

LINKING SOLUTIONS FOR QUASILINEAR EQUATIONS AT CRITICAL GROWTH INVOLVING THE “1-LAPLACE” OPERATOR

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ABSTRACT. We show that the problem at critical growth, involving the 1-Laplace operator and obtained by relaxation of $-\Delta_1 u = \lambda|u|^{-1}u + |u|^{1^*-2}u$, admits a nontrivial solution $u \in BV(\Omega)$ for any $\lambda \geq \lambda_1$. Nonstandard linking structures, for the associated functional, are recognized.

1. INTRODUCTION AND MAIN RESULT

Let Ω be a bounded domain in \mathbb{R}^n , $n \geq 2$, with Lipschitz boundary. We are interested in the existence of nontrivial solutions u to the problem which comes from the relaxation of

$$(1.1) \quad \begin{cases} -\operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) = \lambda \frac{u}{|u|} + |u|^{1^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda \in \mathbb{R}$ and $1^* = n/(n-1)$ is the critical Sobolev exponent for the embedding of $W_0^{1,1}(\Omega)$ in $L^q(\Omega)$.

Problem (1.1) looks as the formal limit, as $p \rightarrow 1^+$, of the problem at critical growth

$$(1.2) \quad \begin{cases} -\operatorname{div} (|\nabla u|^{p-2} \nabla u) = \lambda |u|^{p-2} u + |u|^{p^*-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p^* = np/(n-p)$. Let us set, whenever $1 \leq p < n$,

$$S = \inf \left\{ \frac{\int_{\mathbb{R}^n} |\nabla u|^p dx}{\left(\int_{\mathbb{R}^n} |u|^{p^*} dx \right)^{p/p^*}} : u \in C_c^\infty(\mathbb{R}^n) \setminus \{0\} \right\},$$

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$$\lambda_1 = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} |u|^p dx} : u \in C_c^\infty(\Omega) \setminus \{0\} \right\}.$$

Problem (1.2) has received much attention in the last years, starting from the celebrated paper of BREZIS-NIRENBERG [5], where it was shown that, for $p = 2$, problem (1.2) admits a positive solution u for every $\lambda \in]0, \lambda_1[$ and $n \geq 4$. The result has been extended by EGNELL, GARCIA AZORERO-PERAL ALONSO, GUEDDA-VERON [19, 22, 25], who have proved that (1.2) admits a positive solution u for any $\lambda \in]0, \lambda_1[$, provided that $p > 1$ and $n \geq p^2$. Such a solution u can be obtained via the Mountain pass theorem of AMBROSETTI-RABINOWITZ [1] applied to the C^1 -functional $f : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined as

$$f(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\lambda}{p} \int_{\Omega} |u|^p dx - \frac{1}{p^*} \int_{\Omega} |u|^{p^*} dx$$

and satisfies

$$(1.3) \quad 0 < f(u) < \frac{1}{n} S^{n/p}.$$

When $\lambda \geq \lambda_1$, it is still meaningful to look for nontrivial solutions u , but the situation is quite different in the two cases $p = 2$ and $p \neq 2$. If $p = 2$, it has been proved by CAPOZZI-FORTUNATO-PALMIERI [7] that problem (1.2) has a nontrivial solution u for any $\lambda \geq \lambda_1$, provided that $n \geq 5$ (see also GAZZOLA-RUF [23, Corollary 1]). Such a solution can be obtained via the Linking theorem of RABINOWITZ (see e.g. [31, Theorem 5.3]) applied to the functional f and still satisfies (1.3).

On the other hand, when $p \neq 2$ there is in general no direct sum decomposition of $W_0^{1,p}(\Omega)$, which allows to recognize a linking structure in a standard way, unless λ belongs to a suitable right neighborhood $[\lambda_1, \bar{\lambda}[$ of λ_1 , as shown in ARIOLI-GAZZOLA [3], where it is proved that, for any $p > 1$, problem (1.2) has a nontrivial solution u for any $\lambda \in [\lambda_1, \bar{\lambda}[$, provided that $\frac{n^2}{n+1} > p^2$. Nevertheless, the result of CAPOZZI-FORTUNATO-PALMIERI has been recently extended, via a nonstandard linking construction, in DEGIOVANNI-LANCELOTTI [13], where it is shown that the result of ARIOLI-GAZZOLA actually holds for any $\lambda \geq \lambda_1$.

Coming to the case $p = 1$, let us first give a precise relaxed formulation of (1.1). First of all, denote by $\|\cdot\|_p$ the usual norm in L^p and by \mathcal{H}^k the k -dimensional Hausdorff measure. For every $u \in BV(\Omega)$ (see e.g. [2, 24]), let us set

$$|Du|(\Omega) := \sup \left\{ \int_{\Omega} u \operatorname{div} v dx : v \in C_c^\infty(\Omega; \mathbb{R}^n), \|v\|_\infty \leq 1 \right\}.$$

Then, according to KAWOHL-SCHURICHT [28], we mean that we are looking for $u \in BV(\Omega)$ such that

$$(1.4) \quad \begin{cases} \text{there exist } z \in L^\infty(\Omega; \mathbb{R}^n) \text{ and } \gamma \in L^\infty(\Omega) \text{ such that} \\ \|z\|_\infty \leq 1, \operatorname{div} z \in L^n(\Omega), -\int_\Omega u \operatorname{div} z \, dx = |Du|(\Omega) + \int_{\partial\Omega} |u| \, d\mathcal{H}^{n-1}, \\ \|\gamma\|_\infty \leq 1, \gamma|u| = u \quad \text{a.e. in } \Omega, \\ -\operatorname{div} z = \lambda\gamma + |u|^{1^*-2}u \quad \text{a.e. in } \Omega, \end{cases}$$

(n is the exponent conjugate to 1^*). Other equivalent formulations can be obtained applying the next Proposition 3.1. Since $u = 0$ is a solution for any λ (take $(z, \gamma) = (0, 0)$), we say that $u = 0$ is the *trivial* solution of (1.4). Let us also define a locally Lipschitz functional $f : BV(\Omega) \rightarrow \mathbb{R}$ by

$$f(u) = |Du|(\Omega) + \int_{\partial\Omega} |u| \, d\mathcal{H}^{n-1} - \lambda \int_\Omega |u| \, dx - \frac{1}{1^*} \int_\Omega |u|^{1^*} \, dx.$$

The result of BREZIS-NIRENBERG has been extended also to this setting by DEMEN-
GEL [17], who has proved that (1.4) admits a nonnegative, nontrivial solution u satisfying

$$(1.5) \quad 0 < f(u) < \frac{1}{n} S^n$$

for any $\lambda \in]0, \lambda_1[$. The argument is based on an approximation procedure from the case $p > 1$.

Our purpose is to cover the case $\lambda \geq \lambda_1$, in the line of the result of CAPOZZI-FORTUNATO-PALMIERI, by a direct approach. Our result is the following

Theorem 1.1. *Let Ω be a bounded open subset of \mathbb{R}^n , $n \geq 2$, with Lipschitz boundary. Then, for every $\lambda \geq \lambda_1$, problem (1.4) admits a nontrivial solution $u \in BV(\Omega) \cap L^\infty(\Omega)$ satisfying (1.5).*

For the proof, we will apply (nonsmooth) variational methods to the functional f . A first idea could be to apply the approach of CHANG [8] to the locally Lipschitz functional f defined on $BV(\Omega)$. However, it has been already observed that, in such a setting, the Palais-Smale condition fails even in the subcritical case, as the norm-convergence of BV cannot be usually obtained for a Palais-Smale sequence (see MARZOCCHI [29] and DEGIOVANNI-MARZOCCHI-RĂDULESCU [15]). For this reason, it is more convenient to extend the functional f to $L^{1^*}(\Omega)$ with value $+\infty$ outside $BV(\Omega)$. In this setting, the nonsmoothness increases, as f is only lower semicontinuous, but the techniques of CORVELLEC-DEGIOVANNI-MARZOCCHI, IOFFE-SCHWARTZMAN, KATRIEL [11, 26, 27] can be applied, in particular as specified in DEGIOVANNI-SCHURICHT [16]. On the other

hand, we have more compactness and in Theorem 5.3 we will show that f satisfies $(PS)_c$ whenever $c < (1/n)S^n$, as one may expect from the case $p > 1$ (see [25, Theorem 3.4]).

A second difficulty, typical in the case $p \neq 2$ when $\lambda \geq \lambda_1$, is that there is no direct sum decomposition which allows to recognize a linking structure in a standard way. Therefore, as in [13], we will apply the Linking theorem of [12], in which linear subspaces are substituted by cones.

In the next section we recall mainly from [16] some tools of nonsmooth analysis. In Section 3 we specify our functional framework, taking advantage of the results of [28]. In Section 4 we build the cones which have to substitute linear subspaces in the linking structure. Section 5 is devoted to the Palais-Smale condition, while in the last section we prove the main result.

2. TOOLS OF NONSMOOTH ANALYSIS

Let Y be a metric space endowed with the distance d and let $f : Y \rightarrow [-\infty, +\infty]$ be a function. We also consider

$$\text{epi}(f) = \{(u, s) \in Y \times \mathbb{R} : f(u) \leq s\}$$

endowed with the topology induced by $Y \times \mathbb{R}$. The next definition, equivalent to that of [14], is taken from [6].

Definition 2.1. For every $u \in Y$ with $f(u) \in \mathbb{R}$, we denote by $|df|(u)$ the supremum of the σ 's in $[0, +\infty[$ such that there exist a neighborhood W of $(u, f(u))$ in $\text{epi}(f)$, $\delta > 0$ and a continuous map $\mathcal{H} : W \times [0, \delta] \rightarrow Y$ satisfying

$$d(\mathcal{H}((v, s), t), v) \leq t, \quad f(\mathcal{H}((v, s), t)) \leq s - \sigma t,$$

whenever $(v, s) \in W$ and $t \in [0, \delta]$.

The extended real number $|df|(u)$ is called the *weak slope* of f at u .

Definition 2.2. An element $u \in Y$ is said to be a (*lower*) *critical point* of f , if $f(u) \in \mathbb{R}$ and $|df|(u) = 0$. A real number c is said to be a (*lower*) *critical value* of f , if there exists a (*lower*) *critical point* u of f with $f(u) = c$.

Definition 2.3. A *Palais-Smale sequence* (*(PS)-sequence*, for short) for f is a sequence (u_k) in Y such that

$$\sup_k |f(u_k)| < +\infty$$

and such that $|df|(u_k) \rightarrow 0$.

Given a real number c , a *Palais-Smale sequence at level c* (*(PS)_c-sequence*, for short) is a *(PS)-sequence* (u_k) such that $f(u_k) \rightarrow c$.

The function f is said to satisfy $(PS)_c$, if every $(PS)_c$ -sequence admits a convergent subsequence in Y .

Assume now that X is a real Banach space, whose dual space will be denoted by X' . In the following, $\partial f(u)$ will denote the CLARKE-ROCKAFELLAR subdifferential and $f^0(u; v)$ the associated generalized directional derivative [10, 32].

Let $f_0 : X \rightarrow]-\infty, +\infty]$ be a convex, lower semicontinuous function and $f_1, g : X \rightarrow \mathbb{R}$ two locally Lipschitz continuous functions. Let also $f = f_0 + f_1$ and

$$M = \{u \in X : g(u) = 0\}.$$

We are first interested in a (nonsmooth) extension of the Linking theorem, in which linear subspaces are substituted by symmetric cones. If $A \subseteq X \setminus \{0\}$ is symmetric, we denote by $\text{Index}(A)$ the \mathbb{Z}_2 -cohomological index of FADELL-RABINOWITZ [20, 21]. Let us recall that $\gamma^+(A) \leq \text{Index}(A) \leq \gamma^-(A)$, where, according to [9],

$$\gamma^+(A) = \sup\{m \in \mathbb{N} : \text{there exists an odd continuous map } \psi : \mathbb{R}^m \setminus \{0\} \rightarrow A\},$$

$$\gamma^-(A) = \inf\{m \in \mathbb{N} : \text{there exists an odd continuous map } \psi : A \rightarrow \mathbb{R}^m \setminus \{0\}\}.$$

Theorem 2.4. *Let X_-, X_+ be two symmetric cones in X such that X_+ is closed in X ,*

$$X_- \cap X_+ = \{0\},$$

$$\text{Index}(X_- \setminus \{0\}) = \text{Index}(X \setminus X_+) < \infty.$$

Let also $e \in X \setminus X_-$, $0 < r_+ < r_-$,

$$S_+ = \{v \in X_+ : \|v\| = r_+\},$$

$$Q = \{te + u : t \geq 0, u \in X_-, \|te + u\| \leq r_-\},$$

$$P = \{u \in X_- : \|u\| \leq r_-\} \cup \{te + u : t \geq 0, u \in X_-, \|te + u\| = r_-\}$$

be such that

$$\sup_P f < \inf_{S_+} f, \quad \sup_Q f < +\infty.$$

Then f admits a $(PS)_c$ -sequence with

$$\inf_{S_+} f \leq c \leq \sup_Q f.$$

In particular, if f satisfies $(PS)_c$, then c is a critical value of f .

Proof. If $f : X \rightarrow \mathbb{R}$ is of class C^1 , by [12, Corollary 2.9] the assertion is a particular case of [12, Theorem 2.2]. If $f : X \rightarrow \mathbb{R}$ is continuous, the proof is exactly the same, by the Deformation theorem of [11]. The case we are treating can be reduced to the continuous one arguing, as in [16], on the continuous function $\mathcal{G}_f : \text{epi}(f) \rightarrow \mathbb{R}$ defined by $\mathcal{G}_f(u, s) = s$. \square

We also need an information in the constrained case.

Theorem 2.5. *Assume that f and g are even with $g(0) \neq 0$ and that*

$$\text{Index}(\{u \in M : f(u) < +\infty\}) = \infty.$$

Suppose also that $f|_M$ is bounded from below, satisfies $(PS)_c$ for any $c \in \mathbb{R}$ and that, for every $u \in M$ with $f(u) < +\infty$, there exist $u_{\pm} \in X$ such that $f(u_{\pm}) < +\infty$ and

$$g^0(u; u_- - u) < 0, \quad g^0(u; u - u_+) < 0.$$

For every $m \geq 1$, let

$$c_m = \inf \left\{ \sup_A f : A \subseteq M, A \text{ is symmetric and } \text{Index}(A) \geq m \right\}.$$

Then $c_m \rightarrow +\infty$ and, for every $m \geq 1$ and c with $c_m \leq c < c_{m+1}$, we have

$$\text{Index}(\{u \in M : f(u) \leq c\}) = m.$$

Proof. In the C^1 setting, the assertion follows from the Deformation theorem (see e.g. [12, Theorem 3.2]). For the extension to the nonsmooth case we are treating, we may argue as in the previous proof. \square

Finally, let us recall from [16, Theorem 3.5] two results which connect the metric notion of weak slope with that of subdifferential.

Theorem 2.6. *Let $u \in X$ with $f(u) < +\infty$ and $|df|(u) < +\infty$. Then there exist $\alpha \in \partial f_1(u)$ and $w \in X'$ such that $\|w\| \leq |df|(u)$ and $-\alpha + w \in \partial f_0(u)$, i.e.*

$$f_0(v) \geq f_0(u) - \langle \alpha, v - u \rangle + \langle w, v - u \rangle, \quad \forall v \in X.$$

Theorem 2.7. *Let $u \in M$ with $f(u) < +\infty$ and $|d(f|_M)|(u) < +\infty$. Assume also that there exist $u_{\pm} \in X$ such that $f(u_{\pm}) < +\infty$ and*

$$g^0(u; u_- - u) < 0, \quad g^0(u; u - u_+) < 0.$$

Then there exist $\alpha \in \partial f_1(u)$, $\beta \in \partial g(u)$, $\lambda \in \mathbb{R}$ and $w \in X'$ such that $\|w\| \leq |d(f|_M)|(u)$ and $-\alpha + \lambda\beta + w \in \partial f_0(u)$, i.e.

$$f_0(v) \geq f_0(u) - \langle \alpha, v - u \rangle + \lambda \langle \beta, v - u \rangle + \langle w, v - u \rangle, \quad \forall v \in X.$$

3. THE FUNCTIONAL FRAMEWORK

Let Ω be a bounded open subset of \mathbb{R}^n , $n \geq 2$, with Lipschitz boundary and let $\lambda \in \mathbb{R}$. According to [28], let us define a convex, lower semicontinuous functional $f_0 : L^{1^*}(\Omega) \rightarrow]-\infty, +\infty]$ by

$$f_0(u) = \begin{cases} |Du|(\Omega) + \int_{\partial\Omega} |u| d\mathcal{H}^{n-1} & \text{if } u \in BV(\Omega), \\ +\infty & \text{if } u \in L^{1^*}(\Omega) \setminus BV(\Omega), \end{cases}$$

and two locally Lipschitz continuous functionals $f_1, g : L^{1^*}(\Omega) \rightarrow \mathbb{R}$ by

$$f_1(u) = -\lambda \int_{\Omega} |u| dx - \frac{1}{1^*} \int_{\Omega} |u|^{1^*} dx,$$

$$g(u) = \int_{\Omega} |u| dx - 1.$$

As usual, the dual of $L^{1^*}(\Omega)$ will be identified with $L^{(1^*)'}(\Omega) = L^n(\Omega)$. Moreover, f_0 is a norm on $BV(\Omega)$ equivalent to the canonical one and, according to [17, 28], we have

$$(3.1) \quad S = \min \left\{ \frac{f_0(u)}{\|u\|_{1^*}} : u \in BV(\Omega) \setminus \{0\} \right\},$$

$$(3.2) \quad \lambda_1 = \min \left\{ \frac{f_0(u)}{\|u\|_1} : u \in BV(\Omega) \setminus \{0\} \right\}.$$

In particular, contrary to the case $p > 1$, the constant S is achieved, for instance on characteristic functions of balls contained in Ω (see [4]).

We are interested in the application of variational methods to $f = f_0 + f_1$ on the whole space $L^{1^*}(\Omega)$ and to f_0 constrained on

$$M = \{u \in L^{1^*}(\Omega) : g(u) = 0\}.$$

In order to apply the results of the previous section, let us first recall from [28] the next

Proposition 3.1. *Let $u \in BV(\Omega)$ and $w \in L^n(\Omega)$. Then the following facts are equivalent:*

- (a) we have $w \in \partial f_0(u)$;
- (b) we have

$$\int_{\Omega} uw dx = |Du|(\Omega) + \int_{\partial\Omega} |u| d\mathcal{H}^{n-1}$$

and there exists $z \in L^\infty(\Omega; \mathbb{R}^n)$ such that $\|z\|_\infty \leq 1$ and $-\operatorname{div} z = w$;

- (c) there exists $z \in L^\infty(\Omega; \mathbb{R}^n)$ such that $\|z\|_\infty \leq 1$, $-\operatorname{div} z = w$ and

$$\int_{\Omega} uw\varphi dx - \int_{\Omega} uz \cdot \nabla\varphi dx = \sup \left\{ \left| \int_{\Omega} u \operatorname{div} \psi dx \right| : \psi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n), |\psi| \leq \varphi \right\}$$

for every $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi \geq 0$.

Proof. It is enough to combine [28, Proposition 4.23] with [28, Proposition A.12] and recall that the function defined as

$$\begin{cases} u & \text{on } \Omega, \\ 0 & \text{on } \mathbb{R}^n \setminus \Omega, \end{cases}$$

belongs to $BV(\mathbb{R}^n)$. □

In general, the graph of the subdifferential of a convex, lower semicontinuous functional is strong-weak* closed. In our case, we have a better property which will be useful later.

Proposition 3.2. *Let (u_k) be a sequence in $BV(\Omega)$ and (w_k) a sequence in $L^n(\Omega)$ such that (u_k) is weakly convergent to u in $L^{1^*}(\Omega)$, (w_k) is weakly convergent to w in $L^n(\Omega)$ and $w_k \in \partial f_0(u_k)$ for every $k \in \mathbb{N}$.*

Then $u \in BV(\Omega)$ and $w \in \partial f_0(u)$.

Proof. For every $h > 0$, let $T_h, R_h : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $T_h(s) = \min\{\max\{s, -h\}, h\}$, $R_h(s) = s - T_h(s)$. By [2, Theorem 3.99] we have

$$|Du|(\Omega) = |D(T_h(u))|(\Omega) + |D(R_h(u))|(\Omega),$$

hence

$$(3.3) \quad f_0(u) = f_0(T_h(u)) + f_0(R_h(u)), \quad \forall u \in BV(\Omega).$$

First of all, from the inequality

$$0 = f_0(0) \geq f_0(u_k) - \int_{\Omega} w_k u_k \, dx$$

we see that (u_k) is bounded in $BV(\Omega)$. It follows that $u \in BV(\Omega)$ and that $(T_h(u_k))$ is strongly convergent to $T_h(u)$ in $L^{1^*}(\Omega)$, for every $h > 0$.

We also have

$$\begin{aligned} f_0(v) + f_0(R_h(u_k)) &\geq f_0(v + R_h(u_k)) \geq f_0(u_k) + \int_{\Omega} w_k(v + R_h(u_k) - u_k) \, dx \\ &= f_0(T_h(u_k)) + f_0(R_h(u_k)) + \int_{\Omega} w_k(v - T_h(u_k)) \, dx, \end{aligned}$$

whence

$$f_0(v) \geq f_0(T_h(u_k)) + \int_{\Omega} w_k(v - T_h(u_k)) \, dx.$$

Passing to the limit as $k \rightarrow \infty$ and taking into account the lower semicontinuity of f_0 , we get

$$f_0(v) \geq f_0(T_h(u)) + \int_{\Omega} w(v - T_h(u)) \, dx.$$

Passing to the limit as $h \rightarrow \infty$, the assertion follows. □

Let us also prove a simple regularity property. A related result is contained in [18, Proposition 7].

Proposition 3.3. *Let $u \in BV(\Omega)$ with $\partial f_0(u) \neq \emptyset$. Then $u \in L^\infty(\Omega)$.*

Proof. Let $w \in L^n(\Omega)$ with $w \in \partial f_0(u)$. For every $h > 0$, we have

$$f_0(T_h(u)) \geq f_0(u) + \int_{\Omega} w(T_h(u) - u) dx.$$

By (3.1), (3.3) and Hölder’s inequality, it follows

$$S \|R_h(u)\|_{1^*} \leq f_0(R_h(u)) \leq \int_{\Omega} w R_h(u) dx \leq \left(\int_{\{|u|>h\}} |w|^n dx \right)^{1/n} \|R_h(u)\|_{1^*}.$$

If h is large enough to guarantee that

$$\left(\int_{\{|u|>h\}} |w|^n dx \right)^{1/n} < S,$$

we infer that $\|R_h(u)\|_{1^*} = 0$ and the assertion follows. \square

Finally, from [28] we have the

Proposition 3.4. *Let $u \in BV(\Omega)$ with $|df|(u) < +\infty$. Then $u \in L^\infty(\Omega)$ and there exist $\gamma \in L^\infty(\Omega)$ and $w \in L^n(\Omega)$ such that $\|\gamma\|_\infty \leq 1$, $\gamma|u| = u$ a.e. in Ω , $\|w\|_n \leq |df|(u)$ and*

$$f_0(v) \geq f_0(u) + \lambda \int_{\Omega} \gamma(v - u) dx + \int_{\Omega} |u|^{1^*-2} u(v - u) dx + \int_{\Omega} w(v - u) dx, \quad \forall v \in BV(\Omega).$$

Proof. It is enough to combine Theorem 2.6 with Proposition 3.3 and [28, Proposition 4.23]. \square

Corollary 3.5. *If $u \in L^{1^*}(\Omega)$ is a critical point of f , then $u \in BV(\Omega) \cap L^\infty(\Omega)$ and u is a solution of (1.4).*

Proof. It is enough to combine Proposition 3.1 with Proposition 3.4. \square

4. SYMMETRIC CONES RELATED TO THE 1-LAPLACE OPERATOR

In this section we show how to build, for the 1-Laplace operator, two cones X_-, X_+ with the properties required in Theorem 2.4. The construction is based on a sequence of eigenvalues for the 1-Laplace operator. We refer the reader to MILBERS-SCHURICHT [30] for a slightly different construction of such a sequence.

Proposition 4.1. *The following facts hold:*

(a) for every $u \in BV(\Omega) \cap M$, there exist $u_{\pm} \in BV(\Omega)$ such that

$$g^0(u; u_- - u) < 0, \quad g^0(u; u - u_+) < 0;$$

(b) for every $u \in BV(\Omega) \cap M$ with $|d(f_{0|M})|(u) < +\infty$, we have $u \in L^\infty(\Omega)$ and there exist $\lambda \in \mathbb{R}$, $\gamma \in L^\infty(\Omega)$ and $w \in L^n(\Omega)$ such that $\|\gamma\|_\infty \leq 1$, $\gamma|u| = u$ a.e. in Ω , $\|w\|_n \leq |d(f_{0|M})|(u)$ and

$$f_0(v) \geq f_0(u) + \lambda \int_{\Omega} \gamma(v - u) dx + \int_{\Omega} w(v - u) dx, \quad \forall v \in BV(\Omega);$$

(c) the functionals f_0 and g are even with $g(0) \neq 0$ and $\text{Index}(BV(\Omega) \cap M) = \infty$ with respect to the topology of $L^{1^*}(\Omega)$; moreover, $f_{0|M}$ is bounded from below and satisfies $(PS)_c$ for any $c \in \mathbb{R}$.

Proof. In the proof of [28, Theorem 4.6] it is shown that (a) holds. Then assertion (b) follows from Theorem 2.7, Proposition 3.3 and [28, Proposition 4.23]. Since $BV(\Omega)$ has infinite dimension, it is obvious that $\gamma^+(BV(\Omega) \cap M) = \infty$, also with respect to the topology of $L^{1^*}(\Omega)$. Therefore $\text{Index}(BV(\Omega) \cap M) = \infty$.

If (u_k) is a (PS) -sequence for $f_{0|M}$, by (b) we have

$$f_0(v) \geq f_0(u_k) + \lambda_k \int_{\Omega} \gamma_k(v - u_k) dx + \int_{\Omega} w_k(v - u_k) dx, \quad \forall v \in BV(\Omega)$$

with $\lambda_k \in \mathbb{R}$, $\gamma_k \in L^\infty(\Omega)$ and $w_k \in L^n(\Omega)$ satisfying $\|\gamma_k\|_\infty \leq 1$, $\gamma_k|u_k| = u_k$ a.e. in Ω and $\|w_k\|_n \rightarrow 0$. Since f_0 is an equivalent norm in $BV(\Omega)$, up to a subsequence (u_k) is convergent to $u \in BV(\Omega)$ weakly in $L^{1^*}(\Omega)$ and strongly in $L^1(\Omega)$, while (γ_k) is convergent to γ in the weak* topology of $L^\infty(\Omega)$. Moreover, by Proposition 3.1 we have

$$f_0(u_k) = \lambda_k \int_{\Omega} \gamma_k u_k dx + \int_{\Omega} w_k u_k dx = \lambda_k \int_{\Omega} |u_k| dx + \int_{\Omega} w_k u_k dx = \lambda_k + \int_{\Omega} w_k u_k dx.$$

Therefore, also (λ_k) is bounded, hence convergent, up to a subsequence, to some λ . From Proposition 3.2 it follows that $\lambda\gamma \in \partial f_0(u)$, whence, by Proposition 3.1,

$$\lim_k f_0(u_k) = \lim_k \left(\lambda_k \int_{\Omega} \gamma_k u_k dx + \int_{\Omega} w_k u_k dx \right) = \lambda \int_{\Omega} \gamma u dx = f_0(u).$$

From [15, Theorem 4.10] we conclude that (u_k) is strongly convergent to u in $L^{1^*}(\Omega)$.

The other assertions contained in (c) are obvious. \square

For every $m \geq 1$, let

$$\lambda_m = \inf \left\{ \sup_A f_0 : A \subseteq M, A \text{ is symmetric and } \text{Index}(A) \geq m \right\}.$$

Since $\text{Index}(A) = 0$ only for $A = \emptyset$, it is clear that the definition of λ_1 agrees with (3.2).

Theorem 4.2. *We have that $\lambda_m \rightarrow +\infty$. Moreover, for every $m \geq 1$ and μ with $\lambda_m \leq \mu < \lambda_{m+1}$, we have*

$$\text{Index} \left(\left\{ u \in BV(\Omega) \setminus \{0\} : |Du|(\Omega) + \int_{\partial\Omega} |u| d\mathcal{H}^{n-1} \leq \mu \int_{\Omega} |u| dx \right\} \right) = m$$

with respect to the topology of $L^{1^}(\Omega)$.*

Proof. Since f_0 and $\|\cdot\|_1$ are both positively homogeneous of degree 1, it is enough to combine Theorem 2.5 with Proposition 4.1. \square

In view of the application of Theorem 2.4, let us see a first possible choice of X_-, X_+ .

Theorem 4.3. *Let $m \geq 1$ and let $\lambda_m < \mu < \lambda_{m+1}$. Then there exist a symmetric cone X_- in $BV(\Omega)$ and a symmetric cone X_+ in $L^{1^*}(\Omega)$ such that X_- is closed in $L^1(\Omega)$, X_+ is closed in $L^{1^*}(\Omega)$ and:*

(a) *we have*

$$X_- \subseteq \left\{ u \in BV(\Omega) : |Du|(\Omega) + \int_{\partial\Omega} |u| d\mathcal{H}^{n-1} \leq \lambda_m \int_{\Omega} |u| dx \right\} \cap L^\infty(\Omega);$$

(b) *$X_- \cap M$ is bounded in $L^\infty(\Omega)$ and strongly compact in $L^1(\Omega)$;*

(c) *we have*

$$X_+ \cap BV(\Omega) \subseteq \left\{ u \in BV(\Omega) : |Du|(\Omega) + \int_{\partial\Omega} |u| d\mathcal{H}^{n-1} \geq \mu \int_{\Omega} |u| dx \right\};$$

(d) *we have $\text{Index}(X_- \setminus \{0\}) = \text{Index}(L^{1^*}(\Omega) \setminus X_+) = m$ with respect to the topology of $L^{1^*}(\Omega)$.*

Proof. Let

$$\tilde{X}_- = \left\{ u \in BV(\Omega) : |Du|(\Omega) + \int_{\partial\Omega} |u| d\mathcal{H}^{n-1} \leq \lambda_m \int_{\Omega} |u| dx \right\}.$$

Since $\tilde{X}_- \cap M$ is an odd deformation retract of $\tilde{X}_- \setminus \{0\}$, by Theorem 4.2 we have that $\text{Index}(\tilde{X}_- \cap M) = m$. Moreover, $\tilde{X}_- \cap M$ is strongly compact in $L^1(\Omega)$.

Let T_h, R_h be defined as before. First of all, we claim that there exists $h > 0$ such that

$$(4.1) \quad f_0(T_h(u)) \leq \lambda_m \int_{\Omega} |T_h(u)| dx, \quad \forall u \in \tilde{X}_- \cap M;$$

$$(4.2) \quad \int_{\Omega} |T_h(u)| dx \geq \frac{1}{2}, \quad \forall u \in \tilde{X}_- \cap M.$$

Actually, for every $u \in BV(\Omega)$ Hölder's inequality and (3.1) yield

$$\int_{\Omega} |u| dx \leq \mathcal{L}^n(\{u \neq 0\})^{\frac{1}{n}} \left(\int_{\Omega} |u|^{1^*} dx \right)^{\frac{1}{1^*}} \leq \frac{1}{S} \mathcal{L}^n(\{u \neq 0\})^{\frac{1}{n}} f_0(u).$$

Since for every $u \in BV(\Omega) \cap M$ we have $R_h(u) \in BV(\Omega)$ and

$$1 = \int_{\Omega} |u| \, dx \geq \int_{\{R_h(u) \neq 0\}} |u| \, dx \geq h \mathcal{L}^n(\{R_h(u) \neq 0\}),$$

it follows

$$S h^{\frac{1}{n}} \int_{\Omega} |R_h(u)| \, dx \leq f_0(R_h(u)) \quad \forall u \in BV(\Omega) \cap M.$$

Then, if h is large enough, we have

$$\lambda_m \int_{\Omega} |R_h(u)| \, dx \leq f_0(R_h(u)) \quad \forall u \in BV(\Omega) \cap M$$

and (4.1) follows from (3.3). Moreover, if $u \in \tilde{X}_- \cap M$, we also have

$$S h^{\frac{1}{n}} \int_{\Omega} |R_h(u)| \, dx \leq f_0(R_h(u)) \leq f_0(u) \leq \lambda_m.$$

Then (4.2) also follows, provided that h is large enough.

With this choice of h , let

$$X_- = \left\{ t T_h(u) : t \geq 0, u \in \tilde{X}_- \cap M \right\}.$$

Then X_- is a symmetric cone in $BV(\Omega) \cap L^\infty(\Omega)$. From (4.1) it follows that $X_- \subseteq \tilde{X}_-$, while (4.2) implies that

$$\|v\|_\infty \leq 2h \|v\|_1, \quad \forall v \in X_-.$$

In particular, $X_- \cap M$ is bounded in $L^\infty(\Omega)$. Since the surjective map

$$\begin{aligned} \tilde{X}_- \cap M &\longrightarrow X_- \cap M \\ u &\longmapsto \frac{T_h(u)}{\|T_h(u)\|_1} \end{aligned}$$

is odd and continuous with respect to the topology of $L^{1^*}(\Omega)$, we have

$$\text{Index}(X_- \setminus \{0\}) \geq \text{Index}(X_- \cap M) \geq \text{Index}(\tilde{X}_- \cap M) = m.$$

Actually, equality holds, as $X_- \subseteq \tilde{X}_-$. Finally, the above map is also continuous with respect to the topology of $L^1(\Omega)$. Therefore $X_- \cap M$ is strongly compact in $L^1(\Omega)$ and X_- is closed in $L^1(\Omega)$.

Again from Theorem 4.2 we know that

$$\text{Index} \left(\left\{ u \in BV(\Omega) \cap M : |Du|(\Omega) + \int_{\partial\Omega} |u| \, d\mathcal{H}^{n-1} \leq \mu \right\} \right) = m.$$

Let U be a symmetric open neighborhood of such a set satisfying $\text{Index}(U) = m$. Then

$$X_+ = L^{1^*}(\Omega) \setminus \{tu : t > 0, u \in U\}$$

has the required properties. \square

5. THE PALAIS-SMALE CONDITION

Lemma 5.1. *Let (u_k) be a (PS) sequence for f and let $u \in BV(\Omega)$. Assume that (u_k) is bounded in $BV(\Omega)$ and weakly convergent to u in $L^{1^*}(\Omega)$.*

Then we have

$$\lim_k (f_0(u_k) - \|u_k\|_{1^*}^{1^*}) = f_0(u) - \|u\|_{1^*}^{1^*},$$

$$\limsup_k (f_0(R_h(u_k)) - \|R_h(u_k)\|_{1^*}^{1^*}) \leq f_0(R_h(u)) - \|R_h(u)\|_{1^*}^{1^*}, \quad \forall h > 0.$$

Proof. By Proposition 3.4, there exist (γ_k) in $L^\infty(\Omega)$ and (w_k) in $L^n(\Omega)$ such that $\|\gamma_k\|_\infty \leq 1$, $\gamma_k|u_k| = u_k$ a.e. in Ω , $\|w_k\|_n \rightarrow 0$ and $\lambda\gamma_k + |u_k|^{1^*-2}u_k + w_k \in \partial f_0(u_k)$. Moreover, (u_k) is also strongly convergent to u in $L^1(\Omega)$ and, up to a subsequence, (γ_k) is convergent to some γ in the weak* topology of $L^\infty(\Omega)$. By Proposition 3.2 it follows $\lambda\gamma + |u|^{1^*-2}u \in \partial f_0(u)$. Then by Proposition 3.1 we have

$$(5.1) \quad \begin{aligned} f_0(u_k) &= \lambda \int_\Omega \gamma_k u_k \, dx + \int_\Omega |u_k|^{1^*} \, dx + \int_\Omega w_k u_k \, dx \\ &= \lambda \int_\Omega |u_k| \, dx + \int_\Omega |u_k|^{1^*} \, dx + \int_\Omega w_k u_k \, dx, \\ f_0(u) &= \lambda \int_\Omega \gamma u \, dx + \int_\Omega |u|^{1^*} \, dx, \end{aligned}$$

whence

$$\begin{aligned} \lim_k \left(f_0(u_k) - \int_\Omega |u_k|^{1^*} \, dx \right) &= \lim_k \left(\lambda \int_\Omega \gamma_k u_k \, dx + \int_\Omega w_k u_k \, dx \right) \\ &= \lambda \int_\Omega \gamma u \, dx = f_0(u) - \int_\Omega |u|^{1^*} \, dx. \end{aligned}$$

By (3.3) we also have

$$f_0(R_h(u_k)) - \|R_h(u_k)\|_{1^*}^{1^*} = (f_0(u_k) - \|u_k\|_{1^*}^{1^*}) - f_0(T_h(u_k)) + (\|u_k\|_{1^*}^{1^*} - \|R_h(u_k)\|_{1^*}^{1^*}).$$

On the other hand, $(T_h(u_k))$ is convergent to $T_h(u)$ in $L^{1^*}(\Omega)$ and we have that

$$0 \leq |s|^{1^*} - |R_h(s)|^{1^*} \leq \varepsilon |s|^{1^*} + C_{h,\varepsilon}, \quad \forall \varepsilon > 0.$$

From [12, Lemma 4.2] it follows that

$$\lim_k (\|u_k\|_{1^*}^{1^*} - \|R_h(u_k)\|_{1^*}^{1^*}) = (\|u\|_{1^*}^{1^*} - \|R_h(u)\|_{1^*}^{1^*}).$$

By the lower semicontinuity of f_0 , the second assertion also follows. \square

Lemma 5.2. *Each (PS) sequence for f is bounded in $BV(\Omega)$.*

Proof. Let (u_k) be a (PS) sequence for f and assume, for a contradiction, that $f_0(u_k) \rightarrow +\infty$. If we set

$$v_k = \frac{u_k}{f_0(u_k)},$$

up to a subsequence (v_k) is strongly convergent in $L^1(\Omega)$ to some $v \in BV(\Omega)$. Since

$$\frac{f(u_k)}{f_0(u_k)} = 1 - \lambda \|v_k\|_1 - \frac{1}{1^*} (f_0(u_k))^{1^*-1} \|v_k\|_{1^*}^{1^*},$$

from the boundedness of $(f(u_k))$ we deduce that (v_k) is strongly convergent to 0 in $L^{1^*}(\Omega)$. On the other hand, as before it holds (5.1) with $\|w_k\|_n \rightarrow 0$. It follows

$$f(u_k) = \frac{1}{n} [f_0(u_k) - \lambda \|u_k\|_1] + \frac{1}{1^*} \int_{\Omega} w_k u_k \, dx,$$

namely

$$\frac{f(u_k)}{f_0(u_k)} = \frac{1}{n} [1 - \lambda \|v_k\|_1] + \frac{1}{1^*} \int_{\Omega} w_k v_k \, dx.$$

Passing to the limit as $k \rightarrow \infty$, we get $0 = 1/n$ and a contradiction follows. \square

Theorem 5.3. *For every $\lambda \in \mathbb{R}$, the functional f satisfies $(PS)_c$ whenever $c < (1/n)S^n$.*

Proof. Let (u_k) be a $(PS)_c$ sequence with $c < (1/n)S^n$. We already know that (u_k) is bounded in $BV(\Omega)$, hence convergent, up to a subsequence, to some $u \in BV(\Omega)$ weakly in $L^{1^*}(\Omega)$ and strongly in $L^1(\Omega)$. From (5.1) it also follows that

$$f(u_k) = \frac{1}{n} \|u_k\|_{1^*}^{1^*} + \int_{\Omega} w_k u_k \, dx,$$

with $\|w_k\|_n \rightarrow 0$, whence

$$\lim_k \|u_k\|_{1^*}^{1^*-1} = (nc)^{1/n} < S.$$

Given $\varepsilon > 0$, let $h > 0$ be such that

$$f_0(R_h(u)) - \|R_h(u)\|_{1^*}^{1^*} < \varepsilon (S - (nc)^{1/n}).$$

Then we have

$$\limsup_k \|R_h(u_k)\|_{1^*}^{1^*-1} \leq (nc)^{1/n}$$

and, by (3.1),

$$(S - \|R_h(u_k)\|_{1^*}^{1^*-1}) \|R_h(u_k)\|_{1^*} \leq f_0(R_h(u_k)) - \|R_h(u_k)\|_{1^*}^{1^*}.$$

From Lemma 5.1 it follows

$$\limsup_k \|R_h(u_k)\|_{1^*} < \varepsilon,$$

whence $\|R_h(u)\|_{1^*} < \varepsilon$. Since $(T_h(u_k))$ is strongly convergent to $T_h(u)$ in $L^{1^*}(\Omega)$, we have

$$\begin{aligned} \limsup_k \|u_k - u\|_{1^*} &\leq \limsup_k \|T_h(u_k) - T_h(u)\|_{1^*} \\ &\quad + \limsup_k \|R_h(u_k)\|_{1^*} + \|R_h(u)\|_{1^*} \leq 2\varepsilon \end{aligned}$$

and the assertion follows by the arbitrariness of ε . \square

6. PROOF OF THE MAIN RESULT

Let $x_0 \in \Omega$ and let

$$e_\varrho = n^{n-1} \varrho^{1-n} \chi_{B_\varrho(x_0)}.$$

Then it is well known (see [4]) that $e_\varrho \in BV(\mathbb{R}^n)$ and

$$(6.1) \quad |De_\varrho|(\mathbb{R}^n) = \int_{\mathbb{R}^n} |e_\varrho|^{1^*} dx = S^n,$$

$$(6.2) \quad \int_{\mathbb{R}^n} |u_\varrho| dx = n^{n-1} \mathcal{L}^n(B_1(0)) \varrho.$$

Let $\lambda \geq \lambda_1$, let $m \geq 1$ be such that $\lambda_m \leq \lambda < \lambda_{m+1}$ and let $\lambda < \mu < \lambda_{m+1}$. Let X_-, X_+ be as in Theorem 4.3. Let also

$$v_\varrho = \chi_{\mathbb{R}^n \setminus B_{2\varrho}(x_0)} v, \quad \forall v \in X_-;$$

$$X_-^\varrho = \{v_\varrho : v \in X_-\}.$$

Lemma 6.1. *There exist $C, \bar{\varrho} > 0$ such that $\overline{B_{2\bar{\varrho}}(x_0)} \subseteq \Omega$ and*

$$(6.3) \quad f_0(v_\varrho) \leq f_0(v) + C\varrho^{n-1} \left(\int_\Omega |v|^{1^*} dx \right)^{1/1^*},$$

$$(6.4) \quad \int_\Omega |v_\varrho|^{1^*} dx \geq \int_\Omega |v|^{1^*} dx - C\varrho^n \int_\Omega |v|^{1^*} dx,$$

$$(6.5) \quad \int_\Omega |v_\varrho| dx \geq \int_\Omega |v| dx - C\varrho^n \left(\int_\Omega |v|^{1^*} dx \right)^{1/1^*},$$

$$(6.6) \quad e_\varrho \notin X_-^\varrho \text{ and } X_-^\varrho \text{ is closed in } L^1(\Omega),$$

$$(6.7) \quad X_-^\varrho \cap X_+ = \{0\}, \quad \text{Index}(X_-^\varrho \setminus \{0\}) = \text{Index}(L^{1^*}(\Omega) \setminus X_+) = m,$$

for every $v \in X_-$ and $\varrho \in]0, \bar{\varrho}]$.

Proof. Let first $\bar{\varrho} > 0$ be such that $\overline{B_{2\bar{\varrho}}(x_0)} \subseteq \Omega$ and let $0 < \varrho \leq \bar{\varrho}$. According to [2] and Theorem 4.3, we have

$$f_0(v_\varrho) \leq f_0(v) + \|v\|_\infty |D\chi_{B_{2\varrho}(x_0)}|(\Omega) \leq f_0(v) + C \varrho^{n-1} \|v\|_{1^*},$$

whence (6.3). The proof of (6.4) and (6.5) is similar and even simpler.

It is clear that $e_\varrho \notin X_-^\varrho$. From (6.3), (6.5) and Theorem 4.3 it also follows that

$$f_0(v_\varrho) \leq \frac{1}{2} (\lambda_m + \mu) \int_\Omega |v_\varrho| dx, \quad \forall v \in X_-,$$

provided that ϱ is small enough. Therefore $X_-^\varrho \cap X_+ = \{0\}$. Moreover, for every $v \in X_-$ we have

$$\begin{aligned} \int_\Omega |v| dx &\leq \mathcal{L}^n(B_{2\varrho}(x_0))^{\frac{1}{n}} \left(\int_\Omega |v|^{1^*} dx \right)^{\frac{1}{1^*}} + \int_{\Omega \setminus B_{2\varrho}(x_0)} |v| dx \\ &\leq S^{-1} \mathcal{L}^n(B_{2\varrho}(x_0))^{\frac{1}{n}} f_0(v) + \int_{\Omega \setminus B_{2\varrho}(x_0)} |v| dx \\ &\leq S^{-1} \lambda_m \mathcal{L}^n(B_{2\varrho}(x_0))^{\frac{1}{n}} \int_\Omega |v| dx + \int_{\Omega \setminus B_{2\varrho}(x_0)} |v| dx. \end{aligned}$$

If ϱ is small enough, we get

$$\int_\Omega |v| dx \leq C \int_{\Omega \setminus B_{2\varrho}(x_0)} |v| dx \quad \text{for every } v \in X_-.$$

First of all, it follows that we have $v_\varrho = 0$ only for $v = 0$. Since $\{v \mapsto v_\varrho\}$ is continuous and odd with respect to the topology of $L^{1^*}(\Omega)$ from $X_- \setminus \{0\}$ to $X_-^\varrho \setminus \{0\}$, we get

$$\text{Index}(X_-^\varrho \setminus \{0\}) \geq \text{Index}(X_- \setminus \{0\}) = \text{Index}(L^{1^*}(\Omega) \setminus X_+) = m.$$

Actually, equality holds, as $X_-^\varrho \setminus \{0\} \subseteq L^{1^*}(\Omega) \setminus X_+$. Finally, let $(v^{(k)})$ be a sequence in X_- with $(v_\varrho^{(k)})$ convergent to some u in $L^1(\Omega)$. Then $(v^{(k)})$ is bounded in $L^1(\Omega \setminus B_{2\varrho}(x_0))$, hence in $L^1(\Omega)$, hence in $BV(\Omega)$. Up to a subsequence, $(v^{(k)})$ is $L^1(\Omega)$ -convergent to some element of X_- , whence $u \in X_-^\varrho$. Therefore, X_-^ϱ is closed in $L^1(\Omega)$. \square

Lemma 6.2. *There exist $\bar{\varrho}, \delta > 0$ such that*

$$(6.8) \quad \sup \left\{ f(te_\varrho + u) : t \geq 0, u \in X_-^\varrho \right\} \leq \frac{1}{n} S^n (1 - \delta \varrho)^n, \quad \forall \varrho \in]0, \bar{\varrho}].$$

Proof. Let $\bar{\varrho} > 0$ be first such that the assertion of Lemma 6.1 holds and let $0 < \varrho \leq \bar{\varrho}$. Since X_-^ϱ is a cone, it is easily seen that

$$\begin{aligned} & \sup \left\{ f(te_\varrho + u) : t \geq 0, u \in X_-^\varrho \right\} \\ &= \frac{1}{n} \left[\sup \left\{ \frac{f_0(e_\varrho + u) - \lambda \|e_\varrho + u\|_1}{\|e_\varrho + u\|_{1^*}} : u \in X_-^\varrho \right\} \right]^n \\ &= \frac{1}{n} \left[\sup \left\{ \frac{(f_0(e_\varrho) - \lambda \|e_\varrho\|_1) + (f_0(u) - \lambda \|u\|_1)}{(\|e_\varrho\|_{1^*}^{1^*} + \|u\|_{1^*}^{1^*})^{1/1^*}} : u \in X_-^\varrho \right\} \right]^n, \end{aligned}$$

as e_ϱ and u have disjoint supports. Writing $u = v_\varrho$ with $v \in X_-$, the assertion we need to prove takes the form

$$\sup \left\{ \frac{(f_0(e_\varrho) - \lambda \|e_\varrho\|_1) + (f_0(v_\varrho) - \lambda \|v_\varrho\|_1)}{(\|e_\varrho\|_{1^*}^{1^*} + \|v_\varrho\|_{1^*}^{1^*})^{1/1^*}} : v \in X_- \right\} \leq S(1 - \delta\varrho).$$

If we set $\sigma = n^{n-1} \mathcal{L}^n(B_1(0))$, by (6.1), (6.2), Lemma 6.1 and the fact that $\lambda_m \leq \lambda$, we have

$$\frac{(f_0(e_\varrho) - \lambda \|e_\varrho\|_1) + (f_0(v_\varrho) - \lambda \|v_\varrho\|_1)}{(\|e_\varrho\|_{1^*}^{1^*} + \|v_\varrho\|_{1^*}^{1^*})^{1/1^*}} \leq \frac{(S^n - \sigma\varrho) + (C\varrho^{n-1}\|v\|_{1^*} + \lambda C\varrho^n\|v\|_{1^*})}{(S^n + \|v\|_{1^*}^{1^*} - C\varrho^n\|v\|_{1^*}^{1^*})^{1/1^*}}.$$

Now, arguing by contradiction, let $\delta = 1/k$, let $\varrho_k \rightarrow 0^+$ and let $v^{(k)} \in X_-$ be such that

$$\frac{(f_0(e_{\varrho_k}) - \lambda \|e_{\varrho_k}\|_1) + (f_0(v_{\varrho_k}^{(k)}) - \lambda \|v_{\varrho_k}^{(k)}\|_1)}{(\|e_{\varrho_k}\|_{1^*}^{1^*} + \|v_{\varrho_k}^{(k)}\|_{1^*}^{1^*})^{1/1^*}} > S \left(1 - \frac{\varrho_k}{k}\right).$$

It follows

$$\frac{(S^n - \sigma\varrho_k) + (C\varrho_k^{n-1}\|v_k\|_{1^*} + \lambda C\varrho_k^n\|v_k\|_{1^*})}{(S^n + \|v_k\|_{1^*}^{1^*} - C\varrho_k^n\|v_k\|_{1^*}^{1^*})^{1/1^*}} > S \left(1 - \frac{\varrho_k}{k}\right).$$

Up to subsequences, it is enough to consider the three cases:

- (i) $\|v_k\|_{1^*} \rightarrow +\infty$,
- (ii) $\|v_k\|_{1^*} \rightarrow \ell \in]0, +\infty[$,
- (iii) $\|v_k\|_{1^*} \rightarrow 0$.

In case (i) we get

$$\frac{(S^n - \sigma\varrho_k) + (C\varrho_k^{n-1}\|v_k\|_{1^*} + \lambda C\varrho_k^n\|v_k\|_{1^*})}{(S^n + \|v_k\|_{1^*}^{1^*} - C\varrho_k^n\|v_k\|_{1^*}^{1^*})^{1/1^*}} \rightarrow 0$$

while in case (ii) we obtain

$$\frac{(S^n - \sigma\varrho_k) + (C\varrho_k^{n-1}\|v_k\|_{1^*} + \lambda C\varrho_k^n\|v_k\|_{1^*})}{(S^n + \|v_k\|_{1^*}^{1^*} - C\varrho_k^n\|v_k\|_{1^*}^{1^*})^{1/1^*}} \rightarrow \frac{S^n}{(S^n + \ell^{1^*})^{1/1^*}} < S.$$

In both cases, a contradiction follows. In case (iii) we have, eventually as $k \rightarrow \infty$,

$$\begin{aligned} & \frac{(S^n - \sigma \varrho_k) + (C \varrho_k^{n-1} \|v_k\|_{1^*} + \lambda C \varrho_k^n \|v_k\|_{1^*})}{(S^n + \|v_k\|_{1^*}^{1^*} - C \varrho_k^n \|v_k\|_{1^*}^{1^*})^{1/1^*}} \\ & \leq \frac{(S^n - \sigma \varrho_k) + (C \varrho_k^{n-1} \|v_k\|_{1^*} + \lambda C \varrho_k^n \|v_k\|_{1^*})}{S^{n-1}} \\ & = S - S^{1-n} \varrho_k (\sigma - C \varrho_k^{n-2} \|v_k\|_{1^*} - \lambda C \varrho_k^{n-1} \|v_k\|_{1^*}). \end{aligned}$$

Then a contradiction follows also in this case. \square

Proof of Theorem 1.1. Let $\lambda \geq \lambda_1$, let $m \geq 1$ be such that $\lambda_m \leq \lambda < \lambda_{m+1}$ and let $\lambda < \mu < \lambda_{m+1}$. Let X_-, X_+ be as in Theorem 4.3 and let $\bar{\varrho} > 0$ be small enough to guarantee that the assertions of Lemmata 6.1 and 6.2 hold.

Since $\lambda < \mu$, for every $u \in X_+$ we have

$$f(u) \geq \left(1 - \frac{\lambda}{\mu}\right) S \|u\|_{1^*} - \frac{1}{1^*} \|u\|_{1^*}^{1^*}.$$

Therefore, there exist $r_+, \alpha > 0$ such that $f(u) \geq \alpha$ for every $u \in X_+$ with $\|u\|_{1^*} = r_+$. On the other hand, since $\lambda \geq \lambda_m$, by Lemma 6.1 we also have, for every $v \in X_-$,

$$f(v_\varrho) \leq C \varrho^{n-1} \|v\|_{1^*} + \lambda C \varrho^n \|v\|_{1^*} - \frac{1}{1^*} \|v\|_{1^*}^{1^*} + \frac{C}{1^*} \varrho^n \|v\|_{1^*}^{1^*} \leq \frac{\alpha}{2} - \frac{1}{2 \cdot 1^*} \|v\|_{1^*}^{1^*},$$

provided that $\varrho > 0$ is small enough. Combining this fact with Lemmata 6.1 and 6.2, we see that there exists $\varrho > 0$ such that $e_\varrho \notin X_-^e$, X_-^e is closed in $L^1(\Omega)$ and

$$X_-^e \cap X_+ = \{0\}, \quad \text{Index}(X_-^e \setminus \{0\}) = \text{Index}(L^{1^*}(\Omega) \setminus X_+) = m,$$

$$\sup \left\{ f(te_\varrho + u) : t \geq 0, u \in X_-^e \right\} < \frac{1}{n} S^n,$$

$$\sup \left\{ f(u) : u \in X_-^e \right\} \leq \frac{\alpha}{2}.$$

Since X_-^e is closed in $L^1(\Omega)$, hence in $L^{1^*}(\Omega)$, there exists $b > 0$ such that

$$\|te_\varrho\|_{1^*} + \|u\|_{1^*} \leq b \|te_\varrho + u\|_{1^*} \quad \text{for every } t \in \mathbb{R} \text{ and } u \in X_-^e$$

(see also [12]). Consequently, there exists $b' > 0$ such that

$$f_0(u) \leq b' \|u\|_{1^*} \quad \text{for every } u \in \mathbb{R}e_\varrho + X_-^e,$$

whence

$$f(u) \rightarrow -\infty \quad \text{whenever } \|u\|_{1^*} \rightarrow \infty \text{ with } u \in \mathbb{R}e_\varrho + X_-^e.$$

In particular, there exists $r_- > r_+$ such that $f(u) \leq 0$ whenever $u \in \mathbb{R}e_\varrho + X_-^e$ with $\|u\|_{1^*} = r_-$.

From Theorems 2.4 and Theorem 5.3 we deduce that f admits a critical value c with $0 < c < \frac{1}{n} S^n$. By Corollary 3.5, there exists a solution $u \in BV(\Omega) \cap L^\infty(\Omega)$ of (1.4) with

$$0 < f(u) < \frac{1}{n} S^n.$$

Of course, u is a nontrivial solution. □

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