Nonlinear wave and Schrödinger equations on compact Lie groups and homogeneous spaces

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Abstract. We prove the existence of Cantor families of small amplitude periodic solutions for wave and Schrödinger equations on compact Lie groups and homogeneous spaces with merely differentiable nonlinearities. The NLS equation on homogeneous spaces arises as a mean field approximation of condensates of many-body lattice problems. The highly degenerate eigenvalues of the Laplace Beltrami operator give rise to huge clusters of "small divisors". We apply a Lyapunov-Schmidt decomposition and an abstract Nash-Moser implicit function theorem. We provide a new algebraic framework to prove the key tame estimates for the inverse linearized operators along Banach scales of Sobolev functions. We exploit properties of the eigenvalues and eigenfunctions of the Laplace Beltrami operator on Lie groups.

1 Introduction

We consider autonomous nonlinear wave equations

$$(\mathbf{NLW}) \qquad u_{tt} - \Delta u + \mu u = f(x,u)\,, \quad x \in \mathcal{M}\,, \tag{1.1}$$

and Schrödinger equations

$$(\mathbf{NLS}) \qquad \mathrm{i}u_t - \Delta u + \mu u = f(x,|u|^2)u\,, \quad x \in \mathcal{M}\,, \tag{1.2}$$

on a compact, connected, Riemannian manifold $\mathcal{M}$ without boundary, where $\mathcal{M}$ is a Lie group (Definition 2.1), or, more generally, a homogeneous space, namely a manifold on which a Lie group acts transitively and smoothly (Definition 2.8). Homogeneous spaces are also called in the literature "coset manifolds".

Classical compact connected Lie groups are the standard torus $\mathbb{T}^n$, the special orthogonal group $SO(n)$ (with $n > 1$) of all the $n \times n$ real orthogonal matrices of determinant 1, the special unitary group $SU(n)$ of all $n \times n$ unitary complex matrices of determinant 1, etc, see [35]. Actually they are completely classified as stated in Theorem 2.2. Examples of compact homogeneous spaces are the spheres S^n (which are completely described by the orbits of $SO(n+1)$), the real and complex Grassmanians (as well as Grassmanians on other classical groups), the "moving frames" (the manifold of the k-ples of orthonormal vectors in $\mathbb{R}^n$ with the natural action of the orthogonal group $O(n)$), the symmetric spaces. We refer to [23], [11], [36] for many other examples.

The operator Δ is the Laplace-Beltrami operator defined with respect to a Riemannian metric compatible with the group structure (see section 2.2), $\mu > 0$ is a constant, the nonlinearity is finitely many times differentiable and vanishes at $u = 0$ at least quadratically.

The importance of the Schrödinger equation on Lie groups has been highlighted since the works of Weyl [40] (incidentally, this was a main motivation for studying representation theory of Lie groups). The understanding of the rotational modes of atoms and molecules leads to linear Schrödinger equations on compact groups, see e.g. [34]. Time independent linear Schrödinger equations on compact homogeneous spaces appear in various contexts of high energy physics, for example in connection with the quantum Hall effect and the Laundau equation, see [38], [30] or [26] and references therein.

Non linear Schrödinger equations appear usually in many body quantum physics as "Hartree-Fock" mean field approximations to describe condensated states, see for instance [19]. Several many body lattice problems described by continuous Lie groups are studied in physics, such as the XY or Heisenberg model (generalizations of the classical Ising model). Then these models lead in natural way to NLS on S^1, or spheres S^n, $SO(3)$, $SU(n)$, or, more generally, on homogeneous spaces.

Wave equations on compact Lie groups and symmetric spaces have also been investigated, see [23]-[25].

In this paper we prove, for any compact homogeneous space $\mathcal{M}$, the existence of Cantor families of small amplitude periodic solutions for the wave and Schrödinger equations (1.1)-(1.2) with merely differentiable nonlinearities, see Theorems 1.1-1.2.

These solutions are the continuations of normal modes oscillations of the linearized equations $\mathrm{i}u_t - \Delta u + \mu u = 0$ and $u_{tt} - \Delta u + \mu u = 0$. The normal frequencies are, for NLS, the eigenvalues of $-\Delta + \mu$ on $\mathcal{M}$, and, for NLW, their square roots. Theorem 1.1 can be considered as a PDE extension for NLW of the classical Lyapunov resonant center theorem, and Theorem 1.2 of the Weinstein-Moser theorem for the completely resonant NLS equation, see e.g. [2].

This type of problems has received much attention in the last twenty years. Actually, building on the experience gained from the qualitative study of finite dimensional dynamical systems, the existence of periodic and quasi-periodic solutions gives an important insight toward better understanding the complex flow evolution of Hamiltonian PDEs. Interestingly, many tools used for proving our Theorems 1.1-1.2 below for NLS and NLW resembles those required for Birkhoff normal form stability results (e.g. [1] for Zoll manifolds) and seem connected to the techniques used in well-posedness results of the initial value problem on compact manifolds, see e.g. [12]-[13].

Up to now, the majority of the existence results of invariant tori for PDEs have been proved when $\mathcal{M}$ is a torus $\mathbb{T}^n$ (or a rectangle). In this case the eigenfunctions of the Laplacian are the exponentials.

The main difficulty for the existence proof of invariant tori comes from arbitrarily "small divisors" in their perturbative expansion series. Such "small divisors" are due to complex resonance phenomena between the frequencies of the quasi-periodic solutions and the normal mode frequencies. The properties of the eigenfunctions are important, from a dynamical point of view, in order to control the exchange of energy among the normal modes. The crucial properties of the exponential basis is the multiplication property $\exp(\mathrm{i}jx)\exp(\mathrm{i}kx) = \exp(\mathrm{i}(j+k)x)$.

The first pionering results of Kuksin [27] and Wayne [39] applied to 1-dimensional, analytic, PDEs on an interval with Dirichlet boundary conditions. Their method was based on KAM theory (see also the works by Kuksin and Pöschel [32]-[28]-[29]) and strongly exploited that the eigenvalues of the Laplacian are simple. Already for periodic boundary conditions (i.e. $\mathcal{M} = \mathbb{T}^1$) the eigenvalues are asymptotically double and the KAM approach was adapted later in Chierchia-You [14] for the wave equation.

A direct bifurcation approach, which does not exclude multiplicity for the normal frequencies, had been previously developed by Craig and Wayne [16] who introduced the Lyapunov-Schimdt decomposition and solved the "small divisor" problem, for periodic solutions of 1-dimensional PDEs, with a Newton-type iterative scheme. The key step of this approach is to prove the invertibility of the linearized equations in a neighborhood of the equilibrium, togheter with estimates of the inverses in analytic norm. The main difficulty comes from the fact that these linear PDEs have non-constant coefficients. The Craig-Wayne approach [16], see also [15], is based on a Frölich-Spencer [22] coupling type technique. The key ingredients are:

(i) the "singular sites" –namely the Fourier indexes of the small divisors– are more and more "separated at infinity"

(ii) the eigenfunctions are "well-localized with respect to the exponentials".

These two properties, together with the analyticity of the functions, imply a very weak "interaction between the singular sites", and allow to prove estimates for the inverse linearized operators in analytic norms. Property (ii) implies, in particular, that the multiplication operator for an analytic function is represented in the eigenfunctions basis as a matrix with exponentially fast off-diagonal decay.

Property (ii) has been proved in [16] to hold for the eigenfunctions of 1-dimensional sturm Liouville operators $-\partial_{xx} + V(x)$ on $\mathbb{T}^1$. Concerning property (i), in [16] the clusters of the singular sites have a uniform size. This holds for 1-dimensional wave and Schrödinger equations.

For higher dimensional PDEs, a major further difficulty is that the dimensions of the eigenspaces of the Laplacian increase to infinity. Generalizing the Craig-Wayne approach, the first existence results of periodic solutions for analytic wave and Schrödinger equations on $\mathbb{T}^n$, $n \geq 2$, have been proved by Bourgain in [8] and [9] (actually [9] proves also the existence of quasi-periodic solutions when $n = 2$). The key point is to relax the separation property (i), proving:
$(i)'$ a partition of the singular sites in huge *clusters* (with unbounded dimensions) separated at infinity.
This is achieved exploiting that the eigenvalues of the Laplacian on $\mathbb{T}^n$ are sum of integer squares and, in the case of NLW, assuming also strong non-resonance assumptions on the frequency.

Recently this result has been generalized by Berti and Bolle [4] for merely differentiable nonlinearities and assuming weaker non-resonance conditions. The proof is based on a Nash-Moser implicit function iterative scheme, generalized into an abstract setting in [5]. A key role in the proof is played by a Frölich-Spencer coupling technique in Sobolev spaces. Here, the multiplication operator for a Sobolev function is represented in the exponential basis as a Töplitz matrix with polynomially fast off-diagonal decay.

Existence of periodic solutions for analytic PDEs on $\mathbb{T}^n$ (or rectangles) has been also proved by Gentile and Procesi [20], solving the small divisor problems with Lindstedt series techniques.

Before concluding this brief survey of results, we mention that existence of also quasi-periodic solutions for wave and Schrödinger equations with "Fourier multipliers" on $\mathbb{T}^n$ has been proved by Bourgain in [10]. Moreover, also the KAM approach has been extended by Eliasson and Kuksin [18] to prove the existence of elliptic tori for Schrödinger equations on $\mathbb{T}^n$ with a convolution potential. The simplification introduced by these models is that the normal modes eigenfunctions are exactly the exponentials.

For any compact Riemannian manifold the spectrum of the Laplace-Beltrami operator Δ is discrete, its eigenvalues tend to infinity (with an asymptotics given by Weyl theorem) and the eigenfunctions form an L^2-orthonormal basis. For a manifold without boundary, the eigenfunctions are orthogonal also in any Sobolev space $H^s(\mathcal{M})$. Then it follows that the matrix coefficients representing the multiplication operator for a smooth function decay as inverse powers of the differences between the eigenvalues of $\sqrt{-\Delta}$. However, these eigenvalues are in general not sufficiently separated (as it happens in dimension 1) and these informations alone do not seem sufficient for proving the existence of periodic solutions of the nonlinear equations (1.1)-(1.2). Some finer properties of the spectrum and of the eigenfunctions seem required for controlling the small divisors effects.

When $\mathcal{M} = S^n$, or, more generally, a Zoll manifold (i.e. the geodesic flow is periodic), the spectrum of the Laplace-Beltrami operator is contained in disjoint intervals, growing linearly to infinity, as for 1-dimensional PDEs. Moreover properties of "localizations of the eigenfunctions" of the type (ii), although much weaker than in [16], hold (when $\mathcal{M} = S^n$ they are the spherical harmonics). These properties have been recently used by Bambusi, Delort, Grebert, Szeftel [1], [17] to prove Birkhoff normal form stability results for wave equations. These could also be used to prove existence of small amplitude periodic solutions (however this result has been recently achieved in [5] with simpler methods). Note that spheres and Zoll manifolds are particular symmetric spaces of rank one.

It is a general fact that the presence of continuous symmetries expressed via a Lie group action on a manifold $\mathcal{M}$ places strong constraints on its geometry and "harmonic analysis". In this paper we strongly exploit this basic idea. If the action is transitive, then, up to isomorphism,

$$\mathcal{M} = (G \times \mathbb{T}^n)/N$$

where G is a simply connected compact Lie group, $\mathbb{T}^n$ is a torus and N is a closed subgroup of $G \times \mathbb{T}^n$ (see (2.16)). The manifold $\mathcal{M}$ is called an homogeneous space. The functions on $\mathcal{M}$ can be seen as functions defined on the Lie group $G \times \mathbb{T}^n$ and invariant under the action of N,

$$L^2(\mathcal{M}) = L^2(G \times \mathbb{T}^n/N) = \Big\{ f \in L^2(G \times \mathbb{T}^n) \mid f(xg) = f(x)\,,\ \forall x \in G \times \mathbb{T}^n,\ g \in N \Big\}\,. \tag{1.3}$$

Also, if we endow $\mathcal{M}$ with an appropriate metric structure (Theorem 2.17), the Laplace-Beltrami operator on $\mathcal{M}$ can be identified with the Δ operator on the Lie group, acting on the functions invariant under N.

Then we "lift" the equations (1.1)-(1.2) on the Lie group $G \times \mathbb{T}^n$ (of dimension greater that $\mathcal{M}$). This allows to use the tools of harmonic analysis on Lie groups.

The analogue of the Fourier analysis on the non-commutative group G is provided by the Peter-Weyl Theorem 2.5. It states that $L^2(G)$ is the Hilbert sum of the subspaces generated by all the irreducible unitary representations of the group. Defining on G a "canonical" Riemaniann metric (given by minus the Killing form, Theorem 2.8), each subspace is an eigenspace of the corresponding Laplace-Beltrami operator, see Theorem 2.10. Precise informations on the eigenvalues and eigenfunctions of the Laplacian on the simply connected group G are obtained decomposing G as a product of simply connected compact Lie groups of simple type (Theorem 2.1). A key step is to use the "highest weight theory" of Lie groups which fully describes the eigenvalues and eigenspaces of the Laplace operator, see [33]. A complete set of eigenfunctions of Δ on $G \times \mathbb{T}^n$ are then obtained multiplying for the usual exponential basis.

In order to deal with these highly degenerate eigenvalues we find convenient to follow a Lyapunov-Schmidt procedure and to use a Nash-Moser implicit function theorem. The main problem is to prove the invertibility of the linearized equations togheter with "tame" estimates of the inverses in Sobolev norm.

In order to get the separation property $(i)'$ for the singular sites, we exploit that the eigenvalues of Δ on $\mathcal{M}$ (see (1.4)) are essentially like the eigenvalues of the Laplacian restricted to a maximal connected commutative subgroup of $G \times \mathbb{T}^n$, i.e. a torus (whose dimension is the rank of the group). Moreover they are all in $\mathbb{Z}/D$ for some $D \in \mathbb{N}$. This yields the required separation properties $(i)'$ reasoning, for NLW, as in [8], [4], and, for NLS, as in [9].

The other property we need for the inversion of the linearized operators is:

$(ii)'$ The multiplication operator $u \mapsto bu$ for a Sobolev function $b \in H^s(\mathcal{M})$ is represented in the eigenfunctions basis as a block-matrix with the off-diagonal decay (4.13).

The block structure of the matrix takes into account the (large) multiplicity of the degenerate eigenvalues of Δ on $\mathcal{M}$. We remark that, however, several blocks could correspond to the same eigenvalue (this block-decomposition follows by the highest weight theory). On each block we consider the operatorial L^2-matrix norm. Then the (polynomial) decay property $(ii)'$ -proved in Lemma 4.7- follows essentially by the informations on multiplication of the eigenfunctions of Δ on Lie groups (see Lemma 2.14).

Once property $(ii)'$ has been guessed the main new technique to get the tame estimates for the linearized operators in the Sobolev scale, is to embed the matrix which represents the multiplication operator into a suitable algebra of quasi-Töplitz block-matrices satisfying interpolation inequalities (sections 4.1 and 7.2). Then, product and inverse of quasi-Töplitz matrices will exhibit the same block off-diagonal decay. This abstract procedure simplifies considerably also the construction in [4], which is valid for the exponential basis, and provides an intrinsic procedure for the Craig-Wayne-Bourgain approach.

Before concluding this introduction we mention that Theorems 1.1-1.2 hold for fixed nonlinearities, not only for parameter dependent PDEs. Then we need to solve the bifurcation equation (infinite dimensional for the completely resonant NLS equation) and, in particular, we show that it possesses non-degenerate solutions. These computations require many tools of Fourier analysis on Lie groups. As examples, we deal in section 5.1 with NLW on $SU(n)$ and, in section 8.1, with NLS on $SO(3)$.

We now state precisely our main results.

1.1 Main results

We first recall some basic results of harmonic analysis on the homogeneous space $\mathcal{M} = (G \times \mathbb{T}^n)/N$. The eigenvalues and the eigenfunctions of the Laplacian on a simply connected compact group G are

$$-|j_1 + \rho|^2 + |\rho|^2 \,, \quad \mathbf{e}_{j_1,\sigma}(x_1)\,, \; x_1 \in G\,, \; j_1 \in \Lambda^+(G)\,, \; \sigma = 1, \ldots, d_{j_1}\,,$$

where $\Lambda^+(G)$ is the cone generated by the natural combinations of the "fundamental weights" $w_i \in \mathbb{R}^r$, $i = 1, \ldots, r$ (r is the rank of the group) and $\rho := \sum_{i=1}^r w_i$ (see Theorems 2.9 and 2.10). The degeneracy of the eigenvalues satisfies $d_{j_1} \leq |j_1 + \rho|^{\dim(\mathrm{G}) - r}$. Moreover there exists $D := D(G) \in \mathbb{N}$ such that $-|j_1 + \rho|^2 + |\rho|^2 \in \mathbb{Z} D^{-1}$ for all $j_1 \in \Lambda^+(G)$, see Lemma 2.15.

By Fubini theorem $L^2(G \times \mathbb{T}^n) = L^2(G) \otimes L^2(\mathbb{T}^n)$ and a complete set of eigenfunctions of Δ on $G \times \mathbb{T}^n$ are the products of the eigenfunctions of the Laplacian on G times the exponentials. By (1.3), we find a basis of eigenfunctions of Δ on $\mathcal{M} = (G \times \mathbb{T}^n)/N$ selecting the N-invariant eigenfunctions. In conclusion, the eigenvalues and the eigenfunctions of $-\Delta + \mu$ on $\mathcal{M}$ are

$$\omega_j^2 := |j_1 + \rho|^2 - |\rho|^2 + |j_2|^2 + \mu \,, \quad \mathbf{e}_{j,\sigma}(x) := \mathbf{e}_{j_1,\sigma}(x_1) e^{\mathrm{i} j_2 \cdot x_2} \,, \; x = (x_1, x_2) \in G \times \mathbb{T}^n \,, \tag{1.4}$$

where the index $j = (j_1, j_2)$ is restricted to a subset $\Lambda_{\mathcal{M}} \subset \Lambda^+(G) \times \mathbb{Z}^n$, see (2.18), and $\sigma \subset [1, \mathrm{d}_j]$ with $\mathrm{d}_j \leq d_{j_1}$, see Theorem 2.19.

1.1.1 The NonLinear Wave Equation

The solutions of the linearized equation $u_{tt} - \Delta u + \mu u = 0$ on $\mathcal{M}$ are the normal modes

$$v = \sum_{l = \pm 1} \sum_{j \in \Lambda_{\mathcal{M}}} \sum_{\sigma = 1}^{\mathrm{d}_j} v_{l,j,\sigma} e^{\mathrm{i} l \omega_j t} \mathbf{e}_{j,\sigma} \,, \quad v_{l,j,\sigma} \in \mathbb{C} \,.$$

Fixed $j_0 \in \Lambda_{\mathcal{M}}$, we want to prove the existence of small amplitude periodic solutions of the nonlinear equation (1.1) with frequencies close to ω_{j_0}. Assuming that μ is an irrational number, the normal mode frequencies ω_j commensurable with ω_{j_0}, satisfy $\omega_j = \omega_{j_0}$. Actually, if μ is diophantine, i.e. there are constants $\gamma_0 > 0$, $\tau_0 > 1$, such that

$$|\mu m + n| \geq \frac{\gamma_0}{\langle m \rangle^{\tau_0}} \,, \quad \langle m \rangle := \max(1, |m|) \,, \quad \forall (m, n) \in \mathbb{Z}^2 \setminus \{(0,0)\} \,, \tag{1.5}$$

then the following "first order Melnikov" non-resonance condition holds (proved at the end of section 2).
Lemma. *Let μ be diophantine. There exist $\gamma_1, \tau_1 > 0$ such that*

$$|\omega_{j_0}^2 l^2 - \omega_j^2| \geq \frac{\gamma_1}{\langle l \rangle^{\tau_1}} \,, \quad \forall \, (l, j) \in \mathbb{Z} \times \Lambda_{\mathcal{M}} \,, \; (|l|, \omega_j) \neq (1, \omega_{j_0}) \,. \tag{1.6}$$

Concerning the nonlinearity, we assume that $f \in C^k(\mathcal{M} \times \mathbb{R}, \mathbb{R})$, $k > 2$, and, for some $m \in [2, k]$,

$$f(x, 0) = \ldots = (\partial_u^{m-1} f)(x, 0) = 0, \; (\partial_u^m f)(x, 0) \neq 0 \,. \tag{1.7}$$

Rescaling in (1.1) amplitude and time $u(t, x) \mapsto \delta u(\omega t, x)$, $\delta > 0$, we look for 2π-periodic solutions of

$$\omega^2 u_{tt} - \Delta u + \mu u = \varepsilon \, \mathfrak{f}(\delta, u) \,, \qquad \varepsilon := \delta^{m-1} \,, \tag{1.8}$$

where

$$\mathfrak{f}(\delta, u) := \frac{f(x, \delta u)}{\delta^m} = a(x) u^m + O(\delta) \,, \qquad a(x) := \frac{(\partial_u^m f)(x, 0)}{m!} \,,$$

in the real Sobolev space $H^s_{\mathrm{even}} := \{ u \in H^s(\mathbb{T} \times \mathcal{M}, \mathbb{R}) \mid u(-t, x) = u(t, x) \}$ for some $s \leq k$.

We define the "resonant" indices $\mathcal{Q} \subset \mathbb{Z} \times \Lambda_{\mathcal{M}}$ as

$$\mathcal{Q} := \Big\{ (l,j) \in \mathbb{Z} \times \Lambda_{\mathcal{M}} \text{ such that } (|l|, \omega_j) = (1, \omega_{j_0}) \Big\} \quad \text{and} \quad \mathcal{P} := (\mathbb{Z} \times \Lambda_{\mathcal{M}}) \setminus \mathcal{Q} . \tag{1.9}$$

Then we perform a Lyapunov-Schmidt reduction according to the decomposition

$$H^s_{\text{even}} = (Q \cap H^s_{\text{even}}) \oplus (P \cap H^s_{\text{even}})$$

where

$$Q := \Big\{ q = \sum_{(l,j)\in\mathcal{Q}} e^{\mathrm{i}lt} \sum_{\sigma=1}^{\mathrm{d}_j} q_{l,j,\sigma}\, \mathbf{e}_{j,\sigma} \in H^0_{\text{even}} \Big\} \qquad \text{and} \qquad P := Q^{\perp} .$$

To highlight the relationship between the frequency and the amplitude we set

$$\omega^2 - \omega^2_{j_0} = \varepsilon \chi \qquad \text{with} \qquad \chi \in \{-1, +1\}$$

chosen later in (1.13). We project equation (1.8) on Q and P. Writing $u = q + p$, $q \in Q$, $p \in P$, using that $q_{tt} = -q$, equation (1.8) is equivalent to

$$\begin{cases} q = -\chi\, \Pi_Q \mathfrak{f}(\delta, q+p) & \text{bifurcation equation} \\ \omega^2 p_{tt} - \Delta p + \mu p = \varepsilon \Pi_P \mathfrak{f}(\delta, q+p) & \text{range equation} \end{cases} \tag{1.10}$$

where Π_Q, Π_P denote the orthogonal projectors onto Q, P.

For $\delta = 0$ and $p = 0$ the bifurcation equation reduces to

$$q = -\chi\, \Pi_Q (a(x) q^m) \tag{1.11}$$

and we suppose that

$$\Pi_Q (a(x) q^m) \not\equiv 0 . \tag{1.12}$$

Taking

$$\chi := \begin{cases} -1 & \text{if } \exists\, q \in Q \text{ such that } \int_{\mathbb{T}\times\mathcal{M}} a(x) q^{m+1} > 0 \\ 1 & \text{if } \exists\, q \in Q \text{ such that } \int_{\mathbb{T}\times\mathcal{M}} a(x) q^{m+1} < 0 \end{cases} \tag{1.13}$$

equation (1.11) possesses at least a "mountain pass" critical point $\bar{q} \in Q \setminus \{0\}$ for the functional

$$\Phi(q) := \int_{\mathbb{T}\times\mathcal{M}} \frac{q^2}{2} + \chi a(x) \frac{q^{m+1}}{m+1}\, dt\, dx .$$

We assume the following non degeneracy condition:

(ND) There is an invariant subspace $\mathcal{H}^s \subset H^s_{\text{even}}$ of the NLW equation (1.8) and a solution $\bar{q} \in Q \cap \mathcal{H}^s \setminus \{0\}$ of (1.11) which is non degenerate, namely $h = 0$ is the unique solution of the linearized equation

$$h + \chi \Pi_Q (m\, a(x) \bar{q}^{m-1} h) = 0 , \quad h \in Q \cap \mathcal{H}^s . \tag{1.14}$$

We give in section 2 an explicit example where (ND) is satisfied for $\mathcal{M} = SU(n)$.

Theorem 1.1. (Wave equation) *Suppose $\mu > 0$ is diophantine and fix $j_0 \in \Lambda_{\mathcal{M}}$. There is $s_0 := s_0(\mathcal{M})$, $k := k(\mathcal{M}) \in \mathbb{N}$ such that:*
$\forall f \in C^k$ satisfying (1.7) for some $m \in [2, k]$ and (ND), there exist $\delta_0 > 0$, a curve

$$u \in C^1([0, \delta_0]; \mathcal{H}^{s_0}) \qquad \text{with} \qquad \|u(\delta) - \delta \bar{q}\|_{s_0} = O(\delta^2) ,$$

and a positive measure Cantor set $\mathcal{C} \subset [0, \delta_0]$ such that, $\forall \delta \in \mathcal{C}$, $u(\delta)(\omega t, x)$ is a solution of (1.1) with $\omega^2 = \omega^2_{j_0} + \chi\, \delta^{m-1}$ and χ is chosen in (1.13).
This result holds for $SU(n)$, $n \neq 2, 4$, $f = u^3 + r(x, u)$ where $r(x, u)$ satisfies (1.7) for some $m > 3$ and $r(x, u) = r(x^{-1}, u)$.

1.1.2 The NonLinear Schrödinger Equation

Under the transformation $u \mapsto e^{\mathrm{i}\mu t} u$, equation (1.2) transforms into the completely resonant PDE

$$\mathrm{i}u_t - \Delta u = f(x, |u|^2)u \tag{1.15}$$

where the normal mode frequencies $|j_1 + \rho|^2 - |\rho|^2 + |j_2|^2 \in \mathbb{Z}/D$, $D \in \mathbb{N}$, are all commensurable.

Fixed $j_0 \in \Lambda_{\mathcal{M}}$, we want to prove the existence of small amplitude periodic solutions of the nonlinear Schrödinger equation (1.15), with frequencies close to $\omega_{j_0}^2$, of the form of "wave packets". Note that (1.15) always possesses standing wave solutions $e^{\mathrm{i}\omega t}U(x)$ where $U(x)$ satisfies the elliptic PDE

$$-\Delta U - \omega U = f(x, |U|^2)U\,.$$

On the contrary, the more complex "wave packet" periodic solutions of (1.15) proved in Theorem 1.2 are not obtained by separation of variables, and correspond, in general, to quasi-periodic solutions of (1.2). We assume that $f(x, y) \in C^k(\mathcal{M} \times \mathbb{R}, \mathbb{R})$ and, for some $m \in [1, k]$,

$$f(x, 0) = \ldots = (\partial_y^{m-1} f)(x, 0) = 0\,,\ (\partial_y^m f)(x, 0) \neq 0\,. \tag{1.16}$$

Rescaling amplitude and time $u(t, x) \mapsto \delta u(\omega t, x)$ we look for solutions in $H^s := H^s(\mathbb{T} \times \mathcal{M}, \mathbb{C})$ of

$$\mathrm{i}\omega u_t - \Delta u = \varepsilon\, \mathfrak{f}(\delta, |u|^2)u\,, \qquad \varepsilon := \delta^{2m}\,, \tag{1.17}$$

where $\mathfrak{f}(\delta, |u|^2) := f(x, \delta^2|u|^2)/\varepsilon = a(x)|u|^{2m} + O(\delta^2)$ and $a(x) := (\partial_y^m f)(x, 0)/m!$

We define the infinite dimensional resonant subspace

$$Q := \Big\{ q = \sum_{(l,j) \in \mathcal{Q}} e^{\mathrm{i}lt} \sum_{\sigma=1}^{\mathrm{d}_j} q_{l,j,\sigma} \mathbf{e}_{j,\sigma} \in H^0 \Big\} \quad \text{where} \quad \mathcal{Q} := \Big\{ (l, j) \in \mathbb{Z} \times \Lambda_{\mathcal{M}} \mid \omega_{j_0}^2 l - \omega_j^2 = 0 \Big\}$$

and we perform a Lyapunov-Schmidt reduction along Q and $P := Q^\perp$ as for the wave equation. We set the "frequency-amplitude" relation $\omega = \omega_{j_0}^2 + \chi\varepsilon$ for some $\chi \in \{-1, +1\}$. The bifurcation equation at $\delta = 0$, $p = 0$, reduces, using $\mathrm{i}q_t = \omega_{j_0}^{-2} \Delta q$, to

$$\mathrm{i}q_t = \chi \Pi_Q (a(x)|q|^{2m} q)\,. \tag{1.18}$$

We suppose that:

(ND) There exists an invariant subspace $\mathcal{H}^s \subset H^s$ and a solution $q_0 \in Q \cap \mathcal{H}^s \setminus \{0\}$ of (1.18) for some $\chi \in \{-1, 1\}$ satisfying $q_0(t, x) \neq e^{\mathrm{i}lt}U(x)$ for all $l \in \mathbb{N}$. The solution q_0 is non-degenerate in $Q \cap \mathcal{H}^s$, namely the linear operator

$$J_0[h] := -\mathrm{i}h_t + \chi \Pi_Q a(x) \Big((m+1)|q_0|^{2m} h + m|q_0|^{2(m-1)} q_0^2 \bar{h} \Big)\,, \quad h \in Q \cap \mathcal{H}^s\,,$$

is invertible and the "Töplitz" norm $|J_0^{-1}|_s \leq C(s)$ for all $s \geq 0$ (see Definition 4.2).

If $q_0 = e^{ilt}U(x)$ for some $l \in \mathbb{N}$ the periodic solution found in Theorem 1.2 would just reduce to a standing wave solution. The last assumption on J_0^{-1} is required since Q is infinite dimensional.

Theorem 1.2. (Schrödinger equation) *Fix $j_0 \in \Lambda_{\mathcal{M}}$. There is $s_0 := s_0(\mathcal{M})$, $k := k(\mathcal{M}) \in \mathbb{N}$, such that:*
$\forall f \in C^k$ satisfying (1.16) for some $m \in [1, k]$ and (ND), there exist $\delta_0 > 0$, a curve

$$u \in C^1([0, \delta_0]; \mathcal{H}^{s_0}) \qquad \text{with} \qquad \|u(\delta) - \delta q_0\|_{s_0} = O(\delta^2)\,,$$

and a positive measure Cantor set $\mathcal{C} \subset [0, \delta_0]$ such that, $\forall \delta \in \mathcal{C}$, $u(\delta)(\omega t, x)$ is a solution of (1.15) with $\omega = \omega_{j_0}^2 + \chi\, \delta^{2m}$. Then $e^{\mathrm{i}\mu t}u(\delta)$ is a (quasi-periodic) solution of (1.2).
This result holds for $SO(3)$, $f = |u|^2 + r(|u|^2)$ where $r(y)$ satisfies (1.16) for some $m > 1$.

The paper is organized as follows. In section 2 we introduce the basic notions of Fourier analysis on groups, and we describe the main properties of the eigenfunctions and eigenvalues of the Laplace-Beltrami operator on Lie groups and homogeneous spaces. Section 3 presents the abstract Nash-Moser Theorem of [5]. In section 4 we prove the block off-diagonal decay property (4.13) for the multiplication operator in the Sobolev scale H^s. Then we embed it into an algebra of quasi-Töplitz block matrices, satisfying the interpolation inequalities. In section 5 we solve the bifurcation equation for NLW on $SU(n)$. In Sections 6-7 we prove the tame estimates for the inverse linearized operator in the Sobolev scale. Finally, in section 8 we prove the existence of periodic solutions for NLS.

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2 Lie groups

In this section we summarize the main definitions and results on Lie groups that are required in the paper. For proofs and details we refer to [21], [35] and [33].

Definition 2.1. *A* **Lie group** *G is a group which is also a C^∞ manifold and the group operations $(x,y) \in G \times G \to xy \in G$ and $x \mapsto x^{-1}$ are C^∞. The corresponding* **Lie algebra** *$\mathfrak{g}$ is the linear space of the left invariant vector fields $X(x)$ on G.*

The Lie algebra $\mathfrak{g}$ is closed with respect to the Lie bracket[1] operation $[\ ,\ ]$ which assigns to any two vector fields X, Y on G the vector field $[X,Y] := XY - YX$. The map $\Psi : X(\cdot) \mapsto X(e)$, where e is the identity element of G, is an isomorphism of $\mathfrak{g}$ onto the tangent space T_eG.

A Lie algebra $\mathfrak{g}$ is said simple if $\mathfrak{g}$ has no proper ideals and $\mathfrak{g}$ is non abelian. A Lie group G is of "simple type", if its Lie algebra is simple.

Theorem 2.1. **([11])** *The simply connected compact Lie groups of simple type are:*
- *the special unitary group $SU(n)$, $n \geq 2$,*
- *$Spin(n)$, the double cover of the special orthogonal group $SO(n)$,*
- *$SP(n) := Sp(n;\mathbb{C}) \cap U(2n)$ where $Sp(n;\mathbb{C})$ is the group of the $2n \times 2n$ complex symplectic matrices,*
- *the five exceptional groups called G_2, F_4, E_6, E_7, E_8.*

Incidentally, $Spin(3) = SU(2) = SP(1)$. Connected compact Lie groups are classified as follows.

Theorem 2.2. **(Classification)** *Up to global isomorphisms, each connected compact Lie group $\mathcal{G} = (G \times \mathbb{T}^n)/N$ where $\mathbb{T}^n$ is a torus, G is simply connected and N is a finite central*[2] *subgroup of $G \times \mathbb{T}^n$. Moreover G is a product of the Lie groups listed in Theorem 2.1.*

2.1 Harmonic analysis on compact groups

Let $L^2(G) := L^2(G,\mathbb{C})$ be the Lebesgue space defined with respect to the normalized Haar measure μ of any compact topological group G, that is a left and right invariant Borel measure, i.e. $\mu(xE) = \mu(E) = \mu(Ex)$ for any Borel set $E \subset G$, $x \in G$, normalized with $\mu(G) = 1$. Such a measure exists and it is unique on any compact topological group.

In order to extend the classical Fourier analysis for functions defined on the (non commutative) group G, we introduce some basic definitions about the representations of G on a finite dimensional complex vector space V with a given Hilbert space structure (we can limit to study finite dimensional unitary representations because G is compact).

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[1] A Lie bracket is a bilinear, antisymmetric map which satisfies the Jacobi identity.

[2] Namely $\mathrm{n}g = g\mathrm{n}$, $\forall \mathrm{n} \in N, g \in G \times \mathbb{T}^n$.

Definition 2.2. *A* **unitary representation** ρ_V *of G on the representation space V is a continuous homomorphism $x \mapsto \rho_V(x)$ which maps G into the group of unitary transformations $U(V) \subset \mathrm{End}(V)$.*

More precisely, a representation should be denoted by the pair (ρ_V, V). When no ambiguity exists it is customary to refer to a representation simply as ρ_V or as V.

Definition 2.3. *The representations ρ_V and ρ_W of G on V and W respectively, are* **equivalent** *if there is an isomorphism of vector spaces $\psi : V \to W$ such that $\rho_V(x) = \psi^{-1} \circ \rho_W(x) \circ \psi$ for each $x \in G$. In such a case, the representation spaces V and W are* **equivalent***.*

Given one or two representations of a group it is possible to form many new representations using standard constructions of linear algebra.

Definition 2.4. *The* **direct sum** *representation $\rho_V \oplus \rho_W$ of G on $V \oplus W$ is defined by*

$$(\rho_V \oplus \rho_W)(x)[v, w] := [\rho_V(x)[v], \rho_W(x)[w]] \,, \quad \forall x \in G, \, v \in V, \, w \in W \,.$$

The **tensor product** *representation $\rho_V \otimes \rho_W$ of G on $V \otimes W$ is defined by*

$$(\rho_V \otimes \rho_W)(x)(v \otimes w) := (\rho_V)(x)[v] \otimes (\rho_V)(x)[w] \,, \quad \forall \, x \in G, \, v \in V, \, w \in W \,.$$

The **dual** *representation ρ_{V^*} of G on the dual space V^* is defined by*

$$\rho_{V^*}(x)[f] := f \circ \rho_V^{-1}(x) = f \circ \rho_V^{\dagger}(x) \,, \quad \forall x \in G \,, \, f \in V^* \,.$$

Definition 2.5. *An* **invariant** *of a representation ρ_V is a vector $v \in V$ such that $\rho_V(x)v = v$, $\forall x \in G$. A subspace $W \subset V$ is invariant for a representation ρ_V of G if $\rho_V(x)(W) \subset W$, $\forall x \in G$. If a representation space V has no non-trivial invariant subspaces, then V and ρ_V are* **irreducible***. Finally, V and ρ_V are* **completely reducible** *if they are equivalent to a direct sum of irreducible representations ρ_{V_j}, namely*

$$V = \oplus_j V_j^{\oplus c_j} \,, \; c_j \in \mathbb{N} \,, \quad V_j^{\oplus c_j} := \underbrace{V_j \oplus \ldots \oplus V_j}_{c_j - \text{times}} \,, \quad \rho_V = \oplus_j \rho_{V_j}^{\oplus c_j} \,, \quad \rho_{V_j}^{\oplus c_j} := \underbrace{\rho_{V_j} \oplus \ldots \oplus \rho_{V_j}}_{c_j - \text{times}} \,.$$

In such a case we say that "V contains c_j copies of V_j".

If $W \subset V$ is an invariant subspace for a unitary representation ρ_V, then the orthogonal $W^{\perp}$ is also invariant, and we deduce that every unitary representation ρ_V of a compact group is completely reducible. Given an orthonormal basis $\{v_1, \ldots, v_n\}$ of V, a unitary representation ρ_V of G is equivalent to the representation on $\mathbb{C}^n$ described by the unitary matrices

$$U(x) := \{U_{l,k}(x)\} = \{(\rho_V(x)v_l, v_k)\} \,, \quad l, k = 1, \ldots, n \,, \quad n := \dim(V) \,. \tag{2.1}$$

Note that, if a representation is given by unitary matrices $U(x)$ on $\mathbb{C}^n$, then its dual representation on $(\mathbb{C}^n)^*$ is given by the complex conjugated matrix $\bar{U}(x)$.

Let $\mathscr{M}_V$ denote the vector space generated by the matrix-coefficients $U_{l,k}(x)$, namely

$$\mathscr{M}_V := \Big\{ x \mapsto \mathrm{tr}(BU(x)) \,, \; B \in \mathrm{Mat}(\mathbb{C}^n) \Big\} = \Big\{ x \mapsto \mathrm{tr}(A\rho_V(x)) \,, \; A \in \mathrm{End}(V) \Big\} \,. \tag{2.2}$$

Note that any function in $\mathscr{M}_V$ is continuous.

Theorem 2.3. (Schur orthogonality relations) *Suppose that ρ_V, ρ_W are non equivalent irreducible unitary representations of G. Then $\mathscr{M}_V$ and $\mathscr{M}_W$ are orthogonal subspaces of $L^2(G)$, and*

$$\int_G \mathrm{tr}(A\rho_V(x)) \, \overline{\mathrm{tr}(B\rho_V(x))} d\mu(x) = \frac{\mathrm{tr}(AB^{\dagger})}{\dim(V)} \,, \quad \forall A, B \in \mathrm{End}(V) \,. \tag{2.3}$$

In particular the functions $\sqrt{\dim(V)}(\rho_V(x)v_l, v_k)$, $l, k = 1, \ldots, n$, are an orthonormal basis of $\mathscr{M}_V$.

By the previous Theorem, if ρ_V is irreducible, the map $A \mapsto \mathrm{tr}\,(A\rho_V(x))$ is injective -and then an isomorphism- between the vectors spaces $\mathrm{End}(V)$ and $\mathscr{M}_V$. Then we have

Lemma 2.4. *If ρ_V is irreducible, any function in $\mathscr{M}_V$ can be uniquely represented as in (2.2).*

In general, given two representations ρ_V, ρ_W, and $A \in \mathrm{End}(V \oplus W)$, we have

$$\mathrm{tr}(A\,\rho_{V\oplus W}(x)) = \mathrm{tr}(A_V\rho_V(x)) + \mathrm{tr}(A_W\rho_W(x))\,, \quad \forall x \in G\,, \tag{2.4}$$

where $A_V := \Pi_V A|_V \in \mathrm{End}(V)$ and $A_W := \Pi_W A|_W \in \mathrm{End}(W)$. Moreover, $\forall A \in \mathrm{End}(V)$, $B \in \mathrm{End}(W)$,

$$\mathrm{tr}(A\rho_V(x))\,\mathrm{tr}(B\rho_W(x)) = \mathrm{tr}((A\otimes B)(\rho_V\otimes\rho_W)(x))\,, \quad \forall x \in G\,, \tag{2.5}$$

where $A\otimes B \in \mathrm{End}(V\otimes W)$ is defined by $(A\otimes B)(v\otimes w) := Av\otimes Bw$, $\forall v \in V$, $w \in W$.

As a consequence, the product of two functions of $\mathscr{M}_V$ and $\mathscr{M}_W$ is in $\mathscr{M}_{V\otimes W}$.

Theorem 2.5. (Peter-Weyl) *Let $\hat{G}$ be the set of equivalence classes of irreducible unitary representations of the compact group G and, for each $j \in \hat{G}$, let $\mathscr{M}_j := \mathscr{M}_{V_j}$. We have the Hilbert decomposition*

$$L^2(G) = \widehat{\oplus}_{j\in\hat{G}}\mathscr{M}_j\,.$$

For $f \in L^2(G)$, we have the L^2-convergent "Fourier series"

$$f(x) = \sum_{j\in\hat{G}} \mathrm{tr}\,(\mathbf{f}_j\,\mathbf{e}_j(x))\,, \qquad \mathbf{f}_j := \int_G f(x)\bar{\mathbf{e}}_j(x)d\mu\,, \tag{2.6}$$

where $\mathbf{e}_j(x) := (\dim V_j)^{\frac{1}{2}}U_j(x)$ and $U_j(x)$ are defined in (2.1). The $\mathbf{f}_j$ are called the "Fourier coefficients".

By the Schur orthogonality relations, the matrix coefficients $\mathbf{e}_{j,\sigma}(x)$, $\sigma = 1,\ldots,\dim V_j^2$, are an L^2-orthonormal basis for $\mathscr{M}_j$.

Note that the **trivial** representation of G on $V_0 := \mathbb{C}$, defined by $\rho_{\mathbb{C}}(x) = I$, is irreducible. We denote by $\mathscr{M}_0$ the corresponding space in the Peter-Weyl Theorem. The functions of $\mathscr{M}_0$ are the constants.

For a product group $G = G_1\times G_2$ we apply the previous analysis to each $L^2(G_i)$, $i = 1,2$ separately. By Fubini theorem

$$L^2(G_1\times G_2) = L^2(G_1)\otimes L^2(G_2) \tag{2.7}$$

and all the irreducible representation spaces are $V_{j_1}\otimes V_{j_2}$ with $j_1 \in \hat{G}_1$ and $j_2 \in \hat{G}_2$.

Definition 2.6. *The* **character** *of a representation ρ_V of G is the function $\chi_V(x) := \mathrm{tr}(\rho_V(x))$.*

By (2.4)-(2.5), the character of the direct sum and tensor product representations are respectively

$$\chi_{V_1\oplus V_2}(x) = \chi_{V_1}(x) + \chi_{V_2}(x)\,, \quad \chi_{V_1\otimes V_2}(x) = \chi_{V_1}(x)\,\chi_{V_2}(x)\,.$$

Moreover, the character of the dual representation is

$$\chi_{V^*}(x) = \bar{\chi}_V(x)\,. \tag{2.8}$$

The character of a representation ρ_V is a central function, namely it satisfies $f(gxg^{-1}) = f(x)$, $\forall x, g \in G$. Actually any central function in M_V is a multiple of the character χ_V and, by the Peter-Weyl Theorem, we have the following theorem.

Theorem 2.6. *The characters χ_{V_j}, $j \in \hat{G}$, form an Hilbert basis for the central functions of $L^2(G)$.*

By Schur's lemma we directly deduce the following lemma.

Lemma 2.7. *Given $i, j \in \hat{G}$, the representation space $V_i\otimes V_j^*$ contains no invariants if $i \neq j$ and one independent invariant if $i = j$. Then, given a representation space $V = \oplus_{j\in\hat{G}}V_j^{\oplus c_j}$, the space $V\otimes V_j^*$ contains c_j-independent invariants.*

All the previous results are valid for any compact topological group. We now specialize to *Lie* groups.

2.2 Laplace operator on compact Lie groups and homogeneous spaces

By the classification Theorem 2.2 any connected compact Lie group $\mathcal{G} = (G \times \mathbb{T}^n)/N$. Moreover the simply connected Lie group G is product of groups of simple type listed in Theorem 2.1. Then, to understand harmonic analysis for any $\mathcal{G}$, we first define and study the Laplace-Beltrami operator on simply connected groups of simple type. Then we deal with the product space $G \times \mathbb{T}^n$, using (2.7). At the end of this section we discuss how the Laplacian passes to the quotient space.

Theorem 2.8. *Let G be a simply connected compact Lie group of simple type. Up to a scalar factor there exists a unique Riemannian metric on G which is invariant under left and right multiplication (bi-invariant). A scalar product on the Lie algebra $\mathfrak{g}$ is*

$$(X, Y) := -\mathrm{tr}(Ad(X) \circ Ad(Y)), \quad \forall X, Y \in \mathfrak{g}$$

where $Ad(X)(\cdot) := [X, \cdot\,]$ is a linear map in $\mathfrak{g}$.

The bilinear form $-(X, Y)$ is called the Killing form. The Laplace Beltrami operator Δ on G is defined with respect to the Riemannian metric of Theorem 2.8 (the operator Δ is called the “Casimir” in most books on representation theory). On a product of Lie groups, the Killing form, and hence the Laplace operator, is defined component-wise. This defines Δ on any simply-connected compact Lie group.

All the irreducible representations of a compact Lie group G are parametrized by a discrete cone.

Theorem 2.9. *([33] p. 349, Proposition 1) For a simply connected compact Lie group G of rank*[3] *r, there is a one-to-one correspondence between the set of equivalence classes $\hat{G}$ of irreducible unitary representations and a discrete cone*

$$\Lambda^+ := \Lambda^+(G) := \Big\{ j = \sum_{i=1}^{r} n_i w_i \,, \ n_i \in \mathbb{N} \Big\} \subset \mathbb{R}^r$$

generated by r independent vectors $w_i \in \mathbb{R}^r$. The $\{w_1, \ldots, w_r\}$ are called the **fundamental weights** *of the group and Λ^+ the cone of* **dominant weights***. The irreducible representation of G corresponding to the dominant weight $j = 0$ is the trivial representation on $V_0 = \mathbb{C}$.*

The matrix coefficients of an irreducible representation are eigenfunctions of the Laplacian.

Theorem 2.10. *([33] p. 367, Lemma 1) Each $\mathscr{M}_j$ is an eigenspace of the Laplace Beltrami operator Δ with eigenvalue $-|j+\rho|^2 + |\rho|^2$, $\rho := \sum_{i=1}^r w_i$. We have $d_j := \dim(\mathscr{M}_j) \le |j+\rho|^{\dim(G)-r}$.*

Since the Laplacian is a real operator, for any eigenvector $\mathbf{e}_{j,\sigma}(x) \in \mathscr{M}_j$ also $\bar{\mathbf{e}}_{j,\sigma}(x)$ is an eigenvector of Δ with the same eigenvalue. Therefore $\bar{\mathbf{e}}_{j,\sigma}(x) \in \mathscr{M}_{\tilde{j}}$ for some $\tilde{j} \in \Lambda^+$. Moreover $V_{\tilde{j}} = V_j^*$ because the matrix $\bar{\mathbf{e}}_j(x)$ is the dual representation on V_j^* of the matrix $\mathbf{e}_j(x)$.

To describe in an effective way the eigenvalues and the eigenfunctions of the Laplace operator a major role is played by the positive simple roots of the group. We refer to [33] for the definition and several properties. The positive simple roots $\alpha_1, \ldots, \alpha_r \in \mathbb{R}^r$ of G satisfy the relations

$$(w_i, \alpha_j) = \frac{1}{2} \delta_{ij} |\alpha_j|^2 \,, \quad \forall i, j = 1, \ldots, r \,, \tag{2.9}$$

where δ_{ij} is the Kronecker symbol. We define the cone

$$\mathscr{R}^+ := \Big\{ \alpha = \sum_{i=1}^{r} n_i \alpha_i \,, \ n_i \in \mathbb{N} \Big\} \subset \mathbb{R}^r$$

generated by the natural combinations of the positive simple roots.

---

[3] The rank of G is the dimension of any maximal connected commutative subgroup of G (maximal torus).

Lemma 2.11. *The cones Λ^+, $\mathscr{R}^+ \subset \mathbb{R}^r$ intersect the hyperplane $\langle \rho \rangle^\perp \subset \mathbb{R}^r$, $\rho := \sum_{i=1}^r w_i$, only in 0 and the other elements are contained in the halfspace $(\rho, y) > 0$, $y \in \mathbb{R}^r$.*

Proof. By the theory of roots $\rho := \sum_{i=1}^r w_i = \sum_{i=1}^r N_i \alpha_i$ where the N_i are given positive integers (which depend on the group). Then the result follows from (2.9). ∎

We define the lattice

$$\Lambda := \Big\{ j = \sum_{i=1}^r n_i w_i \,, \ n_i \in \mathbb{Z} \Big\} \subset \mathbb{R}^r$$

generated by the fundamental weights $w_1, \dots, w_r$ (see Theorem 2.9) and $\Lambda^{++} := \rho + \Lambda^+$.

Definition 2.7. (Dominance order) *For $i, j \in \Lambda$, we say that $i \geq j$ if $i = j + \alpha$ for some $\alpha \in \mathscr{R}^+$.*

We shall need the following Lemma.

Lemma 2.12. *There is $c \in (0,1)$ such that $\forall i \geq j$, $i \in \Lambda^+$, $j \in \Lambda$ with $(\rho, j) > 0$, one has $|i+\rho| \geq c|j+\rho|$.*

Proof. We claim that there is $a \in (0,1)$ such that, $\forall j \in \Lambda$ with $(\rho, j) > 0$ and $\forall \alpha \in \mathscr{R}^+$, one has

$$(j + \rho, \alpha) \geq -a|j+\rho||\alpha| \,. \tag{2.10}$$

Calling $x := (j+\rho)|j+\rho|^{-1}$ and $y := \alpha|\alpha|^{-1}$ let $-a := \min_{X \times Y}(x, y)$ on the compact region $X \times Y$ where X is the intersection of the unitary sphere with the half space $(x, \rho) \geq 0$ and Y is the intersection of the unitary sphere with the prisma spanned by the w_i. By Lemma 2.11 the cone and the plane $(x, \rho) = 0$ do not have any common line and then $a \neq 1$. If $i \geq j$, namely, by Definition 2.7, $i = j + \alpha$ for some $\alpha \in \mathscr{R}^+$, then $|i+\rho|^2 \geq (1-a^2)|j+\rho|^2$, by (2.10) and minimizing on $|\alpha|$. ∎

The product of two eigenfunctions of the Laplace operator can be expressed as a finite sum of eigenfunctions. The next theorem provides a quantitative statement of this fact. It is a key ingredient for proving the decay properties of the multiplication operator.

Theorem 2.13. (Cartan multiplication) *([33], p. 345) Given two irreducible representations V_i, V_j one has $V_i \otimes V_j = V_{i+j} \oplus V'$ where V' is sum of irreducibles with dominant weights $j_1 \in \Lambda^+$, $j_1 < i + j$.*

Theorem 2.14. (Multiplication of eigenfunctions) *Let $a \in \mathscr{M}_i$, $b \in \mathscr{M}_j$. Then $a\, b \in \oplus_{j_1 \leq i+j} \mathscr{M}_{j_1}$.*

Proof. Since $V_i \otimes V_j$ is completely reducible we write $V_i \otimes V_j = \oplus_{j_1 \in \Lambda^+} V_{j_1}^{c_{ij}^{j_1}}$ for some $c_{ij}^{j_1}$. By Theorem 2.13 we have $c_{ij}^{j_1} = 0$ if $j_1 > i + j$. ∎

Many informations on the simple roots are encoded in the non-symmetric Cartan matrix

$$\mathcal{C} := \mathcal{C}(G) := (\mathcal{C}_{i,j})_{i,j=1,\dots,r} \qquad \text{where} \qquad \mathcal{C}_{i,j} := 2\frac{(\alpha_i, \alpha_j)}{|\alpha_j|^2} \,. \tag{2.11}$$

It is sufficient to describe the Cartan matrix of the Lie groups of simple type of Theorem 2.1, because the Cartan matrix of the product of a Lie group $G_1 \times G_2$ is the direct sum $\mathcal{C}(G_1) \oplus \mathcal{C}(G_2)$.

Lemma 2.15. *For any simply connected Lie group G, there is $D \in \mathbb{N}$ such that $(w_i, w_j) \in D^{-1}\mathbb{Z}$. Hence*

$$|j|^2, \ |j+\rho|^2, \ (\rho, j) \in D^{-1}\mathbb{Z} \,, \quad \forall j \in \Lambda^+ \,. \tag{2.12}$$

Proof. By (2.9), (2.11) we get $(w_i, w_j) = |\alpha_i|^2 (\mathcal{C}^{-T})_{i,j}/2$. Then (2.12) follows because the matrix elements of the Cartan matrix satisfy $\mathcal{C}_{i,j} \in \mathbb{Z}$ (actually $|\mathcal{C}_{i,j}| \leq 3$) and $|\alpha_j|^2 \in \mathbb{N}/2$, see [33]. ∎

The eigenspaces of the Laplace operator on $G \times \mathbb{T}^n$ are $\mathscr{M}_{j_1} e^{\mathrm{i} j_2 x_2}$ with $x_1 \in G$, $x_2 \in \mathbb{T}^n$, $(j_1, j_2) \in \Lambda^+ \times \mathbb{Z}^n$, the eigenfunctions $\mathbf{e}_{j_1,\sigma}(x_1) e^{\mathrm{i} j_2 x_2}$, $1 \leq \sigma \leq d_j$, and the eigenvalues $-|j_1 + \rho|^2 + |\rho|^2 - |j_2|^2$.

We finally need to understand how to pass to the quotient space (see [23]-[24] or [33] pp. 81-85).

Theorem 2.16. *Let H be a closed subgroup of a Lie group $\mathcal{G}$. Then there is a unique manifold structure on the quotient space $\mathcal{G}/H$ so that the projection map $\pi : \mathcal{G} \to \mathcal{G}/H$ is a smooth submersion.*

Theorem 2.17. *Given a bi-invariant metric on $\mathcal{G}$, the projection π induces on $\mathcal{G}/H$ a Riemannian structure such that the Laplace-Beltrami operator on $C^\infty(\mathcal{G}/H, \mathbb{C})$ is identified with the Laplace-Beltrami operator on*

$$C^\infty_{\mathrm{inv}}(\mathcal{G}, \mathbb{C}) := \Big\{ f \in C^\infty(\mathcal{G}, \mathbb{C}) \text{ such that } f(x) = f(xg)\,, \ \forall x \in \mathcal{G}\,, \ g \in H \Big\} \tag{2.13}$$

and the diagram commutes

$$\begin{tikzcd} C^\infty(\mathcal{G}/H, \mathbb{C}) \arrow[r, "\pi^*"] \arrow[d, "\Delta_{\mathcal{G}/H}"'] & C^\infty_{\mathrm{inv}}(\mathcal{G}, \mathbb{C}) \arrow[d, "\Delta_G"] \\ C^\infty(\mathcal{G}/H, \mathbb{C}) \arrow[r, "\pi^*"] & C^\infty_{\mathrm{inv}}(\mathcal{G}, \mathbb{C}) \end{tikzcd} \tag{2.14}$$

See for instance [36] for a discussion.

Given an irreducible representation ρ_V of $\mathcal{G}$ we consider the subspace

$$W := \Big\{ w \in V \mid \rho_V(g) w = w\,, \ \forall g \in H \Big\} \subset V$$

and the subspace of functions $\mathscr{N}_V \subset \mathscr{M}_V$ defined by

$$\mathscr{N}_V := \mathrm{Span}\Big\{ (\rho_V(x) w_k, v_l)\,, \ k = 1, \ldots, \dim(W), \ \ l = 1, \ldots, \dim(V) \Big\} \tag{2.15}$$

where $(v_l)_{l=1,\ldots,\dim(V)}$ is a basis of V and $(w_k)_{k=1,\ldots,\dim(W)}$ of W.

Lemma 2.18. *The $\mathscr{N}_V$ are the functions of $\mathscr{M}_V$ satisfying (2.13). Moreover* $\dim \mathscr{N}_V = \dim V \dim W$.

Proof. For any $w, v \in V$, $v \neq 0$, the matrix coefficient $(\rho_V(x) w, v)$ satisfies (2.13) if and only if

$$(\rho_V(xg) w, v) = (\rho_V(x) \rho_V(g) w, v) = (\rho_V(x) w, v)\,, \ \forall x \in \mathcal{G}\,, g \in H\,,$$

and therefore $(\rho_V(x)(\rho_V(g)w - w), v) = 0$, $\forall x \in \mathcal{G}$, $g \in H$. This implies –by the irreducibility of ρ_V– that $\rho_V(g) w - w = 0$, $\forall g \in H$, i.e. $w \in W$. ■

Let $(g, x) \mapsto gx$ denote the action $\mathcal{G} \times X \to X$ of a group $\mathcal{G}$ on a set X. We recall that the orbit of x is $\mathcal{O}(x) := \{gx \in X\,, \ g \in \mathcal{G}\}$. The stabilizer of x is the subgroup $\mathcal{G}_x := \{g \in \mathcal{G}\,, \ gx = x\}$. The action of a group $\mathcal{G}$ on X is called transitive if, $\forall x \in X$, the orbit $\mathcal{O}(x) = X$.

Definition 2.8. *A compact manifold $\mathcal{M}$ is said homogeneous if there is a compact Lie group $\mathcal{G}$ which acts on $\mathcal{M}$ transitively and differentiably, i.e. for each $g \in \mathcal{G}$, the map $x \mapsto gx$ is differentiable in $\mathcal{M}$.*

By Theorem 2.2 the group $\mathcal{G} = (G \times \mathbb{T}^n)/N_1$ where N_1 is a finite central subgroup. The action of $G \times \mathbb{T}^n$ on any $p \in \mathcal{M}$ induces a diffeomorphism

$$\mathcal{M} \overset{\phi}{\leftrightarrow} (G \times \mathbb{T}^n)/N \qquad \text{where} \qquad N := N_1 \mathcal{G}_p \tag{2.16}$$

and $\mathcal{G}_p$ is the stabilizer of p (note that $\mathcal{G}_p$ is closed and see Theorem 2.16). By Theorem 2.17 a bi-invariant metric on $G \times \mathbb{T}^n$ induces a metric on $(G \times \mathbb{T}^n)/N$ and, then, on $\mathcal{M}$, see [6].

Remark 2.1. *If $\mathcal{M}$ is also a Riemannian manifold with a $\mathcal{G}$-invariant metric, it is natural to ask if it can be derived by submersion from a bi-invariant metric on $G \times \mathbb{T}^n$. In general, this is not the case. However this is true for the sphere S^n with the canonical metric (see [24]) and in several applications arising from physics, see e.g. [36] or [26] and [31] or [26] for an application to the Landau equation.*

For each index $j = (j_1, j_2) \in \Lambda^+ \times \mathbb{Z}^n$ of the irreducible representations of $G \times \mathbb{T}^n$ we consider the subspace -defined in (2.15)- of N-invariant functions

$$\mathscr{N}_j \subset \mathscr{M}_j := \mathscr{M}_{j_1} e^{\mathrm{i} j_2 \cdot x_2} . \tag{2.17}$$

Each $\mathscr{N}_j$ is an eigenspace of the Laplacian with dimension $\dim \mathscr{N}_j \leq \dim \mathscr{M}_j$.

It could happen that for some $j \in \Lambda^+ \times \mathbb{Z}^n$ the subspace $\mathscr{N}_j = \{0\}$. The set

$$\Lambda_{\mathcal{M}} := \Big\{ j \in \Lambda^+ \times \mathbb{Z}^n \ \textit{such that} \ \mathscr{N}_j \neq \{0\} \Big\} \tag{2.18}$$

is closed under sum. By the Peter-Weyl Theorem 2.5 we deduce the spectral theory of the Laplace Beltrami operator on a compact homogeneous space.

Theorem 2.19. *$L^2(\mathcal{M}) = \widehat{\oplus}_{j \in \Lambda_{\mathcal{M}}} \mathscr{N}_j$. A basis for $\mathscr{N}_j \subset \mathscr{M}_j$ is, up to a reordering of the index σ, $\mathbf{e}_{j,\sigma}(x) = \mathbf{e}_{j_1,\sigma}(x_1) \exp(\mathrm{i} j_2 x_2)$, $\sigma = 1, \ldots, \mathrm{d}_j$ for some $1 \leq \mathrm{d}_j \leq d_j$.*

To conclude, we prove the non-resonance property (1.6).

Proof of (1.6). Let $j_0 = (j_{01}, j_{02}) \in \Lambda_{\mathcal{M}}$. $\forall (l, j) \in \mathbb{Z} \times \Lambda_{\mathcal{M}}$ such that $(|l|, \omega_j^2) \neq (1, \omega_{j_0}^2)$, by (1.4), (1.5) and (2.12),

$$|\omega_{j_0}^2 l^2 - \omega_j^2| = \Big| \Big(|j_{01} + \rho|^2 l^2 - |j_1 + \rho|^2 + |j_{02}|^2 l^2 - |j_2|^2 + |\rho|^2 (1 - l^2) \Big) + \mu (1 - l^2) \Big| \geq \frac{c'}{|1 - l^2|^\alpha} \geq \frac{\gamma_1}{\langle l \rangle^{2\alpha}}$$

for some constant $\gamma_1 > 0$.

3 Abstract Nash-Moser Theorem with parameters

We recall now the abstract Nash-Moser Theorem [5] (with a few notations adapted to the present setting).

Let $(X_s, \| \ \|_s)_{s \geq 0}$ be a scale of Banach spaces such that

$$\forall s \leq s', \quad X_{s'} \subseteq X_s , \qquad \|u\|_s \leq \|u\|_{s'} , \quad \forall u \in X_{s'} .$$

Let $(H^{(N)})_{N \geq 0}$ be an increasing family of closed subspaces of $\bigcap_{s \geq 0} X_s$ with projectors $\Pi^{(N)} : X_0 \to H^{(N)}$ satisfying the "smoothing" properties: $\forall s \geq 0$, $\forall d \geq 0$,

$$(\mathbf{S1}) \qquad \|\Pi^{(N)} u\|_{s+d} \leq N^d \|u\|_s , \quad \forall u \in X_s \tag{3.1}$$

$$(\mathbf{S2}) \qquad \|(I - \Pi^{(N)}) u\|_s \leq N^{-d} \|u\|_s , \quad \forall u \in X_{s+d} . \tag{3.2}$$

We consider a C^2 map $F : [0, \delta_0) \times X_{s_0 + \nu} \to X_{s_0}$ where $s_0 \geq 0$, $\nu > 0$, $\delta_0 > 0$, satisfying

- **(F1)** $F(0,0) = 0$,

and the "tame" properties: $\exists\, S \in (s_0, \infty]$ such that $\forall s \in [s_0, S)$, $\forall u \in X_{s+\nu}$ with $\|u\|_{s_0} \leq 2$, $\forall \delta \in [0, \delta_0)$,

- **(F2)** $\|\partial_\delta F(\delta, u)\|_s \leq C(s)(1 + \|u\|_{s+\nu})$, $\|D_u F(\delta, 0)[h]\|_s \leq C(s) \|h\|_{s+\nu}$
- **(F3)** $\|D_u^2 F(\delta, u)[h, v]\|_s \leq C(s)(\|u\|_{s+\nu} \|h\|_{s_0} \|v\|_{s_0} + \|v\|_{s+\nu} \|h\|_{s_0} + \|h\|_{s+\nu} \|v\|_{s_0})$
- **(F4)** $\|\partial_\delta D_u F(\delta, u)[h]\|_s \leq C(s)(\|h\|_{s+\nu} + \|u\|_{s+\nu} \|h\|_{s_0})$.

The main assumption concerns the invertibility of the linear operators $L^{(N)}(\delta, u) := \Pi^{(N)} D_u F(\delta, u)_{|H^{(N)}}$. We consider two parameters $\kappa \geq 0$, $\sigma \geq 0$, such that

$$\sigma > 4(\kappa + \nu) \, , \qquad \bar{s} := s_0 + 4(\kappa + \nu + 1) + 2\sigma < S \ . \tag{3.3}$$

For all $\gamma > 0$, we define appropriate subsets

$$J^{(N)}_{\gamma,\kappa} \subseteq \Big\{ (\delta, u) \in [0, \delta_0) \times H^{(N)} \mid L^{(N)}(\delta, u) \text{ is invertible and } \forall s \in \{s_0, \bar{s}\}, \\ \| L^{(N)}(\delta, u)^{-1}[h] \|_s \leq \frac{N^\kappa}{\gamma} (\|h\|_s + \|u\|_s \|h\|_{s_0}) \, , \ \forall h \in H^{(N)} \Big\} \, . \tag{3.4}$$

Given $\mathtt{k} > 0$, we define $\mathcal{U}^{(N)}_{\mathtt{k}} := \{ u \in C^1([0, \delta_0), H^{(N)}) \mid \|u\|_{s_0} \leq 1, \, \|\partial_\delta u\|_{s_0} \leq \mathtt{k} \}$ and, for $u \in \mathcal{U}^{(N)}_{\mathtt{k}}$,

$$\mathcal{G}^{(N)}_{\gamma,\kappa}(u) := \Big\{ \delta \in [0, \delta_0) \mid (\delta, u(\delta)) \in J^{(N)}_{\gamma,\kappa} \Big\} \, . \tag{3.5}$$

- **Assumption (L)** There exist $\sigma \geq 0$, $\kappa \geq 0$ satisfying (3.3), $\bar{\gamma} > 0$, $M \in \mathbb{N}$, $C > 0$, such that:

  $$\textbf{i)}\, \forall \gamma \in (0, \bar{\gamma}], \ \forall \delta \in (0, \delta_0], \ |(\mathcal{G}^{(M)}_{\gamma,\kappa}(0))^c \cap [0, \delta)| \leq C \gamma \delta \, . \tag{3.6}$$

  ii) $\forall \gamma \in (0, \bar{\gamma}]$, $\bar{\mathtt{k}} > 0$, $\exists \tilde{\delta} := \tilde{\delta}(\gamma, \bar{\mathtt{k}}) \in (0, \delta_0]$ such that, $\forall \delta \in (0, \tilde{\delta}]$, $N' \geq N \geq M$, $u_1 \in \mathcal{U}^{(N)}_{\bar{\mathtt{k}}}$, $u_2 \in \mathcal{U}^{(N')}_{\bar{\mathtt{k}}}$ with $\|u_2 - u_1\|_{s_0} \leq N^{-\sigma}$,

  $$\Big| \Big(\mathcal{G}^{(N')}_{\gamma,\kappa}(u_2) \Big)^c \setminus \Big(\mathcal{G}^{(N)}_{\gamma,\kappa}(u_1) \Big)^c \bigcap [0, \delta) \Big| \leq C \frac{\gamma \delta}{N} \, . \tag{3.7}$$

The following Nash-Moser type Theorem holds:

Theorem 3.1. **[5]** *$\exists C > 0$ and, $\forall \gamma \in (0, \bar{\gamma})$, there is $\delta_0 > 0$ and a C^1 map $u : [0, \delta_0) \to X_{s_0+\nu}$ such that $u(0) = 0$ and $F(\delta, u(\delta)) = 0$ except in a set $\mathcal{C}_\gamma$ of Lebesgue measure $|\mathcal{C}_\gamma| \leq C \gamma \delta_0$. Moreover, for all $\delta \in (0, \delta_0)$, $|\mathcal{C}_\gamma \cap [0, \delta)| \leq C \gamma \delta$.*

4 The Sobolev scale H^s

Consider an homogeneous space $\mathcal{M} = (G \times \mathbb{T}^n)/N$, where G is a simply connected Lie group of dimension d and rank r.

The Peter-Weyl Theorem 2.5, combined with the standard Fourier theory in the time variable $t \in \mathbb{T}$, implies the orthogonal decomposition

$$L^2 := L^2(\mathbb{T} \times \mathcal{M}, \mathbb{C}) = \oplus_{k \in E^+} \mathcal{N}_k \quad \text{where} \quad k := (l, j) \, , \ E^+ := \mathbb{Z} \times \Lambda_{\mathcal{M}} \, , \ \mathcal{N}_k := \langle e^{\mathrm{i}lt} \rangle \otimes \mathscr{N}_j \, . \tag{4.1}$$

The Fourier series of $u \in L^2$ is defined by

$$u = \sum_{k \in E^+} u_k \qquad \text{where} \qquad u_k := \Pi_{\mathcal{N}_k} u \qquad \text{and} \qquad \Pi_{\mathcal{N}_k} : L^2 \to \mathcal{N}_k$$

are the spectral projectors. In components

$$u(t, x) = \sum_{(l,j) \in E^+} e^{\mathrm{i}lt} \sum_{\sigma=1}^{\mathrm{d}_j} u_{l,j,\sigma} \mathbf{e}_{j,\sigma} \qquad \text{and} \qquad \|u\|^2_{L^2} = 2\pi \sum_{(l,j) \in E^+} \sum_{\sigma=1}^{\mathrm{d}_j} |u_{l,j,\sigma}|^2 \, . \tag{4.2}$$

For $s \geq 0$, we define the Sobolev scale of Hilbert spaces

$$H^s := H^s(\mathbb{T} \times \mathcal{M}, \mathbb{C}) := \Big\{ u = \sum_{k \in E^+} u_k \ \Big| \ \ \|u\|_s^2 := \sum_{k \in E^+} |k + \vec{\rho}|^{2s} \|u_k\|_{L^2}^2 < +\infty \Big\}$$

where $\vec{\rho} = (0, \rho, 0) \in \mathbb{Z} \times \Lambda^+ \times \mathbb{Z}^n$. We have $H^0 = L^2$. Since $\mathbb{T} \times \mathcal{M}$ is a compact C^∞-Riemannian manifold without boundary, for any $s \in \mathbb{N}$, H^s is equivalent to the usual Sobolev space $H^s = \{u \in L^2 \mid D^\alpha u \in L^2, \ \forall |\alpha| \leq s, \ \|u\|_s^2 = \sum_{|\alpha| \leq s} \|D^\alpha u\|_{L^2}^2 \}$.

Lemma 4.1. *For* $s \geq \mathfrak{s}_0 > (\dim(\mathcal{M}) + 1)/2$, $\forall u, v \in H^s$,

1. **(Sobolev embedding)** $\|u\|_{L^\infty} \leq C(s) \|u\|_s$
2. **(Algebra)** $\|uv\|_s \leq C(s) \|u\|_s \|v\|_s$
3. **(Interpolation)** $\|uv\|_s \leq C(s, \mathfrak{s}_0)(\|u\|_s \|v\|_{\mathfrak{s}_0} + \|u\|_{\mathfrak{s}_0} \|v\|_s)$.

PROOF. These inequalities are well known for C^∞-functions with compact support in $\mathbb{R}^{\dim(\mathcal{M})+1}$, see e.g. [37], and, therefore, by a partition of unity argument, they still hold true for any compact C^∞-Riemannian manifold without boundary. ■

We now introduce a preorder relation on $E := \mathbb{Z} \times \Lambda \times \mathbb{Z}^n$.

Definition 4.1. (Dominance order) *Given* $k = (l, j_1, j_2), k' = (l', j_1', j_2') \in E$ *we say that* $k \geq k'$ *if* $|l| \geq |l'|$, $j_1 \geq j_1'$ *(see Definition 2.7) and* $|j_2| \geq |j_2'|$.

Lemma 4.2. (Multiplication of eigenfunctions) *Given* $a \in \mathcal{N}_k$ *and* $b \in \mathcal{N}_{k'}$ *then*

$$ab \in \oplus_{k_1 \in D(k,k')} \mathcal{N}_{k_1} \quad \text{where} \quad D(k, k') := \Big\{ k'' \in E^+ \ \Big| \ l'' = l + l' \,, \ \ j_1'' \leq j_1 + j_1' \,, \ j_2'' = j_2 + j_2' \Big\}$$

where $k = (l, j_1, j_2)$, $k' = (l', j_1', j_2')$, $k'' = (l'', j_1'', j_2'')$.

PROOF. The conditions $l'' = l + l'$, $j_2'' = j_2 + j_2'$ are the multiplication rules for the exponentials $e^{\mathrm{i}lt} e^{\mathrm{i}l't} = e^{\mathrm{i}(l+l')t}$, $e^{\mathrm{i}j_2 \cdot x_2} e^{\mathrm{i}j_2' \cdot x_2} = e^{\mathrm{i}(j_2 + j_2') \cdot x_2}$. The condition $j_1'' \leq j_1 + j_1'$ comes from Theorem 2.14. ■

The next lemma is the key for proving the decay of the matrix representing the multiplication operator.

Lemma 4.3. *Let* $s > (d + n + 1)/2$. *If* $u \in \oplus_{k \geq k_0, k \in E^+} \mathcal{N}_k$ *with* $k_0 := (l_0, j_0) \in E$, $j_0 = (j_{01}, j_{02}) \in \Lambda^+(G) \times \mathbb{Z}^n$ *and* $(\rho, j_{01}) \geq 0$, *then*

$$\|u\|_{L^\infty} \leq \frac{C(s) \|u\|_s}{\langle k_0 \rangle^{s - (d+n+1)/2}} \qquad \text{where} \qquad \langle k \rangle := \max(1, |k|) \,. \tag{4.3}$$

PROOF. By (4.2), Definition 4.1, and Cauchy inequality, we have

$$|u(t, x)| \leq \sum_{\substack{(l,j) \in E^+ \\ (l,j) \geq (l_0, j_0)}} \Big| \sum_{\sigma = 1, \ldots, \mathrm{d}_j} u_{l,j,\sigma} \mathbf{e}_{j,\sigma} \Big| \leq \sum_{\substack{(l,j) \in E^+ \\ (l,j) \geq (l_0, j_0)}} \|\Pi_{\mathcal{N}_{l,j}} u\|_0 \Big(\sum_{\sigma = 1, \ldots, \mathrm{d}_j} |\mathbf{e}_{j,\sigma}|^2 \Big)^{1/2} . \tag{4.4}$$

Now $\mathbf{e}_{j,\sigma}(x) = \mathbf{e}_{j_1,\sigma}(x_1) e^{\mathrm{i}j_2 \cdot x_2}$ and, since $\mathbf{e}_{j_1}(x_1)$ is a unitary matrix, then

$$\sum_{\sigma = 1, \ldots, \mathrm{d}_j} |\mathbf{e}_{j,\sigma}|^2 \leq \mathrm{tr}\,(\mathbf{e}_{j_1} \mathbf{e}_{j_1}^\dagger) = n_{j_1}^2 = d_{j_1} \tag{4.5}$$

(where $\mathbf{e}_{j_1}^\dagger$ denotes the adjoint matrix). Applying again Cauchy inequality in (4.4),

$$\begin{aligned}\|u\|_{L^\infty}^2 &\leq \sum_{\substack{(l,j)\in E^+\\(l,j)\geq(l_0,j_0)}} \|\Pi_{\mathcal{N}_{l,j}} u\|_0^2 \Big(|l|^2+|j_1+\rho|^2+|j_2|^2\Big)^s \sum_{\substack{(l,j)\in E^+\\(l,j)\geq(l_0,j_0)}} \frac{d_{j_1}}{(l^2+|j_1+\rho|^2+|j_2|^2)^s} \\ &= \|u\|_s^2 \sum_{\substack{(l,j)\in E^+\\(l,j)\geq(l_0,j_0)}} \frac{d_{j_1}}{(l^2+|j_1+\rho|^2+|j_2|^2)^s} \leq \|u\|_s^2 \sum_{\substack{(l,j)\in E^+\\(l,j)\geq(l_0,j_0)}} \frac{|j_1+\rho|^{\dim(G)-r}}{(l^2+|j_1+\rho|^2+|j_2|^2)^s} \qquad (4.6)\end{aligned}$$

by Theorem 2.10. By Lemma 2.12, $j \geq j_0$ implies $|j_1+\rho| > c|j_{01}+\rho|$ with $c \in (0,1)$, and we estimate the last term in (4.6) by

$$\sum_{\substack{k\in E^+\\k\geq k_0}} \frac{1}{|k+\vec{\rho}|^{2s-d+r}} \leq \int_{|x|>c|k_0+\vec{\rho}|} \frac{d^{r+n+1}x}{|x|^{2s-d+r}} \leq C \int_{y>c|k_0+\vec{\rho}|} \frac{dy}{y^{2s-d-n}} \leq \frac{C_2(s)}{1+|k_0|^{2s-(d+n+1)}}$$

deducing (4.3). ■

Note that the previous Theorem provides a proof of the embedding $H^s \hookrightarrow C(\mathbb{T}\times\mathcal{M})$ for $s > \dim(\mathbb{T}\times G\times\mathbb{T}^n)/2 > \dim(\mathbb{T}\times\mathcal{M})/2$. We did not attempt to recover the optimal bound on s given in Lemma 4.1, being useless for the purposes of this paper.

4.1 Quasi-Töplitz block matrices

According to the splitting $H^s = \oplus_{k\in E^+}\mathcal{N}_k$ we shall identify a linear operator A acting on H^s with its matrix representation $A = (A_k^{k'})_{k,k'\in E^+}$ with blocks $A_k^{k'} \in \mathcal{L}(\mathcal{N}_{k'},\mathcal{N}_k)$.

Definition 4.2. (Quasi-Töplitz) *We define the quasi-Töplitz block matrices*

$$\mathcal{A}_s := \Big\{ A = (A_k^{k'})_{k,k'\in E^+} \ : \ |A|_s^2 := \sup_{k\in E^+} \sum_{k'\in E^+} \langle k-k'\rangle^{2s} \|A_k^{k'}\|_0^2 < \infty \Big\}$$

where $\|A_k^{k'}\|_0 := \sup_{u\in\mathcal{N}_{k'},\|u\|_0=1} \|A_k^{k'}u\|_0$ *is the* L^2*-operator norm in* $\mathcal{L}(\mathcal{N}_{k'},\mathcal{N}_k)$*.*

If $s' > s$ then $\mathcal{A}_{s'} \subset \mathcal{A}_s$ because $|\cdot|_{s'} > |\cdot|_s$. Let $\mathfrak{s}_0 > (r+n+1)/2$.

Lemma 4.4. (Algebra) *For* $s > \mathfrak{s}_0$*,* $\mathcal{A}_s$ *is an algebra:*

$$|AB|_s \leq C(s)|A|_s|B|_s\,, \quad \forall A,B\in\mathcal{A}_s\,. \qquad (4.7)$$

A similar matrix algebra has been used in [7]. Applying (4.7) iteratively, we get, $\forall m \in \mathbb{N}$,

$$|A^m|_s \leq C(s)^{m-1}|A|_s^m\,. \qquad (4.8)$$

The algebra property of $\mathcal{A}_s$ is implied by the more general interpolation inequality (4.9).

Lemma 4.5. (Interpolation) $\forall s \geq \mathfrak{s}_0$, $\forall A,B \in \mathcal{A}_s$,

$$|AB|_s \leq C(s)\Big(|A|_s|B|_{\mathfrak{s}_0} + |A|_{\mathfrak{s}_0}|B|_s\Big) \qquad (4.9)$$

and, $\forall u \in H^s$,

$$\|Au\|_s \leq C(s)\Big(|A|_s\|u\|_{\mathfrak{s}_0} + |A|_{\mathfrak{s}_0}\|u\|_s\Big)\,. \qquad (4.10)$$

Proof. By definition

$$|AB|_s^2 = \sup_{k\in E^+} \sum_{k'\in E^+} \langle k-k'\rangle^{2s} \|(AB)_k^{k'}\|_0^2 \le \sup_{k\in E^+} \sum_{k'\in E^+} \langle k-k'\rangle^{2s} \Big(\sum_{k_1\in E^+} \|A_k^{k_1}\|_0 \|B_{k_1}^{k'}\|_0 \Big)^2 = S_1 + S_2$$

where

$$S_1 := \sup_{k\in E^+} \sum_{k'\in E^+} \Big(\sum_{k_1\in E^+} \|A_k^{k_1}\|_0 \langle k_1-k\rangle^{\mathfrak{s}_0} \|B_{k_1}^{k'}\|_0 \langle k'-k_1\rangle^{s} \frac{\langle k-k'\rangle^s}{\langle k_1-k\rangle^{\mathfrak{s}_0}\langle k'-k_1\rangle^s} \Big)^2 \quad \text{if} \quad \frac{\langle k'-k\rangle}{\langle k'-k_1\rangle} \le 2$$

and

$$S_2 := \sup_{k\in E^+} \sum_{k'\in E^+} \Big(\sum_{k_1\in E^+} \|A_k^{k_1}\|_0 \langle k_1-k\rangle^{s} \|B_{k_1}^{k'}\|_0 \langle k'-k_1\rangle^{\mathfrak{s}_0} \frac{\langle k-k'\rangle^s}{\langle k_1-k\rangle^{s}\langle k'-k_1\rangle^{\mathfrak{s}_0}} \Big)^2 \quad \text{if} \quad \frac{\langle k'-k\rangle}{\langle k'-k_1\rangle} > 2$$

and note that in this second case $\frac{\langle k-k'\rangle}{\langle k-k_1\rangle} \le 2$. By Hölder inequality, and exchanging the order of sum,

$$\begin{aligned} S_1 &\le \sup_{k\in E^+} \sum_{k'\in E^+} \Big(\sum_{k_1\in E^+} \|A_k^{k_1}\|_0^2 \langle k_1-k\rangle^{2\mathfrak{s}_0} \|B_{k_1}^{k'}\|_0^2 \langle k'-k_1\rangle^{2s} \sum_{k_1\in E^+} \frac{\langle k'-k\rangle^{2s}}{\langle k_1-k\rangle^{2\mathfrak{s}_0}\langle k'-k_1\rangle^{2s}} \Big) \\ &\le \sup_{k\in E^+} \sum_{k_1\in E^+} \|A_k^{k_1}\|_0^2 \langle k_1-k\rangle^{2\mathfrak{s}_0} \Big(\sum_{k'\in E^+} \|B_{k_1}^{k'}\|_0^2 \langle k'-k_1\rangle^{2s} \Big) \sum_{k_1\in E^+} \frac{C(s)}{\langle k_1\rangle^{2\mathfrak{s}_0}} \le |B|_s^2 |A|_{\mathfrak{s}_0}^2 K(s) \end{aligned}$$

for $\mathfrak{s}_0 > (r+n+1)/2$. Similarly $S_2 \le |B|_{\mathfrak{s}_0}^2 |A|_s^2 K(s)$, proving (4.9). The (4.10) follows similarly. ■

Applying (4.9) iteratively, we obtain, $\forall m \in \mathbb{N}$,

$$|A^m|_s \le m(C(s)|A|_{\mathfrak{s}_0})^{m-1} |A|_s \,. \tag{4.11}$$

Lemma 4.6. *Let $A \in \mathcal{A}_s$, $\Omega_1, \Omega_2 \subset E^+$, $\Omega_1 \cap \Omega_2 = \emptyset$. Then $\|A_{\Omega_2}^{\Omega_1}\|_0 \le C(s)|A|_s / \mathrm{d}(\Omega_1,\Omega_2)^{s-(r+n+1)/2}$.*

Proof. For any $u = \sum_{k'\in\Omega_1} u_{k'}$ we have

$$\begin{aligned} \|A_{\Omega_2}^{\Omega_1} u\|_0^2 &= \sum_{k\in\Omega_2} \Big\| \sum_{k'\in\Omega_1} A_k^{k'} u_{k'} \Big\|_0^2 \le \sum_{k\in\Omega_2} \Big(\sum_{k'\in\Omega_1} \|A_k^{k'}\| \langle k-k'\rangle^s \|u_{k'}\|_0 \frac{1}{\langle k-k'\rangle^s} \Big)^2 \\ &\le \sum_{k'\in\Omega_1} \|u_{k'}\|_0^2 \Big(\sum_{k\in\Omega_2} \|A_k^{k'}\|^2 \langle k-k'\rangle^{2s} \Big) \Big(\sum_{k'\in\Omega_1} \frac{1}{\langle k-k'\rangle^{2s}} \Big) \\ &\le \|u\|_0^2 C(s)^2 |A|_s^2 / \mathrm{d}(\Omega_1,\Omega_2)^{2s-(r+n+1)} \end{aligned}$$

where $r+n+1$ is the dimension of E^+. ■

Since H^s is an algebra, any function $b \in H^s$ defines the multiplication operator

$$u(t,x) \mapsto b(t,x)u(t,x)\,, \qquad \forall u \in H^s\,, \tag{4.12}$$

which is represented by $(B_k^{k'})_{k,k'\in E^+}$ with $B_k^{k'} := \Pi_{\mathcal{N}_k} b(t,x)_{| \mathcal{N}_{k'}} \in \mathcal{L}(\mathcal{N}_{k'}, \mathcal{N}_k)$.

Lemma 4.7. (multiplication operator) *If $b \in H^s$ is real, then the matrix $(B_k^{k'})_{k,k'\in E^+}$ is self-adjoint, i.e. $(B_k^{k'})^\dagger = B_{k'}^k$, and, $\forall s > (d+n+1)/2$,*

$$\|B_k^{k'}\|_0 \le \frac{C(s)\|b\|_s}{\langle k-k'\rangle^{s-(d+n+1)/2}}\,. \tag{4.13}$$

PROOF. The self-adjointness is a consequence that b is real. In order to prove (4.13) we first suppose that $k = (l, j)$ and $k' = (l', j')$ satisfy $(\rho, j_1 - j'_1) \geq 0$. We split

$$b = b^{\geq k-k'} + b^{\ngeq k-k'} \quad \text{where} \quad b^{\geq k-k'} := \sum_{k_1 \geq k-k', k_1 \in E^+} \Pi_{\mathcal{N}_{k_1}} b\,, \; b^{\ngeq k-k'} := \sum_{k_1 \ngeq k-k', k_1 \in E^+} \Pi_{\mathcal{N}_{k_1}} b$$

(note that the relation $\geq$ is only a partial order, see Definition 4.1). By Lemma 4.2 and Definition 4.1

$$\Pi_{\mathcal{N}_k}(b^{\ngeq k-k'} u_{k'}) = 0\,, \qquad \forall u_{k'} \in \mathcal{N}_{k'}\,,$$

and therefore

$$\|B_k^{k'}\|_0 := \sup_{\substack{\|u_{k'}\|_0 = 1 \\ u_{k'} \in \mathcal{N}_{k'}}} \|\Pi_{\mathcal{N}_k}(b u_{k'})\|_0 = \sup_{\substack{\|u_{k'}\|_0 = 1 \\ u_{k'} \in \mathcal{N}_{k'}}} \|b^{\geq k-k'} u_{k'}\|_0 \leq \|b^{\geq k-k'}\|_{L^\infty} \overset{(4.3)}{\leq} \frac{C(s)\|b\|_s}{\langle k - k' \rangle^{s-(d+n+1)/2}}$$

applying Lemma 4.3 because $(\rho, j - j') \geq 0$. If $(\rho, j - j') < 0$ then $(\rho, j' - j) > 0$ and

$$\|B_k^{k'}\|_0 = \|(B_{k'}^k)^\dagger\|_0 = \|B_{k'}^k\|_0 \leq \frac{C(s)\|b\|_s}{\langle k' - k \rangle^{s-(d+n+1)/2}}$$

by the previous estimate. Then (4.13) holds in any case. ■

By Lemma 4.7 we have the following corollary:

Lemma 4.8. *For any real $b \in H^{s+a}$ with $a \geq (d + r + 2n + 3)/2$, the matrix $B = (B_k^{k'})_{k,k' \in E^+}$ which represents the multiplication operator (4.12) is self-adjoint, it belongs to the algebra of quasi-Töplitz matrices $\mathcal{A}_s$ and*

$$|B|_s \leq K(s)\|b\|_{s+a}\,. \tag{4.14}$$

We shall need to consider restrictions of quasi-Töplitz matrices. Given a set of indexes $\mathcal{I} \subset E^+$, let

$$\mathcal{A}_s(\mathcal{I}) := \Big\{ A = (A_k^{k'})_{k,k' \in \mathcal{I}} \; : \; (A_k^{k'})^\dagger = A_{k'}^k \;\text{ and }\; |A|_s^2 := \sup_{k \in \mathcal{I}} \sum_{k' \in \mathcal{I}} \langle k - k' \rangle^{2s} \|A_k^{k'}\|_0^2 < \infty \Big\}\,.$$

Lemma 4.9. *For $A := (A_k^{k'})_{k,k' \in E^+} \in \mathcal{A}_s$ its restriction $A_{\mathcal{I}} := (A_k^{k'})_{k,k' \in \mathcal{I}} \in \mathcal{A}_s(\mathcal{I})$ satisfies $|A_{\mathcal{I}}|_s \leq |A|_s$. On the other hand, any $A := (A_k^{k'})_{k,k' \in \mathcal{I}} \in \mathcal{A}_s(\mathcal{I})$ can be extended to a matrix in $\mathcal{A}_s$ setting $A_k^{k'} = 0$ for $k, k' \notin \mathcal{I}$ without changing the norm $|A|_s$.*

Therefore all the properties (algebra, interpolation ...) hold for $\mathcal{A}_s(\mathcal{I})$ with constants which do not depend on $\mathcal{I}$. Moreover, the matrices $I_{\mathcal{I}}$ which represent the projectors

$$\Pi_{\mathcal{I}} : H^s \to H_{\mathcal{I}} := \oplus_{k \in \mathcal{I} \cap E^+} \mathcal{N}_k \qquad \text{satisfy} \qquad |I_{\mathcal{I}}|_s = 1\,, \; \forall s \geq 0\,. \tag{4.15}$$

4.2 Composition operators

We conclude this section stating a technical result for the composition operator

$$u(t, x) \mapsto \mathfrak{f}(\delta, u)(t, x) := \delta^{-m} f(x, \delta u(t, x))$$

acting between the Sobolev spaces H^s, see section 2 of [3].

Lemma 4.10. (Composition operator) *Fix $\mathfrak{s}_0 > 1 + (d + n + 1)/2$. Assume that $f \in C^k$ satisfies (1.7) for some $0 \leq m \leq k - 4$. Then, $\forall\, s \in [\mathfrak{s}_0, k - 2]$, the map*

$$(\delta, u) \mapsto \mathfrak{f}(\delta, u) \text{ is in } C^2([0, \delta_0] \times H^s; H^s)$$

and satisfies the "tame" estimates

$$\|\mathfrak{f}(\delta, u)\|_s, \|(\partial_u \mathfrak{f})(\delta, u)\|_s, \|(\partial_\delta \mathfrak{f})(\delta, u)\|_s, \|(\partial^2_{\delta,u} \mathfrak{f})(\delta, u)\|_s, \|(\partial_{uu} \mathfrak{f})(\delta, u)\|_s \leq C(s, \|u\|_{\mathfrak{s}_0})(1 + \|u\|_s)\,. \tag{4.16}$$

Notice that if f is real on real then $(\delta, u) \mapsto \mathfrak{f}(\delta, u)$ is in $C^2([0, \delta_0] \times H^s_{\text{even}}; H^s_{\text{even}})$.

5 The bifurcation equation

By the hypothesis (ND) the subspace $\mathcal{H}^s \subset H^s_{\rm even}$ is invariant for the NLW equation (1.8) and there is a non-degenerate solution of the equation (1.14) in $Q \cap \mathcal{H}^s$. Then, by the standard implicit function Theorem, we solve in $\mathcal{H}^s$ the bifurcation equation (1.10).

Lemma 5.1. *There exist $\delta_0, R_0 > 0$ and a C^2 map $(\delta, p) \mapsto q(\delta, p)$ defined on*

$$[0, \delta_0] \times \{p \in P \cap \mathcal{H}^s \, : \, \|p\|_s < R_0\} \to Q \cap \mathcal{H}^s \, , \quad \forall \, s \in [\mathfrak{s}_0, k-2] \, ,$$

such that $q(0,0) = \bar{q}$ and $q(\delta, p)$ solves the bifurcation equation in (1.10)*. Moreover*

$$\partial_p q(\delta, p)[h] = -\chi J^{-1} \Pi_Q \Big((\partial_u \mathfrak{f})(\delta, q(\delta, p) + p) \, h \Big) \, , \quad \forall h \in Q \cap \mathcal{H}^s \tag{5.1}$$

where $J : Q \to Q$ is the linear operator

$$J[H] := J(\delta, p)[H] := H + \chi \Pi_Q \Big((\partial_u f)(\delta, q(\delta, p) + p) H \Big) \, , \quad \forall H \in Q \, , \tag{5.2}$$

and

$$\|q(\delta, p)\|_s, \|\partial_\delta q(\delta, p)\|_s, \|\partial_p q(\delta, p)\|_s, \|\partial^2_{\delta, p} q(\delta, p)\|_s, \|\partial_{pp} q(\delta, p)\|_s \leq C(s)(1 + \|p\|_s) \, . \tag{5.3}$$

Proof. Note that the operator $J(\delta, p)$ in (5.2) reduces, for $\delta = 0$, $p = 0$, to $h \mapsto h + \chi \Pi_Q (m a(x) \bar{q}^{m-1} h)$ which is invertible in $Q \cap \mathcal{H}^s$ by (ND). Finally (5.3) holds simply because Q is finite dimensional. ■

5.1 The non-degeneracy condition

We now consider $\mathcal{M} = SU(n)$.

Lemma 5.2. *Let $n \neq 2, 4$. If $f(x, u) = u^3 + r(x, u)$ where $r(x, u)$ satisfies (1.7) for some $m > 3$ and $r(x, u) = r(x^{-1}, u)$ then the non-degeneracy condition (ND) holds in the invariant subspace*

$$\mathcal{H}^s := \Big\{ u(t, x) \in H^s_{\rm even} \mid u(t, x) = u(t, x^{-1}) \Big\} \, . \tag{5.4}$$

The remainder of this section is devoted to the proof of Lemma 5.2.

Remark 5.1. *A real function $u(x)$ satisfies $u(x) = u(x^{-1})$ if and only if its Fourier coefficients $\mathbf{u}_j := \int_{SU(n)} u(x) \bar{\mathbf{e}}_j(x) = \mathbf{u}_j^\dagger$ (defined in (2.6)) are self-adjoint. We use that $x \mapsto x^{-1}$ is measure preserving.*

We choose $j_0 = w_1 \in \Lambda^+$ to be the first fundamental weight of $SU(n)$. It corresponds to the least non-zero eigenvalue of the Laplacian and to the standard representation $\rho_{\mathbb{C}^n}(x) = x$ of $SU(n)$ on $\mathbb{C}^n$. Its dual representation on $(\mathbb{C}^n)^*$ is the complex conjugated matrix $\bar{x}$ and corresponds to the fundamental weight w_{n-1}. We shall denote by V the representation space $\mathbb{C}^n$ endowed with the standard basis $\{e_i\}_{i=1}^n$.

The bifurcation equation (1.11) reduces to

$$q = \Pi_Q q^3 \, , \quad q \in Q \cap \mathcal{H}^s \, , \tag{5.5}$$

because $a(x) = 1$ and $\chi = -1$ by (1.13).

Lemma 5.3. $Q \cap \mathcal{H}^s := \{ q_A := (\cos t) {\rm tr}(A(x \oplus \bar{x}))$ where $A := A_1 \oplus \bar{A}_1 \, , A_1 \in {\rm Mat}(\mathbb{C}^n), A_1 = A_1^\dagger \}$.

PROOF. The dominant weights $j \in \Lambda^+(SU(n))$ such that $\omega_j = \omega_{j_0}$ are $j = w_1, w_{n-1}$. Then any complex function with Fourier indices in $\mathcal{Q}$ defined in (1.9) is uniquely represented by

$$q_A(t,x) = (\cos t)(\mathrm{tr}(A_1 x) + \mathrm{tr}(A_2 \bar{x})) \overset{(2.4)}{=} (\cos t)\,\mathrm{tr}(A(x \oplus \bar{x}))\,, \quad A := A_1 \oplus A_2\,,$$

where $A_1,\ A_2 \in \mathrm{Mat}(\mathbb{C}^n)$, and, by Lemma 2.4, $\mathrm{tr}(A_1 x) \in \mathscr{M}_{w_1}$ and $\mathrm{tr}(A_2 x) \in \mathscr{M}_{w_{n-1}}$. Then $q_A(t,x) = \bar{q}_A(t,x)$ is real iff $A_1 = \bar{A}_2$ and $q_A(t,x) = q_A(t,x^{-1})$ if and only if $A_1^\dagger = A_1$ (see remark 5.1). ■

In order to solve (5.5) we have to compute $\Pi_Q q_A^3$. By (2.5) we have

$$q_A^3 = \Big(\mathrm{tr}(A(x \oplus \bar{x}))\Big)^3 (\cos t)^3 = \mathrm{tr}(A^{\otimes 3}(x \oplus \bar{x})^{\otimes 3})(\cos t)^3 = \mathrm{tr}(\mathcal{A}\,\rho_U(x))(\cos t)^3 \tag{5.6}$$

where $\mathcal{A} := A^{\otimes 3} := A \otimes A \otimes A \in \mathrm{End}(U)$, $\rho_U(x) := (x \oplus \bar{x})^{\otimes 3}$ and

$$U := (V \oplus V^*)^{\otimes 3} \quad = \quad V^{\otimes 3} \oplus (V \otimes V \otimes V^*) \oplus (V \otimes V^* \otimes V) \oplus (V^* \otimes V \otimes V) \oplus \text{duals}\,. \tag{5.7}$$

We decompose in irreducible subspaces each representation space in the direct sum (5.7) (see Def. 2.4).

Lemma 5.4. *Let $n \neq 2, 4$. In the direct sum decomposition of*
(i) $V^{\otimes 3}$, $(V^*)^{\otimes 3}$ there are no copies neither of V nor of V^*.
(ii) $V \otimes V \otimes V^*$, $V \otimes V^* \otimes V$, $V^* \otimes V \otimes V$ there are exactly 2 copies of V and none of V^*. Hence their duals have 2 copies of V^* and none of V.

PROOF. The main ingredient is the "First fundamental Theorem of invariant Theory", see [33]. For $SU(n)$ it states that the only invariant polynomial functions $F(v_1, \ldots, v_k, \phi_1, \ldots, \phi_h)$, $v_i \in V$, $\phi_j \in V^*$ are polynomials in the generators

$$F(\phi_i, v_j) = \phi_i(v_j)\,, F(v_{i_1}, \ldots, v_{i_n}) = \det(v_{i_1}, \ldots, v_{i_n}),\ F(\phi_{i_1}, \ldots, \phi_{i_n}) = \det(\phi_{i_1}, \ldots, \phi_{i_n})\,.$$

Now, a representation space W contains k copies of V if and only if $W \otimes V^*$ contains k independent invariants (see Lemma 2.7). Since there is a one to one correspondence between the invariant multilinear functions and invariants, the only invariants in $V \otimes V \otimes V^* \otimes V^*$ are $\sum_{i,k} e_i \otimes e_k \otimes e^i \otimes e^k$ and $\sum_{i,k} e_i \otimes e_k \otimes e^k \otimes e^i$. ■

Remark 5.2. *For $SU(4)$ the only difference is that there is one copy of V in $V^{\otimes 3}$. For $SU(2)$ there is only one fundamental weight and $V^{\otimes 3}$ contains 2 copies of V. We deal this case for NLS in section 8.1.*

By (5.7) and the previous lemma we get the decomposition

$$U = (V \oplus V^*)^{\oplus 6} \oplus_{j \neq w_1, w_{n-1}} V_j^{\oplus c_j}\,. \tag{5.8}$$

An orthonormal basis for the six copies of V (resp. V^*) in U is explicitly given below.

Lemma 5.5. *Consider the 2 orthogonal subspaces $\mathcal{V}_j := \mathrm{Span}_{k=1,\ldots,n} E_k^{(j)} \subset V^* \otimes V \otimes V$, $j = 1, 2$, where*

$$E_k^{(1)} := \frac{e_k^{(1)}}{\sqrt{n}}\,, \quad E_k^{(2)} := \frac{n e_k^{(2)} - e_k^{(1)}}{\sqrt{n^3 - n}}\,, \quad e_k^{(1)} := \sum_i e^i \otimes e_i \otimes e_k\,, \quad e_k^{(2)} := \sum_i e^i \otimes e_k \otimes e_i\,,$$

and $e_i \in V$, $e^i \in V^*$, $i = 1, \ldots, n$, are the standard basis of V and V^*.

Similarly, consider the 2 orthogonal subspaces $\mathcal{V}_j := \mathrm{Span}_{k=1,\ldots,n} E_k^{(j)} \subset V \otimes V^* \otimes V$, $j = 3, 4$, where

$$E_k^{(3)} := \frac{e_k^{(3)}}{\sqrt{n}}\,, \quad E_k^{(4)} := \frac{n e_k^{(4)} - e_k^{(3)}}{\sqrt{n^3 - n}}\,, \quad e_k^{(3)} := \sum_i e_i \otimes e^i \otimes e_k\,, \quad e_k^{(4)} := \sum_i e_k \otimes e^i \otimes e_i\,.$$

Finally, consider the 2 orthogonal subspaces $\mathcal{V}_j := \mathrm{Span}_{k=1,\ldots,n} E_k^{(j)} \subset V \otimes V \otimes V^*$, $j = 5, 6$,

$$E_k^{(5)} = \frac{e_k^{(5)}}{\sqrt{n}} \,, \quad E_k^{(6)} = \frac{n e_k^{(6)} - e_k^{(5)}}{\sqrt{n^3 - n}} \,, \quad e_k^{(5)} := \sum_i e_k \otimes e_i \otimes e^i \,, \quad e_k^{(6)} := \sum_i e_i \otimes e_k \otimes e^i \,.$$

The representation spaces $\mathcal{V}_j$ *are equivalent to* V*, then the dual spaces* $\mathcal{V}_j^*$ *are equivalent to* V^**. Finally the* $\{E_k^{(j)}\}_{k=1,\ldots,n}$ *are an orthonormal basis in* $\mathcal{V}_j$*.*

For $\mathcal{A} \in \mathrm{End}(U)$, we represent its restriction $\Pi_{\mathcal{V}_j} \mathcal{A}_{|\mathcal{V}_j}$ to $\mathcal{V}_j$ by the $n \times n$ matrix $\mathcal{A}_j$ with entries $(\mathcal{A}_j)_{h,k} = (E_h^{(j)}, \mathcal{A} E_k^{(j)})$. By (2.4) we have

$$\Pi_{\mathscr{M}_{w_1}} \mathrm{tr}(\mathcal{A} \rho_U(x)) = \sum_{j=1}^{6} \mathrm{tr}(\mathcal{A}_j x) \,, \qquad \Pi_{\mathscr{M}_{w_{n-1}}} \mathrm{tr}(\mathcal{A} \rho_U(x)) = \sum_{j=1}^{6} \mathrm{tr}(\bar{\mathcal{A}}_j \bar{x}) \tag{5.9}$$

where $\bar{\mathcal{A}}_j$ is the matrix which represents $\Pi_{\mathcal{V}_j^*} \mathcal{A}_{|\mathcal{V}_j^*}$.

Lemma 5.6. *The function* $(\sqrt{2}/3)\,(\cos t)\,\chi_{V \oplus V^*}(x)$ *is a real non-degenerate solution of (5.5).*

Proof. The caracter $\chi_{V \oplus V^*}(x) = \mathrm{tr}\,(x \oplus \bar{x}) = \mathrm{tr}\,(x) + \mathrm{tr}\,(\bar{x})$. The function $\alpha q_I = \alpha (\cos t) \chi_{V \oplus V^*}$ with $\alpha := \sqrt{2}/3$ is a solution of (5.5) in $Q \cap \mathcal{H}^s$. Indeed, by (5.6) and (5.9) with $\mathcal{A}_j = I$, we get

$$\Pi_Q (\alpha q_I)^3 = \alpha^3 \, \Pi_{\langle \cos t \rangle} (\cos t)^3 \, \Pi_{\mathscr{M}_{w_1} \oplus \mathscr{M}_{w_{n-1}}} (\mathrm{tr}\,(x) + \mathrm{tr}\,(\bar{x}))^3 = \alpha^3 \frac{3}{4} (\cos t) 6 (\mathrm{tr}\,(x) + \mathrm{tr}\,(\bar{x})) = \alpha q_I \,.$$

We have to prove that the linearized equation $h_A - 3\alpha^2 \Pi_Q q_I^2 h_A = 0$, $h_A \in Q \cap \mathcal{H}^s$, has the only solution $h_A = 0$. This amounts to show that

$$2(\mathrm{tr}\,(A_1 x) + \mathrm{tr}\,(\bar{A}_1 \bar{x})) - \Pi_{\mathscr{M}_{w_1} \oplus \mathscr{M}_{w_{n-1}}} \Big((\mathrm{tr}\,(x) + \mathrm{tr}\,(\bar{x}))^2 \, \mathrm{tr}\,((A_1 \oplus \bar{A}_1)(x \oplus \bar{x})) \Big) = 0 \tag{5.10}$$

has the only solution $A = A_1 \oplus \bar{A}_1 = 0$. By (2.4)-(2.5), we have

$$\Big(\mathrm{tr}\,((I \oplus I)(x \oplus \bar{x})) \Big)^2 \mathrm{tr}\,(A(x \oplus \bar{x})) = \mathrm{tr}\,(\mathcal{B} \rho_U) \quad \text{where} \quad \mathcal{B} := (I \oplus I)^{\otimes 2} \otimes (A_1 \oplus \bar{A}_1) \,.$$

As in (5.9), we find

$$\Pi_{\mathscr{M}_{w_1}} \mathrm{tr}\,(\mathcal{B} \rho_U) = \sum_{j=1}^{6} \mathrm{tr}\,(\mathcal{B}_j x) \,, \quad \Pi_{\mathscr{M}_{w_{n-1}}} \mathrm{tr}\,(\mathcal{B} \rho_U) = \sum_{j=1}^{6} \mathrm{tr}\,(\bar{\mathcal{B}}_j \bar{x})$$

where $\mathcal{B}_j$ represents $\Pi_{\mathcal{V}_j} \mathcal{B}_{|\mathcal{V}_j}$ and $\bar{\mathcal{B}}_j$ represents $\Pi_{\mathcal{V}_j^*} \mathcal{B}_{|\mathcal{V}_j^*}$. Since V, V^* are irreducibles, the (5.10) becomes

$$2A_1 = \sum_{j=1}^{6} \mathcal{B}_j \,, \qquad 2\bar{A}_1 = \sum_{j=1}^{6} \bar{\mathcal{B}}_j \,. \tag{5.11}$$

Lemma 5.7. *We have* $\mathcal{B}_1 = \mathcal{B}_3 = A_1$, $\mathcal{B}_4 = \mathcal{B}_2$,

$$\mathcal{B}_2 = \frac{n}{n^2 - 1} \mathrm{tr}\,(A_1) I - \frac{A_1}{n^2 - 1} \,, \ \mathcal{B}_5 = \frac{\mathrm{tr}\,(A_1)}{n} I \,, \ \mathcal{B}_6 = \frac{n^2 + 1}{n^3 - n} \mathrm{tr}\,(A_1) I - \frac{2A_1}{n^2 - 1} \,.$$

Proof. The coefficients $(E_h^{(j)}, \mathcal{B} E_k^{(j)})$ can be computed using lemma 5.5 and recalling that $A_1 = A_1^\dagger$. ∎

By the previous lemma the equations (5.11) reduce to $A_1 = n \mathrm{tr}\,(A_1) I$. Then we get $\mathrm{tr}\,(A_1)(n^2 - 1) = 0$, and, for $n \neq 1$, $\mathrm{tr}\,(A_1) = 0$ and $A_1 = 0$. ∎

The case $f = u^p$ with p odd is similar, because $(V \oplus V^*)^{\otimes p}$ always contains copies of V.

Remark 5.3. *For the other groups the main point is to have the analogue of Lemma 5.4. For E_6 one can proceed essentially as above. For the other groups we have a unique fundamental representation W with minimum eigenvalue of the Laplace operator and W and W^* are equivalent. This allows us to construct an invariant tensor $J \in W \otimes W$ which, under the isomorphism between W and W^* mapping $W \otimes W$ to $W \otimes W^* = \mathrm{End}(W)$, corresponds to the identity map. By the theory of symmetry on tensor spaces we see that we have copies of the representation on each odd tensor power $W^{\otimes 2k+1}$. One copy of W is $J^{\otimes k} \otimes W$ and other copies are obtained by permuting the tensors. In particular, for $W^{\otimes 3}$ we can exhibit the copies of the representation W appearing.*

6 The range equation and proof of Theorem 1.1

We solve the range equation

$$F(\delta, p) := \mathbb{D}(\omega)p - \varepsilon \mathcal{F}(\delta, p) = 0 \tag{6.1}$$

where $\omega := \omega(\delta) := (\omega_{j_0}^2 + \chi \delta^{m-1})^{1/2}$, $\varepsilon := \delta^{m-1}$,

$$\mathbb{D}(\omega) := \omega^2 \partial_{tt} - \Delta + \mu \qquad \text{and} \qquad \mathcal{F}(\delta, p) := \Pi_P \mathfrak{f}(\delta, q(\delta, p) + p)\,, \tag{6.2}$$

applying the abstract Nash-Moser implicit function Theorem 3.1 in the scale $X_s := H^s_{\mathrm{even}} \cap P$. To simplify notations we solve the range equation in the larger space $H^s_{\mathrm{even}} \cap P$ (not in the subspace $\mathcal{H}^s \cap P$).

For all $N \in \mathbb{N}$ we consider the finite dimensional subspaces

$$H^{(N)} := \oplus_{k \in E^+_N \cap \mathcal{P}} \mathcal{N}_k \cap H^0_{\mathrm{even}} \subset \bigcap_{s \geq 0} X_s \qquad \text{where} \qquad E^+_N := \{k \in E^+ \mid |k + \vec{\rho}| \leq N\}$$

($\mathcal{P}$ is defined in (1.9) and $\mathcal{N}_k$ in (4.1)) and the corresponding projectors $\Pi^{(N)} : X_s \to H^{(N)}$, $\Pi^{(N)} u := \sum_{k \in E^+_N \cap \mathcal{P}} \Pi_{\mathcal{N}_k} u$. The smoothing properties (S1)-(S2) in (3.1)-(3.2) hold.

Lemma 6.1. *Let $\mathfrak{s}_0 > 1 + (d + n + 1)/2$ and $f \in C^k$ with $k > \mathfrak{s}_0 + 2$. For all $s \in [\mathfrak{s}_0, k-2]$ the map $F \in C^2([0, \delta_0) \times X_{s+2}, X_s)$, $F(0,0) = 0$, and "tame" properties (F2)-(F4) hold* (*with $\nu = 2$ and $S = k-2$*).

Proof. The composition operator $(\delta, u) \mapsto \mathfrak{f}(\delta, u)$ is C^2 by Lemma 4.10, and, by Lemma 5.1, also the map $(\delta, q) \mapsto q(\delta, p)$ is C^2. Properties (F2)-(F4) follow by (4.16) and (5.3). ■

The main issue is to prove assumption (L) concerning the invertibility of the linearized operators

$$L^{(N)}(\delta, p) := \Pi^{(N)}(\mathbb{D}(\omega) - \varepsilon D_p \mathcal{F}(\delta, p))_{|H^{(N)}}\,. \tag{6.3}$$

It will be a consequence of the following lemma proved in the next section.

Lemma 6.2. (**Invertibility**) *For all $\tau > 1$, there is $s_0 := s_0(\mathcal{M}, \tau)$ such that, if*

$$|\omega^2 m - n| \geq \frac{\gamma}{\langle n \rangle^{3/2}}\,,\ \gamma \in (0,1)\,, \quad \forall (m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}\,, \tag{6.4}$$

$\varepsilon(\|p\|_{s_0} + 1) \leq c(\gamma, k)$ is small enough, and

$$\forall 1 \leq K \leq N\,, \qquad \left\|(L^{(K)}(\delta, p))^{-1}\right\|_0 \leq \frac{4K^\tau}{\gamma}\,, \tag{6.5}$$

then $L^{(N)}(\delta, p)$ is invertible and, $\forall s \in [s_0, k-1]$,

$$\left\|(L^{(N)}(\delta, p))^{-1} h\right\|_s \leq \frac{N^\kappa}{\gamma}(\|h\|_s + \|p\|_s \|h\|_{s_0})\,,\ \forall h \in H^{(N)}\,, \tag{6.6}$$

with $\kappa := (4\tau + 3r + 2n + d + 7)/2$.

We now complete the proof of property (L). We fix in (6.5)

$$\tau \geq \max\Big\{\tau_1, 2+d+n+\frac{m-2}{m-1}(\tau_1+2)\Big\}\,, \quad \gamma \in (0,\bar\gamma]\,,\ \bar\gamma \leq \gamma_1\,, \tag{6.7}$$

(τ_1 and γ_1 are given in (1.6)) and, correspondingly, the parameter

$$\sigma := \max\{4(\kappa+2), d+n+3\}+1$$

(the first condition is the first inequality in (3.3) with $\nu = 2$). We then take $f \in C^k$ with

$$k > s_0 + 4(\kappa+3) + 2\sigma + 2 \tag{6.8}$$

which is the second condition in (3.3) with $S := k-2$.

For $\delta_0 > 0$ small enough, by Lemma 6.2, defining

$$J^{(N)}_{\gamma,\kappa} := \Big\{(\delta,p) \in [0,\delta_0) \times H^{(N)} \mid \|p\|_{s_0} \leq 1\,,\ \omega(\delta) \text{ satisfies (6.4) and (6.5) holds}\Big\}\,,$$

the inclusion (3.4) is satisfied. Given a function $\delta \mapsto p(\delta)$ in $\mathcal{U}^{(N)}_{\mathtt{k}}$, the set $\mathcal{G}^{(N)}_{\gamma,\kappa}$ defined in (3.5) is

$$\mathcal{G}^{(N)}_{\gamma,\kappa}(p) = \bigcap_{1\leq K\leq N} \mathtt{G}_K \bigcap \mathcal{G} \tag{6.9}$$

where

$$\mathtt{G}_K := \Big\{\delta \in [0,\delta_0) \mid \Big\|\Big(L^{(K)}(\delta,p(\delta))\Big)^{-1}\Big\|_0 \leq 4\frac{K^\tau}{\gamma}\Big\} \quad \text{and} \quad \mathcal{G} := \Big\{\delta \in [0,\delta_0) \mid \omega(\delta) \text{ satisfies (6.4)}\Big\}\,.$$

Lemma 6.3. *Let μ be diophantine. If $\varepsilon_0\gamma_1^{-1}M^{\tau_1+2}$ is small enough, then $\mathcal{G}^{(M)}_{\gamma,\kappa}(0) = \mathcal{G}$. Moreover $\mathcal{G}$ satisfies (3.6).*

PROOF. The eigenvalues of $L^{(K)}$ have the form $-\omega^2 l^2 + \omega_j^2 + O(\varepsilon)$ ($l \in \mathbb{Z}$, $(j_1,j_2) \in \Lambda^+ \times \mathbb{Z}^n$) with $|(l, j_1+\rho, j_2)| \leq K$. Hence, by (1.6), if $\varepsilon_0\gamma_1^{-1}K^{\tau_1+2}$ is small enough, all the eigenvalues of $L^{(K)}$ have modulus $\geq \gamma_1/(4K^{\tau_1}) \geq \gamma/(4K^\tau)$. Then $\mathtt{G}_K = [0,\delta_0)$. The measure estimate for $\mathcal{G}$ is standard. ■

Lemma 6.4. *The measure estimate (3.7) holds.*

PROOF. In the Appendix. ■

All the assumption of the Nash-Moser Theorem 3.1 are satisfied and we deduce Theorem 1.1.

7 Inversion of the linearized equation: proof of Lemma 6.2

It is notationally more convenient to prove Lemma 6.2 in the larger scale of complex Sobolev spaces $X_s = H^s \cap \oplus_{k\in\mathcal{P}}\mathcal{N}_k$. The same estimates will hold in $H^s_{\text{even}} \cap P$.

According to the orthogonal decomposition $H^{(N)} = \oplus_{k\in E^+_N\cap\mathcal{P}}\mathcal{N}_k$ we represent

$$h \mapsto L^{(N)}[h] := \Pi^{(N)}\Big(\mathbb{D}(\omega) + \varepsilon b(t,x)(h + (\partial_p q)[h])\Big)\,, \quad \forall h \in H^{(N)}\,,$$

where $b(t,x) := (\partial_u \mathfrak{f})(\delta, q(\delta,p)+p) \in H^s_{\text{even}}$, $\partial_p q := \partial_p q(\delta,p)$, by the block matrix

$$L^{(N)} = D + \varepsilon T$$

where $D := \mathrm{diag}_{k\in E^+_N\cap\mathcal{P}}(\mathfrak{D}_k I_k)$ with small divisors

$$\mathfrak{D}_k := -\omega^2 l^2 + |j_1+\rho|^2 - |\rho|^2 + |j_2|^2 + \mu\,, \quad k := (l, j_1, j_2) \in \mathbb{Z}\times\Lambda^+\times\mathbb{Z}^n\,, \tag{7.1}$$

I_k is the identity map in $\mathcal{N}_k$, and

$$T := (T_k^{k'})_{k,k'\in E^+_N\cap\mathcal{P}}\,, \qquad T_k^{k'} := \Pi_{\mathcal{N}_k} L^{(N)}{}_{|\mathcal{N}_{k'}} \in \mathcal{L}(\mathcal{N}_{k'}, \mathcal{N}_k)\,. \tag{7.2}$$

The main properties to establish are
(i) separation of the singular sites (section 7.1)
(ii) Töplitz estimates for the block matrix T (section 7.2).

7.1 Regular and Singular sites

Definition 7.1. *Fixed $\eta\in(0,1)$ we define the* **regular sites** *R and the* **singular sites** *S, as*

$$R := \Big\{k\in E^+\cap\mathcal{P} \mid |\mathfrak{D}_k|\geq\eta\Big\} \quad \text{and} \quad S = R^c = \Big\{k\in E^+\cap\mathcal{P} \mid |\mathfrak{D}_k|<\eta\Big\}\,. \tag{7.3}$$

We shall assume that there is a partition of the singular sites

$$S = \bigcup_{\alpha\in\mathbb{N}} \Omega_\alpha \tag{7.4}$$

where the "clusters" $\Omega_\alpha \subset E^+$ are pairwise disjoint and satisfy

- (H1) **(dyadic)** $M_\alpha \leq 2m_\alpha$, $\forall\alpha$, where $M_\alpha := \max_{k\in\Omega_\alpha}|k+\vec\rho|$, $m_\alpha := \min_{k\in\Omega_\alpha}|k+\vec\rho|$.
- (H2) **(separation)** $\exists\lambda$, $\bar C>0$ such that $\mathrm{d}(\Omega_\alpha,\Omega_\beta)\geq\bar C(M_\alpha+M_\beta)^\lambda$, $\forall\alpha\neq\beta$

where $\mathrm{d}(\Omega_\alpha,\Omega_\beta) := \min_{k\in\Omega_\alpha, k'\in\Omega_\beta}|k-k'|$ and λ depends only on $\mathcal{M}$.

We shall denote with the same symbols S, R, Ω_α also their restrictions to E^+_N, for each N.

Actually, the Diophantine condition (6.4), (7.1) and (2.12), imply the existence of such a partition:

Lemma 7.1. *Assume (6.4). There exists $\eta_0(\gamma)$ such that, if $\eta\in(0,\eta_0(\gamma)]$, there exists a partition of the singular sites S like (7.4) satisfying (H1)-(H2).*

The proof, which follows essentially the scheme of [4], is in the Appendix.

7.2 Töplitz estimates

We now consider the operator $B_0 : H^s\cap P\to H^s\cap P$ defined by

$$B_0(h) := \Pi_P\Big(b(t,x)(\partial_p q)[h]\Big) \overset{(5.1)}{=} -\chi\Pi_P\Big(b(t,x)J^{-1}\Pi_Q(b(t,x)h)\Big)\,.$$

Lemma 7.2. *The matrix which represents B_0 is self-adjoint, belongs to $\mathcal{A}_s(\mathcal{P})$, and $|B_0|_s\leq K(s)|B|_s$.*

Proof. The fact that B_0 is self-adjoint is immediate writing

$$B_0(h) = -\chi\Pi_P\Big(b(t,x)\Pi_Q J^{-1}\Pi_Q(b(t,x)\Pi_P h)\Big)\,.$$

The matrix which represents J is in $\mathcal{A}_s(\mathcal{Q})$ and $\mathcal{Q}$ is finite dimensional (see (1.9)). By the algebra property $J^{-1}\in\mathcal{A}_s(\mathcal{Q})$ as well, and $|J^{-1}|_s\leq C(s)$. If B represents the multiplication by $b(t,x)$, by Lemma 4.8 the matrix $B\in\mathcal{A}_s$, as well as the matrices which represent the projections Π_P, Π_Q, see (4.15). Applying Lemmata 4.4 and 4.5 we deduce $|B_0|_s\leq C(s)|B|_s|B|_{\mathfrak{s}_0}$. ■

By Lemmata 4.8 and 7.2 we finally get:

Lemma 7.3. *Consider a real* $b \in H^{s+a}_{\rm even}$ *with* $a \geq (d+r+2n+3)/2$. *The matrix* $T = (T_k^{k'})_{k,k' \in E_N^+}$ *defined in (7.2) is self-adjoint and belongs to the algebra of quasi-Töplitz matrices* $\mathcal{A}_s(E_N^+)$ *with*

$$|T|_s \leq K(s)\|b\|_{s+a}\,. \tag{7.5}$$

Note also that, since the index $|k+\rho|, |k'+\rho| \leq N$, then, $\forall s > a$,

$$|T|_s \leq (2N)^a |T|_{s-a} \overset{(7.5)}{\leq} K'(s) N^a \|b\|_s\,. \tag{7.6}$$

7.3 Reduction along the regular sites

According to the decomposition

$$H^{(N)} := H_R \oplus H_S \qquad \text{where} \qquad H_R := \oplus_{k \in R \cap E_N^+} \mathcal{N}_k\,,\ H_S := \oplus_{k \in S \cap E_N^+} \mathcal{N}_k$$

we represent $L^{(N)}$ as the self-adjoint block matrix

$$L^{(N)} = \begin{pmatrix} L_R & L_R^S \\ L_S^R & L_S \end{pmatrix} \qquad \text{where} \qquad L_R^S = (L_S^R)^\dagger\,,\ L_R^\dagger := L_R\,,\ L_S^\dagger = L_S\,.$$

For the sequel we fix

$$s_1 := \frac{(2+\lambda)(r+n+1)+2\tau+1}{2\lambda} \tag{7.7}$$

and we shall always assume that

$$\varepsilon\gamma^{-1}\eta^{-1}(1+|T|_{s_1}) \leq c(k) \quad \text{is small enough}\,. \tag{7.8}$$

Moreover we consider the index $s \in [s_1, k-1]$.

Lemma 7.4. *The operator* L_R *is invertible,* $|L_R^{-1}|_{s_1} \leq 2\eta^{-1}$ *and, for all* $s \in [s_1, k-1]$,

$$|L_R^{-1}|_s \leq C(s)\eta^{-1}(1+\varepsilon\eta^{-1}|T|_s)\,, \tag{7.9}$$

$$\|L_R^{-1}h\|_s \leq C(s)\eta^{-1}\Big((1+\varepsilon\eta^{-1}|T|_s)\,\|h\|_{s_1} + \|h\|_s\Big)\,, \quad \forall h \in H^{(N)}. \tag{7.10}$$

Proof. By (7.3), the diagonal matrix D_R satisfies $|D_R^{-1}|_s \leq \eta^{-1}$, $\forall s \geq 0$. By (4.7), the Neumann series

$$L_R^{-1} = \sum_{m \geq 0} (-\varepsilon)^m (D_R^{-1}T_R)^m D_R^{-1} \tag{7.11}$$

is totally convergent in $|\cdot|_{s_1}$, with $|L_R^{-1}|_{s_1} \leq 2\eta^{-1}$, taking $\varepsilon\eta^{-1}|T|_{s_1} \leq c(s_1)$ small enough.
Proof of (7.9). We obtain, $\forall m \in \mathbb{N}$,

$$\begin{aligned}
\varepsilon^m |(D_R^{-1}T_R)^m D_R^{-1}|_s &\overset{(4.7)}{\leq} \varepsilon^m C(s) |(D_R^{-1}T_R)^m|_s |D_R^{-1}|_s \\
&\overset{(4.11)}{\leq} C(s)\varepsilon^m m (C(s)|D_R^{-1}T_R|_{s_1})^{m-1} |D_R^{-1}T_R|_s \eta^{-1} \\
&\overset{(4.7)}{\leq} C'(s)\varepsilon m \eta^{-2}\Big(\varepsilon C(s)\eta^{-1}|T|_{s_1}\Big)^{m-1} |T|_s\,.
\end{aligned}$$

Then (7.9) follows by (7.11), for $\varepsilon\eta^{-1}|T|_{s_1} < c(s_1)$ small enough.

Proof of (7.10). By the interpolation estimates (4.10) and (7.9):

$$\|L_R^{-1}h\|_s \le C(s)(|L_R^{-1}|_s\|h\|_{s_1} + |L_R^{-1}|_{s_1}\|h\|_s) \le C'(s)\Big(\eta^{-1}(1+\varepsilon\eta^{-1}|T|_s)\|h\|_{s_1} + 2\eta^{-1}\|h\|_s\Big)$$

since $|L_R^{-1}|_{s_1} \le 2\eta^{-1}$. ■

The invertibility of $L^{(N)}$ is reduced to that of the "quasi-singular" matrix

$$\mathcal{L} := L_S - L_S^R L_R^{-1} L_R^S \tag{7.12}$$

via the "resolvent type" identity

$$(L^{(N)})^{-1} = \begin{pmatrix} I & -L_R^{-1}L_R^S \\ 0 & I \end{pmatrix} \begin{pmatrix} L_R^{-1} & 0 \\ 0 & \mathcal{L}^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -L_S^R L_R^{-1} & I \end{pmatrix}. \tag{7.13}$$

Note that $\mathcal{L} \in \mathcal{A}_s(S)$ because $\mathcal{L}$ is the restriction to S of the quasi-Töplitz matrix

$$I_S\Big(L - I_S L I_R \tilde{L}^{-1} I_R L I_S\Big) I_S \in \mathcal{A}_s \qquad \text{where} \qquad \tilde{L} = \begin{pmatrix} I & 0 \\ 0 & L_R \end{pmatrix} \in \mathcal{A}_s$$

and I_S, $I_R \in \mathcal{A}_s$ are the projection matrices on S, R, see (4.15).

7.4 The "quasi-singular" matrix $\mathcal{L}$

By (7.4) the singular sites restricted to E_N^+ are

$$S = \bigcup_{\alpha \in \mathcal{I}_N} \Omega_\alpha \quad \text{where} \quad \mathcal{I}_N := \Big\{\alpha \in \mathbb{N} \ \mid \ m_\alpha \le N\Big\}$$

and $\Omega_\alpha \equiv \Omega_\alpha \cap E_N^+$. According to the decomposition $H_S := \oplus_{\alpha \in \mathcal{I}_N} H_\alpha$ where $H_\alpha := \oplus_{k \in \Omega_\alpha} \mathcal{N}_k$, we represent $\mathcal{L}$ as the block matrix $\mathcal{L} = (\mathcal{L}_\alpha^\beta)_{\alpha,\beta \in \mathcal{I}_N}$ where $\mathcal{L}_\alpha^\beta := \Pi_{H_\alpha} \mathcal{L}_{|H_\beta}$. We write

$$\mathcal{L} = \mathcal{D} + \mathcal{T} \qquad \text{where} \qquad \mathcal{D} := \text{diag}_{\alpha \in \mathcal{I}_N}(\mathcal{L}_\alpha), \quad \mathcal{L}_\alpha := \mathcal{L}_\alpha^\alpha, \quad \mathcal{T} := (\mathcal{L}_\alpha^\beta)_{\alpha \ne \beta}.$$

Lemma 7.5. (Decay of $\mathcal{T}$) $\forall \alpha \ne \beta$

$$\|\mathcal{L}_\alpha^\beta\|_0 \le \frac{C(s)\varepsilon|T|_s}{\text{d}(\Omega_\alpha, \Omega_\beta)^{s - \frac{r+n+1}{2}}}. \tag{7.14}$$

Proof. Since T and L_R^{-1} belong to the algebra of quasi-Töplitz matrices $\mathcal{A}_s$, also the matrix $\mathcal{T} = \varepsilon T_S - \varepsilon^2 T_S^R L_R^{-1} T_R^S$ belongs to $\mathcal{A}_s$. Then (7.14) follows by Lemma 4.6 once we prove that

$$|\mathcal{T}|_s \le \varepsilon|T|_s + \varepsilon^2 |T_S^R L_R^{-1} T_R^S|_s \le C(s)\varepsilon|T|_s. \tag{7.15}$$

The last bound in (7.15) follows from

$$\begin{aligned} |T_S^R L_R^{-1} T_R^S|_s &\overset{(4.9)}{\le} C(s)\Big(|T_S^R|_{s_1}|L_R^{-1}T_R^S|_s + |T_S^R|_s|L_R^{-1}T_R^S|_{s_1}\Big) \\ &\overset{(4.9),(4.7)}{\le} C'(s)\Big(|T|_{s_1}(|L_R^{-1}|_{s_1}|T|_s + |L_R^{-1}|_s|T|_{s_1}) + |T|_s|L_R^{-1}|_{s_1}|T|_{s_1}\Big) \\ &\overset{(7.9)}{\le} C_1''(s)\eta^{-1}|T|_{s_1}|T|_s \end{aligned}$$

using $|L_R^{-1}|_{s_1} \le 2\eta^{-1}$, $|T|_{s_1} \le |T|_s$, and $\varepsilon\eta^{-1}|T|_{s_1} \le 1$. ■

By the assumption (6.5) we deduce the following lemma (proved in the Appendix).

Lemma 7.6. *$\forall \alpha \in \mathcal{I}_N$, $\mathcal{L}_\alpha$ is invertible and $\|\mathcal{L}_\alpha^{-1}\|_0 \le C\gamma^{-1}M_\alpha^{\tau}$.*

Then the block diagonal matrix $\mathcal{D}$ is invertible. Moreover:

Lemma 7.7. (Inversion of D) *$\forall s \ge 0$, $\|\mathcal{D}^{-1}\|_s \le C(s)\gamma^{-1}N^{\tau}$.*

Proof. Let $u = \sum_{\alpha\in\mathcal{I}_N} u_\alpha$, $u_\alpha \in H_\alpha$. By Lemma 7.6, and since each Ω_α is dyadic (see (H1)),

$$\begin{aligned}\|\mathcal{D}^{-1}u\|_s^2 \quad &= \quad \sum_{\alpha\in\mathcal{I}_N}\|\mathcal{L}_\alpha^{-1}u_\alpha\|_s^2 \overset{(3.1)}{\le} \sum_{\alpha\in\mathcal{I}_N} M_\alpha^{2s}\|\mathcal{L}_\alpha^{-1}u_\alpha\|_0^2 \le \frac{C}{\gamma^2}\sum_{\alpha\in\mathcal{I}_N} M_\alpha^{2(s+\tau)}\|u_\alpha\|_0^2 \\ &\overset{(3.2)}{\le} \quad \frac{C}{\gamma^2}\sum_{\alpha\in\mathcal{I}_N} M_\alpha^{2(s+\tau)}\frac{\|u_\alpha\|_s^2}{m_\alpha^{2s}} \overset{(H1)}{\le} \frac{C}{\gamma^2}\sum_{\alpha\in\mathcal{I}_N} M_\alpha^{2(s+\tau)}2^s\frac{\|u_\alpha\|_s^2}{M_\alpha^{2s}} \le \frac{C(s)}{\gamma^2}N^{2\tau}\|u\|_s^2\end{aligned}$$

because $M_\alpha \le N$. ■

We now prove that $\mathcal{L}$ is invertible and $\mathcal{L}^{-1}$ satisfies an interpolation inequality in high Sobolev norm.

Lemma 7.8. (Inversion of $\mathcal{L}$) *$\mathcal{L}$ is invertible, $\|\mathcal{L}^{-1}\|_{s_1} \le C\gamma^{-1}N^{\tau}$, and, $\forall s \in [s_1, k-1]$,*

$$\|\mathcal{L}^{-1}u\|_s \le \frac{K(s)}{\gamma}N^{\kappa_1}\Big(|T|_s\|u\|_{s_1} + |T|_{s_1}\|u\|_s\Big), \quad \forall u \in H_S\,, \tag{7.16}$$

where $\kappa_1 := 2\tau + r + n + 1$.

Proof. If $C(s_1)\|\mathcal{D}^{-1}\mathcal{T}\|_{s_1} < 1/2$ then the Neumann series

$$\mathcal{L}^{-1} = (I + \mathcal{D}^{-1}\mathcal{T})^{-1}\mathcal{D}^{-1} = \sum_{m\ge 0}(-1)^m(\mathcal{D}^{-1}\mathcal{T})^m\mathcal{D}^{-1} \tag{7.17}$$

is convergent in operator norm $\|\ \|_{s_1}$ and, using Lemma 7.7, $\|\mathcal{L}^{-1}\|_{s_1} \le C\gamma^{-1}N^{\tau}$.

Lemma 7.9. *$C(s_1)\|\mathcal{D}^{-1}\mathcal{T}\|_{s_1} < 1/2$.*

Proof. For all $u = \sum_{\beta\in\mathcal{I}_N} u_\beta \in H_S$, $u_\beta \in H_\beta$,

$$\begin{aligned}\|\mathcal{D}^{-1}\mathcal{T}u\|_{s_1}^2 \quad &= \quad \sum_{\alpha\in\mathcal{I}_N}\Big\|\sum_{\beta\in\mathcal{I}_N}\mathcal{L}_\alpha^{-1}\mathcal{L}_\alpha^\beta u_\beta\Big\|_{s_1}^2 \overset{(3.1)}{\le} \sum_{\alpha\in\mathcal{I}_N} M_\alpha^{2s_1}\Big\|\sum_{\beta\in\mathcal{I}_N}\mathcal{L}_\alpha^{-1}\mathcal{L}_\alpha^\beta u_\beta\Big\|_0^2 \\ &\le \quad \frac{C}{\gamma^2}\sum_{\alpha\in\mathcal{I}_N} M_\alpha^{2s_1}\Big(\sum_{\beta\in\mathcal{I}_N} M_\alpha^{\tau}\|\mathcal{L}_\alpha^\beta\|_0\|u_\beta\|_0\Big)^2\end{aligned} \tag{7.18}$$

using Lemma 7.6. Since $\|u_\beta\|_0 \overset{(3.2)}{\le} \|u_\beta\|_{s_1}/m_\beta^{s_1}$, Cauchy inequality in (7.18) gives

$$\begin{aligned}\|\mathcal{D}^{-1}\mathcal{T}u\|_{s_1}^2 \quad &\le \quad \gamma^{-2}\sum_{\alpha\in\mathcal{I}_N} M_\alpha^{2(s_1+\tau)}\sum_{\beta\in\mathcal{I}_N}\frac{\|\mathcal{L}_\alpha^\beta\|_0^2}{m_\beta^{2s_1}}\sum_{\beta\in\mathcal{I}_N}\|u_\beta\|_{s_1}^2 \\ &\overset{(7.14),(H2)}{\le} \quad \gamma^{-2}\sum_{\alpha,\beta\in\mathcal{I}_N}\frac{M_\alpha^{2(s_1+\tau)}}{m_\beta^{2s_1}}\frac{C(s_1)^2\varepsilon^2|T|_{s_1}^2}{(M_\alpha + M_\beta)^{\lambda(2s_1-(r+n+1))}}\|u\|_{s_1}^2 \\ &\overset{(7.7)}{\le} \quad \gamma^{-2}C(s_1)^2\varepsilon^2|T|_{s_1}^2\|u\|_{s_1}^2\sum_{\alpha\in\mathbb{N}}\frac{1}{M_\alpha^{r+n+2}}\sum_{\beta\in\mathbb{N}}\frac{1}{m_\beta^{r+n+2}} \\ &\le \quad \gamma^{-2}K(s_1)^2\varepsilon^2|T|_{s_1}^2\|u\|_{s_1}^2\end{aligned} \tag{7.19}$$

where $K(s_1)^2 = C(s_1)^2(\sum_{k\in E^+}\langle k\rangle^{-(r+n+2)})^2$ is independent of N. By (7.8) we prove the lemma. ■

Lemma 7.10. $\|\mathcal{D}^{-1}\mathcal{T}u\|_s \leq \varepsilon K(s)\gamma^{-1}(N^{\kappa_0}|T|_s\|u\|_{s_1} + |T|_{s_1}\|u\|_s)$ *where* $\kappa_0 := \tau + r + n + 1$.

Proof. We estimate $\|\mathcal{D}^{-1}\mathcal{T}u\|_s^2$ as in (7.18) (with s_1 substituted by s) as follows:

$$\begin{aligned}\|\mathcal{D}^{-1}\mathcal{T}u\|_s^2 &\leq \gamma^{-2}2^s\Big(\sum_{\alpha\in\mathcal{I}_N} M_\alpha^{2s}\big(\sum_{\beta\in\mathcal{I}_N : M_\alpha\leq 5m_\beta} M_\alpha^\tau\|\mathcal{L}_\alpha^\beta\|_0\|u_\beta\|_0\big)^2 \\ &\qquad + \sum_{\alpha\in\mathcal{I}_N} M_\alpha^{2s}\big(\sum_{\beta\in\mathcal{I}_N : M_\alpha > 5m_\beta} M_\alpha^\tau\|\mathcal{L}_\alpha^\beta\|_0\|u_\beta\|_0\big)^2\Big) =: \gamma^{-2}2^s(S_1+S_2)\,.\end{aligned}$$

Estimate of (S_1). Proceeding as in (7.19) with s_1 substituted by s, we obtain

$$S_1 \leq K_1^2\varepsilon^2|T|_{s_1}^2\|u\|_s^2$$

where K_1 is independent of N, because

$$\begin{aligned}K_1^2 &= \sum_{\alpha\in\mathcal{I}_N}\frac{M_\alpha^{2(s+\tau)}}{M_\alpha^{\lambda(2s_1-(r+n+1))}}\sum_{\beta\in\mathcal{I}_N : M_\alpha\leq 5m_\beta}\frac{C(s_1)^2}{m_\beta^{2s}} \leq \sum_{\alpha\in\mathcal{I}_N}\frac{M_\alpha^{2(s+\tau)}}{M_\alpha^{\lambda(2s_1-(r+n+1))}}\sum_{k\in E^+ : \langle k\rangle\geq M_\alpha/5}\frac{C(s_1)^2}{\langle k\rangle^{2s}} \\ &\leq \sum_{\alpha\in\mathcal{I}_N}\frac{M_\alpha^{2(s+\tau)}}{M_\alpha^{\lambda(2s_1-(r+n+1))}}\frac{C_1(s)}{M_\alpha^{2s-(r+n+1)}} \leq C_1(s)\sum_{\alpha\in\mathcal{I}_N}\frac{M_\alpha^{2\tau+r+n+1}}{M_\alpha^{\lambda(2s_1-(r+n+1))}} \overset{(7.7)}{\leq} K(s)\,.\end{aligned}$$

Estimate of (S_2). Applying Cauchy inequality in S_2 we get

$$\begin{aligned}S_2 &\overset{(3.2)}{\leq} \sum_{\alpha\in\mathcal{I}_N} M_\alpha^{2(s+\tau)}\sum_{\beta\in\mathcal{I}_N : M_\alpha>5m_\beta}\frac{\|\mathcal{L}_\alpha^\beta\|_0^2}{m_\beta^{2s_1}}\sum_{\beta\in\mathcal{I}_N}\|u_\beta\|_{s_1}^2 \\ &\overset{(7.5)}{\leq} \sum_{\alpha\in\mathcal{I}_N} M_\alpha^{2(s+\tau)}\sum_{\beta\in\mathcal{I}_N : M_\alpha>5m_\beta}\frac{\varepsilon^2|T|_s^2C(s)^2}{m_\beta^{2s_1}\mathrm{d}(\Omega_\alpha,\Omega_\beta)^{2s-(r+n+1)}}\|u\|_{s_1}^2 \\ &\leq \sum_{\alpha\in\mathcal{I}_N} M_\alpha^{2(s+\tau)}\sum_{\beta\in\mathcal{I}_N}\frac{\varepsilon^2|T|_s^2C(s)^2}{m_\beta^{2s_1}M_\alpha^{2s-(r+n+1)}}\|u\|_{s_1}^2\end{aligned}$$

because $M_\alpha > 5m_\beta$ implies $m_\alpha > 5M_\beta/4$ and $\mathrm{d}(\Omega_\alpha,\Omega_\beta) > M_\alpha/10$. Then

$$S_2 \leq C(s)^2\varepsilon^2|T|_s^2\|u\|_{s_1}^2\sum_{\alpha\in\mathcal{I}_N} M_\alpha^{2\tau+r+n+1}\sum_{\beta\in\mathbb{N}}\frac{1}{m_\beta^{2s_1}} \leq C'(s)^2\varepsilon^2|T|_s^2\|u\|_{s_1}^2N^{2(\tau+r+n+1)}$$

using that $|\mathcal{I}_N| \leq CN^{r+n+1}$ and $2s_1 > r+n+2$. ■

Lemma 7.11. *Letting* $A := \mathcal{D}^{-1}\mathcal{T}$, *we have* $\forall m\in\mathbb{N}$

$$\|A^mu\|_s \leq (\varepsilon\gamma^{-1}K(s))^m\Big(mN^{\kappa_0}|T|_s|T|_{s_1}^{m-1}\|u\|_{s_1} + |T|_{s_1}^m\|u\|_s\Big)\,. \tag{7.20}$$

Proof. We proceed by induction. Lemma 7.10 implies (7.20) for $m=1$. Then, again by Lemma 7.10,

$$\begin{aligned}\|A^mu\|_s &\leq \varepsilon\gamma^{-1}K(s)(N^{\kappa_0}|T|_s\|A^{m-1}u\|_{s_1} + |T|_{s_1}\|A^{m-1}u\|_s) \\ &\overset{(7.19)}{\leq} \varepsilon\gamma^{-1}K(s)(N^{\kappa_0}|T|_s(K(s_1)\varepsilon\gamma^{-1}|T|_{s_1})^{m-1}\|u\|_{s_1} + |T|_{s_1}\|A^{m-1}u\|_s) \\ &\overset{(7.20)}{\leq} (\varepsilon\gamma^{-1}K(s))^m(mN^{\kappa_0}|T|_s|T|_{s_1}^{m-1}\|u\|_{s_1} + |T|_{s_1}^m\|u\|_s)\end{aligned}$$

using that $K(s) \geq K(s_1)$. ■

Proof of Lemma 7.8 completed. By (7.17)

$$\begin{aligned}\|\mathcal{L}^{-1}u\|_s &\leq \|\mathcal{D}^{-1}u\|_s + \sum_{p\geq 1} \|(\mathcal{D}^{-1}\mathcal{T})^m(\mathcal{D}^{-1}u)\|_s \leq \|\mathcal{D}^{-1}u\|_s + \|\mathcal{D}^{-1}u\|_s \sum_{m\geq 1}(\varepsilon\gamma^{-1}K(s)|T|_{s_1})^m \\ &\overset{(7.20)}{+} N^{\kappa_0}K(s)\varepsilon\gamma^{-1}|T|_s\|\mathcal{D}^{-1}u\|_{s_1}\sum_{m\geq 1} m(K(s)\varepsilon\gamma^{-1}|T|_{s_1})^{m-1} \\ &\overset{(7.8)}{\leq} \gamma^{-1}N^{\kappa_0+\tau}K_1(s)(\|u\|_s + \varepsilon|T|_s\|u\|_{s_1})\end{aligned}$$

and using Lemma 7.7. ■

Proof of Lemma 6.2. By (6.4) and Lemma 7.1, taking $\eta := \eta_0(\gamma)$, we deduce the existence of a partition like in (7.4) satisfying (H1)-(H2).

By (7.5), the smallness condition (7.8) is implied taking $s_0 := s_1 + a$ (with s_1 defined in (7.7) and $a := (d + r + 2n + 3)/2$) and $\varepsilon(1 + \|p\|_{s_0}) \leq c(\gamma, k)$ small enough. Note that s_0 depends only on $\mathcal{M}$ and τ because λ, a depend only on $\mathcal{M}$ (see (H2) and Lemma 7.1).

By the "resolvent identity" (7.13), setting $u = u_S + u_R$ with $u_S \in H_S$, $u_R \in H_R$,

$$\|(L^{(N)})^{-1}u\|_s \leq \|L_R^{-1}u_R + L_R^{-1}L_S^R\mathcal{L}^{-1}(u_S + L_R^SL_R^{-1}u_R)\|_s + \|\mathcal{L}^{-1}(u_S + L_R^SL_R^{-1}u_R)\|_s\,.$$

To prove (6.6), we apply repeatedly the interpolation estimates (4.9) and (4.10). We apply Lemma 7.4 to bound $\|L_R^{-1}\|_{s_1}$ and $\|L_R^{-1}u\|_s$, we use Lemma 7.8 for $\|\mathcal{L}^{-1}u\|_s$ and (4.10) to estimate $\|L_R^Su\|_s$. By applying (7.6), we finally obtain

$$\|(L^{(N)}(\delta,p))^{-1}u\|_s \leq \frac{K_1(s)}{\gamma}N^{\kappa}(\|u\|_s + \varepsilon\|D_p\mathcal{F}(\delta,p)\|_s\|u\|_{s_1}) \leq \frac{N^{\kappa}}{\gamma}(\|u\|_s + \|p\|_s\|u\|_{s_0})$$

with $\kappa := \kappa_1 + a + 1 = 2\tau + r + n + a + 2$ and N sufficiently large.

8 The NLS equation

We look for periodic solutions of the "vector" NLS equations

$$\begin{cases} \mathrm{i}\omega u_t^+ - \Delta u^+ = & \varepsilon\mathfrak{f}(\delta, u^+u^-)u^+ \\ -\mathrm{i}\omega u_t^- - \Delta u^- = & \varepsilon\mathfrak{f}(\delta, u^+u^-)u^- \end{cases} \tag{8.1}$$

where $\mathbf{u} := (u^+, u^-) \in \mathbf{H}^s := H^s \oplus H^s$. Note that (8.1) reduces to the scalar NLS equation (1.17) in the invariant subspace

$$V^s := \Big\{\mathbf{u} := (u^+, u^-) \in \mathbf{H}^s \mid \bar{u}^+ = u^-\Big\}\,. \tag{8.2}$$

Remark 8.1. *The reason for doubling the equations, instead of working directly with the scalar NLS, is that the matrix which represent $u \mapsto \bar{u}$ is not Töplitz.*

The Fourier analysis of section 4 can be applied to $\mathbf{u} := (u^+, u^-) \in L^2(\mathbb{T} \times \mathcal{M}, \mathbb{C}^2)$ expanding

$$\mathbf{u} = \sum_{k\in E^+}\mathbf{u}_k \qquad \text{where} \qquad \mathbf{u}_k := \Pi_{\mathcal{N}_k}\mathbf{u} = (\Pi_{\mathcal{N}_k}u^+, \Pi_{\mathcal{N}_k}u^-)\,.$$

We then consider the linear operators on $\mathbf{H}^s$ represented by quasi-Töplitz matrices defined as in section 4.1, with the only difference that now the blocks $A_k^{k'} \in \mathcal{L}(\mathcal{N}_{k'} \oplus \mathcal{N}_{k'}, \mathcal{N}_k \oplus \mathcal{N}_k)$. All the results of section 4 remain valid. We point out that Lemma 4.7 has to be stated as follows.

Lemma 8.1. (multiplication operator) *Given $f(t,x), g(t,x) \in H^s$ with $f(t,x) \in \mathbb{R}$ real, then*

$$B := \begin{pmatrix} f(t,x) & g(t,x) \\ \bar{g}(t,x) & f(t,x) \end{pmatrix}$$

is self-adjoint and, $\forall s > (d+n+1)/2$,

$$\|B_k^{k'}\| \leq C(s) \frac{(\|f\|_s + \|g\|_s)}{\langle k - k' \rangle^{s-(d+n+1)/2}} \,. \tag{8.3}$$

Proof. The B is self-adjoint since f is real. By definition

$$B_k^{k'} := \begin{pmatrix} \Pi_{\mathcal{N}_k} f(t,x) \Pi_{\mathcal{N}_{k'}} & \Pi_{\mathcal{N}_k} g(t,x) \Pi_{\mathcal{N}_{k'}} \\ \Pi_{\mathcal{N}_k} \bar{g}(t,x) \Pi_{\mathcal{N}_{k'}} & \Pi_{\mathcal{N}_k} f(t,x) \Pi_{\mathcal{N}_{k'}} \end{pmatrix} .$$

Recall that $k = (l, j_1, j_2)$ with $l \in \mathbb{Z}$, $j_1 \in \Lambda^+$ and $j_2 \in \mathbb{Z}^n$. If $(\rho, j_1 - j_1') > 0$ we proceed for each component as in Lemma 4.7, obtaining (8.3). The bound for $B_k^{k'}$ with $(\rho, j_1 - j_1') < 0$ is obtained by the relation $B_k^{k'} = (B_{k'}^k)^\dagger$. ■

The resonant subspace for the vector NLS equation (8.1) is

$$\mathbf{Q} := Q \oplus Q^* \quad \text{where} \quad Q^* := \Big\{ q = \sum_{(l,j) \in \mathbb{Z} \times \Lambda_{\mathcal{M}}} e^{\mathrm{i}lt} \sum_{\sigma=1}^{d_j} q_{l,j,\sigma} \mathbf{e}_{j,\sigma} \in H^0 \mid \omega_{j_0}^2 l + \omega_j^2 = 0 \Big\} .$$

We perform the Lyapunov-Schmidt reduction as in the previous sections along $\mathbf{Q}$ and $\mathbf{P} := \mathbf{Q}^\perp$. In view of (ND) we restrict further the NLS equation in the invariant subspace $\mathcal{H}^s \oplus \mathcal{H}^s$. Restricting $\mathbf{p} := (p^+, p^-)$ to V^s (defined in (8.2)), the implicit function theorem, and the non-degeneracy assumption (ND) (lifted to V^s), imply the existence of a C^2 solution $\mathbf{q}(\delta, \mathbf{p}) \in V^s \cap \mathbf{Q}$ of the bifurcation equation. The tame properties as in Lemma 5.1 follow by the assumption $|J_0^{-1}|_s \leq C(s)$.

It remains to solve the range equation. Setting $\mathbf{u}(\delta, \mathbf{p}) := \mathbf{p} + \mathbf{q}(\delta, \mathbf{p})$, $\mathbf{p} \in V^s$, we define

$$\mathbb{D}(\omega) := \begin{pmatrix} \mathrm{i}\omega\partial_t - \Delta & 0 \\ 0 & -\mathrm{i}\omega\partial_t - \Delta \end{pmatrix}, \qquad \mathcal{F}(\delta, \mathbf{p}) := \mathfrak{f}(\delta, |u^+(\delta, \mathbf{p})|^2) \mathbf{u}(\delta, \mathbf{p})$$

and we apply the abstract Nash-Moser Theorem 3.1 to

$$F(\delta, \mathbf{p}) := \mathbb{D}(\omega)\mathbf{p} - \varepsilon \mathcal{F}(\delta, \mathbf{p}) = 0 \,.$$

We have to prove assumption (L) concerning the linearized operators

$$L^{(N)}(\delta, \mathbf{p}) := \Pi^{(N)} \Big(\mathbb{D}(\omega) - \varepsilon D_{\mathbf{p}} \mathcal{F}(\delta, \mathbf{p}) \Big)_{|\mathbf{H}^{(N)}}$$

where

$$D_{\mathbf{p}} \mathcal{F}(\delta, \mathbf{p}) := \varepsilon \begin{pmatrix} \mathfrak{f}(\delta, |u^+|^2) + \mathfrak{f}'(\delta, |u^+|^2)|u^+|^2 & \mathfrak{f}'(\delta, |u^+|^2)(u^+)^2 \\ \mathfrak{f}'(\delta, |u^+|^2)(u^-)^2 & \mathfrak{f}(\delta, |u^+|^2) + \mathfrak{f}'(\delta, |u^+|^2)|u^+|^2 \end{pmatrix} \Big(1 + D_{\mathbf{p}} \mathbf{q}(\delta, \mathbf{p}) \Big) .$$

Property (L) is a consequence of the following lemma, similar to section 6.

Lemma 8.2. *(***Invertibility***) For all $\tau > 1$, there are constants $\kappa > 0$, $s_0 := s_0(\mathcal{M}, \tau)$ such that, if $\varepsilon(\|\mathbf{p}\|_{s_0} + 1) \leq c(\gamma, k)$ is small enough, and*

$$\forall 1 \leq K \leq N \,, \qquad \Big\| (L^{(K)}(\delta, \mathbf{p}))^{-1} \Big\|_0 \leq \frac{4K^\tau}{\gamma} \,, \tag{8.4}$$

then $L^{(N)}(\delta, \mathbf{p})$ is invertible and an estimate like (6.6) holds.

The proof of Lemma 8.2 requires only small modifications in the proof of Lemma 6.2. First remark that the diophantine hypothesis (6.4) is not necessary to get a partition of the singular sites as in (7.4). It can be proved for any ω as in the case $\mathcal{M} = \mathbb{T}^n$ by "convexity" arguments as in [9], or [20]. The key observation is that the eigenvalues ω_j^2 in (1.4) are similar to those of a torus and the small divisors appear close to the "parabolid" $\omega l - \omega_j^2 = 0$.

Then, following subsection 7.2 and using Lemma 8.1 we obtain the bound (7.5). The remaining subsections 7.3 and 7.4 remains valid because are based on purely algebraic techniques: they only require that the operator T belongs to the quasi-Töplitz matrix algebra.

We can finally prove property (L). The sets $\mathcal{G}^{(N)}_{\gamma,\kappa}(\mathbf{p})$ are defined as in (6.9) and $\mathcal{G} \equiv [0, \delta_0)$. The analogue Lemma 6.3 states that, if $\varepsilon_0 M$ is small, then $\mathcal{G}^{(M)}_{\gamma,\kappa}(p) = \mathcal{G}$. We then follow Appendix A.1 word by word. To prove (A.3) we proceed as follows.

We consider the parameter $\lambda := \omega^{-1}$ and

$$\mathscr{L}^{(K)}(\lambda, \mathbf{p}) := L^{(K)}(\delta(\lambda), \mathbf{p})\lambda$$

which is a self-adjoint perturbation of

$$\Pi^{(K)} \begin{pmatrix} \mathrm{i}\partial_t - \lambda\Delta & 0 \\ 0 & -\mathrm{i}\partial_t - \lambda\Delta \end{pmatrix}_{|\mathbf{H}^{(K)}} .$$

We consider frequencies ω close to $\omega_{j_0} > 0$. Note that $\delta \mapsto \lambda(\delta) = \omega^{-1}(\delta)$ is a diffeomorphism in $[\delta_1/2, \delta_1]$ with $\lambda'(\delta) \geq c\delta_1^{2m-1}$. Now, for ε small enough, the matrix $\partial_\lambda \mathscr{L}^{(K)}(\lambda, \mathbf{p})$ is positive definite and we evaluate the measure of

$$\Big\{\lambda \in \Big[\omega_{j_0}^{-1}, \frac{1}{\omega_{j_0} - \delta_1^{2m}}\Big] \mid \exists \text{ an eigenvalue of } \mathscr{L}^{(K)}(\lambda, \mathbf{u}) \text{ with modulus in } \Big[\omega_{j_0}^{-1}\gamma K^{-\tau}, \frac{\gamma K^{-\tau} + C\delta_1^{2m} N^{-\sigma}}{\omega_{j_0} - \delta_1^{2m}}\Big]\Big\} ,$$

following Lemma 3.2 of [4], see remark A.1.

8.1 The NLS-equation on $SO(3)$

The group $SO(3)$ is isomorphic to $SU(2)\backslash\{\pm e\}$. The compact group $SU(2)$ is of simple type, it has rank 1, fundamental weight $w_1 = 1/2$ and so $\Lambda^+ = \mathbb{N}/2$. It is customary to denote the dominant weights $\lambda \in \Lambda^+$ by $\lambda = j/2$, $j \in \mathbb{N}$, so that the eigenvalues of $-\Delta$ on $SU(2)$ are

$$\omega_j^2 = \frac{1}{4}\Big((j+1)^2 - 1\Big) .$$

All the unitary representations ρ_{V_j} of $SU(2)$ are self dual, namely $\bar{\rho}_{V_j} = \rho_{V_j}$ and hence the characters $\chi_j := \chi_{V_j}$ are real, see (2.8). Moreover, the following "Clebsch-Gordan" multiplication rules hold:

$$\chi_i \, \chi_j = \sum_{l=0}^{\min(i,j)} \chi_{i+j-2l} \, . \tag{8.5}$$

Since $SO(3) = SU(2)\backslash\{\pm e\}$ is a homogeneous space (see Definition 2.8) we consider the set $\Lambda_{SO(3)} \subset \Lambda^+ = \mathbb{N}/2$ defined as in (2.18). It results

$$\Lambda_{SO(3)} = \{j/2 \, , \ j \in 2\mathbb{N}\} = \mathbb{N} \, .$$

We want to prove the existence of small amplitude periodic solutions of the NLS-equation

$$\mathrm{i}u_t - \Delta u = f(|u|^2)u \, , \quad f(|u|^2) = |u|^2 + r(|u|^2) \, , \quad r(0) = r'(0) = 0 \, , \tag{8.6}$$

with frequency close to least non zero eigenvalue $\omega_{j_0}^2 = 1/4$ of $-\Delta$ on $SU(2)$.

Since the nonlinearity is x-independent, we look for solutions of the corresponding rescaled equation (1.17) in the invariant subspace of central functions with real fourier coefficients

$$\mathcal{H}^s = \Big\{ u(t,x) = \sum_{(l,j)\in\mathbb{Z}\times\Lambda_{SO(3)}} u_{l,j}\, e^{\mathrm{i}lt}\chi_j(x) \in H^s\,,\ u_{l,j}\in\mathbb{R} \Big\}\,.$$

Since $\mathrm{i}\partial_t$ and Δ leave $\mathcal{H}^s$ invariant, we only have to prove the following lemma:

Lemma 8.3. *If $u \in \mathcal{H}^s$ then $f(|u|^2)u \in \mathcal{H}^s$.*

PROOF. The space $\mathcal{H}^s$ is closed under product (use the multiplication rule (8.5)) and complex conjugation. Then we only need to show that $f(|u|^2) \in \mathcal{H}^s$ for all $u \in \mathcal{H}^s$. The main point is that the space of real valued functions in $\mathcal{H}^s$ coincides with the real valued functions $f(t,x)$ which are central and even i.e. $f(t,x) = f(-t,x^{-1})$. Indeed, for $f \in \mathcal{H}^s$ we have

$$\overline{f(t,x)} \quad = \sum_{(l,j)\in\mathbb{Z}\times SO(3)} f_{l,j}e^{-\mathrm{i}lt}\mathrm{tr}\,\overline{(\rho_j(x))} = \sum_{(l,j)\in\mathbb{Z}\times SO(3)} f_{l,j}e^{-\mathrm{i}lt}\mathrm{tr}\,(\rho_j(x^{-1})) = f(-t,x^{-1})$$

using the definition of unitary representation. Hence, if $f \in \mathcal{H}^s$ is real then it is even.

Conversely, if $f(t,x)$ is real valued, central and even, then $f_{l,j} = \overline{f_{l,j}}$. Now if $u \in \mathcal{H}^s$ then $|u|^2 = u\bar{u}$ it is in $\mathcal{H}^s$ and it is real valued. By the above arguments $|u|^2$ is even. In conclusion $f(|u|^2)$ is real valued, central and even and hence in $\mathcal{H}^s$. ■

The bifurcation equation (1.18) (where we choose $\chi = -1$) reduces to

$$\mathrm{i}q_t + \Pi_Q|q|^2q = 0\,, \qquad Q\cap\mathcal{H}^s = \Big\{ q(t,x) = \sum_{j\in 2\mathbb{N}} q_j e^{\mathrm{i}4\omega_j^2 t}\chi_j(x) \Big\}\,. \tag{8.7}$$

The remaining part of this section is devoted to prove the following lemma.

Lemma 8.4. *The bifurcation equation (8.7) admits a non-degenerate solution of the form*

$$q_0(t,x) = q_6 e^{i48t}\chi_6(x) + q_8 e^{i80t}\chi_8(x)\,. \tag{8.8}$$

In Fourier coefficients the equation (8.7) is

$$4\omega_h^2 q_h = \sum_{j_1,j_2,j_3} q_{j_1}q_{j_2}\bar{q}_{j_3}c(h,j_1,j_2,j_3) \quad \text{such that} \quad \omega_{j_1}^2 + \omega_{j_2}^2 - \omega_{j_3}^2 = \omega_h^2 \tag{8.9}$$

where $c(h,j_1,j_2,j_3) := \int_{SO(3)} \chi_h\chi_{j_1}\chi_{j_2}\chi_{j_3}$. We find solutions of (8.9) in finite dimensional subspaces.

Lemma 8.5. *For all $j > 4$ the subspace*

$$Q_j := \Big\{ q(t,x) = q_j e^{4\mathrm{i}\omega_j^2 t}\chi_j(x) + q_{j+2}e^{4\mathrm{i}\omega_{j+2}^2 t}\chi_{j+2}(x) \Big\} \subset Q\cap\mathcal{H}^s \tag{8.10}$$

is invariant under the map $q \mapsto \Pi_Q|q|^2q$. The bifurcation equation (8.9) restricted to Q_j is

$$\begin{cases} \big((j+1)^2-1\big)q_j &= q_j\big((j+1)|q_j|^2 + 2(j+1)|q_{j+2}|^2\big) \\ \big((j+3)^2-1\big)q_{j+2} &= q_{j+2}((j+3)|q_{j+2}|^2 + 2(j+1)|q_j|^2) \end{cases} \tag{8.11}$$

which has a unique solution $q_0(t,x)$.

Proof. By (8.9), in order to prove the invariance of Q_j it is sufficient to show that, if $j_1, j_2, j_3 \in \{j, j+2\}$, then $\omega_{j_1}^2 + \omega_{j_2}^2 - \omega_{j_3}^2 = \omega_h^2$ is verified only if $h \in \{j, j+2\}$. This holds true because, for $j \neq 4$,

$$4(2\omega_j^2 - \omega_{j+2}^2) = (j-1)^2 - 9 \neq (h+1)^2 - 1\,, \quad 4(2\omega_{j+2}^2 - \omega_j^2) = (j+5)^2 - 9 \neq (h+1)^2 - 1\,,$$

using that the only natural solutions of $A^2 - B^2 = 8$ are $(A, B) = (3, 1)$. We claim that

$$c(h, h, k, k) = \min\{h, k\} + 1 \tag{8.12}$$

which, by (8.9), implies (8.11). By (8.5) we have

$$\chi_h^2 \chi_k = \sum_{i=0}^{h} \chi_{2(h-i)} \chi_k = \sum_{i=0}^{h} \sum_{j=0}^{\min(k, 2(h-i))} \chi_{2(h-i-j)+k}\,.$$

Now $2(h-i-j)+k = k$ only if $j = h-i$ which is smaller than $2(h-i)$. If $h \leq k$ then there are solutions for all $0 \leq i \leq h$. If $h > k$ then $h - i \leq k$ holds for $h - k \leq i \leq h$. This proves (8.12). ■

We now prove that the solution $q_0(t, x) \in Q_j$ found in Lemma 8.5 is non-degenerate for $j = 6$.

Lemma 8.6. *Let $j = 6$. The unique solution $h \in Q \cap \mathcal{H}^s$ of*

$$J_0[h] := \mathrm{i} h_t + 2|q_0(t, x)|^2 h + (q_0(t, x))^2 \bar{h} = 0 \tag{8.13}$$

is $h = 0$. Moreover J_0 is 2×2-block diagonal and so $|J_0^{-1}|_s \leq C(s)$.

Proof. By the definition of $\mathcal{H}^s$, q_6, q_8 and for $h_k \in \mathbb{R}$, equation (8.13) writes

$$\begin{aligned}
&\Pi_Q \Big(\sum_{k \in 2\mathbb{N}} e^{4\mathrm{i}\omega_k^2 t} 4\omega_k^2 h_k e^{4\mathrm{i}\omega_k^2 t} \chi_k - 2 e^{4\mathrm{i}\omega_k^2 t} \big(q_6^2 \chi_6^2 + q_8^2 \chi_8^2 \big) h_k \chi_k - 2 e^{4\mathrm{i}(\omega_6^2 - \omega_8^2 + \omega_k^2)t} q_8 q_6 h_k \chi_6 \chi_8 \chi_k \\
&-2 e^{4\mathrm{i}(\omega_8^2 - \omega_6^2 + \omega_k^2)t} q_8 q_6 h_k \chi_6 \chi_8 \chi_k - e^{4\mathrm{i}(2\omega_6^2 - \omega_k^2)t} q_6^2 h_k \chi_6^2 \chi_k - e^{4\mathrm{i}(2\omega_8^2 - \omega_k^2)t} q_8^2 h_k \chi_8^2 \chi_k \\
&-2 e^{4\mathrm{i}(\omega_6^2 + \omega_8^2 - \omega_k^2)t} q_6 q_8 h_k \chi_6 \chi_8 \chi_k \Big) = 0\,.
\end{aligned} \tag{8.14}$$

It results that the unique solution with $h, k \in 2\mathbb{N}$ of $\omega_6^2 - \omega_8^2 + \omega_k^2 = \omega_h^2$ is $(h, k) = (6, 8)$, and, of $\omega_8^2 - \omega_6^2 + \omega_k^2 = \omega_h^2$, is $(h, k) = (8, 6)$. Moreover, for all $h, k \in 2\mathbb{N}$, $2\omega_6^2 - \omega_k^2 \neq \omega_h^2$, $2\omega_8^2 - \omega_k^2 \neq \omega_h^2$. Finally

$$\omega_8^2 + \omega_6^2 - \omega_k^2 = \omega_h^2 \quad \text{has the only solutions } (h, k) = (6, 8), (8, 6), (2, 10), (10, 2)\,.$$

This implies that the equation for the h_k is diagonal for all $k \neq 2, 6, 8, 10$ and has two 2×2 blocks for the harmonics $6, 8$ and $2, 10$.

By (8.12) we have $c(6, 6, k, k) = \min(6, k) + 1$, $c(8, 8, k, k) = \min(8, k) + 1$. With similar computations using the Clebsh-Gordan multiplication rules (8.5) it results $c(8, 6, 2, 10) = 3$.

Then equation (8.14) gives, for $k \neq 2, 6, 8, 10$,

$$\Big[(k+1)^2 - 1 - 2\Big((\min(6, k) + 1) q_6^2 + (\min(8, k) + 1) q_8^2 \Big) \Big] h_k = 0$$

and, for (h_2, h_{10}) and (h_6, h_8) the two by two equations

$$\begin{cases} \Big(4 - 3(q_6^2 + q_8^2)\Big) h_2 - 3 q_6 q_8 h_{10} = 0 \\ \Big(60 - (9 q_8^2 + 7 q_6^2)\Big) h_{10} - 3 q_6 q_8 h_2 = 0 \end{cases} \qquad \begin{cases} 48 - 7(3 q_6^2 + 2 q_8^2) h_6 - 28 q_6 q_8 h_8 = 0 \\ 80 - (27 q_8^2 + 14 q_6^2) h_8 - 28 q_6 q_8 h_6 = 0\,. \end{cases}$$

A direct computation, using that, by (8.11), $q_6^2 = (48/7) - (32/19)$ and $q_8^2 = 16/19$, shows that $h_k = 0$, $\forall k$. ■

A Appendix

A.1 The measure estimate: Proof of Lemma 6.4

As in the proof of Lemma 6.3, for all $N, N' \le N_{\varepsilon_0} := (c\gamma/\varepsilon_0)^{1/(\tau_1+2)}$, it results $\mathcal{G}^{(N)}_{\gamma,\kappa}(u_1) = \mathcal{G}^{(N')}_{\gamma,\kappa}(u_2) = \mathcal{G}$ and thus (3.7) is trivially satisfied in such cases. In order to prove (3.7) in the other cases, it is sufficient to prove that, $\forall \delta_1 \in [0, \delta_0]$,

$$\Big| \Big(\mathcal{G}^{(N')}_{\gamma,\kappa}(u_2)\Big)^c \setminus \Big(\mathcal{G}^{(N)}_{\gamma,\kappa}(u_1)\Big)^c \bigcap \Big[\frac{\delta_1}{2}, \delta_1\Big) \Big| \le C \frac{\gamma \delta_1}{N} . \tag{A.1}$$

Indeed we can decompose $[0, \delta_0] = \cup_{n \ge 1} [\delta_0 2^{-n}, \delta_0 2^{-(n-1)}]$. For $N' \ge N$, let us consider δ_1 fixed and consider always the complementary sets in $[\frac{\delta_1}{2}, \delta_1\Big)$.

$$\begin{aligned} \Big(\mathcal{G}^{(N')}_{\gamma,\kappa}(u_2)\Big)^c \setminus \Big(\mathcal{G}^{(N)}_{\gamma,\kappa}(u_1)\Big)^c &= \Big(\mathcal{G}^{(N')}_{\gamma,\kappa}(u_2)\Big)^c \cap \mathcal{G}^{(N)}_{\gamma,\kappa}(u_1) \\ &\subset \Big[\cup_{K \le N} \Big(\mathtt{G}^c_K(u_2) \cap \mathtt{G}_K(u_1) \cap \mathcal{G} \Big) \Big] \bigcup \Big[\cup_{K > N} \mathtt{G}^c_K(u_2) \cap \mathcal{G} \Big] . \end{aligned}$$

As we have just seen, if $K \le N_{\varepsilon_1}$ then $\mathtt{G}^c_K(u_2) \cap \mathcal{G} = \emptyset$. Hence it is enough to prove that, if $\|u_1 - u_2\|_{s_0} \le N^{-\sigma}$, then

$$\mathcal{B} := \sum_{N_\varepsilon < K \le N} |\mathtt{G}^c_K(u_2) \cap \mathtt{G}_K(u_1)| + \sum_{K > \max\{N, N_\varepsilon\}} |\mathtt{G}^c_K(u_2)| \le \bar{C} \frac{\gamma \delta_1}{N} . \tag{A.2}$$

Since $L^{(K)}(u)$ is self-adjoint $\|(L^{(K)}(u))^{-1}\|_0$ is the inverse of the eigenvalue of smallest modulus.

Since $\|L^{(K)}(u_2) - L^{(K)}(u_1)\|_0 = O(\varepsilon \|u_2 - u_1\|_{s_0}) = O(\varepsilon N^{-\sigma})$, if one of the eigenvalues of $L^{(K)}(u_2)$ is in $[-4\gamma K^{-\tau}, 4\gamma K^{-\tau}]$ then, by the variational characterization of the eigenvalues of $L^{(K)}(u)$, one of the eigenvalues of $L^{(K)}(u_1)$ is in $[-4\gamma K^{-\tau} - C\varepsilon N^{-\sigma}, 4\gamma K^{-\tau} + C\varepsilon N^{-\sigma}]$. As a result

$$\begin{aligned} \mathtt{G}^c_K(u_2) \cap \mathtt{G}_K(u_1) \quad \subset \quad & \Big\{ \delta \in [\delta_1/2, \delta_1] \mid \exists \text{ at least one eigenvalue of } L^{(K)}(\delta, u_1) \\ & \qquad \text{with modulus in } [4\gamma K^{-\tau}, 4\gamma K^{-\tau} + C\varepsilon N^{-\sigma}] \Big\} . \end{aligned}$$

We now use Lemma 3.2 of [4] which is based on a simple eigenvalue variation argument: if ε is small enough and I is a compact interval in $[-\gamma, \gamma]$ of length $|I|$, then

$$\Big| \{ \delta \in [\delta_1/2, \delta_1] \text{ s.t. at least one eigenvalue of } L^{(K)}(\delta, u_1) \text{ belongs to } I \} \Big| \le C \frac{K^{d+n+1} |I|}{\delta_1^{m-2}} . \tag{A.3}$$

Remark A.1. *The main point of the lemma is that $\partial_\omega L^{(K)}$ is positive definite.*

As a consequence $|\mathtt{G}^c_K(u_2) \cap \mathtt{G}_K(u_1)| \le C K^{d+n+1} \delta_1^{-(m-2)} \varepsilon N^{-\sigma} \le C \delta_1 N^{-\sigma} K^{d+n+1}$. Moreover, still by (A.3), $|\mathtt{G}^c_K(u_2)| \le C K^{d+n+1} \gamma K^{-\tau} \delta_1^{-m+2}$. Hence $\mathcal{B}$ defined in (A.2) satisfies

$$\begin{aligned} \mathcal{B} &\le C\delta_1 \Big(\sum_{K \le N} K^{d+n+1} \Big) N^{-\sigma} + C \frac{\gamma}{\delta_1^{m-2}} \Big(\sum_{K > \max\{N, N_\varepsilon\}} K^{d+n+1-\tau} \Big) \\ &\le C\delta_1 N^{d+n+2-\sigma} + C' \delta_1^{-(m-2)} \gamma (\max\{N, N_\varepsilon\})^{d+n+2-\tau} \le \bar{C} \gamma \delta_1 N^{-1} , \end{aligned}$$

for (6.7). This proves the measure estimate (A.1).

A.2 Separation properties of the singular sites

Consider the bilinear symmetric form $\varphi_\omega : \mathbb{R}^{r+n+1} \times \mathbb{R}^{r+n+1} \to \mathbb{R}$ defined by

$$\varphi_\omega(x, x') := J \cdot J' - \omega^2 l\, l' , \qquad \forall\, x = (l, J) ,\ x' = (l', J') \in \mathbb{R} \times \mathbb{R}^{r+n}$$

and the corresponding quadratic form $Q_\omega(x) = \varphi_\omega(x, x) := |J|^2 - \omega^2 l^2$.

Note that for $\vec{\rho} = (0, \rho, 0)$ and for all $k = (l, j_1, j_2) \in \mathbb{Z} \times \Lambda^+ \times \mathbb{Z}^n$ one has $Q_\omega(k + \vec{\rho}) = \mathfrak{D}_k + |\rho|^2$, where $\mathfrak{D}_k$ are the small divisors. For notational convenience we will denote $x = k + \vec{\rho} = (l, J)$ where $x \in \mathbb{Z} \times \Lambda^{++} \times \mathbb{Z}^n$ since $j_1 \in \Lambda^+$ and $\Lambda^{++} = \rho + \Lambda^+$.

A vector $x = (l, J) \in \mathbb{Z} \times \Lambda^{++} \times \mathbb{Z}^n$ is said "weakly singular" if $|Q_\omega(x)| \le C$ for some constant C fixed once and for all. In particular if $(l, j) \in \mathbb{Z} \times \Lambda^+ \times \mathbb{Z}^n$ is singular according to Definition 7.3 then $x = (l, j + \rho) \in \mathbb{Z} \times \Lambda^{++} \times \mathbb{Z}^n$ is "weakly singular" with $|Q_\omega(x)| \le 1 + m + |\rho|^2$.

Definition A.1. *A sequence $x_0, \ldots, x_K \in \mathbb{Z} \times \Lambda^{++} \times \mathbb{Z}^n$ of distinct, weakly singular vectors satisfying, for some $B \ge 2$, $|x_{k+1} - x_k| \le B$, $\forall k = 0, \ldots, K-1$, is called a B-chain of length K.*

Theorem A.1. *If ω^2 satisfies (6.4), then any B-chain has length $K \le B^C/\gamma^p$ for some $C := C(G) > 0$ and $p := p(G) > 0$.*

For the proof of Theorem A.1 we proceed with the same strategy of [4].

Given lattice vectors $f_i \in \mathbb{Z} \times \Lambda \times \mathbb{Z}^n$, $i = 1, \ldots, n$, $1 \le m \le r + n + 1$, linearly independent on $\mathbb{R}$, we consider the subspace $F := \mathrm{Span}_{\mathbb{R}}\{f_1, \ldots, f_m\}$ of $\mathbb{R}^{r+n+1}$ and the restriction $\varphi_{\omega|F}$ of the bilinear form φ_ω to F, which is represented by the symmetric matrix $A_\omega := \{\varphi_\omega(f_i, f_{i'})\}_{i,i'=1}^m$. Introducing the symmetric bilinear forms $\mathcal{R}(x, x') := J \cdot J'$ and $\mathcal{S}(x, x') := l\, l'$, we write

$$\varphi_\omega = \mathcal{R} - \omega^2 \mathcal{S} \quad \text{and} \quad A_\omega = R - \omega^2 S$$

where $R := \{\mathcal{R}(f_i, f_{i'})\}_{i,i'=1}^m = (R_1, \ldots, R_m)$, $S := \{\mathcal{S}(f_i, f_{i'})\}_{i,i'=1}^m = (S_1, \ldots, S_m)$ are the matrices that represent respectively $\mathcal{R}_{|F}$ and $\mathcal{S}_{|F}$ in the basis $\{f_1, \ldots, f_m\}$. Here R_i, S_i, $i = 1, \ldots, m$, denote the column vectors respectively of R and S. The main lemma is the following:

Lemma A.2. *If $\mathcal{M} = (G \times \mathbb{T}^n)/N$ then the matrices R, S have coefficients in $D^{-1}\mathbb{Z}$ for some $D \in \mathbb{N}$.*

PROOF. Let us represent each $f_i \in \mathbb{Z} \times \Lambda \times \mathbb{Z}^n$ by the components $\{f_{i,j}\}_{j=0}^{r+n+1}$ with $f_{i,j} \in \mathbb{Z}$, in such a way that $f_{i,0}$ is the projection on $\mathbb{Z}$, $\sum_{j=1}^r f_{i,j} w_j$ is the projection on Λ and $(f_{i,j})_{j=r+1}^{r+n}$ is the projection on $\mathbb{Z}^n$. By definition $S_{i,i'} = f_{i,0} f_{i',0}$ and hence integer valued for all i, i'. Similarly one has:

$$R_{i,i'} = \sum_{j,j'=1}^{r} f_{i,j} f_{i',j'} (w_j, w_{j'}) + \sum_{j=r+1}^{r+n} f_{i,j}^2$$

and by (2.12) one has that $(w_j, w_{j'}) \in \mathbb{Z}D^{-1}$. ■

Lemma A.3. *Assume that ω^2 satisfies (6.4). Then A_ω is invertible and*

$$\|A_\omega^{-1}\| \le \frac{c(m, D)}{\gamma} \Big(\max_{i=1,\ldots,m} |f_i| \Big)^{5m-2} . \tag{A.4}$$

PROOF. The matrix S has rank at most 1 because it represents the restriction to F of a bilinear form of rank 1. Since any two columns of S are colinear, the development in ω^2 of $\det A_\omega$ reduces to

$$\det A_\omega = \det(R_1 - \omega^2 S_1, \ldots, R_m - \omega^2 S_m) = \det(R_1, \ldots, R_m) - \omega^2 \sum_{i=1}^m \det(R_1, \ldots, S_i, \ldots, R_m). \tag{A.5}$$

Therefore $D^m \det A_\omega = P(\omega^2)$ is a polynomial in ω^2 of degree at most 1, with integer coefficients since $\det R$, $\det(R_1, \ldots, S_i, \ldots, R_m) \in D^{-m}\mathbb{Z}$ by Lemma A.2. Furthermore $P(\cdot)$ is not identically zero because $P(-1) = D^m \det(R+S)$ is positive (the matrix $R+S$ being positive definite).

By (A.5), if $\det R = 0$ then $|\det A_\omega| \geq \omega^2 D^{-m}$, and, if $\det R \neq 0$, since ω^2 satisfies (6.4), $|\det A_\omega| \geq \gamma D^{5m/2} |\det R|^{-3/2}$. Since $R + S = {}^t\mathcal{F}\mathcal{F}$ with $\mathcal{F} = (f_1, \ldots, f_m)$, we have $0 \leq \det R \leq \det(R+S) = (\det \mathcal{F})^2 \leq |f_1|^2 \ldots |f_m|^2 \leq M^{2m}$, where $M := \max_{i=1,\ldots,m} |f_i|$. Hence

$$|\det A_\omega| \geq \frac{D^{5m/2}\gamma}{M^{3m}}\,. \tag{A.6}$$

By (6.4), $\omega^2 \geq \gamma$, and (A.6) holds in both cases. Applying the Cramer rule

$$|(A_\omega^{-1})_{i,i'}| \leq \frac{c(m,D)}{|\det A_\omega|}(M^2)^{(m-1)} \overset{(A.6)}{\leq} \frac{c'(n,D)}{\gamma} M^{3m+2(m-1)}$$

whence (A.4) follows. ■

The remainder of the proof follows [4] word by word and we omit it.

A.3 Proof of Lemma 7.6

It is sufficient to prove that $\|\mathcal{L}_\alpha w\|_0 \geq \|w\|_0 \gamma/(cM_\alpha^\tau)$, $\forall w \in H_\alpha$. For all $w \in H_\alpha \subset H_S$, we have

$$\mathcal{L}_\alpha w + \sum_{\beta \neq \alpha} \mathcal{L}_\beta^\alpha w = \mathcal{L} w \overset{(7.12)}{=} L_S w - L_S^R L_R^{-1} L_R^S w = \Pi_{H_S} L^{(N)} h \text{ where } \ h := w - L_R^{-1} L_R^S w\,. \tag{A.7}$$

Step 1: $\sum_{\beta \neq \alpha} \|\mathcal{L}_\beta^\alpha w\|_0 \leq \varepsilon |T|_{s_1} C(s_1) M_\alpha^{r+n+1-\nu} \|w\|_0$ where $\nu := \lambda(s_1 - (r+n+1)/2) > 0$.

$$\sum_{\beta \neq \alpha} \|\mathcal{L}_\beta^\alpha w\|_0 \leq \sum_{\beta \neq \alpha} \frac{C\varepsilon |T|_{s_1} \|w\|_0}{\mathrm{d}(\Omega_\alpha, \Omega_\beta)^{s_1 - \frac{r+n+1}{2}}} \leq C\varepsilon |T|_{s_1} \|w\|_0 \sum_{\beta \neq \alpha} \frac{1}{(M_\alpha + M_\beta)^\nu}\,.$$

Then, for $\nu > r+n+1$, Step 1 follows by

$$\sum_{\beta \neq \alpha} \frac{1}{(M_\alpha + M_\beta)^\nu} \leq \sum_{k \in E^+} \frac{1}{(M_\alpha + |k + \vec{\rho}|)^\nu} \leq C \int_1^{+\infty} \frac{y^{r+n} dy}{(M_\alpha + y)^\nu} \leq \frac{C(\nu)}{(1 + M_\alpha)^{\nu - (r+n+1)}}\,.$$

Step 2: $\|L^{(N)} h\|_0 \geq 2^{-\tau-1} \gamma M_\alpha^{-\tau} \|w\|_0$.
Decompose $h = h' + h''$ with $h' := \Pi^{(K)} h$, $h'' := h - \Pi^{(K)} h$ and $K := 2M_\alpha$. We have

$$\|\Pi_{H_S} L^{(N)} h\|_0 \geq \|\Pi^{(K)} L^{(N)} h\|_0 \geq \|L^{(K)} h'\|_0 - \varepsilon \|T_{E_K^+}^{(E_K^+)^c} h''\|_0 \geq \frac{\gamma}{(2M_\alpha)^\tau} \|h'\|_0 - \varepsilon C |T|_{s_1} \|h''\|_0, \tag{A.8}$$

by assumption (6.5) and Lemma 7.3. Moreover since $h = w - {L_R}^{-1} L_R^S w$ and $w \in H_\alpha \subset H^{(K)}$,

$$h'' = -(1 - \Pi^{(K)}) L_R^{-1} L_R^S w = -[L_R^{-1}]_{R \cap (E_K^+)^c}^R L_R^\alpha w\,.$$

Now $\mathrm{d}(\Omega_\alpha, R \cap (E_K^+)^c) \geq M_\alpha$, and we derive that

$$\|h''\|_0 \leq \frac{C\varepsilon |T|_{s_1}}{M_\alpha^{s_1 - (r+n+1)/2}} \|w\|_0\,. \tag{A.9}$$

Furthermore, since $w \in H_\alpha$ and $h' - w = -\Pi^{(K)} L_R^{-1} L_R^S w \in H_R$, we have $\|h'\|_0 = (\|h' - w\|_0^2 + \|w\|_0^2)^{1/2} \geq \|w\|_0$ and, (A.8), (A.9), imply

$$\|L^{(N)} h\|_0 \geq \frac{\gamma \|w\|_0}{(2M_\alpha)^\tau} - \frac{C\varepsilon^2 |T|_{s_1}^2}{M_\alpha^{s_1 - (r+n+1)/2}} \|w\|_0 \geq \frac{\gamma \|w\|_0}{(2M_\alpha)^\tau} \Big(1 - \frac{C'\varepsilon^2 |T|_{s_1}^2}{\gamma M_\alpha^{s_1 - \tau - (r+n+1)/2}}\Big) \geq 2^{-\tau-1} \frac{\gamma}{M_\alpha^\tau} \|w\|_0$$

because $s_1 > \tau + (r+n+1)/2$, and provided that $\varepsilon\gamma^{-1}$ is small enough.

References

[1] Bambusi D., Delort J.M., Grebert B., Szeftel J., *Almost global existence for Hamiltonian semilinear Klein-Gordon equations with small Cauchy data on Zoll manifolds*, Comm. Pure Appl. Math. 60, 11, 1665-1690, 2007.

[2] Berti M., *Nonlinear Oscillations of Hamiltonian PDEs*, Progr. Nonlinear Differential Equations Appl. 74, H. Brézis, ed., Birkhäuser, Boston, 1-181, 2008.

[3] Berti M., Bolle P., *Cantor families of periodic solutions of wave equations with C^k nonlinearities*, Nodea, 15, 247-276, 2008.

[4] Berti M., Bolle P., *Sobolev periodic solutions for nonlinear wave equations in higher spatial dimensions*, Archive Rational Mechanics, published on line 21-1-09, 2009.

[5] Berti M., Bolle P., Procesi M., *An abstract Nash-Moser Theorem with parameters and applications to PDEs*, preprint 2009.

[6] Besse, A., *Einstein manifolds*, Classics in Mathematics. Springer-Verlag, 2008.

[7] Biasco L., Di Gregorio L., *A Birkhoff-Lewis type Theorem for the nonlinear wave equation*, preprint.

[8] Bourgain J., *Construction of periodic solutions of nonlinear wave equations in higher dimension*, Geom. Funct. Anal. 5, 629-639, 1995.

[9] Bourgain J., *Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations*, Ann. of Math. 148, 363-439, 1998.

[10] Bourgain J., *Green's function estimates for lattice Schrödinger operators and applications*, Annals of Mathematics Studies 158, Princeton University Press, Princeton, 2005.

[11] Bröcker, T., tom Dieck T., *Representations of compact Lie groups*, Graduate Texts in Mathematics, 98, Springer-Verlag, New York, 1985.

[12] Burq, N., Gérard, P., Tzvetkov, N., *Multilinear eigenfunction estimates and global existence for the three dimensional nonlinear Schrödinger equations*, Ann. Sci. École Norm. Sup. 4, 38, 255-301, 2005.

[13] Burq, N., Gérard, P., Tzvetkov, N., *Bilinear eigenfunction estimates and the nonlinear Schrödinger equation on surfaces*, Invent. Math., 159, 187-223, 2005.

[14] Chierchia L., You J., *KAM tori for 1D nonlinear wave equations with periodic boundary conditions*, Comm. Math. Phys. 211, 497-525, 2000.

[15] Craig W., *Problèmes de petits diviseurs dans les équations aux dérivées partielles*, Panoramas et Synthèses, 9, Société Mathématique de France, Paris, 2000.

[16] Craig W., Wayne C. E., *Newton's method and periodic solutions of nonlinear wave equation*, Comm. Pure Appl. Math. 46, 1409-1498, 1993.

[17] Delort, J.-M., Szeftel, J. *Long-time existence for semi-linear Klein-Gordon equations with small Cauchy data on Zoll manifolds*, Amer. J. Math. 128, 5, 1187-1218, 2006.

[18] Eliasson L. H., Kuksin S., *KAM for the nonlinear Schrödinger equation*, in print Annals of Math..

[19] Erdös L., Yau H.-T., *Derivation of the nonlinear Schrödinger equation from many body Coulomb system.*, Adv. Theor. Math. Phys. 5, 1169-1205, 2001.

[20] Gentile G., Procesi M., *Periodic solutions for a class of nonlinear partial differential equations in higher dimension*, to appear in Comm. Math. Phys..

[21] Faraut J., *Analysis on Lie groups, an introduction*, Cambridge studies in adv. math., 110, 2008.

[22] Fröhlich J., Spencer T., *Absence of diffusion in the Anderson tight binding model for large disorder or low energy*, Comm. Math. Phys. 88, 151-184., 1983.

[23] Helgason S., *Geometric analysis on Symmetric Spaces* , Math. monographs, AMS, 39, 1994.

[24] Helgason S., *Groups and geometric analysis, Integral geometry, invariant differential operators, and spherical functions*, Corrected reprint of the 1984 original. Mathematical Surveys and Monographs, 83. American Mathematical Society, Providence, 2000.

[25] Helgason S., *Wave equations on Homogeneous Spaces* , in Lie Group Representations III. Lecture Notes in Math. 1077, 254-287, Springer Verlag, New York, 1984.

[26] Hu L., *The Laplacian on homogeneous spaces*, J. Math. Phys., 49, 2008.

[27] Kuksin S., *Hamiltonian perturbations of infinite-dimensional linear systems with imaginary spectrum*, Funktsional Anal. i Prilozhen., 21, 22-37, 95, 1987.

[28] Kuksin S., *Analysis of Hamiltonian PDEs*, Oxford Lecture series in Math. and its applications, 19, Oxford University Press, 2000.

[29] Kuksin S., Pöschel J., *Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation*, Ann. of Math. (2), 143, 149-179, 1996.

[30] Meng G., *Geometric construction of the quantum Hall effect in all even dimensions*, J. Phys. A: Math. Gen. 36, 2003.

[31] Negro J., Del Olmo M.A., Rodriguez-Marco A., *Landau quantum systems: an approach based on symmetry*, J. Phys. A: Math. Gen. , 2002.

[32] Pöschel J., *A KAM-Theorem for some nonlinear PDEs*, Ann. Scuola Norm. Sup. Pisa, Cl. Sci., 23, 119-148, 1996.

[33] Procesi C., *Lie groups. An approach through invariants and representations*, Springer, 2007.

[34] Racah, *Group Theory and Spectroscopy*, Springer Tracts in Modern Physics, Volume 37, 1965.

[35] Sepansky M., *Compact Lie groups*, graduate text in Mathematics, 235, 2007.

[36] Taniguchi Y., *Normal homogeneous manifolds and their spectra* Osaka J. Math. 18, 555-576, 1981.

[37] Taylor M., *Partial Differential equations*, Vol. III, Springer-Verlag, New York, 1997.

[38] Zhang S. C., Hu J. P., *A Four-Dimensional Generalization of the Quantum Hall Effect*, Science, 294, 823, 2001.

[39] Wayne E., *Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory*, Comm. Math. Phys. 127, 479-528, 1990.

[40] Weyl H., *The theory of groups and quantum mechanics*, Dover Publications, New York, 1950.

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