q2)\)-equation (for \( \delta = 0, w = 0 \)), \( d\Phi_0(v_1 + v_2(0, v_1, 0))[k] = 0, \forall k \in V_2 \). Moreover, since \( \forall h \in V_1, D_{v_1}v_2(0, v_1, 0)[h] \in V_2 \),

\[
d\Psi_0(v_1)[h] = d\Phi_0(v_1 + v_2(0, v_1, 0)) [h + D_{v_1}v_2(0, v_1, 0)[h]] = d\Phi_0(v_1 + v_2(0, v_1, 0))[h]
\]

where \( h \in V_1 \) and \( \Phi_0 \) is introduced in Theorem 1.2. Then \( v_1 := \Pi_{V_1} \bar{v} \in B(R, V_1) \) is a non-degenerate solution of \([98]\).

**Proof.** By Lemma \([2.11]\), \( \Pi_{V_2} \bar{v} = v_2(0, \bar{v}, 0) \). Hence, since \( \bar{v} \) solves \([11]\), \( v_1 \) solves \([98]\). Now assume that \( h_1 \in V_1 \) is a solution of the linearized equation at \( \bar{v} \) of \([98]\). This means

\[
-\Delta h_1 = s^*\Pi_{V_1} \left( p a_p(x)(\bar{v}_1 + v_2(0, \bar{v}_1, 0))^{p-1}(h_1 + h_2) \right)
\]

where \( h_2 := D_{v_1}v_2(0, \bar{v}_1, 0)[h_1] \in V_2 \). Now, by the definition of the map \( v_2 \), we have

\[
-\Delta v_2(0, v_1, 0) = s^*\Pi_{V_2} \left( a_p(x)(v_1 + v_2(0, v_1, 0))^p \right), \quad \forall v_1 \in B(2R, V_1),
\]

from which we derive, taking the differential at \( \bar{v} \),

\[
-\Delta h_2 = s^*\Pi_{V_2} \left( p a_p(x)(\bar{v}_1 + v_2(0, \bar{v}_1, 0))^{p-1}(h_1 + h_2) \right).
\]

Summing \([101]\) and \([102]\), we obtain that \( h = h_1 + h_2 \) is a solution of the linearized form at \( \bar{v} \) of equation \([11]\). Since \( \bar{v} \) is a non-degenerate solution of \([11]\), \( h = 0 \), hence \( h_1 = 0 \). As a result, \( \bar{v}_1 := \Pi_{V_1} \bar{v} \) is a non-degenerate solution of \([98]\). \( \square \)

### 5.2 Proof of Theorem 1.2

By assumption, \( \bar{v} \) is a non-degenerate solution of equation \([11]\). Hence, by Lemma 5.1, \( \bar{v}_1 := \Pi_{V_1} \bar{v} \in B(R, V_1) \) is a non-degenerate solution of \([98]\).

Since the map \( (\delta, v_1) \to -\Delta v_1 - \Pi_{V_1} g(\delta, v_1) \) is in \( C^\infty([0, \delta_0) \times V_1; V_1) \), by the Implicit Function Theorem, there is a \( C^\infty \) path

\[
\delta \mapsto v_1(\delta) \in B(2R, V_1)
\]

such that \( v_1(\delta) \) is a solution of \([97]\) and \( v_1(0) = \bar{v}_1 \).

By Theorem 3.1, the function

\[
\bar{u}(\delta) := \delta \left[ v_1(\delta) + v_2(\delta, v_1(\delta)), \bar{w}(\delta, v_1(\delta)) \right] \in X_{p/2, s}
\]

is a solution of equation \([3]\) if \( \delta \) belongs to the Cantor-like set

\[
C := \left\{ \delta \in [0, \delta_0) \mid (\delta, v_1(\delta)) \in B_\infty \right\}.
\]

By Proposition 3.2, the smoothness of \( v_1(\cdot) \) implies that the Cantor set \( C \) has full density at the origin, i.e. satisfies the measure estimate \([4]\).

Finally, by \([103]\), since \( \bar{v} = \bar{v}_1 + v_2(0, \bar{v}_1, 0) \),

\[
\left\| \bar{u}(\delta) - \delta \bar{v} \right\|_{p/2, s} = \delta \left\| (v_1(\delta) - \bar{v}_1) + (v_2(\delta, v_1(\delta)), \bar{w}(\delta, v_1(\delta))) - v_2(0, \bar{v}_1, 0) + \bar{w}(\delta, v_1(\delta)) \right\|_{p/2, s}
\]

\[
\leq \delta \left( \left\| v_1(\delta) - \bar{v}_1 \right\|_{p/2, s} + \left\| v_2(\delta, v_1(\delta)), \bar{w}(\delta, v_1(\delta)) \right\|_{p/2, s} - v_2(0, \bar{v}_1, 0) \right\|_{p/2, s}
\]

and \( O(\delta^2) \).
by \([7]\).

This proves Theorem 1.2 in the case when \(\pi\) is non-degenerate in the whole space \(V\).

Now, we can look for \(2\pi/n\) time-periodic solutions of \([8]\) as well (they are particular \(2\pi\) periodic solutions). Let

\[
X_{\pi,s,n} := \left\{ u \in X_{\pi,s} \mid u \text{ is } \frac{2\pi}{n} \text{ time - periodic} \right\} = V_n \oplus W_n
\]

where \(V_n\) (defined in \((14)\) and \(W_n\) are the subspaces of \(V\) and \(W\) formed by the functions \(2\pi/n\)-periodic in \(t\).

Introducing an appropriate finite dimensional subspace \(V_{1,n} \subset V_n\) we split \(V_n = V_{1,2} \oplus V_{2,n}\) and we obtain associated \((Q1),(Q2),(P)\)-equations as in \((15)\).

With the arguments of sections 2 and 3 we can solve the \((Q2)\) and \((P)\)-equations exactly as in the case \(n = 1\).

The 0th-order bifurcation equation is again equation \((11)\), but in \(V_n\), and the corresponding functional is just the restriction of \(\Phi_0\) to \(V_n\).

The main assumption of Theorem 1.2—that at least one of the critical points of \((\Phi_0|_{V_n})\), called \(\pi\), is non-degenerate—allows to find a \(C^\infty\)-path \(\delta \mapsto \nu_1(\delta)\) in \(V_{1,n}\) of solutions of equation \((97)\).

As above this implies the conclusions of Theorem 1.2

### 6 Proof of Theorem 1.1

For this section we define the linear map \(H_n : V \to V\) by:

\[
\text{for } v(t, x) = \eta(t + x) - \eta(t - x) \in V, \quad (H_n v)(t, x) := \eta(n(t + x)) - \eta(n(t - x))
\]

so that \(V_n = H_n V\).

#### 6.1 Case \(f(x, u) = a_3(x)u^3 + O(u^4)\)

**Lemma 6.1** Let \(\langle a_3 \rangle := (1/\pi) \int_0^\pi a_3(x) dx \neq 0\). Taking \(s^* = \text{sign}(a_3)\), \(\exists n_0 \in \mathbb{N}\) such that \(\forall n \geq n_0\) the 0th order bifurcation equation \((16)\) has a solution \(\pi \in V_n\) which is non-degenerate in \(V_n\).

**Proof.** Equation \((16)\) is the Euler-Lagrange equation of

\[
\Phi_0(v) = \frac{\|v\|_{H^1}^2}{2} - s^* \int_\Omega a_3(x) v^4 \frac{4}{4}.
\]

The functional \(\Phi_n(v) := \Phi_0(H_n v)\) has the following development: for \(v(t, x) = \eta(t + x) - \eta(t - x) \in V\) we obtain, using that \(\int_\Omega v^4 = \int_\Omega (H_n v)^4\),

\[
\Phi_n(v) = 2\pi n^2 \int_\Omega \eta^2(t) dt - s^* \langle a_3 \rangle \int_\Omega v^4 \frac{4}{4} - s^* \int_\Omega \left( a_3(x) - \langle a_3 \rangle \right)(H_n v)^4 \frac{4}{4}.
\]

We choose \(s^* = \text{sign}(a_3)\) so that \(s^* \langle a_3 \rangle > 0\). To simplify notations take \(\langle a_3 \rangle > 0\) so \(s^* = 1\).

\[
\Phi_n\left(\sqrt[2\pi n]{\langle a_3 \rangle} v\right) = \frac{8\pi n^4}{\langle a_3 \rangle} \frac{1}{2} \int_\Omega \eta(t) ds - \frac{1}{8\pi} \int_\Omega v^4 + \frac{1}{8\pi} \int_\Omega \left( a_3(x) \langle a_3 \rangle - 1 \right)(H_n v)^4 dt dx
\]

\[
= \frac{8\pi n^4}{\langle a_3 \rangle} \left[ \Psi(\eta) + R_n(v) \right]
\]

where

\[
\Psi(\eta) := \frac{1}{2} \int_\Omega \eta^2(t) dt - \frac{1}{4} \int_\Omega \eta^4(t) dt - \frac{3}{8\pi} \left( \int_\Omega \eta^2(t) dt \right)^2,
\]

\[
R_n(v) := \frac{1}{8\pi} \int_\Omega b(x)(H_n v)^4 dt dx, \quad b(x) := \frac{a_3(x)}{\langle a_3 \rangle} - 1.
\]
Let \( E := \{ \eta \in H^1(\mathbb{T}) \mid \eta \text{ is odd} \} \). It is enough to prove that \( \Psi : E \to \mathbb{R} \) has a non-degenerate critical point \( \eta \) and that \( \mathcal{R}_n \) is small for large \( n \), see Lemma 6.2. Indeed the operator \( \Psi''(\eta) \) has the form \( \text{Id} + \text{Compact} \) so that if its kernel is 0 then \( \Psi''(\eta) \) is invertible. Hence, by the implicit function theorem, for \( n \) large enough, \( \Phi_n \) too (hence \( \Phi_{0|\mathcal{R}_n} \)) has a non-degenerate critical point.

The critical points of \( \Psi \) in \( E \) are the 2\( \pi \)-periodic odd solutions of

\[
\ddot{\eta} + \eta^3 + 3(\eta^2)\eta = 0. \tag{105}
\]

By \cite{2} it is known that there exists a solution of \( \text{(105)} \) which is a non-degenerate critical point of \( \Psi \) in \( E \). It remains to prove Lemma 6.2.

\section*{Lemma 6.2}

There holds

\[
\| D\mathcal{R}_n(v) \|, \| D^2\mathcal{R}_n(v) \| \to 0 \quad \text{as} \quad n \to +\infty \tag{106}
\]

uniformly for \( v \) in bounded sets of \( E \).

\textbf{Proof.} We shall prove the estimate only for \( D^2\mathcal{R}_n \). We have

\[
D^2\mathcal{R}_n(v)[h,k] = \frac{3}{2\pi} \left( \int_0^\pi b(x)(\mathcal{H}_n v)^2(\mathcal{H}_n h) (\mathcal{H}_n k) \right) = \frac{3}{2\pi} \int_0^\pi b(x)g(nx) \, dx
\]

where \( g(y) \) is the \( \pi \)-periodic function defined by

\[
g(y) := \int_T (\eta(t+y) - \eta(t-y))^2(\beta(t+y) - \beta(t-y)) (\gamma(t+y) - \gamma(t-y)) \, dt,
\]

\( \beta \) and \( \gamma \) being associated with \( h \) and \( k \) as \( \eta \) is with \( v \). Developing in Fourier series \( g(y) = \sum_{l \in \mathbb{Z}} g_l \exp(2\pi i y) \) we have \( g(nx) = \sum_{l \in \mathbb{Z}} g_l \exp(2\pi i n x) \). Extending \( b(x) \) to a \( \pi \)-periodic function, we also write \( b(x) = \sum_{l \in \mathbb{Z}} b_l \exp(2\pi i x) \), with \( b_0 = \langle b \rangle = 0 \). Therefore

\[
\| D^2\mathcal{R}_n(v)[h,k] \| \leq \frac{3}{2} \left( \sum_{l \neq 0} \left| g_l b_l \right| \right) \leq \frac{3}{2} \left( \sum_{l \neq 0} g_l^2 \right)^{1/2} \left( \sum_{l \neq 0} b_l^2 \right)^{1/2} \leq \frac{3}{2} \| g \|_{L^2(0,\pi)} \left( \sum_{l \neq 0} b_l^2 \right)^{1/2} \leq C \| n \|_{\infty} \| \beta \|_{\infty} \| \gamma \|_{\infty} \left( \sum_{l \neq 0} b_l^2 \right)^{1/2} \leq C \| v_0 \|_{H^4} \| h \|_{H^4} \| k \|_{H^4} \left( \sum_{l \neq 0} b_l^2 \right)^{1/2}.
\]

Since \( \left( \sum_{l \neq 0} b_l^2 \right)^{1/2} \to 0 \) as \( n \to \infty \) it proves \( (106) \). With a similar calculus we can prove that \( D\mathcal{R}_n(v) \to 0 \) as \( n \to +\infty \).

\section{6.2 Case \( f(x,u) = a_2 u^2 + O(u^4) \)}

With the frequency-amplitude relation (17) system (7) with \( p = 2 \) becomes

\[
\begin{align*}
-\Delta v &= -\delta^{-1} \Pi_V g_\delta(x,v+w) \quad \text{(Q)} \\
L_\omega w &= \delta \Pi_W g_\delta(x,v+w) \quad \text{(P)}
\end{align*}
\tag{107}
\]

where

\[
g_\delta(x,u) = \frac{f(x,u)}{\delta^2} = a_2 u^2 + \delta^2 a_4(x) u^4 + \ldots.
\tag{108}
\]

With the further rescaling

\[
w \to \delta w
\]

and since \( v^2 \in W \), system (107) is equivalent to

\[
\begin{align*}
-\Delta v &= \Pi_V \left( -2a_2 v w - a_2 \delta w^2 - \delta r(\delta, x, v + \delta w) \right) \quad \text{(Q)} \\
L_\omega w &= a_2 v^2 + \delta \Pi_W \left( 2a_2 v w + \delta a_2 w^2 + \delta r(\delta, x, v + \delta w) \right) \quad \text{(P)}
\end{align*}
\tag{109}
\]
where $v(\delta, x, u) = \delta^{-4}(f(x, \delta u) - a_2\delta^2 u^2) = a_4(x)u^4 + ...$

For $\delta = 0$ system (109) reduces to

\[
\begin{aligned}
-\Delta v &= -2a_2\Pi_v(vw) \\
Lw &= a_2v^2
\end{aligned}
\] (110)

where $L := -\partial_{tt} + \partial_{xx}$, and it is equivalent to $w = a_2L^{-1}v^2$, $-\Delta v = -2a_2^2\Pi_v(vL^{-1}v^2)$, namely to the 0th order bifurcation equation (18).

\textbf{Lemma 6.3} If $a_2 \neq 0$, $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ the 0th order bifurcation equation (18) has a solution $v \in V_n$ which is non-degenerate in $V_n$.

\textbf{Proof.} We have to prove that $\Phi_n(v) := \Phi_0(H_n v)$, where $\Phi_0$ is defined in (19), possesses non-degenerate critical points at least for $n$ large.

$\Phi_n$ admits the following development (Lemmas 3.7 and 3.8 in [2]): for $v(t, x) = \eta(t + x) - \eta(t - x)$

\[
\Phi_n(v) = 2\pi n^2 \int_{T} \eta^2(t) dt - \frac{\pi^2 a_2^2}{12} \left( \int_{T} \eta^2(t) dt \right)^2 + \frac{a_2^2}{2n^2} \left( \int_{\Omega} v^2 L^{-1} v^2 + \frac{\pi^2}{6} \left( \int_{T} \eta^2(t) dt \right)^2 \right).
\]

Hence we can write

\[
\Phi_n \left( \frac{\sqrt{12n}}{\sqrt{\pi a_2}} v \right) = \frac{48n^4}{a_2^2} \left[ \frac{1}{2} \int_{T} \eta^2(s) ds - \frac{1}{4} \left( \int_{T} \eta^2(s) ds \right)^2 + \frac{1}{n^2} \mathcal{R}(\eta) \right] = \frac{48n^4}{a_2^2} \left[ \Psi(\eta) + \frac{1}{n^2} \mathcal{R}(\eta) \right] \quad (111)
\]

where

\[
\Psi(\eta) = \frac{1}{2} \int_{T} \eta^2(s) ds - \frac{1}{4} \left( \int_{T} \eta^2(s) ds \right)^2
\]

and $\mathcal{R} : E \to \mathbb{R}$ is a smooth functional defined on $E := \{\eta \in H^1(T) \mid \eta \text{ odd}\}$. By (111), in order to prove that $\Phi_n$ has a non-degenerate critical point for $n$ large enough, it is enough to prove the following Lemma. ■

\textbf{Lemma 6.4} $\Psi : E \to \mathbb{R}$ possesses a non-degenerate critical point.

\textbf{Proof.} The critical points of $\Psi$ in $E$ are the $2\pi$-periodic odd solutions of the equation

\[
\dot{\eta} + \left( \int_{T} \eta^2(t) dt \right) \eta = 0.
\] (112)

Equation (112) has a $2\pi$-periodic solution of the form $\eta(t) = (1/\sqrt{\pi}) \sin t$.

We claim that $\eta$ is non-degenerate. The linearized equation of (112) at $\eta$ is

\[
\dot{h} + h + \frac{2}{\pi} \left( \int_{T} \sin t h(t) dt \right) \sin t = 0.
\] (113)

Developing in time-Fourier series $h(t) = \sum_{k \geq 1} a_k \sin kt$ we find out that any solution of the linearized equation (113) satisfies

\[-k^2 a_k + a_k = 0, \quad \forall k \geq 2, \quad a_1 = 0
\]

and therefore $h = 0$. ■

As in Theorem 1.2 the existence of a solution $v$ of the 0th order bifurcation equation which is non degenerate in some $V_n$ entails the conclusions of Theorem 1.2. To avoid cumbersome notations, we still give the main arguments assuming that $n = 1$.

Since for $\delta = 0$ the solution of the $(P)$-equation in (110) is $w = a_2L^{-1}v^2$, it is convenient to perform the change of variable

\[
w = a_2L^{-1}v^2 + y, \quad y \in W.
\] (114)
Lemma 6.5

System (109) is then written

\[
\begin{cases}
-\Delta v &= -2a_2^2\Pi \nu(vL^{-1}v^2) + \Pi \nu \left(-2a_2y - a_2\delta w^2 - \delta r(\delta, x, v + \delta w)\right) \quad (Q') \\
L_\omega y &= 2a_2\delta^2\mathcal{R}(v^2) + \delta\Pi W \left(2a_2y + \delta a_2w^2 + \delta r(\delta, x, v + \delta w)\right) \quad (P')
\end{cases}
\]

where \( w \) is a function of \( v \) and \( y \) through (114), and the linear operator in \( W \)

\[ \mathcal{R} := (1 - \omega^2)^{-1}(I - L_\omega L^{-1}) = (2\delta^2)^{-1}(I - L_\omega L^{-1}) \]

does not depend on \( \omega \) and can be expressed as

\[ \mathcal{R} \left( \sum_{l \neq j} w_{l,j} \cos(lt) \sin(jx) \right) = \sum_{l \neq j} \frac{l^2}{l^2 - j^2} w_{l,j} \cos(lt) \sin(jx). \]

Since \(|l^2| - j^2|^2 = |l^2| + |j^2|^2|l - j|^2| \leq |l|, \) the operator \( \mathcal{R} \) satisfies the estimate

\[ \forall w \in W, \quad \|\mathcal{R}w\|_{\sigma,s} \leq \|w\|_{\sigma,s+1}. \]

(116)

Splitting \( V = V_1 \oplus V_2, \) the \((Q')\)-equation is divided in two parts: the \((Q')\) and the \((Q')\) equations.

Setting

\[ R := \|\mathcal{R}\|_{0,s} \]

the analogue of Lemma 2.1 is:

**Lemma 6.5** There exist \( N \in \mathbb{N}_+, \sigma = \log 2/N > 0, \delta_0 > 0, \) such that, \( \forall 0 \leq \sigma \leq \sigma, \forall \|v_1\|_{0,s} \leq 2R, \)

\[ \forall \|y\|_{\sigma,s} \leq 1, \forall \delta \in [0, \delta_0), \] there exists a unique solution \( v_2(\delta, v_1, y) \in V_2 \cap X_{\sigma,s} \) of the \((Q')\) equation with

\[ \|v_2(\delta, v_1, y)\|_{\sigma,s} \leq 1. \]

Moreover \( v_2(0, \Pi_{V_1} y, 0) = \Pi_{V_2} y, \) \( v_2(\delta, v_1, y) \in X_{\sigma,s+2} \) and the regularizing property

\[ \left\| D_\omega v_2(\delta, v_1, y)[h]\right\|_{\sigma,s+2} \leq C\|h\|_{\sigma,s} \]

(117)

holds, where \( C \) is some positive constant.

Substituting \( v_2 = v_2(\delta, v_1, y) \) into the \((P')\) equation yields

\[ L_\omega y = \delta\Gamma(\delta, v_1, y) := \delta\tilde\Gamma(\delta, v_1 + v_2(\delta, v_1, y), y) \]

(118)

where

\[ \tilde\Gamma(\delta, v, y) := 2\delta a_2\mathcal{R}(v^2) + \Pi W \left(2a_2y + \delta a_2w(2L^{-1}v^2) + \delta r(\delta, x, v + \delta w)\right) \]

\[ + \delta a_2(2L^{-1}v^2) + \delta r(\delta, x, v + \delta a_2L^{-1}v^2) + y) \).

The \((P')\)-equation (118) can be solved as in sections 3.1 with slight changes that we specify.

**Theorem 6.1** (Solution of the \((P')\)-equation) For \( \delta_0 > 0 \) small enough, there exists a \( C^\infty \)-function \( \tilde{y} : [0, \delta_0) \times B(2R, V_1) \rightarrow W \cap X_{\sigma,s} \) satisfying \( \tilde{y}(0, v_1) = 0, \|\tilde{y}\|_{\sigma,s} = O(\delta), \|D^{k}(\tilde{y})\|_{\sigma,s} = O(1), \) and verifying the following property: let

\[ B_{\infty} := \left\{ (\delta, v_1) \in [0, \delta_0) \times B(2R, V_1) : \left| \omega(\delta)l - j - \frac{M(\delta, v_1, \tilde{y}(\delta, v_1))}{2j} \right| \geq \frac{2\gamma}{(l + j)^{\tau}}, \forall l \geq \frac{1}{3\delta^2}, l \neq j \right\}, \]

where \( \omega(\delta) = \sqrt{1 - 2\delta^2} \) and \( M(\delta, v_1, y) \) is defined in (119). Then \( \forall (\delta, v_1) \in B_{\infty}, \tilde{y}(\delta, v_1) \) solves the \((P')\)-equation (118).
The key point is the inversion, at each step of the iterative process, of a linear operator
\[
\mathcal{L}_n(\delta, v_1, y)[h] = L_n h - \delta P_n \Pi \mathcal{D}_y \Gamma(\delta, v_1, y)[h], \quad h \in W^{(n)}.
\]
We have
\[
\mathcal{D}_y \Gamma(\delta, v_1, y)[h] = \mathcal{D}_y \Gamma(\delta, v_1 + v_2(\delta, v_1, y), y)[h] + \mathcal{D}_y \Gamma(\delta, v_1 + v_2(\delta, v_1, y), y)\mathcal{D}_y v_2(\delta, v_1, y)[h]
\]
and, as it can be directly verified,
\[
\mathcal{D}_y \Gamma(\delta, v, y)[h] = \Pi \mathcal{W} \left( (\partial_u g_3)(x, v + \delta w) \right)
\]
where \( g_3 \) is defined in \([108]\) and \( w \) is given by \([114]\). As in section \([1]\) setting \( a(t, x) := (\partial_u g_3)(x, v(t, x) + \delta w(t, x)) \), we can decompose \( \mathcal{L}_n(\delta, v_1, y) = D - \mathcal{M}_1 - \mathcal{M}_2 \) where (with the notations of section \([3]\))
\[
\begin{align*}
\mathcal{D}h & := L_n h - \delta P_n \Pi \mathcal{W}(a_0(x) h) \\
\mathcal{M}_1 h & := \delta P_n \Pi \mathcal{W}(\pi(t, x) h) \\
\mathcal{M}_2 h & := \delta P_n \Pi \mathcal{W} \mathcal{D}_x \Gamma(\delta, v_1 + v_2(\delta, v_1, y), y)\mathcal{D}_y v_2(\delta, v_1, y)[h].
\end{align*}
\]
As in Lemma \([4.1]\) the eigenvalues of the similarly defined operator \( S_k \) satisfy \( \lambda_{k, j} = j^2 + \delta M(\delta, v_1, y) + O(\delta/j) \), where
\[
M(\delta, v_1, y) := \frac{1}{\Omega} \int_{\Omega} (\partial_u g_3)(x, v_1 + v_2(\delta, v_1, y) + \delta w(t, x)) \, dx dt, \quad w = a_2 L^{-1}(v^2) + y. \tag{119}
\]
The bounds for the operator \( D \) (Lemma \([4.3]\) Corollary \([4.2]\) still hold assuming an analogous non resonance condition, and we can define in the same way the operators \( \mathcal{U}, \mathcal{R}_1, \mathcal{R}_2, \) with \( \|\mathcal{U}^{-1}h\|_{\sigma, s'} = (1 + O(\delta))\|h\|_{\sigma, s'} \). With the same arguments we obtain for \( \mathcal{R}_1 \) the bound
\[
\|\mathcal{R}_1 h\|_{\sigma, s + \frac{7}{4}} \leq \delta^{2-\tau} C \frac{\delta}{\gamma} \|h\|_{\sigma, s + \frac{3}{4}}
\]
which is enough since \( \tau < 2 \).

For the estimate of \( \mathcal{R}_2 \) the most delicate term to deal with is \( \delta^2 |D|^{-1/2} \mathcal{D}_y F |D|^{-1/2} \), where
\[
F(\delta, v_1, y) := \mathcal{R}((v_1 + v_2(\delta, v_1, y))^2),
\]
because the operator \( \mathcal{R} \) induces a loss of regularity, see \([116]\). However, again the regularizing property \([117]\) of the map \( v_2 \) enables to obtain the bound
\[
\|\mathcal{R}_2 h\|_{\sigma, s + \frac{7}{4}} \leq C \frac{\delta}{\gamma} \|h\|_{\sigma, s + \frac{3}{4}}. \tag{120}
\]
The key point is that the loss of \( (\tau - 1) \) derivatives due to \( |D|^{-1/2} \) applied twice, added to the loss of 1 derivative due to \( \mathcal{R} \) in \([116]\) is compensated by the gain of 2 derivatives with \( v_2 \), whenever \( \tau < 2 \). Let us enter briefly into details.
\[
\|\mathcal{D}_y F(\delta, v_1, y)[h]\|_{\sigma, s + 1} = 2 \mathcal{R}((v_1 + v_2) \mathcal{D}_y v_2(\delta, v_1, y)[h])\|_{\sigma, s + 1} \leq 2 \|v_1 + v_2\|_{\sigma, s + 2} \leq C \|v_1 + v_2\|_{\sigma, s + 2}\|\mathcal{D}_y v_2(\delta, v_1, y)[h]\|_{\sigma, s + 2} \leq K(N, R, \|y\|_{\sigma, s})\|h\|_{\sigma, s}
\]
by the regularizing property \([117]\) of \( v_2 \). We can then derive \([120]\) as in the proof of Lemma \([4.9]\) using that \( \tau < 2 \).
Finally, inserting \( y(\delta, v_1) \) in the \((Q1')\)-equation, we get
\[
- \Delta v_1 = \mathcal{G}(\delta, v_1)
\]
where
\[
\mathcal{G}(0, v_1) := -\Pi V_1 \left( 2a_2(v_1 + v_2(0, v_1, 0))L^{-1}(v_1 + v_2(0, v_1, 0))^2 \right).
\]
As in subsection 5.2 since \( \Phi_0 : V \to \mathbb{R} \) possesses a non-degenerate critical point \( \pi \), the equation \(-\Delta v_1 = \mathcal{G}(0, v_1)\) has the non-degenerate solution \( \pi_1 := \Pi V_1 \pi \in B(2R, V_1) \) and, by the Implicit function theorem, there exists a smooth path \( \delta \mapsto v_1(\delta) \in B(2R, V_1) \) of solutions of \([121]\) with \( v_1(0) = \pi \). As in Proposition 3.2 this implies that the set \( \mathcal{C} = \{ \delta \in (0, \delta_0) \mid (\delta, v_1(\delta)) \in B_\infty \} \) has asymptotically full measure at 0.

7 Appendix

Lemma 7.1 If \( q \) is an even integer, then
\[
\int_{\Omega} a(x)v^q(t, x) \, dt \, dx = 0, \quad \forall v \in V \iff \left\{ a(\pi - x) = -a(x), \quad \forall x \in [0, \pi] \right\}.
\]
If \( q \geq 3 \) is an odd integer, then
\[
\int_{\Omega} a(x)v^q(t, x) \, dt \, dx = 0, \quad \forall v \in V \iff \left\{ a(\pi - x) = a(x), \quad \forall x \in [0, \pi] \right\}.
\]

**Proof.** We first assume that \( q = 2s \) is even. If \( a(\pi - x) = -a(x) \) \( \forall x \in (0, \pi) \), then, for all \( v \in V \),
\[
\int_{\Omega} a(x)v^{2s}(t, x) \, dt \, dx = \int_{\Omega} a(\pi - x)v^{2s}(t, \pi - x) \, dt \, dx = \int_{\Omega} -a(x)(-v(t + \pi, x))^{2s} \, dt \, dx
\]
\[
= -\int_{\Omega} a(x)v^{2s}(t, x) \, dt \, dx
\]
and so \( \int_{\Omega} a(x)v^{2s}(t, x) \, dt \, dx = 0. \)

Now assume that \( \Sigma(v) := \int_{\Omega} a(x)v^{2s}(t, x) \, dt \, dx = 0 \) \( \forall v \in V \). Writing that \( D^{2s}\Sigma = 0 \), we get\[
\int_{\Omega} a(x)v_1(t, x) \ldots v_{2s}(t, x) \, dt \, dx = 0, \quad \forall(v_1, \ldots, v_{2s}) \in V^{2s}.
\]
Choosing \( v_{2s}(t, x) = v_{2s-1}(t, x) = \cos lt \sin lx \), we obtain\[
\frac{1}{4} \int_{\Omega} a(x)v_1(t, x) \ldots v_{2(s-1)}(t, x)(\cos(2lt) + 1)(1 - \cos(2lx)) \, dt \, dx = 0
\]
Taking limits as \( l \to \infty \), there results \( \int_{\Omega} a(x)v_1(t, x) \ldots v_{2(s-1)}(t, x) \, dt \, dx = 0 \) \( \forall(v_1, \ldots, v_{2(s-1)}) \in V^{2(s-1)} \).

Iterating this operation, we finally get\[
\forall(v_1, v_2) \in V^2 \quad \int_{\Omega} a(x)v_1(t, x)v_2(t, x) \, dt \, dx = 0, \quad \text{and} \quad \int_{0}^{\pi} a(x) \, dx = 0.
\]
Choosing \( v_1(t, x) = v_2(t, x) = \cos lt \sin lx \) in the first equality, we derive that \( \int_{0}^{\pi} a(x) \sin^2 lx \, dx = 0. \)
Hence \( \forall l \in \mathbb{N} \int_{0}^{\pi} a(x) \cos(2lx) \, dx = 0. \) This implies that \( a \) is orthogonal in \( L^2(0, \pi) \) to \( \mathcal{F} = \{ b \in L^2(0, \pi) \mid \langle b(\pi - x), b(x) \rangle \text{ a.e.} \}. \) Hence \( a(\pi - x) = -a(x) \) a.e., and, since \( a \) is continuous, the identity holds everywhere.

We next assume that \( q = 2s + 1 \) is odd, \( q \geq 3 \). The first implication is derived in a similar way. Now assume that \( \int_{\Omega} a(x)v^q(t, x) \, dt \, dx = 0 \) \( \forall v \in V \). We can prove exactly as in the first part that\[
\forall(v_1, v_2, v_3) \in V^3 \quad \int_{\Omega} a(x)v_1(t, x)v_2(t, x)v_3(t, x) \, dt \, dx = 0.
\]
Choosing \( v_1(t, x) = \cos(l_1 t) \sin(l_1 x) \), \( v_2(t, x) = \cos(l_2 t) \sin(l_2 x) \), \( v_3(t, x) = \cos((l_1 + l_2) t) \sin((l_1 + l_2) x) \) and using the fact that \( \int_0^{2\pi} \cos(l_1 t) \cos(l_2 t) \cos((l_1 + l_2) t) \, dt \neq 0 \), we obtain

\[
\int_0^{\pi} a(x) \left[ \sin^2(l_1 x) \sin(l_2 x) \cos(l_2 x) + \sin^2(l_2 x) \sin(l_1 x) \cos(l_1 x) \right] \, dx = 0
\]

Using the fact that \( \sin(l_1 x) \sin(l_2 x) \) is an eigenvector of \( K \), we obtain

\[
\int_0^{\pi} a(x) \sin(l_1 x) \sin(l_2 x) \sin((l_1 + l_2) x) \, dx = 0.
\]

Letting \( l_2 \) go to infinity and taking limits, \( (122) \) yields \( \int_0^{\pi} (1/2) a(x) \sin(l_1 x) \cos(l_1 x) \, dx = 0 \). Hence \( \int_0^{\pi} a(x) \sin(2l_1 x) = 0, \forall l > 0 \). This implies that, in \( L^2(0, \pi) \), \( a \) is orthogonal to \( G = \{ b \in L^2(0, \pi) \mid b(0, \pi - x) = -b(x) \ \text{a.e.} \} \). Hence \( a(0, \pi - x) = a(x) \ \forall x \in (0, \pi) \). □

**Proof of Lemma 4.1** Let \( K_k(\varepsilon) = S_k^{-1}(\varepsilon) \) be the self-adjoint compact operator of \( F_k \) defined by

\[ (K_k(\varepsilon)u, v)_\varepsilon = (u, v)_{L^2}, \quad \forall u, v \in F_k \]

(in other words \( K_k(\varepsilon)u \)) is the unique weak solution \( z \in F_k \) of \( S_k z := u \).

Note that \( K_k(\varepsilon) \) is a positive operator, i.e. \( (K_k(\varepsilon)u, u)_\varepsilon > 0, \forall u \neq 0 \), and that \( K_k(\varepsilon) \) is also self-adjoint for the \( L^2 \)-scalar product.

By the spectral theory of compact self-adjoint operators in Hilbert spaces, there is a \( \langle \cdot, \cdot \rangle \)-orthonormal basis \( (v_{k,j})_{j \geq 1, j \neq k} \) of \( F_k \), such that \( v_{k,j} \) is an eigenvector of \( K_k(\varepsilon) \) associated to a positive eigenvalue \( \lambda_{k,j}(\varepsilon) \), the sequence \( (\lambda_{k,j}(\varepsilon))_j \) is non-increasing and tends to 0 as \( j \to +\infty \). Each \( v_{k,j}(\varepsilon) \) belongs to \( D(S_k) \) and is an eigenvector of \( S_k \) with associated eigenvalue \( \lambda_{k,j}(\varepsilon) = 1/\nu_{k,j}(\varepsilon) \), with \( (\lambda_{k,j}(\varepsilon))_j \) differentiable and \( K_k(\varepsilon) = -K_k(\varepsilon)MK_k(\varepsilon) \), where \( M := \pi_k(a_0 u) \).

For \( u = \sum_{j \neq k} \alpha_j v_{k,j}(\varepsilon) \in F_k \),

\[ (u, u)_\varepsilon = \sum_{j \neq k} |\alpha_j|^2 \quad \text{and} \quad (u, u)_{L^2} = \sum_{j \neq k} |\alpha_j|^2 / \lambda_{k,j}(\varepsilon). \]

As a consequence,

\[ \lambda_{k,j}(\varepsilon) = \min \left\{ \max_{u \in F_k \| u \|_{L^2} = 1} \langle u, u \rangle_\varepsilon : F \text{ subspace of } F_k \text{ of dimension } j \text{ (if } j < k \text{), } j-1 \text{ (if } j > k) \right\}. \]

It is clear by inspection that \( \lambda_{k,j}(0) = j^2 \) and that we can choose \( \nu_{k,j}(0) = \sqrt{2/\pi} \sin(jx)/j \). Hence, by \( (123) \),

\[ |\lambda_{k,j}(\varepsilon) - j^2| \leq |\varepsilon| \|a_0\|_{\infty} < 1, \]

from which we derive

\[ \forall l \neq j \ |\lambda_{k,l}(\varepsilon) - \lambda_{k,j}(\varepsilon)| \geq (l + j) - 2 \geq 2 \min(l,j) - 1 \geq 1. \]

In particular, the eigenvalues \( \lambda_{k,j}(\varepsilon) \) (\( \nu_{k,j}(\varepsilon) \)) are simple. By the variational characterization \( (123) \) we also see that \( \lambda_{k,j}(\varepsilon) \) depends continuously on \( \varepsilon \), and we can assume without loss of generality that \( \varepsilon \mapsto v_{k,j}(\varepsilon) \) is a continuous map to \( F_k \).

Let \( \varphi_{k,j}(\varepsilon) := \sqrt{\lambda_{k,j}(\varepsilon)} v_{k,j}(\varepsilon) \). \( \varphi_{k,j}(\varepsilon) \) is a \( L^2 \)-orthogonal family in \( F_k \) and

\[ \forall \varepsilon \left\{ \begin{array}{l} K_k(\varepsilon) \varphi_{k,j}(\varepsilon) = \nu_{k,j}(\varepsilon) \varphi_{k,j}(\varepsilon) \\ (\varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon)_{L^2}) = 1 \end{array} \right. \]

We observe that the \( L^2 \)-orthogonality w.r.t. \( \varphi_{k,j}(\varepsilon) \) is equivalent to the \( \langle \cdot, \cdot \rangle \)-orthogonality w.r.t. \( \varphi_{k,j}(\varepsilon) \), and that \( E_{k,j}(\varepsilon) := [\varphi_{k,j}(\varepsilon)]^2 \) is invariant under \( K_k(\varepsilon) \). Using that \( L_{k,j} := (K_k(\varepsilon) - \nu_{k,j}(\varepsilon)I)|_{E_{k,j}(\varepsilon)} \) is invertible, it is easy to derive from the Implicit Function Theorem that the maps \( \varepsilon \mapsto v_{k,j}(\varepsilon) \) and \( \varepsilon \mapsto \varphi_{k,j}(\varepsilon) \) are differentiable.

Denoting by \( P \) the orthogonal projector onto \( E_{k,j}(\varepsilon) \), we have

\[ \varphi'_{k,j}(\varepsilon) = L^{-1}(-PK_k(\varepsilon) \varphi_{k,j}(\varepsilon)) = L^{-1}(PK_kMK_k \varphi_{k,j}(\varepsilon)) = \nu_{k,j}(\varepsilon) L^{-1}K_kPM \varphi_{k,j}(\varepsilon), \]
\[ \nu_{k,j}(\varepsilon) = \left( K_k(\varepsilon) \varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon) \right)_{L^2} = -\left( K_k M K_k \varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon) \right)_{L^2} \\
= -\left( M K_k \varphi_{k,j}(\varepsilon), K_k \varphi_{k,j}(\varepsilon) \right)_{L^2} = -\nu_{k,j}(\varepsilon) \left( M \varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon) \right)_{L^2}. \]

We have
\[ \nu_{k,j} L^{-1} K_k \left( \sum_{l \neq j} \alpha_l v_{k,l} \right) = \sum_{l \neq j} \frac{\nu_{k,j} \nu_{k,l} - \alpha_l v_{k,l}}{\nu_{k,l} - \nu_{k,j}} \sum_{l \neq j} \frac{\alpha_l}{\lambda_{k,j} - \lambda_{k,l}} v_{k,l}. \]

Hence, by (124), \(|\nu_{k,j} L^{-1} K_k P u|_{L^2} \leq |u|_{L^2}/j\). We obtain \(|\varphi'_{k,j}(\varepsilon)|_{L^2} = O(|a_0|_{\infty}/j)\). Hence
\[ \left| \varphi_{k,j}(\varepsilon) - \sqrt{\frac{2}{\pi}} \sin(j \varepsilon) \right|_{L^2} = O(\varepsilon |a_0|_{\infty}). \]

Hence, by (125),
\[ X_{k,j}(\varepsilon) = \left( M \varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon) \right)_{L^2} = \int_0^\pi a_0(x)(\sin(j \varepsilon))^2 dx \]
\[ = \frac{2}{\pi} \int_0^\pi a_0(x)(\sin(j \varepsilon))^2 dx + O\left( \frac{|a_0|_{H^1}}{\varepsilon} \right) \]
Writing \(\sin^2(j \varepsilon) = (1 - \cos(2j \varepsilon))/2\), and since \(\int_0^\pi a_0(x) \cos(2j \varepsilon) dx = -\int_0^\pi (a_0)_x(x) \sin(2j \varepsilon)/2j \ dx\), we get
\[ X_{k,j}(\varepsilon) = \frac{1}{\pi} \int_0^\pi a_0(x) dx + O\left( \frac{|a_0|_{H^1}}{\varepsilon} \right) = M(\delta, v_1, w) + O\left( \frac{|a_0|_{H^1}}{j} \right) \]
Hence \(\lambda_{k,j}(\varepsilon) = j^2 + \varepsilon M(\delta, v_1, w) + O(\varepsilon |a_0|_{H^1}/j)\), which is the first estimate in (80).

References


