Minimal disc-type surfaces embedded in a perturbed cylinder

Mouhamed Moustapha Fall[a] and Carlo Mercuri[b]

February 27, 2009

[a,b]Sissa, Via Beirut 2-4, Trieste, 34014 Italy

Abstract

In the present note we deal with small perturbations of an infinite cylinder in the 3D euclidian space. We find minimal disc-type surfaces embedded in the cylinder and intersecting its boundary perpendicularly. The existence and localization of those minimal discs is a consequence of a non-degeneracy condition for the critical points of a functional related to the oscillations of the cylinder from the flat configuration. *Key words: minimal surfaces, free boundary problems, perturbation methods, nonlinear PDE's. AMS Subject Classifications 53A10, 53C21, 35R35.*

Contents

1 Introduction

We define a perturbed cylindrical surface C_ε parallel to the z-axis parametrized by:

$$C_\varepsilon(\theta, z) := (1 + \varepsilon h(1, \theta, z))(\cos\theta, \sin\theta, 0) + z e_3, \quad \varepsilon > 0$$

[1]E-mail addresses: fall@sissa.it (M.M. Fall), mercuri@sissa.it (C. Mercuri).

where $h = h(r, \theta, z)$ is a smooth real-valued function expressed in the cylindrical coordinates $(r, \theta, z) \in (0, \infty) \times (0, 2\pi) \times \mathbb{R}$ and e_3 is the unit vector parallel to the z-direction. It is woth noticing that if $\varepsilon > 0$ is small, then C_ε is a regular surface of $\mathbb{R}^3$. Hereafter we define $h(\theta, z) := h(1, \theta, z)$ and we denote by Ω_ε the interior of C_ε. We recall that a minimal surface is a surface with mean curvature, $\mathcal{H}$, vanishing everywhere. In particular we are interested in finding disc-type minimal surfaces. We will parametrize the z-level disc-type surface D^z_ε by:

$$D^z_\varepsilon(r, \theta) := r(1 + \varepsilon h(r, \theta, z))(\cos\theta, \sin\theta, 0) + z e_3, \quad r \in (0, 1), \theta \in (0, 2\pi).$$

The boundary is obviously given by the previous equation considering $r = 1$. The aim of the present note is to deform D^z_ε for some z in order to find a surface $\mathcal{D}$ such that, for ε small, $\mathcal{H}(\mathcal{D}) = 0$ and whose boundary intersects C_ε perpendicularly. Fixed z and moving along the e_3-direction, we can define a smooth deformation of D^z_ε, denoted by $\mathcal{D}^z_\varepsilon(t)$, simply by

$$F^z_\varepsilon(r, \theta, t) := r(1 + \varepsilon h(r, \theta, z + t))(\cos\theta, \sin\theta, 0) + (z + t)e_3, \tag{1}$$

defined for $(r, \theta, t) \in (0, 1] \times (0, 2\pi) \times (-t_0, t_0)$. Since

$$F^z_\varepsilon(1, \theta, t) = C_\varepsilon(\theta, z + t),$$

for any fixed t, $F^z_\varepsilon(r, \theta, t)$ parameterizes a disc whose boundary lies on C_ε, therefore $\mathcal{D}^z_\varepsilon(t)$ is an *admissible deformation* of D^z_ε.
Let $\mathcal{C}^{k,\alpha}$ be the standard Hölder Spaces $\mathcal{C}^{k,\alpha}(\bar{B})$. We now state our main result:

Theorem 1.1 *Suppose z_0 is a non-degenerate critical point of the functional*

$$\Gamma(z) := \oint h(\theta, z) d\theta. \tag{2}$$

Then, an $\varepsilon_0 > 0$ exists such that for any $\varepsilon \in (0, \varepsilon_0)$ there exist $\delta_\varepsilon > 0$, a smooth function $\omega_\varepsilon : B \to \mathbb{R}$ on the unit ball $B \subset \mathbb{R}^2$ and z_ε, satisfying $\|\omega_\varepsilon\|_{\mathcal{C}^{2,\alpha}} < \delta_\varepsilon$ and $|z_0 - z_\varepsilon| < \delta_\varepsilon$, such that the set $D^{z_0}_\varepsilon$ can be smoothly deformed, through (1), to a disc-type minimal surface $\mathcal{D}^{z_\varepsilon}_\varepsilon(\omega_\varepsilon(B)) \subset \Omega_\varepsilon$ intersecting C_ε perpendicularly.

We have to point out that the in [18] (Section 3) the existence of closed geodesics for C_ε had been studied with a variant of the Ljapunov-Schmidt reduction scheme. Even in this case the functional Γ defined in (2) determines the location.
A brief comment on the method is in order. Observe that

$$\mathcal{H}(D^z_\varepsilon) = \mathcal{O}(\varepsilon), \quad \text{in } D^z_\varepsilon \tag{3}$$

and

$$\langle N_{D^z_\varepsilon}, N_{C_\varepsilon} \rangle = \mathcal{O}(\varepsilon), \quad \text{on } \partial D^z_\varepsilon. \tag{4}$$

where $N_{D_\varepsilon^z}$ and N_{C_ε} stand for the unit outer normals respectively to D_ε^z and C_ε. Hereafter we use the symbol $\mathcal{O}(\varepsilon^\alpha)$ instead of the Landau symbol $O(\varepsilon^\alpha)$ to indicate that it is also a function defined on B, possibly depending on z (see the notation below). On the other hand we search for some D satisfying

$$\mathcal{H}(D) = 0, \quad \text{in } D \tag{5}$$

and

$$\langle N_D, N_{C_\varepsilon} \rangle = 0, \quad \text{on } \partial D, \tag{6}$$

which correspond to the Euler-Lagrange equation associated to the relative area-functional, let us call it $\mathcal{E} : D \to \text{Area}(D \cap \Omega_\varepsilon)$, restricted to the class of *admissible (and orientable, smooth) surfaces* $\mathcal{S}$, i.e. those $\mathcal{S}$ such that $\mathcal{S} \subset \Omega_\varepsilon$ and $\partial\mathcal{S} \subset C_\varepsilon$.

In order to solve (5) and (6) we localize the problem considering $z \in [a, b]$, and assuming z_0 is a non-degenerate critical point of $\oint h(\theta, z)d\theta$ and $z_0 \in (a, b)$. Because of (3) and (4) the set $Z := \{D_\varepsilon^z \,:\, z \in [a, b]\}$ provides a class of approximate solutions. Consider the map F defined in (1). Because of (1), any admissible set $\mathcal{D}_\varepsilon^z$ nearby D_ε^z can be parametrized by a function $\omega : B \to \mathbb{R}$ such that $\mathcal{D}_\varepsilon^z(\omega) = F_\varepsilon^z(r, \theta, \omega)$. Define $\mathcal{H}(z, \varepsilon, \omega) := \mathcal{H}(\mathcal{D}_\varepsilon^z(\omega))$ and $\mathcal{B}(z, \varepsilon, \omega) := \langle N_{\mathcal{D}_\varepsilon^z(\omega)}, N_{C_\varepsilon}\rangle$. The idea in solving (5)-(6) is then to determine ω such that

$$\mathcal{H}(z, \varepsilon, \omega) = 0 \quad \text{in } \mathcal{D}_\varepsilon^z(\omega)$$

$$\mathcal{B}(z, \varepsilon, \omega) = 0 \quad \text{on } \partial\mathcal{D}_\varepsilon^z(\omega).$$

Denoting $L, \bar{L}$ and $Q, \bar{Q}$ respectively, terms linear and quadratic in ω and its derivatives (see the notation below), the previous problem transforms to

$$\begin{cases} \Delta\omega = \varepsilon L(\omega) + Q(\omega) + \mathcal{O}(\varepsilon^2), & \text{on } B \\ \frac{\partial\omega}{\partial\eta} = \varepsilon h_z(\theta, z) + \varepsilon\bar{L}(\omega) + \bar{Q}(\omega) + \mathcal{O}(\varepsilon^2), & \text{on } \partial B. \end{cases} \tag{7}$$

The problem (7) can be formulated in a weak sense, expressed as

$$(\Phi_{\varepsilon,z}(\omega), \omega') = 0, \quad \forall\omega' \in H^1(B),$$

and splits in a system of two equations, namely the auxiliary+bifurcation equations system (see for example [1]). Indeed we can project $\Phi_{\varepsilon,z}(\omega)$ to the space of average-zero functions and its orthogonal: $\langle 1\rangle \oplus \langle 1\rangle^\perp$. Because of the Poincaré -Wirtinger inequality $\mathcal{L}_{z,\varepsilon}$ defined by $(\mathcal{L}_{z,\varepsilon}\omega, \omega') := \int_B \nabla\omega\nabla\omega'$ is invertible in $\langle 1\rangle^\perp$. The auxiliary equation is then solved by a fixed point argument. Finally, the bifurcation equation reads as a simple equation in $\mathbb{R}$ and can be solved in a small neighborhood of z_0, using an elementary argument based on the Contraction Lemma.
A great deal of work has been devoted on minimal surfaces from the point of view of existence, uniqueness and topological properties of the solutions. We refer to the papers [2], [7], [8], [9], [11], [15], [16], [17], [19], [20] and the references therein.

The method we used is perturbative in nature and the main idea goes back to Ye [21] and it was employed by many authors in the study of constant mean curvature hyper-surfaces, see for example [4], [5], [6], [10], [12], [13] and [14].

Acknowledgments
The authors would like to thank Prof. Antonio Ambrosetti for taking their attention to these questions. They are grateful to Prof. Antonio Ambrosetti and Prof. Andrea Malchiodi at SISSA/ISAS for the helpful discussions. They have been supported by M.U.R.S.T within the PRIN 2008 Variational Methods and Nonlinear Differential Equations.

Notation for error terms: Any expression of the form $L(\omega)$ (resp. $\bar{L}(\omega)$) denotes a linear combination of the function ω together with its derivatives with respect to the function ω up to order 2 (resp. order 1). The coefficients of L or $\bar{L}$ might depend on ε and z but, for all $k \in \mathbb{N}$, there exists a constant $c > 0$ independent of $\varepsilon \in (0,1)$ and $z \in [a,b]$ such that

$$\|L(\omega)\|_{\mathcal{C}^{k,\alpha}} \leq c\,\|\omega\|_{\mathcal{C}^{k+2,\alpha}},$$

and similarly for $\bar{L}$. Furthermore, any expression of the form $Q(\omega)$ (resp $\bar{Q}(\omega)$) denotes a nonlinear operator in the function ω together with its derivatives with respect to ω up to order 2 (resp. 1).
Given $k \in \mathbb{N}$, there exists a constant $c > 0$ independent of $\varepsilon \in (0,1)$ and $z \in [a,b]$ such that

$$\|Q(\omega_1) - Q(\omega_2)\|_{\mathcal{C}^{k,\alpha}} \leq c\,(\|\omega_1\|_{\mathcal{C}^{k+2,\alpha}} + \|\omega_2\|_{\mathcal{C}^{k+2,\alpha}})\ \|\omega_1 - \omega_2\|_{\mathcal{C}^{k+2,\alpha}},$$

provided $\|\omega_i\|_{\mathcal{C}^{k+2,\alpha}} \leq 1$, $i = 1,2$. And similarly for $\bar{Q}$. We also agree that any term denoted by $\mathcal{O}(r^d)$ (with $r \in \mathbb{R}$ may depend on z) is a smooth function on B that might depend on z but satisfies

$$\|\frac{\mathcal{O}(r^d)}{|r|^d}\|_{\mathcal{C}^{k,\alpha}} \leq c$$

for a constant c independent of z.

2 Expansions of the geometrical quantities for $\mathcal{D}^z_\varepsilon(\omega)$

For sake of convenience we define hereafter

$$n := (\cos\theta, \sin\theta, 0)$$

and for every fixed z we use the following parametrization for $\mathcal{D}^z_\varepsilon(\omega)$

$$X(r,\theta) := r(1 + \varepsilon h(r,\theta, z + \omega(r,\theta)))n + (z + \omega(r,\theta))e_3.$$

We can expand the last equation in terms of ω, yielding

$$X = X\Big|_{\omega=0} + \frac{\partial X}{\partial \omega}\Big|_{\omega=0}\omega + \varepsilon Q(\omega).$$

Then we have

$$X(r,\theta) = D^z_\varepsilon(r,\theta) + (e_3 + r\varepsilon h_z(r,\theta,z)n)\omega + \varepsilon Q(\omega),$$

or, equivalently

$$X(r,\theta) = r(1+\varepsilon h(r,\theta,z))n + ze_3 + \omega(r,\theta)e_3 + \varepsilon L(\omega) + Q(\omega).$$

In order to compute the Mean Curvature of $\mathcal{D}^z_\varepsilon(\omega)$, some preliminary computations are in order. We will determine the meaningful geometrical quantities at a first order of approximation in ε. We refer, for example, to the book [3] for notations and terminology.

2.1 The first fundamental form

We compute the following quantities

$$X_r = (1+\varepsilon(h + rh_r))n + \omega_r e_3 + \varepsilon L(\omega) + Q(\omega) \tag{8}$$

$$E := |X_r|^2 = 1 + 2\varepsilon(h + rh_r) + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega) \tag{9}$$

$$X_\theta := r\varepsilon h_\theta n + r(1+\varepsilon h)n_\theta + \omega_\theta e_3 + \varepsilon L(\omega) + Q(\omega) \tag{10}$$

$$G := |X_\theta|^2 = r^2(1+2\varepsilon h) + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega) \tag{11}$$

$$F := \langle X_\theta, X_r\rangle = r\varepsilon h_\theta + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega). \tag{12}$$

Finally we use the First Fundamental Form, E, F, G in terms of the basis X_r, X_θ, to determine the following quantity for later use.

$$EG - F^2 = r^2(1+4\varepsilon h) + 2r^3\varepsilon h_r + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega). \tag{13}$$

2.2 The normal

We are going to expand the Gauss Map for $\mathcal{D}^z_\varepsilon(\omega)$:

$$N := \frac{X_r \wedge X_\theta}{|X_r \wedge X_\theta|} = \frac{X_r \wedge X_\theta}{(EG - F^2)^{1/2}}.$$

Using (8) and (10) we compute

$$\bar{N} := X_r \wedge X_\theta = r(1 + 2\varepsilon h)e_3 - \omega_\theta n_\theta + r^2 \varepsilon h_r e_3 - r\omega_r n + O(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega)$$

and

$$|\bar{N}|^{-1} = \frac{1}{(EG - F^2)^{1/2}} = \frac{1}{r}(1 - 2\varepsilon h) - \frac{r}{2}\varepsilon h_r + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega).$$

Therefore, we have

$$N = \frac{\bar{N}}{|\bar{N}|} = e_3 - \frac{1}{r}\omega_\theta n_\theta - \omega_r n + \frac{r}{2}\varepsilon h_r e_3 + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega). \tag{14}$$

Finally we can compute the derivatives N_r, N_θ :

$$N_r = (\frac{1}{r^2}\omega_\theta - \frac{1}{r}\omega_{\theta r})n_\theta - \omega_{rr} n + (\frac{\varepsilon}{2}h_r + \frac{r}{2}\varepsilon h_{rr})e_3 + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega) \tag{15}$$

$$N_\theta = -(\frac{1}{r}\omega_{\theta\theta} + \omega_r)n_\theta + (\frac{1}{r}\omega_\theta - \omega_{r\theta})n + \frac{r}{2}\varepsilon h_{r\theta} e_3 + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega). \tag{16}$$

2.3 The second fundamental form

As a preliminary step to determine the Mean Curvature for $\mathcal{D}^z_\varepsilon(\omega)$ we expand the Second Fundamental Form given by the following e, f, g quantities. Using (8), (10),(15) and (16) we can determine the Second Fundamental Form:

$$-g := \langle N_\theta, X_\theta \rangle = -r(\frac{1}{r}\omega_{\theta\theta} + \omega_r) + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega),$$

so

$$g = \omega_{\theta\theta} + r\omega_r + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega) \tag{17}$$

$$-e := \langle N_r, X_r \rangle = -\omega_{rr} + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega)$$

$$e = \omega_{rr} + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega) \tag{18}$$

$$-f := \langle N_\theta, X_r \rangle = \frac{1}{r}\omega_\theta - \omega_{r\theta} + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega). \tag{19}$$

2.4 The mean curvature

Using (18),(11), (17), (9), (19) and (12), we have

$$eG = r^2\omega_{rr} + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega)$$

$$gE = \frac{1}{r}\omega_{\theta\theta} + \omega_r + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega)$$

$$-fF = \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega).$$

By equation (13) one has that

$$(EG - F^2)^{-1} = r^{-2}(1 - 2\varepsilon h + r\varepsilon h_r) + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega).$$

We now estimate the Mean Curvature. Through the usual formula

$$\mathcal{H} = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}$$

we finally compute $\mathcal{H}(z, \varepsilon, \omega) := \mathcal{H}(\mathcal{D}^z_\varepsilon(\omega))$,

$$\mathcal{H}(z, \varepsilon, \omega) = \frac{1}{2}(\omega_{rr} + \frac{\omega_r}{r} + \frac{\omega_{\theta\theta}}{r^2}) + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega) \qquad (20)$$

3 Expansion of the orthogonality condition on the boundary

In what follows $h(\theta, z)$ stands for $h(1, \theta, z)$. In order to express the orthogonality condition we compute the normal to $C_\varepsilon(\theta, z)$ with parametrization:

$$Y(z, \theta) := (1 + \epsilon h(\theta, z))n + ze_3.$$

The basis of the tangent space at each point is given by

$$Y_z = \varepsilon h_z n + e_3$$

and

$$Y_\theta = \varepsilon h_\theta n + (1 + \varepsilon h)n_\theta.$$

This yields

$$\bar{N} := Y_z \wedge Y_\theta = \varepsilon h_z e_3 + \varepsilon h_\theta n_\theta - (1 + \varepsilon h)n + \mathcal{O}(\varepsilon^2),$$

hence

$$|\bar{N}| = 1 + \varepsilon h + \mathcal{O}(\varepsilon^2)$$

and finally

$$N(\theta, z) := \frac{\bar{N}}{|\bar{N}|} = \varepsilon h_z(\theta, z)e_3 + \varepsilon h_\theta(\theta, z)n_\theta - (1 + \varepsilon h(\theta, z))n + \mathcal{O}(\varepsilon^2).$$

This yields

$$N(\theta, z+\omega) := \frac{\bar{N}}{|\bar{N}|} = \varepsilon h_z(\theta, z+\omega)e_3 + \varepsilon h_\theta(\theta, z+\omega)n_\theta - (1+\varepsilon h(\theta, z+\omega))n + \mathcal{O}(\varepsilon^2).$$

From (14), we expand the normal of $\mathcal{D}^z_\varepsilon(\omega)$ in $r = 1$:

$$N_{\mathcal{D}^z_\varepsilon(\omega)} = e_3 - \omega_\theta n_\theta - \omega_r n + \frac{\varepsilon}{2} h_r e_3 + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega).$$

Therefore, we get

$$\langle N_{\mathcal{D}^z_\varepsilon(\omega)}, N(\theta, z+\omega)\rangle = \varepsilon h_z(z+\omega) + \omega_r + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega).$$

Finally, since

$$\varepsilon h_z(\theta, z+\omega) = \varepsilon h_z(\theta, z) + \varepsilon L(\omega) + Q(\omega)$$

the last equation becomes

$$\mathcal{B}(z, \varepsilon, \omega) := \omega_r + \varepsilon h_z + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega). \tag{21}$$

4 Proof of Theorem 1.1

Collecting equations (20) and (21) we get the following Nonlinear System of PDE's:

$$\begin{cases} -\Delta\omega = \varepsilon L(\omega) + Q(\omega) + \mathcal{O}(\varepsilon^2), & \text{on } B \\ \frac{\partial\omega}{\partial\eta} = \varepsilon h_z(\theta, z) + \varepsilon \bar{L}(\omega) + \bar{Q}(\omega) + \bar{\mathcal{O}}(\varepsilon^2), & \text{on } \partial B. \end{cases} \tag{22}$$

where $\frac{\partial\omega}{\partial\eta}$ stands for the unit outer normal derivative of ω and B is the unit ball in $\mathbb{R}^2$.

We shall introduce the following operators: for any $\omega, \omega' \in H^1(B)$, let

$$(\mathcal{L}(\omega), \omega') := \int_B \nabla\omega\nabla\omega',$$

for $\omega \in \mathcal{C}^{2,\alpha}$, let

$$(\mathcal{F}_{\varepsilon,z}(\omega), \omega') := \int_B (Q(\omega) + \varepsilon L(\omega) + \mathcal{O}(\varepsilon^2))\omega' +$$

$$\oint_{\partial B} (\varepsilon h_z + \varepsilon \bar{L}(\omega) + \bar{Q}(\omega) + \bar{\mathcal{O}}(\varepsilon^2))\omega', \quad \forall \omega' \in H^1(B).$$

With this notations, (22) is solved provided

$$\mathcal{L}(\omega) = \mathcal{F}_{\varepsilon,z}(\omega). \tag{23}$$

With an abuse of notation, by the Riesz Representation Theorem, we may assume that for any $\omega \in \mathcal{C}^{2,\alpha}$, $\mathcal{F}_{\varepsilon,z}(\omega) \in \mathcal{C}^{0,\alpha}$. Moreover it satisfies

$$(24)\qquad \|\mathcal{F}_{\varepsilon,z}(\omega)\|_{\mathcal{C}^{0,\alpha}} \leq c\varepsilon\left(1+\|\omega\|_{\mathcal{C}^{2,\alpha}}\right) + O\left(\|\omega\|_{\mathcal{C}^{2,\alpha}}\right)\,\|\omega\|_{\mathcal{C}^{2,\alpha}};$$

$$(25)\qquad \begin{aligned}\|\mathcal{F}_{\varepsilon,z}(\omega_1)-\mathcal{F}_{\varepsilon,z}(\omega_2)\|_{\mathcal{C}^{0,\alpha}} \leq c\varepsilon\|\omega_1-\omega_2\|_{\mathcal{C}^{2,\alpha}}+\\ O(\|\omega_1\|_{\mathcal{C}^{2,\alpha}}+\|\omega_2\|_{\mathcal{C}^{2,\alpha}})\|\omega_1-\omega_2\|_{\mathcal{C}^{2,\alpha}}.\end{aligned}$$

Now call

$$H := \{\omega \in H^1(B) \quad : \quad \int_B \omega = 0\}$$

and

$$\mathcal{P}f := f - \frac{1}{\pi}\int_B f, \qquad \forall f \in L^2.$$

From now on, we will assume that $\omega \in \mathcal{PC}^{2,\alpha}$. Therefore, we have to solve the system

$$(26)\qquad \begin{cases} \mathcal{L}(\omega) = \mathcal{P}\mathcal{F}_{\varepsilon,z}(\omega) & \text{auxiliary equation} \\ \\ (Id-\mathcal{P})\mathcal{F}_{\varepsilon,z}(\omega) = 0. & \text{bifurcation equation} \end{cases}$$

With a slight abuse of terminology we refer to the system (26) as the system of auxiliary+bifurcation equations.
Notice that the auxiliary equation is equivalent to the following fixed point problem

$$(27)\qquad \omega = \mathcal{L}^{-1}\mathcal{P}\mathcal{F}_{\varepsilon,z}(\omega), \quad \text{in } \mathcal{PC}^{2,\alpha}.$$

By Poincaré-Wirtinger inequality, $\mathcal{L}$ is coercive in H. Furthermore by elliptic regularity theory, $\mathcal{L} : \mathcal{PC}^{2,\alpha} \to \mathcal{PC}^{0,\alpha}$ is an isomorphism. Hence by (24) and (25), we have that (27) can be readily solved in a small ball in $\mathcal{PC}^{2,\alpha}$. More precisely, there exists a positive constant c such that for any z and $\varepsilon > 0$ small, there exists a unique $\omega_{\varepsilon,z}$ with $\|\omega_{\varepsilon,z}\|_{\mathcal{C}^{2,\alpha}(\bar{B})} \leq c\varepsilon$ solving (27). Moreover reducing ε if necessary, we may assume that $\mathcal{D}^z_\varepsilon(\omega_{\varepsilon,z})$ is embedded in Ω_ε.
By construction we have also that $\omega_{\varepsilon,z}$ smoothly depends on z, furthermore, for $k \geq 0$ we have

$$(28)\qquad \|\frac{\partial^k\omega_{\varepsilon,z}}{\partial z^k}\|_{\mathcal{C}^{2,\alpha}} \leq c_k\varepsilon, \quad z \in [a,b]$$

where c_k is independent on z and on ε small. The bifurcation equation becomes $(Id-\mathcal{P})\,\mathcal{F}_{\varepsilon,z}(\omega_{\varepsilon,z}) = 0$ which is equivalent to

$$\int_B (Q(\omega_{\varepsilon,z}) + \varepsilon L(\omega_{\varepsilon,z}) + \mathcal{O}(\varepsilon^2)) + \oint_{\partial B}(\varepsilon h_z + \varepsilon\bar{L}(\omega_{\varepsilon,z}) + \bar{Q}(\omega_{\varepsilon,z}) + \bar{\mathcal{O}}(\varepsilon^2)) = 0.$$

Namely

$$\oint_{\partial B} h_z(\theta,z)d\theta = g_\varepsilon(z),$$

where g_ε is a smooth function in z such that, because of (28), $\|g_\varepsilon\|_{C^k[a,b]} \leq c'_k\varepsilon$. Define $f_\varepsilon(z) := \oint h_z(\theta,z)d\theta - g_\varepsilon(z)$. If z_0 is a non-degenerate critical point (strict maxima or minima) for $\oint h(\theta,z)d\theta$, then $f'_\varepsilon(z_0)$ is invertible if ε is sufficiently small. Consequently as above, the bifurcation equation is equivalent to the fixed point problem

$$z - z_0 = [f'_\varepsilon(z_0)]^{-1}(f_\varepsilon(z_0) + O(|z-z_0|^2))$$

which has a unique solution z_ε such that $|z_\varepsilon - z_0| < c\varepsilon$. This concludes the proof.

References

[1] Ambrosetti A. and Prodi G., *A Primer of Nonlinear Analysis* , Cambridge Univ. Press, Cambridge Studies in Advanced Mathematics, No.34 (1993).

[2] Bürger W. and Kuwert E., Area-minimizing disks with free boundary and prescribed enclosed volume. Preprint 2005 (to appear in J. Reine Angew. Math.).

[3] Do Carmo M.P., *Differential Geometry of curves and surfaces*, Prentice Hall, Inc. (Englewood Cliffs, New Jersey) 1976.

[4] Fall M.M., Embedded disc-type surfaces with large constant mean curvature and free boundaries, preprint SISSA 2007 (to appear in Com. Cont. Math).

[5] Fall M.M. and Mahmoudi F., Hypersurfaces with free boundary and large constant mean curvature: concentration along submanifolds, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) Vol. VII (2008).

[6] Fall M.M. and Mercuri C., Foliations of small tubes in Riemannian manifolds by capillary minimal discs, preprint SISSA 2008 (to appear in Nonlinear Anal.)

[7] Finn R., *Equilibrium Capillary Surfaces*, Springer-Verlag, New York, 1986

[8] Grüter M. and Jost J., On embedded minimal discs in convex bodies, Annales de l'institut Henri Poincaré (C) Analyse non linéaire, 3, no. 5, (1986), 345-390.

[9] Jost J., Existence results for embedded minimal surfaces of controlled topological type I, Ann. Sci. Sup. Pisa. Classe di Scienze Sr. 4, 13 no. 1 (1986), p. 15-50.

[10] Kapouleas N., Compact constant mean curvature surfaces in Euclidean three-space. J. Differ. Geom. 33, No.3, (1991), 683715.

[11] Lawson H.B., *Lectures on minimal submanifolds*, Vol.I. Second edition. Mathematics Lecture Series, 9. Pulish or Perish, Wimington, Del., 1980.

[12] Mahmoudi, F. and Mazzeo, R., Pacard, F., constant mean curvature hypersurfaces condensing along a submanifold, Geom. funct. anal. Vol. 16 (2006) 924-958.

[13] Mazzeo, R. and Pacard, F., Foliations by constant mean curvature tubes, Comm. Anal. Geom. 13 (2005), no. 4, 633-670.

[14] Pacard F. and Xu X., Constant mean curvature spheres in Riemannian manifolds, preprint (2005) (to appear in Manuscripta Math.).

[15] Ros. A, The isoperimetric problem, Lecture series given during the Caley Mathematics Institute Summer School on the Global Theory of Minimal Surfaces at the MSRI, Berkley, California (2001).

[16] Ros A. and Souam R., On stability of capillary surfaces in a ball, Pacific J. Math. 178 (1997) 345- 361.

[17] Ros A. and Vergasta E., Satability for hypersurfaces of constant mean curvature with free boundary, Geom. Dedicata 56 (1995), no. 1, 19-33.

[18] Secchi S., A note on closed geodesics for a class of non-compact Riemannian manifolds. Adv. Nonlinear Stud. 1 (2001), no. 1, 132-142.

[19] Struwe M., On a free boundary problem for minimal surfaces, Inv. Math. 75 (1984), 547-560.

[20] Struwe M., The existence of surfaces of constant mean curvature with free boundaries, Acta Math. 160 (1988), no. 1-2, 19-64.

[21] Ye R., Foliation by constant mean curvature spheres, Pacific J. Math. 147 (1991), no. 2, 381–396.