Concentration of solutions for some singularly perturbed mixed problems. Part I: existence results

Jesus GARCIA AZORERO[a] Andrea MALCHIODI[b] Luigi MONTORO[c]

Ireneo PERAL[a]

[a]Departamento de Matemáticas, UAM, 28049 Madrid, Spain
[b]SISSA, Via Beirut 2-4, 34014 Trieste, Italy
[c]Dipartimento di Matematica, UNICAL, Ponte Pietro Bucci 31 B, 87036 Arcavacata di Rende, Cosenza, Italy

ABSTRACT. In this paper we study the asymptotic behavior of some solutions to a singularly perturbed problem with mixed Dirichlet and Neumann boundary conditions. We prove that, under suitable geometric conditions on the boundary of the domain, there exist solutions which approach the intersection of the Neumann and the Dirichlet parts as the singular perturbation parameter tends to zero.

Key Words: Singularly Perturbed Elliptic Problems, Finite-dimensional reductions, Local Inversion.

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1 Introduction

We are interested here in the following mixed problem

$$(\tilde{M}_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial_{\mathcal{N}}\Omega; & u = 0 \text{ on } \partial_{\mathcal{D}}\Omega; \\ u > 0 & \text{in } \Omega, \end{cases}$$

where Ω is a smooth bounded subset of $\mathbb{R}^n$, $p \in \left(1, \frac{n+2}{n-2}\right)$, $\varepsilon > 0$ is a small parameter, and $\partial_{\mathcal{N}}\Omega$, $\partial_{\mathcal{D}}\Omega$ are two subsets of the boundary of Ω such that the union of their closures coincides with the whole $\partial\Omega$.

These stationary mixed boundary problems appear in several situations. For instance:

- *Population dynamics.* Assume that a species lives in a bounded region Ω such that the boundary has two parts, $\partial_{\mathcal{N}}\Omega$ and $\partial_{\mathcal{D}}\Omega$, the first one is an obstacle that blocks the pass across, while the second one is killing zone for the population.
- *Nonlinear heat conduction.* In this case $(\tilde{M}_\varepsilon)$ models the heat (for small conductivity) in the presence of a nonlinear source in the interior of the domain, with combined isothermal and isolated regions at the boundary.

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[1] E-mail addresses: jesus.azorero@uam.es, malchiod@sissa.it, montoro@mat.unical.it, ireneo.peral@uam.es

- *Reaction diffusion with semi-permeable boundary.* In this framework we have that the meaning of the Neumann part, $\partial_{\mathcal{N}}\Omega$, is an obstacle to the flux of the matter, while the Dirichlet part, $\partial_{\mathcal{D}}\Omega$, stands for a semipermeable region that allows the outwards transit of the matter produced in the interior of the cell Ω by the reaction $f(u)$.

Partial differential equations as the one in $(\tilde{M}_\varepsilon)$ also appear in the study of reaction-diffusion systems. For single equations with Neumann boundary conditions it known that when Ω is convex the only stable solutions are constants, see [11], [35]. On the other hand, as noticed in [43], reaction-diffusion *systems* with different diffusivities might lead to non-homogeneous stable steady states. A well-know example is the following one (Gierer-Meinhardt)

$$(GM) \qquad \begin{cases} \mathcal{U}_t = d_1 \Delta \mathcal{U} - \mathcal{U} + \frac{\mathcal{U}^p}{\mathcal{V}^q} & \text{in } \Omega \times (0, +\infty), \\ \mathcal{V}_t = d_2 \Delta \mathcal{V} - \mathcal{V} + \frac{\mathcal{U}^r}{\mathcal{V}^s} & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial \mathcal{U}}{\partial \nu} = \frac{\partial \mathcal{V}}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, +\infty), \end{cases}$$

introduced in [19] to describe some biological experiment. The functions $\mathcal{U}$ and $\mathcal{V}$ represent the densities of some chemical substances, the numbers p, q, r, s are non-negative and such that $0 < \frac{p-1}{q} < \frac{r}{s+1}$, and it is assumed that the diffusivities d_1 and d_2 satisfy $d_1 \ll 1 \ll d_2$. In the stationary case of (GM), as explained in [37], when $d_2 \to +\infty$ the function $\mathcal{V}$ is close to a constant (being nearly harmonic and with zero normal derivative at the boundary), and therefore the equation satisfied by $\mathcal{U}$ resembles the first one in $(\tilde{M}_\varepsilon)$. Clearly a similar reduction procedure could be used when mixed boundary conditions are imposed.

Let us now describe some results which concern singularly perturbed problems with Neumann or Dirichlet boundary conditions, and specifically

$$(N_\varepsilon) \quad \begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial_{\mathcal{N}}\Omega; \\ u > 0 & \text{in } \Omega, \end{cases} \qquad (D_\varepsilon) \quad \begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega; \\ u = 0 & \text{on } \partial_{\mathcal{D}}\Omega; \\ u > 0 & \text{in } \Omega. \end{cases}$$

These problems arise also as limits of reaction-diffusion systems different from (GM) (with chemotaxis for example, as shown in [37]). Another motivation comes from the Nonlinear Schrödinger equation $i\hbar \frac{\partial \psi}{\partial t} = -\hbar^2 \Delta \psi + V(x)\psi - |\psi|^{p-1}\psi$ in $\mathbb{R}^n$, where ψ is a complex-valued function (the *wave function*), V is a potential and p is an exponent greater than 1. Indeed, if one looks for *standing waves*, namely solutions of the form $\psi(x,t) = e^{-\frac{i\omega t}{\hbar}} u(x)$, for some real function u, then the latter will satisfy $-\varepsilon^2 \Delta u + V(x) u = u^p$ in $\mathbb{R}^n$, where we have set $\varepsilon = \hbar$ and we absorbed the constant ω into the potential V. Therefore, up to the potential, we still obtain the equation in (N_ε) or (D_ε): about this subject we refer the reader to the (still incomplete) list of papers [1], [2], [6], [17] and to the bibliographies therein.

The typical concentration behavior of solutions u_ε to the above two problems is via a scaling of the variables in the form $u_\varepsilon(x) \sim u_0\left(\frac{x-Q}{\varepsilon}\right)$, where Q is some point of $\overline{\Omega}$, and where U is a solution of

$$(1) \qquad -\Delta U + U = U^p \qquad \text{in } \mathbb{R}^n \quad (\text{or in } \mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \ : \ x_n > 0\}),$$

the domain depending on whether Q lies in the interior of Ω or at the boundary; in the latter case Neumann conditions are imposed. When $p < \frac{n+2}{n-2}$ (and indeed only if this inequality is satisfied), problem (1) admits positive radial solutions which decay to zero at infinity, see [2].

Solutions of $(\tilde{M}_\varepsilon)$ which inherit this profile are called *spike layers*, since they are highly concentrated near some point of $\overline{\Omega}$. There is an extensive literature regarding this type of solutions, beginning from the papers [28], [38], [39]. Indeed their structure is very rich, and we refer for example to the (far from complete) list of references [12], [15], [21], [22], [23], [24], [26], [27], [44], [45].

We want next to describe qualitatively some features of spike layers which are useful in this context. Spike layers solving (N_ε) and sitting on the boundary of Ω are peaked near critical points of the mean curvature of $\partial\Omega$. To see this one can exploit the variational structure of the problem, whose Euler functional is

$$\tilde{I}_{\varepsilon,\mathcal{N}}(u) = \frac{1}{2}\int_\Omega \left(\varepsilon^2|\nabla u|^2 + u^2\right) - \frac{1}{p+1}\int_\Omega |u|^{p+1}; \qquad u \in H^1(\Omega). \tag{2}$$

Plugging into $\tilde{I}_{\varepsilon,\mathcal{N}}$ a function of the form $U_{Q,\varepsilon}(x) = U\left(\frac{1}{\varepsilon}(x-Q)\right)$ with $Q \in \partial\Omega$ one sees that $\tilde{I}_{\varepsilon,\mathcal{N}}(U_{Q,\varepsilon}) = \tilde{C}_0\varepsilon^n - \tilde{C}_1\varepsilon^{n+1}H(Q) + o(\varepsilon^{n+1})$, where $\tilde{C}_0, \tilde{C}_1$ are positive constants depending only on n and p. To obtain this expansion one can use the radial symmetry of U and parameterize $\partial\Omega$ as a normal graph near Q. From the above formula we see that, the bigger is the mean curvature the lower is the energy of this function: roughly speaking, boundary spike layers would tend to move along the gradient of H in order to minimize their energy. Indeed in [38], [39] it was shown that mountain pass solutions to (N_ε) (ground states) concentrate at $\partial\Omega$ near global maxima of the mean curvature.

Concerning instead (D_ε), spike layers with minimal energy concentrate at the interior of the domain, at points which maximize the distance from the boundary, see [40]. The intuitive reason for this is that, if Q is in the interior of Ω and if we want to adapt a function like $U\left(\frac{1}{\varepsilon}(x-Q)\right)$ to the Dirichlet conditions, the adjustment needs an energy which increases as Q becomes closer and closer to $\partial\Omega$. Following the above heuristic argument, we could say that spike layers are *repelled* from the regions where Dirichlet conditions are imposed.

We are interested here in finding boundary spike layers for the mixed problem $(\tilde{M}_\varepsilon)$: our idea is to obtain two compensating effects from the Neumann and the Dirichlet conditions. More precisely, calling $\mathcal{I}_\Omega$ the intersection of the closures of $\partial_{\mathcal{D}}\Omega$ and $\partial_{\mathcal{N}}\Omega$, and assuming that the gradient of H at $\mathcal{I}_\Omega$ points toward $\partial_{\mathcal{D}}\Omega$, a spike layer centered on $\partial_{\mathcal{N}}\Omega$ will be *pushed* toward $\mathcal{I}_\Omega$ by ∇H and will be *repelled* from $\mathcal{I}_\Omega$ by the Dirichlet condition: our main result is the following theorem.

Theorem 1.1 *Suppose* $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, *is a smooth bounded domain, and that* $1 < p < \frac{n+2}{n-2}$ *(*$1 < p < +\infty$ *if* $n = 2$*). Suppose* $\partial_{\mathcal{D}}\Omega$, $\partial_{\mathcal{N}}\Omega$ *are disjoint open sets of* $\partial\Omega$ *such that the union of the closures is the whole boundary of* Ω *and such that their intersection* $\mathcal{I}_\Omega$ *is an embedded hypersurface. Suppose* $\overline{Q} \in \mathcal{I}_\Omega$ *is such that* $H|_{\mathcal{I}_\Omega}$ *is critical and non degenerate at* $\overline{Q}$, *and that* $\nabla H(\overline{Q}) \neq 0$ *points toward* $\partial_{\mathcal{D}}\Omega$. *Then for* $\varepsilon > 0$ *sufficiently small problem* $(\tilde{M}_\varepsilon)$ *admits a solution* u_ε *concentrating at* $\overline{Q}$.

Remark 1.2 *(a) The non degeneracy condition in Theorem 1.1 can be replaced by the condition that* $\overline{Q}$ *is a strict local maximum or minimum of* $H|_{\mathcal{I}_\Omega}$, *or by the fact that there exists an open set* $\mathcal{V}$ *of* $\mathcal{I}_\Omega$ *containing* $\overline{Q}$ *such that* $H(\overline{Q}) < \inf_{\partial\mathcal{V}} H$ *(respectively* $H(\overline{Q}) > \sup_{\partial\mathcal{V}} H$*).*

(b) With more precision, as $\varepsilon \to 0$, *the above solution* u_ε *possesses a unique global maximum point* $Q_\varepsilon \in \partial_{\mathcal{N}}\Omega$, *and* $dist(Q_\varepsilon, \mathcal{I}_\Omega)$ *is of order* $\varepsilon \log\frac{1}{\varepsilon}$ *as* ε *tends to* 0.

The general strategy for proving Theorem 1.1 relies on a finite-dimensional reduction, which is conceptually rather simple and nowadays well understood, see for example the book [2]. One finds first a manifold Z of approximate solutions to the given problem, which in our case are of the form $U\left(\frac{1}{\varepsilon}(x-Q)\right)$, and solve the equation up to a vector (in the Hilbert space) parallel to the tangent plane of this manifold: for this procedure one can use the spectral properties of the linearization of (1), see Lemma 4.3 below. In this way, see Proposition 2.3, one generates a new manifold $\tilde{Z}$ close to Z which represents a natural constraint for the Euler functional of $(\tilde{M}_\varepsilon)$, which is

$$\tilde{I}_\varepsilon(u) = \frac{1}{2}\int_\Omega \left(\varepsilon^2|\nabla u|^2 + u^2\right) dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx; \qquad u \in H^1_{\mathcal{D}}(\Omega). \tag{3}$$

Here $H^1_{\mathcal{D}}(\Omega)$ stands for the space of functions in $H^1(\Omega)$ which have zero trace on $\partial_{\mathcal{D}}\Omega$, and by *natural constraint* we mean a set for which constrained critical points of $\tilde{I}_\varepsilon$ are true critical points.

The main difficulty however is to have a good control of $\tilde{I}_\varepsilon|_{\tilde{Z}}$, which is done improving the accuracy of the functions in the original manifold Z: in fact, the better is the accuracy of these functions, the closer is $\tilde{Z}$ to Z, so the main term in the constrained functional will be given by $\tilde{I}_\varepsilon|_Z$, see Proposition 3.12 and Lemma 4.5 below. To find sufficiently good approximate solutions we start with those constructed in literature the Neumann problem (N_ε) (see Subsection 2.2) which reveal the role of the boundary mean curvature, as in the expansion after (2). However these functions are not zero on $\partial_{\mathcal{D}}\Omega$, and if one tries naively to annihilate them using cut-off functions, the corresponding error turns out to be too large. A method which revealed itself to be useful for (D_ε) is to consider the *projection operator* in $H^1(\Omega)$, which consists in associating to some function in this space its closest element in $H^1_{\mathcal{D}}(\Omega)$. In some previous works, see for example [27] and [46], the asymptotic behavior of this projection has been studied in detail when the limit concentration point lies in the interior of Ω, using the well known limit behavior of the solution U to (1)

$$\lim_{r\to+\infty} e^r r^{\frac{n-1}{2}} U(r) = \alpha_{n,p}, \tag{4}$$

where the positive constant $\alpha_{n,p}$ depends only on n and p, together with

$$\lim_{r\to+\infty} \frac{U'(r)}{U(r)} = -1; \qquad \lim_{r\to+\infty} \frac{U''(r)}{U(r)} = 1. \tag{5}$$

In our case instead, apart from having mixed conditions, the maxima of the spike-layers in Theorem 1.1 tend to the interface $\mathcal{I}_\Omega$, so to better understand the projection we need to work at a scale $d \simeq \varepsilon|\log\varepsilon|$, the order of the distance of the peak from $\mathcal{I}_\Omega$ (see Remark 1.2 (b)). At this scale the boundary of the domain looks nearly flat, so in this step we replace Ω with a non smooth domain $\hat{\Gamma}_D \subseteq \mathbb{R}^n$ such that part of $\partial\hat{\Gamma}_D$ looks like a cut of dimension $n-1$, see the beginning of Section 3. We choose $\hat{\Gamma}_D$ to be even with respect to the coordinate x_n and we study H^1 projections here (with Dirichlet conditions) which are also even in x_n: as a consequence we will find functions which have zero x_n-derivative on $\{x_n = 0\} \setminus \partial\hat{\Gamma}_D$, which mimics the Neumann boundary condition on $\partial_{\mathcal{N}}\Omega$. After analyzing carefully the projection in Subsection 3.1, we define a family of suitable approximate solutions to $(\tilde{M}_\varepsilon)$, which turn out to have a sufficient accuracy for our analysis, estimated in Proposition 3.12.

We can finally apply the above mentioned perturbation method to reduce the problem to a finite dimensional one, and study the functional constrained on $\tilde{Z}$. If z^ε_Q denotes (roughly speaking) an approximate solution peaked at Q, with $dist(Q, \mathcal{I}_\Omega) = d_\varepsilon$, then its energy turns out to be the following

$$\tilde{I}_\varepsilon(z^\varepsilon_Q) = \varepsilon^n \left(\tilde{C}_0 - \tilde{C}_1 \varepsilon H(Q) + e^{-2\frac{d_\varepsilon}{\varepsilon}(1+o(1))} + O(\varepsilon^2) \right).$$

The first two terms in this formula are as in the expansion after (2) for (N_ε), while the third one represents a sort of *potential energy* which decreases with the distance of Q from the interface, consistently with the *repulsive effect* which was described before for (D_ε). From the latter formula it follows that if d_ε remains constant then we recover the expansion corresponding to the Neumann problem. On the other hand even if $d_\varepsilon \to 0$, that is when the concentration points converge to the interface, one can check that the energy has a critical point when $H|_{\mathcal{I}_\Omega}$ is stationary, provided ∇H points toward $\partial_{\mathcal{D}}\Omega$, and $d_\varepsilon \simeq \varepsilon|\log\varepsilon|$, as stated in Theorem 1.1.

In the second part of this work, [18], we will analyze the asymptotic profile of the least energy solutions to $(\tilde{M}_\varepsilon)$ under generic assumptions on the domain and on the interface. We prove that, as for (N_ε), these concentrate at boundary points in the closure of $\partial_{\mathcal{N}}\Omega$ where the mean curvature is maximal. When this constrained maximum is attained on the interface (and if ∇H here is non zero), we can show that the mountain pass has precisely the behavior described in Theorem 1.1. In fact from the contraction mapping theorem one finds that, in a neighborhood of a non degenerate pseudo critical manifold, the critical points of the functional necessarily belong to the natural constraint, see Proposition 2.3.

The plan of the paper is the following. In Section 2 we collect some preliminary material: we recall the abstract variational perturbative scheme and recall known results concerning the Neumann problem (N_ε), adapted to our needs. In Section 3 we then construct a model domain to deal with the interface, analyze here the asymptotics of projections in H^1 and then construct approximate solutions to $(\tilde{M}_\varepsilon)$. Finally in Section 4 we expand the functional on the natural constraint, prove the existence of critical points and deduce Theorem 1.1.

Notation. Generic fixed constants will be denoted by C, and will be allowed to vary within a single line or formula. The symbols $O(t)$ (respectively $o(t)$) will denote quantities for which $\frac{O(t)}{|t|}$ stays bounded (respectively $\frac{o(t)}{|t|}$ tends to zero) as the argument t goes to zero or to infinity. We will often use the notation $d(1+o(1))$, where $o(1)$ stands for a quantity which tends to zero as $d \to +\infty$.

2 Preliminaries

We are interested in finding solutions to $(\tilde{M}_\varepsilon)$ with a specific asymptotic profile, so it is convenient to scale the variables like $x \mapsto \varepsilon x$ and to study $(\tilde{M}_\varepsilon)$ in the dilated domain

$$\Omega_\varepsilon := \frac{1}{\varepsilon}\Omega.$$

After this change of variables the problem becomes

$$(M_\varepsilon) \qquad \begin{cases} -\Delta u + u = u^p & \text{in } \Omega_\varepsilon; \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial_{\mathcal{N}}\Omega_\varepsilon & u = 0 \text{ on } \partial_{\mathcal{D}}\Omega_\varepsilon; \\ u > 0 & \text{in } \Omega_\varepsilon, \end{cases}$$

where $\partial_{\mathcal{N}}\Omega_\varepsilon$ and $\partial_{\mathcal{D}}\Omega_\varepsilon$ stand for the dilations of $\partial_{\mathcal{N}}\Omega$ and $\partial_{\mathcal{D}}\Omega$ respectively.

The Euler functional corresponding to (M_ε) is the following

$$(6) \qquad I_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon} \left(|\nabla u|^2 + u^2\right) dx - \frac{1}{p+1}\int_{\Omega_\varepsilon} |u|^{p+1} dx; \qquad u \in H^1_{\mathcal{D}}(\Omega_\varepsilon),$$

where $H^1_{\mathcal{D}}(\Omega_\varepsilon)$ denotes the family of functions in $H^1(\Omega_\varepsilon)$ with zero trace on $\partial_{\mathcal{D}}\Omega_\varepsilon$.

We next introduce an abstract perturbation method which takes advantage of the variational structure of the problem, allowing to reduce it to a finite dimensional one. We refer the reader mainly to [2] and to the bibliography therein: for our needs we use some small modifications of the arguments in the latter reference and we provide complete proofs for sake of clarity, since they are rather short and simple.

2.1 Perturbation in critical point theory

This subsection treats existence of critical points for a class of functionals which are perturbative in nature. Given a Hilbert space $\mathcal{H}$ (which might depend on the perturbation parameter ε), we want to consider manifolds embedded smoothly in $\mathcal{H}$, precisely for which

i) there exists a smooth d-dimensional manifold $Z_\varepsilon \subseteq \mathcal{H}$ and $C, r > 0$ such that for any $z \in Z_\varepsilon$, the set $Z \cap B_r(z)$ can be parameterized by a map on $B_1^{\mathbb{R}^d}$ whose C^3 norm is bounded by C.

We are also interested in functionals $I_\varepsilon : \mathcal{H} \to \mathbb{R}$ of class $C^{2,\alpha}$ which satisfy the following properties

ii) there exists a continuous function $f : (0, \varepsilon_0) \to \mathbb{R}$ with $\lim_{\varepsilon\to 0} f(\varepsilon) = 0$ such that $\|I'_\varepsilon(z)\| \leq f(\varepsilon)$ for every $z \in Z_\varepsilon$: moreover we require that $\|I''_\varepsilon(z)[q]\| \leq f(\varepsilon)\|q\|$ for every $z \in Z_\varepsilon$ and every $q \in T_z Z_\varepsilon$;

iii) there exist C, $\alpha \in (0,1]$, $r_0 > 0$ such that $\|I''_\varepsilon\|_{C^\alpha} \leq C$ in the subset $\{u \, : \, dist(u, Z_\varepsilon) < r_0\}$;

iv) letting P_z, $z \in Z_\varepsilon$, denote the projection onto the orthogonal complement of $T_z Z_\varepsilon$, there exists $C > 0$ (independent of z and ε) such that $P_z I''_\varepsilon(z)$, restricted to $(T_z Z_\varepsilon)^\perp$, is invertible from $(T_z Z_\varepsilon)^\perp$ into itself, and the inverse operator satisfies $\|(P_z I''_\varepsilon(z))^{-1}\| \leq C$.

First some notation is in order. Let us set $W = (T_z Z_\varepsilon)^\perp$ and let $(q_i)_{1 \leq i \leq d}$ be an orthonormal d-tuple (locally smooth on Z_ε) such that $T_z Z_\varepsilon = span\{q_1, \dots, q_d\}$. In the sequel we will denote by $z = z_\xi$, $\xi \in \mathbb{R}^d$, a smooth local parameterization of Z_ε as in i). Furthermore, we also suppose that $q_i = \partial_{\xi_i} z_\xi / \|\partial_{\xi_i} z_\xi\|$ at a given point of Z_ε.

We will look for critical points of I_ε in the form $u = z + \omega$ with $z \in Z_\varepsilon$ and $\omega \in W$. If $P_z : \mathcal{H} \to W$ is as in iv), the equation $I'_\varepsilon(z + \omega) = 0$ is equivalent to the system

$$\left\{ \begin{array}{ll} P_z I'_\varepsilon(z + \omega) = 0 & (\textit{auxiliary equation}); \\ (Id - P_z) I'_\varepsilon(z + \omega) = 0 & (\textit{bifurcation equation}). \end{array} \right. \tag{7}$$

Proposition 2.1 *Let i)-iv) hold true. Then there exists $\varepsilon_0 > 0$ with the following property: for all $|\varepsilon| < \varepsilon_0$ and for all $z \in Z_\varepsilon$, the auxiliary equation in (7) has a unique solution $\omega = \omega_\varepsilon \in W = (T_z Z_\varepsilon)^\perp$, which is of class C^1 with respect to $z \in Z_\varepsilon$ and such that $\|\omega_\varepsilon(z)\| \leq C_1 f(\varepsilon)$ as $|\varepsilon| \to 0$, uniformly with respect to $z \in Z_\varepsilon$. The derivative of ω with respect to z, ω'_ε, satisfies the bound $\|\omega'_\varepsilon(z)\| \leq C C_1 f(\varepsilon)^\alpha$.*

PROOF. The proof is a refinement of a (by now) standard argument, which can be found for example in [2], Section 2: since however the procedure is rather short, we write here the details for the reader's convenience.

Property iv) allows us to apply the contraction mapping theorem to the auxiliary equation. In fact, by the invertibility of $P_z I''_\varepsilon(z)$ we can rewrite it tautologically as

$$\omega = -(P_z I''_\varepsilon(z))^{-1} \left[P_z I'_\varepsilon(z) + (P_z I'_\varepsilon(z + \omega) - P_z I'_\varepsilon(z) - P_z I''_\varepsilon(z)[\omega]) \right] := G_{\varepsilon,z}(\omega).$$

We claim next that the latter map is a contraction on a suitable metric ball of W. In fact, for the second term of G_ε, using iii) we can write that

$$\|P_z I'_\varepsilon(z + \omega) - P_z I'_\varepsilon(z) - P_z I''_\varepsilon(z)[\omega]\| = \left\| P_z \int_0^1 (I''_\varepsilon(z + s\omega) - I''_\varepsilon(z))[\omega] ds \right\| \leq C \|\omega\|^{1+\alpha},$$

and therefore by ii) and iv) we have

$$\|G_{z,\varepsilon}(\omega)\| \leq C f(\varepsilon) + C^2 \|\omega\|^{1+\alpha}; \qquad \|\omega\| \leq r_0.$$

Similarly, one also finds

$$\|G_{z,\varepsilon}(\omega_1) - G_{z,\varepsilon}(\omega_2)\| \leq C^2 (\|\omega_1\|^\alpha + \|\omega_2\|^\alpha) \|\omega_1 - \omega_2\|.$$

By the last two equations, if we fix $C_1 > 0$ sufficiently large and let $B_\varepsilon = \{\omega \in W \ : \ \|\omega\| \leq C_1 f(\varepsilon)\}$, we can check that $G_{z,\varepsilon}$ is a contraction in B_ε, so for every $z \in Z_\varepsilon$ we obtain a (unique) function ω satisfying the required bound (for brevity, in the sequel the dependence on z will be assumed understood).

Let us now show that also the derivatives of ω with respect to ξ can be controlled by means of $f(\varepsilon)$. Indeed, for the components of ω tangent to Z_ε we can argue as follows: since $(\omega, \partial_\xi z) = 0$ for every ξ, differentiating with respect to ξ we find that $(\partial_\xi \omega, \partial_\xi z) = -(\omega, \partial^2_\xi z)$. Since $\|\omega\| = O(f(\varepsilon))$ and since $\partial^2_\xi z$ is bounded (by i)), the tangent components of $\partial_\xi \omega$ are bounded in norm by $C C_1 f(\varepsilon)$.

About the normal components, we can differentiate the relation $I'_\varepsilon(z + \omega) = \sum_{i=1}^d \overline{\alpha}_i \partial_{\xi_i} z$ (where $\overline{\alpha}_i \in \mathbb{R}$) with respect to ξ to find

$$I''_\varepsilon(z + \omega)[\partial_\xi z] + I''_\varepsilon(z + \omega)[\partial_\xi \omega] = \sum_{i=1}^d (\partial_\xi \overline{\alpha}_i) \partial_{\xi_i} z + \sum_{i=1}^d \overline{\alpha}_i \partial^2_{\xi \xi_i} z. \tag{8}$$

Since I''_ε is locally Hölder continuous (by iii)) and since $\|\omega\| = O(f(\varepsilon))$ projecting on W, for ε small by iv) we have that

$$\|P_z \partial_\xi \omega\| \leq C |\overline{\alpha}| + C \|I''_\varepsilon(z)[\partial_\xi z]\| + C \|\omega\|^\alpha \leq C f(\varepsilon)^\alpha.$$

For the latter inequality we used again $\|\omega\| = O(f(\varepsilon))$ together with the Lipschitzianity of I'_ε (which imply $|\overline{\alpha}| \leq C f(\varepsilon)$) and ii). ■

Remark 2.2 *From formula* (8)*, writing* $I''_\varepsilon(z+\omega)$ *as* $(I''_\varepsilon(z+\omega)-I''_\varepsilon(z))+I''_\varepsilon(z)$ *and using ii), one finds the following more precise estimate*

$$\|P_z\partial_\xi\omega\| \le Cf(\varepsilon) + C\|(I''_\varepsilon(z+\omega)-I''_\varepsilon(z))[\partial_\xi z]\|.$$

Under further regularity assumptions on ω*, the estimate of* $\|P_z\partial_\xi\omega\|$ *can be improved: this will be useful for us to treat the case* $p\in(1,2)$*, see the proof of Proposition 4.5.*

We shall now provide conditions for solving the bifurcation equation in (7). In order to do this, let us define the *reduced functional* $\mathbf{I}_\varepsilon : Z \to \mathbb{R}$ by setting

$$\mathbf{I}_\varepsilon(z) = I_\varepsilon(z+\omega_\varepsilon(z)). \tag{9}$$

Proposition 2.3 *Suppose we are in the situation of Proposition 2.1, and let us assume that* $\mathbf{I}_\varepsilon$ *has, for* $|\varepsilon|$ *sufficiently small, a stationary point* z_ε*: then* $u_\varepsilon = z_\varepsilon + \omega_\varepsilon(z_\varepsilon)$ *is a critical point of* I_ε*. Furthermore, there exist* $\tilde{c},\tilde{r}>0$ *such that if* u *is a critical point of* I_ε *with* $dist(u, Z_{\varepsilon,\tilde{c}}) < \tilde{r}$*, where*

$$Z_{\varepsilon,\tilde{c}} = \{z\in Z_\varepsilon \ : \ dist(z,\partial Z_\varepsilon) > \tilde{c}\},$$

then u *has to be of the form* $z_\varepsilon+\omega_\varepsilon(z_\varepsilon)$ *for some* $z_\varepsilon\in Z_\varepsilon$*.*

PROOF. The first assertion can be proved as follows. Consider the manifold $\tilde{Z}_\varepsilon = \{z+\omega_\varepsilon(z) \ : \ z\in Z_\varepsilon\}$. If z_ε is a critical point of $\mathbf{I}_\varepsilon$, it follows that $u_\varepsilon = z_\varepsilon+\omega(z_\varepsilon)\in\tilde{Z}_\varepsilon$ is a critical point of I_ε constrained on $\tilde{Z}_\varepsilon$ and thus u_ε satisfies $I'_\varepsilon(u_\varepsilon)\perp T_{u_\varepsilon}\tilde{Z}_\varepsilon$. Moreover the definition of ω_ε, see Proposition 2.1, implies that $I'_\varepsilon(u_\varepsilon)\in T_{z_\varepsilon}Z_\varepsilon$. Since, for $|\varepsilon|$ small, $T_{u_\varepsilon}\tilde{Z}_\varepsilon$ and $T_{z_\varepsilon}Z_\varepsilon$ are close, which is a consequence of the smallness of ω'_ε, it follows that $I'_\varepsilon(u_\varepsilon)=0$.

To prove the last statement it is sufficient to notice that the contraction argument in the proof of Proposition 2.1 can be performed in the larger ball $\tilde{B}=\{\omega\in W \ : \ \|\omega\|\le 2\tilde{r}\}$ with $\tilde{r}$ sufficiently small, so one has uniqueness of the solution of the auxiliary equation in this set. The distance condition on $Z_{\varepsilon,\tilde{c}}$ ensures the full applicability of these arguments in $\{dist(u,Z_{\varepsilon,\tilde{c}})<\tilde{r}\}$, so the conclusion follows. ■

2.2 Approximate solutions for (M_ε) with Neumann conditions

In this subsection we exhibit a family of (known) functions which satisfy the equation in (M_ε) up to an error of order ε^2, and whose normal boundary derivative vanishes, up to the same order. After introducing some convenient coordinates which stretch the boundary, we recall some results from [2] concerning approximate solutions to the Neumann problem. Then, using a further change of variables which also stretches the interface, we modify these functions conveniently for our purposes.

Let us describe $\partial\Omega_\varepsilon$ near a generic point $Q\in\partial_{\mathcal{N}}\Omega_\varepsilon$. Without loss of generality, we can assume that $Q=0\in\mathbb{R}^n$, that $\{x_n=0\}$ is the tangent plane of $\partial\Omega_\varepsilon$ (or $\partial\Omega$) at Q, and that $\nu(Q)=(0,\dots,0,-1)$, where $\nu(Q)$ stands for the outer unit normal at Q. In a neighborhood of Q, let $x_n=\psi_Q(x')$ be a local parametrization of $\partial\Omega$, $x'\in\mathbb{R}^{n-1}$. Then on $\partial\Omega$ one has

$$x_n = \psi_Q(x') := \tfrac{1}{2}\langle A_Q x', x'\rangle + O(|x'|^3); \qquad |x'|<\mu_0, \tag{10}$$

where A_Q is the Hessian of ψ at 0, and where μ_0 is some small number depending on Ω. We have clearly $H(Q)=\frac{1}{n-1}trA_Q$. On the other hand, $\partial\Omega_\varepsilon$ is parameterized using the function $\psi^\varepsilon_Q(x') := \frac{1}{\varepsilon}\psi_Q(\varepsilon x')$, for which the following expansions hold

$$\begin{aligned} \psi^\varepsilon_Q(x') &= \frac{\varepsilon}{2}\langle A_Q x', x'\rangle + \varepsilon^2 O(|x'|^3); \\ \partial_i\psi^\varepsilon_Q(x') &= \varepsilon(A_Q x')_i + \varepsilon^2 O(|x'|^2). \end{aligned} \tag{11}$$

Concerning the outer normal ν, we have also

$$\nu = \frac{\left(\frac{\partial \psi_Q^\varepsilon}{\partial x_1}, \dots, \frac{\partial \psi_Q^\varepsilon}{\partial x_{n-1}}, -1\right)}{\left(1 + |\nabla \psi_Q^\varepsilon|^2\right)^{\frac{1}{2}}} = (\varepsilon(A_Q x'), -1) + \varepsilon^2 O(|x'|^2). \tag{12}$$

Since $\partial\Omega_\varepsilon$ is almost flat for ε small and since the function U (see the Introduction) is radial, for $Q \in \partial\Omega_\varepsilon$ we have $\frac{\partial}{\partial\nu} U(\cdot - Q) \sim 0$. Thus $U(\cdot - Q)$ is an approximate solution to (M_ε) if we impose pure Neumann boundary conditions. Hence for the latter problem a natural choice of the manifold Z_ε (see Subsection 2.1) could be the following

$$Z_\varepsilon = \{U(\cdot - Q) := U_Q \ : \ Q \in \partial\Omega_\varepsilon\}.$$

Indeed we need a more accurate expansion, and we will construct better approximate solutions, in particular improving the condition at the boundary. The next lemma is proved in Section 9.2 of [2].

Lemma 2.4 *Let $T = (a_{ij})$ be a $(n-1)\times(n-1)$ symmetric matrix, and consider the following problem*

$$\begin{cases} L_U w = -2\langle Tx', \nabla_{x'} \partial_{x_n} U\rangle - (trT)\partial_{x_n} U & \text{in } \mathbb{R}^n_+; \\ \frac{\partial}{\partial x_n} w = \langle Tx', \nabla_{x'} U\rangle & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{13}$$

where L_U is the operator $L_U u = -\Delta u + u - pU^{p-1}u$. Then (13) *admits a solution $\overline{w}_T$, which is even in the variables x' and satisfies the following decay estimates*

$$|\overline{w}_T(x)| + |\nabla \overline{w}_T(x)| + |\nabla^2 \overline{w}_T(x)| \le C|T|_\infty (1 + |x|^K) e^{-|x|}, \tag{14}$$

where C, K are constants depending only on n and p.

Given μ_0 as in (10), we introduce a new set of coordinates on $B_{\frac{\mu_0}{\varepsilon}}(Q) \cap \Omega_\varepsilon$. Let

$$\tilde{x}' = x'; \qquad \tilde{x}_n = x_n - \psi_Q^\varepsilon(x'). \tag{15}$$

The advantage of these coordinates is that $\partial\Omega_\varepsilon$ identifies with $\{\tilde{x}_n = 0\}$, but the corresponding metric coefficients $\tilde{g}_{ij}$ will not be constant anymore. From (11) it follows that

$$\tilde{g}_{ij} = Id + \varepsilon A^Q + O(\varepsilon^2|\tilde{x}'|^2); \qquad \partial_{\tilde{x}_k}(\tilde{g}_{ij}) = \varepsilon \partial_{\tilde{x}_k} A^Q + O(\varepsilon^2 |\tilde{x}'|), \tag{16}$$

with

$$A^Q = \begin{pmatrix} 0 & A_Q \tilde{x}' \\ (A_Q\tilde{x}')^t & 0 \end{pmatrix}.$$

Here the zero in the upper left corner of A^Q stands for the trivial $(n-1)\times(n-1)$ matrix, while $(A_Q\tilde{x}')^t$ stands for the transpose of the column vector $(A_Q\tilde{x}')$. It is also easy to check that the inverse matrix $(\tilde{g}^{ij})$ is of the form $\tilde{g}^{ij} = Id - \varepsilon A^Q + O(\varepsilon^2|\tilde{x}'|^2)$, and that $\partial_{\tilde{x}_k}(\tilde{g}^{ij}) = -\varepsilon\partial_{\tilde{x}_k} A^Q + O(\varepsilon^2|\tilde{x}'|)$. Moreover, by the expression of the coordinates $\tilde{x}$ one has

$$\det \tilde{g} \equiv 1. \tag{17}$$

Recall that the Laplacian with respect to a general metric is given by

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \partial_j \left(g^{ij} \sqrt{\det g}\right) \partial_i u + g^{ij} \partial^2_{ij} u,$$

so in our situation, by (17), we get

$$\Delta_g u = g^{ij} u_{ij} + \partial_i(g^{ij})\partial_j u.$$

Using (16) we find that, formally

$$\Delta_{\tilde{g}} u = \Delta_{\mathbb{R}^n} u - \varepsilon \left(2\langle A_Q \tilde{x}', \nabla_{\tilde{x}'} \partial_{\tilde{x}_n} u\rangle + (tr A_Q)\partial_{\tilde{x}_n} u\right) + O(\varepsilon^2)(|\nabla u| + |\nabla^2 u|). \tag{18}$$

We also give the expression of the unit outer normal to $\partial\Omega_\varepsilon$, $\tilde{\nu}$, in the new coordinates $\tilde{x}$. Letting ν_i (resp. $\tilde{\nu}_i$) be the components of ν (resp. $\tilde{\nu}$), from $\nu = \sum_{i=1}^n \nu^i \frac{\partial}{\partial x^i} = \sum_{i=1}^n \tilde{\nu}^i \frac{\partial}{\partial \tilde{x}^i}$, we have $\tilde{\nu}_k = \sum_{i=1}^n \nu^i \frac{\partial \tilde{x}^k}{\partial x^i}$. This implies

$$\tilde{\nu}^k = \nu^k, \quad k = 1, \dots, n-1; \qquad\qquad \tilde{\nu}^n = \sum_{i=1}^{n-1} \nu^i \frac{\partial \psi_Q^\varepsilon}{\partial \tilde{x}^i} + \nu^n.$$

From (11) and the subsequent formulas we find

$$\tilde{\nu} = (\varepsilon A_Q \tilde{x}', -1) + \varepsilon^2 O(|\tilde{x}'|^2). \tag{19}$$

Finally the area-element of $\partial\Omega_\varepsilon$ can be expanded as

$$d\sigma = (1 + O(\varepsilon^2 |\tilde{x}'|^2)) d\tilde{x}'. \tag{20}$$

Linearizing the equation in (M_ε) near $U(\tilde{x})$, one sees that the function $U(y) + \varepsilon w(y)$ solves the equation (M_ε) up to an error $o(\epsilon)$, if w satisfies

$$\begin{cases} L_U w = -2\langle A_Q \tilde{x}', \nabla_{\tilde{x}'} \partial_{\tilde{x}_n} U\rangle - (tr A_Q)\, \partial_{\tilde{x}_n} U & \text{in } \mathbb{R}^n_+; \\ \frac{\partial}{\partial \tilde{x}_n} w = \langle A_Q \tilde{x}', \nabla_{\tilde{x}'} U\rangle & \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

Therefore, if $\overline{w}_{A_Q}$ is given by Lemma 2.4 with $T = A_Q$, one sees that, formally at least, the accuracy of the solution improves. To derive rigorous estimates in this spirit, one can choose a cut-off function χ_{μ_0} on $\mathbb{R}^n$ with the properties

$$\begin{cases} \chi_{\mu_0}(\tilde{x}) = 1 & \text{in } B_{\frac{\mu_0}{4}}; \\ \chi_{\mu_0}(\tilde{x}) = 0 & \text{in } \mathbb{R}^n \setminus B_{\frac{\mu_0}{2}}; \\ |\nabla \chi_{\mu_0}| + |\nabla^2 \chi_{\mu_0}| \leq C & \text{in } B_{\frac{\mu_0}{2}}(Q) \setminus B_{\frac{\mu_0}{4}}, \end{cases} \tag{21}$$

and for any $Q \in \partial\Omega$ define the following function, in the coordinates $(\tilde{x}', \tilde{x}_n)$

$$z_{\varepsilon,Q}(\tilde{x}) = \chi_{\mu_0}(\varepsilon \tilde{x})(U(\tilde{x}) + \varepsilon \overline{w}_{A_Q}(\tilde{x})). \tag{22}$$

The next result collects estimates in the statements and the proofs of Lemmas 9.4, 9.7 and 9.8 in [2].

Proposition 2.5 *There exist constants $C, K > 0$, depending only on p, n and Ω such that for ε small the following estimates hold*

$$\left|\frac{\partial z_{\varepsilon,Q}}{\partial \tilde{\nu}}\right|(\tilde{x}') \leq \begin{cases} C\varepsilon^2(1 + |\tilde{x}'|^K)e^{-|\tilde{x}'|} & \textit{for } |\tilde{x}'| \leq \frac{\mu_0}{4\varepsilon}; \\ Ce^{-\frac{1}{C\varepsilon}} & \textit{for } \frac{\mu_0}{4\varepsilon} \leq |\tilde{x}'| \leq \frac{\mu_0}{2\varepsilon}, \end{cases}$$

$$\left|-\Delta_g z_{\varepsilon,Q} + z_{\varepsilon,Q} - z_{\varepsilon,Q}^p\right|(\tilde{x}) \leq \begin{cases} C\varepsilon^2(1 + |\tilde{x}|^K)e^{-|\tilde{x}|} & \textit{for } |\tilde{x}| \leq \frac{\mu_0}{4\varepsilon}; \\ Ce^{-\frac{1}{C\varepsilon}} & \textit{for } \frac{\mu_0}{4\varepsilon} \leq |\tilde{x}| \leq \frac{\mu_0}{2\varepsilon}, \end{cases}$$

$$I_{\varepsilon,\mathcal{N}}(z_{\varepsilon,Q}) = \tilde{C}_0 - \tilde{C}_1 \varepsilon H(\varepsilon Q) + O(\varepsilon^2); \qquad \frac{\partial}{\partial Q} I_{\varepsilon,\mathcal{N}}(z_{\varepsilon,Q}) = -\tilde{C}_1 \varepsilon^2 H'(\varepsilon Q) + o(\varepsilon^2),$$

where

$$\tilde{C}_0 = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^n_+} U^{p+1} dx, \qquad \tilde{C}_1 = \left(\int_0^\infty r^n U_r^2 dr\right) \int_{S^n_+} y_n |y'|^2 d\sigma.$$

To improve the estimate, we need to take into account the effect of the Dirichlet boundary condition. To this end, we make next a further change of variables, in order to stretch also the interface: we claim indeed that, in the coordinates $\tilde{x}$, the latter can be parameterized as $\tilde{x}_1 = d + \tilde{\psi}^{\varepsilon}_{Q}(\tilde{x''})$, $\tilde{x}'' = (\tilde{x}_2, \dots, \tilde{x}_{n-1})$, with $d \in \mathbb{R}$ and $\tilde{\psi}^{\varepsilon}_{Q}$ satisfying estimates similar to (11). To see this, we first claim that the curvature of the interface $\mathcal{I}_{\Omega_\varepsilon}$, in the coordinates $\tilde{x}$, is of order $O(\varepsilon)$. Here we are assuming that the distance of Q from $\mathcal{I}_{\Omega_\varepsilon}$, multiplied by ε, is bounded by a small constant depending on Ω. For showing this, let us consider a curve $\gamma(s)$ in the interface whose geodesic curvature (relative to $\mathcal{I}_{\Omega_\varepsilon}$) vanishes and which is parameterized by arclength: its curvature in $\mathbb{R}^n$ will be therefore of order $O(\varepsilon)$. Let us split γ into its tangent and normal components (with respect to $T_Q \partial\Omega_\varepsilon$) $\gamma = (\gamma^T, \gamma^N)$. Since in our notation $\gamma^N = \psi^{\varepsilon}_{Q}(\gamma^T)$, from (11) and a Taylor expansion we have that

$$|\dot{\gamma}|^2 = |\dot{\gamma}^T|^2 + |\dot{\gamma}^N|^2 \qquad \Rightarrow \qquad |\dot{\gamma}| = |\dot{\gamma}^T|(1 + O(\varepsilon^2)|\gamma^T|^2).$$

The curvature of γ is $k = \frac{1}{|\dot{\gamma}|}\frac{d}{ds}\left(\frac{\dot{\gamma}}{|\dot{\gamma}|}\right)$, which can be written as

$$k = \frac{(1 + O(\varepsilon^2)|\gamma^T|^2)}{|\dot{\gamma}^T|}\frac{d}{ds}\left(\frac{\dot{\gamma}^T}{|\dot{\gamma}^T|}(1 + O(\varepsilon^2)|\gamma^T|^2)\right).$$

Expanding the above expression we obtain

$$k = k^T(1 + O(\varepsilon^2)|\gamma^T|^2) + O(\varepsilon^2)|\gamma^T|.$$

This formula shows that, since k is of order ε, also k^T is of order ε. Therefore, if the point Q is close to the interface (in the sense specified before) and if we choose the $\tilde{x}_1$ axis to be perpendicular to the projection of $\mathcal{I}_{\Omega_\varepsilon}$ onto $T_Q\partial\Omega_\varepsilon$, if the latter projection is at distance d from Q we have that the points in $\mathcal{I}_{\Omega_\varepsilon}$ satisfy

$$\tilde{x}_1 = d + \tilde{\psi}^{\varepsilon}_{Q}(\tilde{x}''); \qquad \tilde{x}'' = (\tilde{x}_2, \dots, \tilde{x}_{n-1}), \tag{23}$$

where $\tilde{\psi}^{\varepsilon}_{Q}$ (which depends on Q and ε) is such that

$$\tilde{\psi}_{\varepsilon}(0) = 0; \qquad \nabla\tilde{\psi}_{\varepsilon}(0) = 0; \qquad \tilde{\psi}^{\varepsilon}_{Q}(\tilde{x}'') = \frac{1}{2}\varepsilon\langle \tilde{A}_Q \tilde{x}'', \tilde{x}''\rangle + O(\varepsilon^2|\tilde{x}''|^2),$$

see also (11). Below, it will be always understood that the symbol d refers to the distance to the scaled interface $\mathcal{I}_{\Omega_\varepsilon}$ and we will never use subscript ε to stress this fact.

Introducing the new coordinates

$$y_1 = \tilde{x}_1 - \tilde{\psi}^{\varepsilon}_{Q}(\tilde{x}''); \qquad y'' = \tilde{x}''; \qquad y_n = \tilde{x}_n, \tag{24}$$

we have that the metric coefficients g^y satisfy again

$$\det g^y \equiv 1, \tag{25}$$

as in (17). Then, similarly to (16) we have that

$$g^y_{ij} = Id + \varepsilon\tilde{A}^Q + O(\varepsilon^2|y'|^2); \qquad \partial_{y_k}(g^y_{ij}) = \varepsilon\partial_{y_k}\tilde{A}^Q + O(\varepsilon^2|y'|) \tag{26}$$

(and similarly $(g^y)^{ij} = Id - \varepsilon\tilde{A}^Q + O(\varepsilon^2|y'|^2)$, $\partial_{y_k}(g^y)^{ij} = -\varepsilon\partial_{y_k}\tilde{A}^Q + O(\varepsilon^2|y'|)$), where

$$\tilde{A}^Q = \begin{pmatrix} 0 & (\tilde{A}_Q y'')^t & A_Q y' \\ \tilde{A}_Q y'' & 0 & \\ (A_Q y')^t & & 0 \end{pmatrix}.$$

Remark 2.6 *(a) We stress that, in the new coordinates y, the origin parameterizes the point Q on $\partial\Omega$, and that functions decaying as $|y| \to +\infty$ will* concentrate *near Q.*

(b) It is also useful to understand how the metric coefficients g^y_{ij} vary with Q. Notice that in (26) the deviation from the Kronecker symbols is of order ε, and that we are working in a domain scaled of $\frac{1}{\varepsilon}$, so a variation of order 1 of Q corresponds to a variation of order ε in the original domain. Therefore, a variation of order 1 in Q yields a difference of order ε^2 in g^y_{ij}, and precisely

$$\frac{\partial g^y_{ij}}{\partial Q} = O(\varepsilon^2 |y'|^2),$$

with a similar estimate for the derivatives of the inverse coefficients $(g^y)^{ij}$. For more details see the end of Subsection 9.2 in [2].

From the latter formulas, the counterpart of (18) becomes

$$\begin{aligned}\Delta_{g^y} u &= \Delta_{\mathbb{R}^n} u - \varepsilon\left(2\langle A_Q y', \nabla_{y'}\partial_{y_n} u\rangle + (trA_Q)\partial_{y_n} u\right) - \varepsilon\left(2\langle \tilde{A}_Q y'', \nabla_{y''}\partial_{y_1} u\rangle + (tr\tilde{A}_Q)\partial_{y_1} u\right) \\ &+ O(\varepsilon^2)\left(|\nabla_y u| + |\nabla^2_y u|\right), \end{aligned} \tag{27}$$

while the analogues of (19) and (20) are

$$\nu^y = (\varepsilon A_Q y', -1) + \varepsilon^2 O(|y'|^2); \qquad d\sigma^y = (1 + O(\varepsilon^2 |y'|^2)) dy'. \tag{28}$$

Reasoning as for the above coordinates $\tilde{x}$ (see the comments after (20)), one sees that the function $U(y) + \varepsilon \tilde{w}(y)$ solves the equation in (M_ε) up to an error $o(\varepsilon)$ if and only if $\tilde{w}$ satisfies

$$\begin{cases} L_U \tilde{w} = -2\langle A_Q y', \nabla_{y'}\partial_{y_n} U\rangle - (trA_Q)\, \partial_{y_n} U - 2\langle \tilde{A}_Q y'', \nabla_{y''}\partial_{y_1} U\rangle - (tr\tilde{A}_Q)\, \partial_{y_1} U & \text{in } \mathbb{R}^n_+; \\ \frac{\partial}{\partial y_n}\tilde{w} = \langle A_Q y', \nabla_{y'} U\rangle & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{29}$$

We have an explicit solution to (29) in terms of the above function $\overline{w}_{A_Q}$ through a simple change of variable. Indeed, in the new coordinates y we can write that formally

$$\begin{aligned} z_{\varepsilon,Q}(\tilde{x}) &\simeq U\left(y_1 + \frac{\varepsilon}{2}\langle \tilde{A}_Q y'', y''\rangle, y'', y_n\right) + \varepsilon \overline{w}_{A_Q}\left(y_1 + \frac{\varepsilon}{2}\langle \tilde{A}_Q y'', y''\rangle, y'', y_n\right) \\ &\simeq U(y) + \frac{\varepsilon}{2}\langle \tilde{A}_Q y'', y''\rangle \partial_1 U(y) + \varepsilon \overline{w}_{A_Q}(y) + o(\varepsilon). \end{aligned}$$

Choosing $\tilde{w}(y) = \tilde{w}_Q(y) := \frac{1}{2}\langle \tilde{A}_Q y'', y''\rangle \partial_1 U(y) + \overline{w}_{A_Q}(y)$ and using the fact that $L_U \partial_{y_i} U(y) = 0$, from an explicit computation we find that

$$L_U \tilde{w}_Q(y) = L_U \overline{w}_{A_Q}(y) - (tr\tilde{A}_Q)\partial_1 U(y) + 2\langle \tilde{A} y'', \nabla_{y''}\partial_1 U(y)\rangle,$$

which is exactly the desired equation. Clearly, by (14), there exists a constant C'_Ω, depending on Ω and on the curvatures of the interface such that

$$|\tilde{w}_Q(y)| + |\nabla \tilde{w}_Q(y)| + |\nabla^2 \tilde{w}_Q(y)| \leq C'_\Omega (1 + |y|^K) e^{-|y|}. \tag{30}$$

In conclusion, choosing a cutoff function as in (21) and defining the new approximate solution $\overline{z}_{\varepsilon,Q}$ as

$$\overline{z}_{\varepsilon,Q}(y) = \chi_{\mu_0}(\varepsilon y)\left(U(y) + \varepsilon \tilde{w}_Q(y)\right), \tag{31}$$

the arguments for the proof of Proposition 2.3 yield the following variant of the above result.

Proposition 2.7 *There exists $C, K > 0$ such that for ε small the following estimates hold*

$$\left|\frac{\partial \overline{z}_{\varepsilon,Q}}{\partial \tilde{\nu}}\right|(y') \leq \begin{cases} C\varepsilon^2 (1 + |y'|^K) e^{-|y'|} & \text{for } |y'| \leq \frac{\mu_0}{4\varepsilon}, \\ C e^{-\frac{1}{C\varepsilon}} & \text{for } \frac{\mu_0}{4\varepsilon} \leq |y'| \leq \frac{\mu_0}{2\varepsilon}; \end{cases}$$

$$\left| -\Delta_g \overline{z}_{\varepsilon,Q} + \overline{z}_{\varepsilon,Q} - \overline{z}^p_{\varepsilon,Q} \right| (y) \leq \begin{cases} C\varepsilon^2 (1+|y|^K) e^{-|y|} & \text{for } |y| \leq \frac{\mu_0}{4\varepsilon}, \\ Ce^{-\frac{1}{C\varepsilon}} & \text{for } \frac{\mu_0}{4\varepsilon} \leq |y| \leq \frac{\mu_0}{2\varepsilon}; \end{cases}$$

$$I_{\varepsilon,\mathcal{N}}(\overline{z}_{\varepsilon,Q}) = \tilde{C}_0 - \tilde{C}_1 \varepsilon H(\varepsilon Q) + O(\varepsilon^2); \qquad \frac{\partial}{\partial Q} I_{\varepsilon,\mathcal{N}}(\overline{z}_{\varepsilon,Q}) = -\tilde{C}_1 \varepsilon^2 H'(\varepsilon Q) + o(\varepsilon^2),$$

where $\tilde{C}_0, \tilde{C}_1$ *are as in Proposition 2.3.*

An immediate consequence of this proposition is that

$$\|I'_\varepsilon(\overline{z}_{\varepsilon,Q})\| \leq C\varepsilon^2 \qquad \text{for all } Q \in \partial_{\mathcal{N}}\Omega_\varepsilon \text{ such that } dist(Q, \mathcal{I}_{\Omega_\varepsilon}) \geq \frac{\mu_0}{\varepsilon}, \tag{32}$$

where $C > 0$ is some fixed constant, and where μ_0 is as at the beginning of Subsection 2.2.

3 Approximate solutions to (M_ε)

The functions $\overline{z}_{\varepsilon,Q}$ constructed in (31) constitute good approximate solutions to (M_ε) when we impose pure Neumann boundary conditions. Nevertheless, we need an expansion which takes into account the parameter $d = \frac{d_\varepsilon}{\varepsilon}$, the distance of the peak point to the interface in the scaled domain (see the notation in the last formula of the introduction), and to this end some relevant modifications are necessary.

As in [40], a useful tool is the *projection operator* onto $H^1_{\mathcal{D}}(\Omega_\varepsilon)$, namely the one which associates to every element in $H^1(\Omega_\varepsilon)$ its closest point in $H^1_{\mathcal{D}}(\Omega_\varepsilon)$. Explicitly, this is constructed subtracting to any given $u \in H^1(\Omega_\varepsilon)$ the solution to

$$\begin{cases} -\Delta v + v = 0 & \text{in } \Omega_\varepsilon; \\ v = u & \text{on } \partial_{\mathcal{D}}\Omega_\varepsilon; \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial_{\mathcal{N}}\Omega_\varepsilon. \end{cases} \tag{33}$$

This solution can be found variationally by looking at the following minimum problem

$$\inf_{v=u \text{ on } \partial_{\mathcal{D}}\Omega_\varepsilon} \left\{ \int_{\Omega_\varepsilon} \left(|\nabla v|^2 + v^2 \right) dx \right\}.$$

Instead of studying (33) directly, it is convenient to modify the domain in order that the region of the boundary near $\mathcal{I}_{\Omega_\varepsilon}$ becomes flat.

For technical reasons we construct a domain in the following way: first in $\mathbb{R}^{n-1}$ we consider the square

$$S = \{|x_1 + 2| < 2\} \cap \{|x''| < 2\}; \qquad x'' = (x_2, \dots, x_{n-1}),$$

and then the set $\tilde{S}$ which is obtained from S rounding off smoothly the corners of S, and in order that

$$(S \cap \{|x_1 + 2| < 1\}) \cup (S \cap \{|x''| < 1\}) = \left(\tilde{S} \cap \{|x_1 + 2| < 1\} \right) \cup \left(\tilde{S} \cap \{|x''| < 1\} \right),$$

see Figure 1. Notice that in this way the set $\{x_1 = 0\} \cap \{|x''| < 1\}$ lies on the boundary of $\tilde{S}$. Below, we will identify $\tilde{S}$ with its natural immersion into $\mathbb{R}^n \cap \{x_n = 0\}$.

Next, in the (x_1, x_n) plane, we consider a C^∞ curve γ which runs on the segment $(t, 0)$, $t \in [0, 4(n+2)]$, which hits the x_n axis horizontally, stays in the quadrant $x_1, x_n > 0$ and stays external to the circle $\{x_1^2 + x_n^2 = 16(n+2)^2\}$, see Figure 2 (there is no specific reason in choosing the number $4(n+2)$: we just need that a sufficiently large cube is contained in the set $\hat{\Gamma}_D$ defined below).

We then consider the set Γ obtained by intersecting the semi-closed quadrant $\{x_1 \geq 0\} \cap \{x_n > 0\}$ with the interior of the curve, together with its reflection across the x_1 axis. Then, we call $\tilde{\Gamma}$ the set generated by the rotation of Γ around the x_n axis, jointly with $\tilde{S}$. We finally define the burger-shaped domain $\hat{\Gamma}$ as

$$\hat{\Gamma} = \tilde{\Gamma} \cup \check{\Gamma}, \tag{34}$$

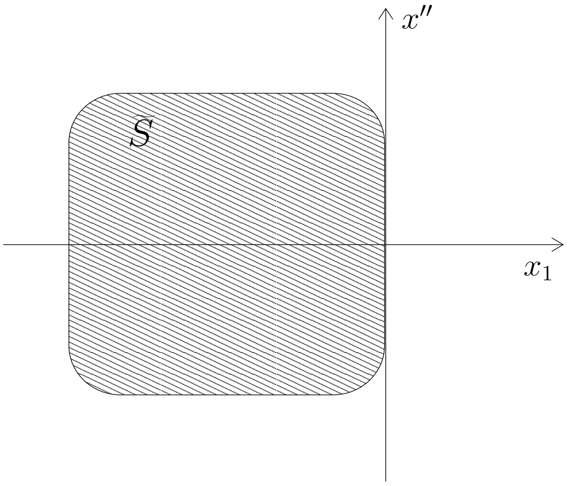

Figure 1: The domain $\tilde{S}$.

where $\check{\Gamma}$ is the reflection of $\tilde{\Gamma}$ across the plane $\{x_n = 0\}$, see Figure 3. For a large number D, we also consider the scaling

$$\hat{\Gamma}_D = D\hat{\Gamma}. \tag{35}$$

The advantage of dealing with this set is that if we solve a Dirichlet problem in $\hat{\Gamma}_D$ with data even in x_n, then for suitable boundary conditions (as in (36) below) the solution in the upper part $\hat{\Gamma}_D \cap \{x_n > 0\}$ will be qualitatively similar to that of (33). Quantitative estimates on the real accuracy of this substitution will be derived in Proposition 3.12, and will turn out to be sufficiently good for our purposes.

Our next goal is to consider the following problem

$$\begin{cases} -\Delta\tilde{\varphi} + \tilde{\varphi} = 0 & \text{in } \hat{\Gamma}_{dD}; \\ \tilde{\varphi} = U(\cdot - dQ_0) & \text{on } \partial\hat{\Gamma}_{dD}, \end{cases} \qquad Q_0 = (-1, 0, \dots, 0). \tag{36}$$

The reason for studying (36) is that, using the coordinates y introduced in Subsection 2.2 (see in particular (24)), the function $U(\cdot - dQ_0)$ stands for the main term in the approximate solution $\overline{z}_{\varepsilon,Q}$, up to a translation in the y_1 axis: this is how we are modeling (33) (in the above coordinates y), when $u = \overline{z}_{\varepsilon,Q}$. By a scaling of the variables, the latter problem is clearly equivalent to

$$\begin{cases} -\frac{1}{d^2}\Delta\varphi + \varphi = 0 & \text{in } \hat{\Gamma}_D; \\ \varphi = U(d(\cdot - Q_0)) & \text{on } \partial\hat{\Gamma}_D. \end{cases} \tag{37}$$

3.1 Asymptotic analysis of (37)

First of all we notice that (37) is solvable: for example one can use the classical method by Perron, provided one knows that barrier functions exist. This is guaranteed by the following result. Its proof could be probably derived from a modification of some arguments in [20]. since however we did not find a direct reference for our case, we give a sketch of the construction.

Lemma 3.1 *The set $\hat{\Gamma}$ admits barrier functions for the operators Δ and $-\Delta + 1$ at all boundary points.*

PROOF. It is clearly sufficient to prove the claim only for the points which are not regular. We begin by considering the Green's function $G(\cdot, \cdot)$ of $-\Delta + 1$ in $\mathbb{R}^n$: when the singularity is at the origin $G(\cdot, 0)$

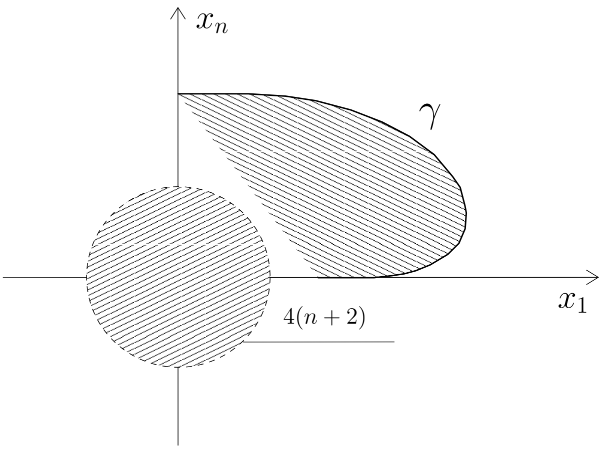

Figure 2: The curve γ in the (x_1, x_n) plane.

is radial and by standard ODE analysis one can show that

$$|x|^{n-2}G(x,0) \to c_n \quad |x| \to 0; \qquad e^{|x|}|x|^{\frac{n-1}{2}}G(x,0) \to c_n' \quad |x| \to +\infty, \tag{38}$$

where c_n, c_n' are positive dimensional constants (see for instance [42], pages 130-133, for more details). Then, given a smooth radial compactly supported non decreasing mollifier function $\varrho(x)$, we can consider the scaled one $\varrho_\lambda(x) = \lambda^n \varrho(\lambda x)$, and for $z \in \partial\hat{\Gamma}_D$ define

$$f_{z,\lambda}(x) = \int_{\partial\hat{\Gamma}_D} \varrho_\lambda(y-z)G(y,x)d\sigma(y), \qquad x \in \hat{\Gamma}_D.$$

By the first formula in (38) and since $\partial\hat{\Gamma}_D$ is $n-1$ dimensional one can show that $f_{z,\lambda}$ is continuous up to the boundary of $\hat{\Gamma}_D$. To sketch the proof of this fact for a singular point, we can ideally substitute $\partial\hat{\Gamma}_D$ with $\mathbb{R}^{n-1} \cap \{y_1' \geq 0\}$ and ϱ_λ with $\chi_{|y'|<1}$. Then the function $f_{z,\lambda}$ becomes

$$f_{z,\lambda}(x) = \int_{\mathbb{R}^{n-1}\cap\{y_1'\geq 0\}} \chi_{B_1(0)}(y')G(y',x)dy',$$

satisfies $-\Delta f_{z,\lambda} + f_{z,\lambda} = 0$ and has clearly a maximum at $x = 0$. Its continuity at the origin follows by dominated convergence: indeed, suppose that $|x_n|^2 + |x'|^2 < R^2$. By (38) we have that the integrand is bounded by

$$c_n' \frac{\chi_{|x'-y'|<1}(|y'|)}{(x_n^2 + |y'|^2)^{\frac{n-2}{2}}} \leq c_n' \frac{\chi_{|y'|<R+1}(|y'|)}{|y'|^{n-2}},$$

which is of class L^1 in $\mathbb{R}^{n-1}$, so continuity follows.

Furthermore, by the exponential decay of G (see the second formula in (38)), if λ is sufficiently large $f_{z,\lambda}$ will be peaked near some point $P_{z,\lambda} \in \partial\hat{\Gamma}_D$ close to z, and $P_{z,\lambda}$ will be its unique maximum. Finally, fixing any $P \in \partial\hat{\Gamma}_D$, we can vary the position of z, depending on λ and P so that $P_{z,\lambda}$ coincides with P: this choice of z, for λ large, will provide a barrier function which is maximal only at P, and which solves $\Delta = u$ in the interior of the domain. With similar constructions one can find barriers which satisfy the same equation and which are minimal at any given point of the boundary. ■

Remark 3.2 *It is worth pointing out that the previous result holds only for domains with cuts since the function in latter formula is only integrable in dimension greater or equal to* $n-1$*. The above argument does not apply to domains with spines as in Lebesgue's counterexample.*

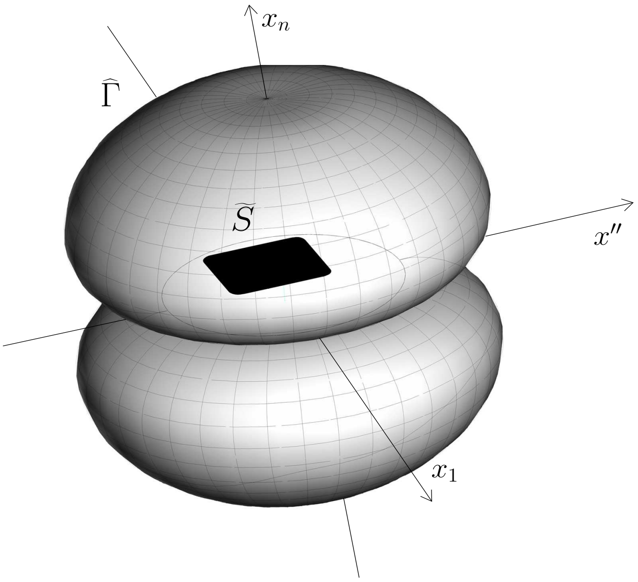

Figure 3: The domain $\hat{\Gamma}$.

As in [40], if we consider the function $\phi = -\frac{1}{d}\log\varphi$, then ϕ satisfies

$$\left\{ \begin{array}{ll} \frac{1}{d}\Delta\phi - |\nabla\phi|^2 + 1 = 0 & \text{in } \hat{\Gamma}_D; \\ \phi = -\frac{1}{d}\log\left(U(d(\cdot - Q_0))\right) & \text{on } \partial\hat{\Gamma}_D, \end{array} \right. \tag{39}$$

where, we recall, $Q_0 = (-1, 0, \dots, 0)$.

Lemma 3.3 *For any fixed constant $D > 0$ we have that*

$$-\frac{1}{d}\log U(d(\cdot - Q_0)) \to |x - Q_0| \qquad \textit{uniformly on } \partial\hat{\Gamma}_D \tag{40}$$

as $d \to +\infty$.

Proof. By (4), the function U satisfies $U(x) = \alpha_{n,p} e^{-|x|} \frac{1}{|x|^{\frac{n-1}{2}}}(1 + o_{|x|}(1))$ as $|x| \to +\infty$, so we find

$$-\frac{1}{d}\log U(d(x - Q_0)) = |x - Q_0| + O\left(\frac{1}{d}|\log(d|x - Q_0|)|\right), \qquad x \in \partial\hat{\Gamma}_D, d \to +\infty.$$

For $x \in \partial\hat{\Gamma}_D$, we have by the construction of $\hat{\Gamma}$ that $1 \leq |x - Q_0| \leq CD$, for a fixed constant C, so we obtain the desired claim provided

$$\log(dD) = o(d),$$

which is the case since D is fixed. ■

By Lemma 3.3, the boundary datum is everywhere close to the function $|x - Q_0|$, so it is useful for us to consider the following auxiliary problem

$$\left\{ \begin{array}{ll} \frac{1}{d}\Delta\phi - |\nabla\phi|^2 + 1 = 0 & \text{in } \hat{\Gamma}_D; \\ \phi = |x - Q_0| & \text{on } \partial\hat{\Gamma}_D. \end{array} \right. \tag{41}$$

Lemma 3.4 *Suppose $D > 1$ is a fixed constant. Then, when $d \to \infty$, problem (41) has a unique solution ϕ^d, which is everywhere positive, and which more precisely satisfies the estimates*

$$1 \leq \phi^d(x) \leq C \qquad \text{in } \hat{\Gamma}_D, \tag{42}$$

where C depends only on D and $\hat{\Gamma}$.

Proof. Proving existence and uniqueness is rather easy: one can apply the transformation inverse to the one at the beginning of this subsection, and study

$$\begin{cases} -\Delta\varphi + \varphi = 0 & \text{in } \hat{\Gamma}_{dD}; \\ \varphi = e^{-|x-dQ_0|} & \text{on } \partial\hat{\Gamma}_{dD}. \end{cases}$$

Existence for the latter problem follows from Lemma 3.1, while uniqueness and positivity of ϕ^d follows from the maximum principle.

To prove (42) we use suitable barrier functions, exactly as in [40], Lemma 4.2: the barrier argument can be applied to (41) even if the domain is singular, by Lemma 3.1. Since $dist(Q_0, \partial\hat{\Gamma}_D) = 1$, the function $\phi^d_- \equiv 1$ in $\hat{\Gamma}_D$ is a subsolution to (41). On the other hand, for C sufficiently large the function $\phi^d_+(x) = C + 2x_1$ is a supersolution, and then our claim follows. ■

We next show some improved pointwise bounds on ϕ^d, which in particular imply a control on the gradient within some region in the boundary of $\hat{\Gamma}_D$. Here and below, many of the arguments in [40] break down if we try to use them for non smooth domains, and we need to perform suitable adaptations. In fact, we obtain gradient bounds only near smooth parts of the boundary, and in general weaker Sobolev norm bounds, see also the beginning of Subsection 3.2.

Lemma 3.5 *Suppose $\tilde{S}$ is as at the beginning of this section, and that D is as in Lemma 3.4. Then there exists a fixed constant $C > 0$ such that for any $0 < \rho < \frac{1}{8}$ there exists $d_\rho > 0$ so large that*

$$|\phi^d(x) - \phi^d(z_x)| \leq C|x - z_x|, \qquad z_x \in \partial\hat{\Gamma}_D, dist(z_x, D\tilde{S}) \geq \rho, |x - z_x| \leq \frac{1}{2}, \quad d \geq d_\rho.$$

In the above formula z_x denotes the point in $\partial\hat{\Gamma}_D$ closest to x (which is unique if D is sufficiently large).

Proof. Let us fix $0 < \rho < \frac{1}{8}$, and let us consider the function $\hat{\phi}^d_+$ defined in this way. For any $x \in \partial\hat{\Gamma}_D$ with $dist(x, D\tilde{S}) \geq \rho$, let n_x denote the unit normal to $\partial\hat{\Gamma}_D$ at x. Since $\partial\hat{\Gamma}_D$ is smooth away from $D\tilde{S}$, since it is flat near this set and since its curvatures tend to zero uniformly as $D \to +\infty$, the sets

$$\Sigma_t := \left\{ z + t n_z \ : \ z \in \partial\hat{\Gamma}_D, dist(z, D\tilde{S}) \geq \rho \right\},$$

where n_z is the inner unit normal at z, constitute a smooth manifold with boundary for any $t \in \left(0, \frac{1}{2}\right]$. Moreover the Σ_t's are flat near their boundary and they are all disjoint as t varies in $\left(0, \frac{1}{2}\right]$. Below in this proof (and only here), we use the convention that the distance function from $\partial D\tilde{S}$ in $\{x_n = 0\}$ is extended inside the interior of $D\tilde{S}$ with negative sign. We then define the function

$$\hat{\phi}^d_+(y) = |z - Q_0| + \theta t; \qquad y \in \left\{ z + t n_z \ : \ z \in \partial\hat{\Gamma}_D, dist(z, D\tilde{S}) \geq \rho, t \in \left[0, \frac{1}{2}\right] \right\},$$

where θ is a large constant so that $\hat{\phi}^d_+(y) > \phi^d(y)$ when $y \in \left\{ z + \frac{1}{2} n_z \ : \ z \in \partial\hat{\Gamma}_D, dist(z, D\tilde{S}) \geq \rho \right\}$. The existence of such constant θ is guaranteed by Lemma 3.4.

Next we consider a smooth non-increasing function $\chi_\rho : \left[-\frac{1}{2}, \rho\right] \to \mathbb{R}$ satisfying the following properties

$$\begin{cases} \chi_\rho(t) = 0 & \text{for } t \in \left[\frac{5}{6}\rho, \rho\right]; \\ \chi'_\rho(t) < 0 & \text{for } t \in \left[-\frac{1}{2}, \frac{2}{3}\rho\right], \end{cases}$$

and another smooth non decreasing cutoff function $\tilde{\chi}_\rho : \left[-\frac{1}{2}, \rho\right]$ such that

$$\begin{cases} \tilde{\chi}_\rho(t) = 0 & \text{for } t \leq \frac{1}{3}\rho; \\ \tilde{\chi}_\rho(t) = 1 & \text{for } t \geq \frac{2}{3}\rho. \end{cases}$$

Finally we extend the function $\hat{\phi}^d_+$ to the set

$$\left\{ z + t n_z \; : \; z \in \mathbb{R}^{n-1}, \; dist(z, D\tilde{S}) \in \left[-\frac{1}{2}, \rho \right] \right\}$$

(we are using the above convention on the distance function) in the following way

$$\hat{\phi}^d_+(z, x_n) = |z - Q_0| + \theta \, |x_n| \, \tilde{\chi}_\rho \left(dist(z, D\tilde{S}) \right) + C_\rho \chi_\rho \left(dist(z, D\tilde{S}) \right). \tag{43}$$

For $dist(z, D\tilde{S}) \geq \frac{2}{3}\rho$, $\tilde{\chi}_\rho \left(dist(z, D\tilde{S}) \right)$ is equal to 1, so the norm of the x_n-component of $\nabla \hat{\phi}^d_+$ is bounded below by θ. Also, for $dist(z, D\tilde{S}) \leq \frac{2}{3}\rho$, $\chi'_\rho \left(dist(z, D\tilde{S}) \right)$ is bounded above by a fixed negative constant. It follows that, for θ and C_ρ sufficiently large, the norm of $\nabla \hat{\phi}^d_+$ can be made arbitrarily big on its domain. By (42), if θ and C_ρ are large then $\hat{\phi}^d_+$ is everywhere bigger than ϕ^d on $\left\{ dist(\cdot, \partial \hat{\Gamma}_D) = \frac{1}{2} \right\}$, so $\hat{\phi}^d_+$ is a supersolution of (41) in $\left\{ dist(\cdot, \partial \hat{\Gamma}_D) < \frac{1}{2} \right\}$.

On the other hand, we claim that the function $\phi^d_- = |x - Q|$ is a subsolution of (41) in the set $\hat{\Gamma}_D \cap \left\{ dist(\cdot, \partial \hat{\Gamma}_D) < \frac{1}{2} \right\}$. In fact, consider first $\hat{\Gamma}_D \setminus B_{\tilde{\delta}(\varepsilon)}(Q_0)$, where $\tilde{\delta}(d)$ is a small positive number depending on d. Here ϕ^d_- satisfies

$$d \Delta \phi^d_- - |\nabla \phi^d_-|^2 + 1 = \frac{n-1}{d|x - Q_0|},$$

and moreover, if we choose $\tilde{\delta}(d)$ sufficiently small, we have that $\phi^d_- < \phi^d$, since ϕ^d is positive. Therefore we obtain that $\phi^d_- \leq \phi^d$ in the closure of $\hat{\Gamma}_D \cap \left\{ dist(\cdot, \partial \hat{\Gamma}_D) < \frac{1}{2} \right\}$.

Finally, since ϕ^d_- and $\hat{\phi}^d_+$ coincide on $\left\{ x \in \partial \hat{\Gamma}_D : dist(x, D\tilde{S}) \geq \rho \right\}$ and have uniform bounds on the gradient here (independently of d), the conclusion is achieved. ■

The gradient estimate which follows from the previous lemma is extended next to a subset of the interior of the domain.

Lemma 3.6 *Suppose D, ϕ^d are as in Lemma 3.4. Then there exists a fixed constant $C > 0$ such that for any $0 < \rho < \frac{1}{8}$ there exists $d_\rho > 0$ so large that*

$$|\nabla \phi^d(x)| \leq C \qquad in \; \left\{ x \in \overline{\hat{\Gamma}}_D : dist(x, D\partial\tilde{S}) \geq \rho \right\}, \quad d \geq d_\rho.$$

Proof. Recall that $\varphi^d := e^{-d\phi^d}$ satisfies the equation $-\frac{1}{d^2}\Delta\varphi^d + \varphi^d = 0$ in $\hat{\Gamma}_D$, and hence the function $\tilde{\varphi}^d := \varphi^d(\cdot/d)$ solves

$$\begin{cases} -\Delta \tilde{\varphi}^d + \tilde{\varphi}^d = 0 & \text{in } \hat{\Gamma}_{dD}; \\ \tilde{\varphi}^d(\cdot) = e^{-|\cdot - dQ_0|} & \text{on } \partial \hat{\Gamma}_{dD}. \end{cases}$$

Notice that $|\nabla \phi^d| \leq C$ is bounded in the desired set if and only if $\frac{|\nabla \tilde{\varphi}^d|}{\tilde{\varphi}^d} \leq C$ in its d-dilation.

By the Harnack inequality and by standard elliptic estimates we obtain immediately that

$$\frac{|\nabla \tilde{\varphi}^d|}{\tilde{\varphi}^d} \leq C_n \qquad \text{in } \left\{ x \in \hat{G}_{dD} : dist(x, \partial \hat{\Gamma}_{dD}) > 1 \right\},$$

where C_n depends only on the dimension n. Therefore we only need to show the estimate in a neighborhood of (a subset of) the boundary, which is done using a blow-up argument. Suppose by contradiction to the statement that we have the following condition

$$\text{for every } m \in \mathbb{N} \text{ there exists } \rho_m \text{ such that for every } \varepsilon > 0 \text{ there exist}$$
$$Q_m \in \hat{\Gamma}_D \text{ with } dist(Q_m, D\tilde{S}) \geq \rho_m \text{ and } d < \varepsilon \text{ with } |\nabla \phi^d(Q_m)| \geq m.$$

If d_{ρ_m} is the constant in Lemma 3.5 corresponding to ρ_m, let us choose d_m such that $d_m = \max\{m, 2d_{\rho_m}\}$. Let us also choose $\tilde{Q}_m$ for which

$$M_m := |\nabla \phi^{d_m}(\tilde{Q}_m)| = \sup_{A_m} |\nabla \phi^{d_m}(Q)|; \qquad A_m = \{Q \in \hat{\Gamma}_D \ : \ dist(Q, D\tilde{S}) \geq \rho_m\}.$$

Consider the functions $\tilde{\varphi}^{d_m}(\cdot) := e^{-d_m \phi^{d_m}(\cdot/d_m)}$, and the new sequence

$$v_m(x) := \frac{\tilde{\varphi}^{d_m}\left(d_m \tilde{Q}_m + \frac{x}{M_m}\right)}{\tilde{\varphi}^{d_m}(d_m \tilde{Q}_m)}. \tag{44}$$

Each v_m satisfies

$$-\Delta v_m + \frac{1}{M_m^2} v_m = 0 \qquad \text{in} \quad M_m d_m(\hat{\Gamma}_D - Q_m), \tag{45}$$

and moreover

$$v_m > 0; \qquad v_m(0) = 1; \qquad |\nabla v_m(0)| = 1; \qquad \sup_{x \in M_m d_m(A_m - Q_m)} \frac{|\nabla v_m(x)|}{v_m(x)} \leq 1. \tag{46}$$

Depending on the asymptotic behavior of $\tilde{Q}_m$, d_m and M_m, we have one of the following two possibilities (up to a subsequence)

(**a**) the component of $M_m d_m(\hat{\Gamma}_D - Q_m)$ containing the origin converges to $\mathbb{R}^n$;

(**b**) the component of $M_m d_m(\hat{\Gamma}_D - Q_m)$ containing the origin converges to a half space $\{x_n > B\}$,

for some $B \leq 0$. On the other hand, we also have one of the following three cases (still, up to a subsequence)

(**c**) the component of $M_m d_m(A_m - Q_m)$ containing the origin converges to $\mathbb{R}^n$;

(**d**) the component of $M_m d_m(A_m - Q_m)$ containing the origin converges to a half space;

(**e**) the component of $M_m d_m(A_m - Q_m)$ containing the origin converges to a quadrant.

In the above alternatives, by *convergence* we mean Hausdorff convergence once we take the intersection with any fixed compact set of $\mathbb{R}^n$. Case (**c**) can only occur when (**a**) holds, and case (**e**) only when (**b**) holds. Calling $\overline{A}$ the limit of the sets $M_m d_m(A_m - Q_m)$, by (46) and by the Ascoli theorem we have convergence (in any smooth sense) of v_m on the compact sets of $\overline{A}$ to some non-negative harmonic function $v : \overline{A} \to \mathbb{R}$ for which $v(0) = 1, |\nabla v(0)| = 1$.

If (**a**) and (**c**) hold, then v must be constant on $\mathbb{R}^n$ by the Liouville theorem, which contradicts the fact that $|\nabla v(0)| = 1$. If (**a**) and (**d**) hold, by (45) we have the Harnack inequality for v_m in any fixed compact set of $\mathbb{R}^n$, provided m is sufficiently large: again by the Ascoli theorem we get convergence to a non-negative entire harmonic function and reach a contradiction as in the previous case.

If (**b**) holds, we can use Lemma 3.5 (including the notation in its statement), the fact that $dz_{x/d} = z_x$ and $\tilde{\phi}^d(\cdot) = e^{-d\phi^d(\cdot/d)}$ to obtain

$$e^{-|z_x - d_m Q_0|} e^{-C|x-z_x|} \leq \tilde{\varphi}^{d_m}(x) \leq e^{-|z_x - d_m Q_0|} e^{C|x-z_x|}; \qquad |x - z_x| \leq \frac{d_m}{2}.$$

Using (44) and the fact that $v_m(0) = 1$, we see that the v_m's converge uniformly to the constant 1 in any given compact set of $\{x_n \geq B\}$. Elliptic regularity results imply indeed convergence in any smooth sense to 1, which is again in contradiction to $|\nabla v_m(0)| = 1$). ■

We are now in position for analyzing the asymptotic behavior of the solutions to (39), whose existence (and uniqueness) can be deduced as in Lemma 3.4.

Proposition 3.7 *For D large but fixed, let Φ^d denote the solution of* (39)*. Then, as $d \to +\infty$, we have*

$$\Phi^d(x) \to \left(\left(1 + \sqrt{x_1^2 + x_n^2} \right)^2 + |x''|^2 \right)^{\frac{1}{2}}$$

uniformly on the compact sets of $\left[\overline{\hat{\Gamma}_D} \setminus (\partial D\tilde{S}) \right] \cap \overline{B}_{\frac{D}{4}}(0)$.

PROOF. Since ϕ^d satisfies (41), we have that $\phi^d + \sup_{x \in \partial\hat{\Gamma}_D} \left| |x - Q_0| + \frac{1}{d} \log U(d(x - Q_0)) \right|$ is a supersolution of (39), while $\phi^d - \sup_{x \in \partial\hat{\Gamma}_D} \left| |x - Q_0| + \frac{1}{d} \log U(d(x - Q_0)) \right|$ is a subsolution. Since Φ^d lies in between these two, by Lemma 3.3 we are reduced to prove the analogous statement for ϕ^d. The proof of the latter fact is a consequence of Lemmas 3.8 and 3.9 below. ■

Lemma 3.8 *If ϕ^d is as in Lemma 3.4 we have that*

$$\phi^d(x) \to \overline{\phi}(x) := \inf_{z \in \partial\hat{\Gamma}_D} \left(|x - z| + |z - Q_0| \right) \qquad \text{as } d \to \infty, \tag{47}$$

uniformly on the compact sets of $\overline{\hat{\Gamma}_D} \setminus \partial D\tilde{S}$.

PROOF. First of all, by Lemma 3.6, in any set compactly contained $\overline{\hat{\Gamma}_D} \setminus \partial D\tilde{S}$ the gradient of ϕ^d is uniformly bounded, provided d is sufficiently large: hence by Ascoli's theorem we know that the ϕ^d's admit limit in the whole closure of $\hat{\Gamma}_D$.

The rest of the proof is a modification of the arguments in Lemma 4.3 of [40]: there it is shown that, when $\hat{\Gamma}_D$ is replaced by a smooth domain Λ, the function on the right-hand side of (47) turns out to be the supremum of all the elements of

$$\mathcal{S}_{\Lambda, |\cdot - Q_0|} = \left\{ v \in W^{1,\infty}(\Lambda) \ : \ v(x) \leq |x - Q_0| \text{ on } \partial\Lambda, |\nabla v| \leq 1 \text{ a.e. in } \Lambda \right\}. \tag{48}$$

We can actually reduce ourselves to this situation. First of all, in $\mathbb{R}^{n-1}$, we have this fact once we take Λ to be $D\tilde{S}$, since its boundary is smooth and since the infimum will be attained on $\partial D\tilde{S}$. Let us call $\overline{\phi}_{D\tilde{S}} : D\tilde{S} \to \mathbb{R}$ the function given by

$$\overline{\phi}_{D\tilde{S}}(\mathrm{x}) := \inf_{z \in \partial D\tilde{S}} \left(|\mathrm{x} - z| + |z - Q_0| \right); \qquad \mathrm{x} \in D\tilde{S},$$

which is a Lipschitz function with Lipschitz constant bounded by 1.

Next, if we restrict ourselves to the set

$$\Lambda_D := \hat{\Gamma}_D \cap \{x_n > 0\},$$

which is now smooth, and if we consider the function $\tilde{\phi}_{D\tilde{S}} : \partial\Lambda_D \to \mathbb{R}$ defined as

$$\tilde{\phi}_{D\tilde{S}}(z) = \begin{cases} |z - Q_0| & z \in \{x_n \geq 0\} \cap \partial\hat{\Gamma}_D; \\ \overline{\phi}_{D\tilde{S}}(z) & z \in D\tilde{S}, \end{cases}$$

we claim that

$$\inf_{z\in\partial\hat{\Gamma}_D} (|x-z| + |z-Q_0|) = \inf_{z\in\partial\Lambda_D} \left(|x-z| + \tilde{\phi}_{D\tilde{S}}\right), \qquad x_n > 0. \tag{49}$$

In fact, suppose z_1 realizes the infimum on the right-hand side. If $z_1 \notin D\tilde{S}$ we are done, since $\tilde{\phi}_{D\tilde{S}}$ and $|\cdot - Q_0|$ coincide on $(\partial\Lambda_D) \setminus D\tilde{S}$. If instead $z_1 \in D\tilde{S}$, by the definition of $\tilde{\phi}_{D\tilde{S}}$ there exists $z_2 \in \partial D\tilde{S}$ such that

$$\tilde{\phi}_{D\tilde{S}}(z_1) = |z_1 - z_2| + |z_2 - Q_0|.$$

Therefore we have

$$\begin{aligned} \inf_{z\in\partial\hat{\Gamma}_D} (|x-z| + |z-Q_0|) &\leq |x-z_2| + |z_2-Q_0| \leq |x-z_1| + |z_1-z_2| + |z_2-Q_0| \\ &= |x-z_1| + \tilde{\phi}_{D\tilde{S}}(z_1) = \inf_{z\in\partial\hat{\Gamma}_D} (|x-z| + |z-Q_0|), \end{aligned}$$

where the last equality is a consequence of our definition and our assumption on z_1, so the claim follows. Hence we can apply the result in [40] in the smooth domain Λ_D, with boundary datum $\tilde{\phi}_{D\tilde{S}}$, and use (49).

Let us now prove next convergence to the function $\overline{\phi}$: similarly to Step 1 in the proof of Lemma 4.3 in [40] (and as noticed before) we can apply Ascoli's theorem (by Lemma 3.6) to guarantee existence of a limit for $d \to \infty$, which indeed must belong to the set $\mathcal{S}_{\hat{\Gamma}_D, |\cdot - Q_0|}$ (see the notation in (48)). We need then to prove only $\lim_{d\to\infty} \phi^d \geq \overline{\phi}$.

Two comments are in order: first of all the solutions are even (by uniqueness and since the boundary datum in (39) is), and we can limit ourselves to take only even functions in the definition of $\overline{\phi}$. Second, we can extend in a Lipschitz manner every $v \in \mathcal{S}_{\hat{\Gamma}_D, |\cdot - Q_0|}$ (even in x_n) to a δ neighborhood of $\hat{\Gamma}_D$: we can for example consider v restricted to the boundary of $\hat{\Gamma}_D$ and extend it constantly in the normal direction (and stop at $x_n = 0$ if we reach this hyperplane at a distance from the boundary smaller than δ). Once we have this extension, we can use convolutions with mollifiers as for Step 2 in the proof of Lemma 4.3 in [40]. ■

Lemma 3.9 *If $\overline{\phi}$ is as in* (47), *then*

$$\overline{\phi}(x) = \left(\left(1 + \sqrt{x_1^2 + x_n^2} \right)^2 + |x''|^2 \right)^{\frac{1}{2}} \qquad x \in \overline{B}_{\frac{D}{4}}(0). \tag{50}$$

Proof. By construction of $\hat{\Gamma}_D$, the points $z \in \hat{\Gamma}_D$ which realize the infimum will necessarily belong to the set $\{\{x_n = 0\} \cup \{x_1 \geq 0\}\}$, so we can write that

$$\overline{\phi}(x) = \inf_{z\in\{\{x_n=0\}\cup\{x_1\geq 0\}\}} (|x-z| + |z-Q_0|).$$

First of all we claim that the latter infimum can be attained on $\{z_1 = 0\}$. In fact, two possibilities may occur: the first is when $x \in \{\{x_n = 0\} \cup \{x_1 \geq 0\}\}$. In this case the points z realizing the infimum are exactly those which belong to the segment connecting x to Q_0, and we can choose the one for which $z_1 = 0$.

In the second case, when $x \notin \{\{x_n = 0\} \cup \{x_1 \geq 0\}\}$, we can argue as follows: for x and Q_0 fixed, the level sets of the function $z \mapsto |x-z| + |z-Q_0|$ are the axially symmetric ellipsoids with focal points x and Q_0. The smaller is the ellipsoid, the smaller is the value of this function so we are reduced to find the smallest ellipsoid which intersects $\{\{x_n = 0\} \cup \{x_1 \geq 0\}\}$: this will happen at a point where $z_1 = 0$, so we can still reduce ourselves to this situation.

After the above claim has been established, it is sufficient to consider the minimum problem

$$\min_{z'' \in \mathbb{R}^{n-2}} \left((x_1^2 + |z'' - x''|^2 + x_n^2)^{\frac{1}{2}} + (1^2 + |z''|^2)^{\frac{1}{2}} \right). \tag{51}$$

By differentiation we obtain that at a minimum point

$$\frac{z'' - x''}{(x_1^2 + |z'' - x''|^2 + x_n^2)^{\frac{1}{2}}} + \frac{z''}{(1^2 + |z''|^2)^{\frac{1}{2}}} = 0,$$

which implies

$$z'' = x'' \frac{\sqrt{x_1^2 + x_n^2} - 1}{x_1^2 + x_n^2 - 1} = \frac{x''}{\sqrt{x_1^2 + x_n^2} + 1}.$$

If we plug in this expression into (51) we obtain the desired conclusion. ■

Remark 3.10 *Since the convergence in* (40) *is indeed uniform in every smooth sense (by* (4)*), a reasoning as in the proof of Lemma 3.6 hold true for the function* Φ^d *(the solution of* (39)*) as well. Therefore, there still exists a fixed constant* $C > 0$ *such that for any* $0 < \rho < \frac{1}{8}$ *there exists* $d_\rho > 0$ *so large that*

$$|\nabla \Phi^d(x)| \leq C \qquad \text{in } \left\{ x \in \overline{\hat{\Gamma}}_D : dist(x, D\tilde{S}) \geq \rho \right\}, \quad d \geq d_\rho. \tag{52}$$

The arguments in [27] (see in particular Proposition 1.4, Lemma 1.5 and Lemma B.1) imply that

$$\nabla \Phi^d \to \nabla \overline{\phi} \qquad \text{uniformly in } \mathcal{O}, \tag{53}$$

where $\mathcal{O}$ *is any set compactly contained in* $\overline{\hat{\Gamma}_D} \setminus (\partial D\tilde{S})$ *on which* $\nabla \overline{\phi} \neq 0$.

Remark 3.11 *It is also useful to make a few comments on the derivative of* φ *with respect to* d*, where* φ *is the solution of* (37)*. Differentiating* (37) *in* d *we obtain*

$$\begin{cases} -\frac{1}{d^2} \Delta \frac{\partial \varphi}{\partial d} + \frac{\partial \varphi}{\partial d} = -\frac{2}{d^3} \Delta \varphi = -\frac{2}{d} \varphi & \text{in } \hat{\Gamma}_D; \\ \frac{\partial \varphi}{\partial d}(x) = \nabla U(d(x - Q_0)) \cdot (x - Q_0) & \text{on } \partial \hat{\Gamma}_D. \end{cases} \tag{54}$$

By the asymptotics in (4)*-*(5) *there exists a positive constant* C_D *such that for* d *large we have the inequality* $\frac{1}{C_D} U(d(x - Q_0)) \leq -\nabla U(d(x - Q_0)) \cdot (x - Q_0) \leq C_D U(d(x - Q_0))$*. Hence, from the latter formula, from the fact that* $\varphi > 0$, (37) *and the maximum principle we obtain that* $\varsigma := -\frac{\partial \varphi}{\partial d} \geq \frac{1}{C_D} \varphi$ *in* $\hat{\Gamma}_D$*. Moreover, as for* (39) *one checks that* $\Upsilon^d := -\frac{1}{d} \log \varsigma$ *satisfies*

$$\begin{cases} \frac{1}{d} \Delta \Upsilon^d - |\nabla \Upsilon^d|^2 + 1 - \frac{\varphi}{d\varsigma} = 0 & \text{in } \hat{\Gamma}_D; \\ \Upsilon^d(x) = -\frac{1}{d} \log\left(-\nabla U(d(x - Q_0)) \cdot (x - Q_0)\right) & \text{on } \partial \hat{\Gamma}_D. \end{cases}$$

Since (as we just remarked) $\frac{\varphi}{\varsigma}$ *stays bounded,* $\frac{\varphi}{d\varsigma}$ *tends to zero as* $d \to +\infty$*. Moreover by* (5) *the boundary datum in the latter formula converges in every smooth sense (where* $\partial \hat{\Gamma}_D$ *is regular) to* $|x - Q_0|$ *as* $d \to +\infty$*. As a consequence, the previous analysis adapts to* Υ^d *and allows to conclude that still*

$$\Upsilon^d \to \overline{\phi} \quad \text{and} \quad \nabla \Upsilon^d \to \nabla \overline{\phi} \qquad \text{uniformly in } \mathcal{O}, \tag{55}$$

where $\mathcal{O}$ *is as in Remark 3.10.*

3.2 Definition of the approximate solutions and study of their accuracy

In this subsection we construct a manifold of approximate solutions to (M_ε), in order to apply the theory in Subsection 2.1.

First of all, we need to collect some preliminary estimates. If Φ^d is the solution of (39), the function

$$\Xi_d(y) = e^{-d\Phi^d\left(\frac{y}{d}+Q_0\right)} \tag{56}$$

solves the problem

$$\begin{cases} -\Delta\Xi_d + \Xi_d = 0 & \text{in } d(\hat\Gamma_D - Q_0); \\ \Xi_d = U(\cdot) & \text{on } d\partial(\hat\Gamma_D - Q_0), \end{cases} \qquad Q_0 = (-1, 0, \dots, 0). \tag{57}$$

As explained at the beginning of Section 3, we can obtain a solution looking at the minimum problem

$$\inf_{v=U \text{ on } d(\partial\hat\Gamma_D - Q_0)} \left\{ \int_{\Omega_\varepsilon} \left(|\nabla v|^2 + v^2\right) dy \right\}. \tag{58}$$

We can easily derive both norm estimates on Ξ_d from (58), and pointwise estimates on Ξ_d, $\nabla\Xi_d$ from Proposition 3.7 and Remark 3.10 respectively.

To derive a norm estimate, one can take a cutoff function $\chi_1 : d(\overline{\hat\Gamma_D} - Q_0) \to \mathbb{R}$ such that

$$\begin{cases} \chi_1(y) = 1 & \text{for } dist(y, d(\partial\hat\Gamma_D - Q_0)) \le \frac{1}{2}; \\ \chi_1(y) = 0 & \text{for } y \in d(\partial\hat\Gamma_D - Q_0),\ dist(y, d(\partial\hat\Gamma_D - Q_0)) \ge 1; \\ |\nabla\chi_1(y)| \le 4 & \text{for all } y, \end{cases}$$

and then consider the function

$$\overline{v}(y) = \chi_1(y)U(y).$$

It is easy to see (using for example polar coordinates centered at the origin and the asymptotic behavior in (4)) that $\|\overline{v}\|_{H^1(d(\hat\Gamma_D - Q_0))} \le e^{-d(1+o(1))}$, so by (58) we also find that

$$\|\Xi_d\|_{H^1(d(\hat\Gamma_D - Q_0))} \le \|\overline{v}\|_{H^1(d(\hat\Gamma_D - Q_0))} \le e^{-d(1+o(1))}. \tag{59}$$

Here and below, we use the notation $d(1+o(1))$ described at the end of the introduction.

To obtain instead pointwise estimates on Ξ_d, we can use Lemma 3.7 to get that

$$\Xi_d(y) = \exp\left[-\left(\left(d + \sqrt{(y_1-d)^2 + |y_n|^2}\right)^2 + |y''|^2\right)^{\frac{1}{2}}\right] \times e^{o(d)}; \qquad y \in d(\mathcal{O} - Q_0), d \to +\infty, \tag{60}$$

where $\mathcal{O}$ is any set compactly contained in $\overline{\hat\Gamma_D} \setminus \partial D\tilde{S}$. On the other hand, Remark 3.10 implies

$$\begin{aligned} \nabla\Xi_d(y) &= -\exp\left[-\left(\left(d + \sqrt{(y_1-d)^2 + |y_n|^2}\right)^2 + |y''|^2\right)^{\frac{1}{2}}\right] \times e^{o(d)} \\ &\quad \times \left(\nabla\overline{\phi}\left(\frac{y}{d} + Q_0\right) + o(1)\right); \qquad y \in d(\mathcal{O} - Q_0), d \to +\infty, \end{aligned} \tag{61}$$

where $\mathcal{O}$ is as before.

We will obtain next similar bounds and estimates for $\frac{\partial\Xi_d}{\partial d}$ and its gradient: recalling that $\Xi_d(y) = \varphi\left(\frac{y}{d} + Q_0\right)$ (and that also φ depends on d) we have that

$$\frac{\partial\Xi_d}{\partial d}(y) = \frac{\partial\varphi}{\partial d}\left(\frac{y}{d} + Q_0\right) - \frac{y}{d^2} \cdot \nabla\varphi\left(\frac{y}{d} + Q_0\right). \tag{62}$$

Using (54) and reasoning as for (59) one can prove that

$$\left\| \frac{\partial \varphi}{\partial d} \left(\frac{\cdot}{d} + Q_0 \right) \right\|_{H^1(d(\hat{\Gamma}_D - Q_0))} \le e^{-d(1+o(1))}. \tag{63}$$

On the other hand, from (57) one finds that the function $\varpi := \frac{y}{d^2} \cdot \nabla \varphi \left(\frac{y}{d} + Q_0 \right) = \frac{y}{d} \cdot \nabla \Xi_d(y)$ satisfies

$$-\Delta \varpi + \varpi = -\frac{2}{d} \Xi_d \qquad \text{in } d(\hat{\Gamma}_D - Q_0).$$

To control the boundary value of ϖ we divide $\partial d(\hat{\Gamma}_D - Q_0)$ into its intersection with $\{y_n = 0\}$ and its complement. In the first region we have simply that $\varpi = \frac{y}{d} \cdot \nabla U(y)$. In the second instead the estimate in (60) holds true, which shows that the L^2 norm of the trace of ϖ on $d(\partial \hat{\Gamma}_D - Q_0)$ is of order $e^{-d(1+o(1))}$. This fact and the latter formula imply that $\|\varpi\|_{H^1(d(\hat{\Gamma}_D - Q_0))}$ satisfies a bound similar to (63), and hence from these two we conclude that

$$\left\| \frac{\partial \Xi_d}{\partial d} \right\|_{H^1(d(\hat{\Gamma}_D - Q_0))} \le e^{-d(1+o(1))}. \tag{64}$$

By Remark 3.11 (which yields $\varphi \le C_D \left| \frac{\partial \varphi}{\partial d} \right|$ and (55)) together with the Harnack inequality (which implies $|\nabla \varphi| \le C d \varphi$ in $d(\mathcal{O} - Q_0)$) one also finds

$$\frac{\partial \Xi_d}{\partial d}(y) = -\exp\left[-\left(\left(d + \sqrt{(y_1 - d)^2 + |y_n|^2} \right)^2 + |y''|^2 \right)^{\frac{1}{2}} \right] \times e^{o(d)} \times \left(1 + O\left(\frac{|y|}{d} \right) \right) \tag{65}$$

for $y \in d(\mathcal{O} - Q_0)$ and $d \to +\infty$, and moreover

$$\left| \nabla \frac{\partial \Xi_d}{\partial d}(y) \right| \le \exp\left[-\left(\left(d + \sqrt{(y_1 - d)^2 + |y_n|^2} \right)^2 + |y''|^2 \right)^{\frac{1}{2}} \right] \times e^{o(d)}; \qquad y \in d(\mathcal{O} - Q_0), d \to +\infty. \tag{66}$$

After these preliminaries, we are now in position to introduce our approximate solutions. Define next two smooth non negative cutoff functions $\chi_D : \mathbb{R}^n \to \mathbb{R}$, $\chi_0 : \mathbb{R} \to \mathbb{R}$ satisfying respectively

$$\begin{cases} \chi_D(y) = 1 & \text{for } |y| \le \frac{dD}{16}; \\ \chi_D(y) = 0 & \text{for } |y| \ge \frac{dD}{8}; \\ |\nabla \chi_D| \le \frac{32}{dD} & \text{on } \mathbb{R}^n, \end{cases} \qquad \begin{cases} \chi_0(y) = 0 & \text{for } y \le -1; \\ \chi_0(y) = 1 & \text{for } y \ge 0; \\ \chi_0 \text{ is non decreasing} & \text{on } \mathbb{R}. \end{cases}$$

Using then the coordinates y in Subsection 2.2 (see also (23) for the relation between Q and d) we define

$$\hat{z}_{\varepsilon,Q}(y) = \chi_{\mu_0}(\varepsilon y) \left[(U(y) - \Xi_d(y)) \chi_D(y) + \varepsilon \tilde{w}_Q(y) \chi_0(y_1 - d) \right]. \tag{67}$$

Notice that, since $\Xi_d(y)$ coincides with $U(y)$ for $y_n = 0$, $y_1 > d$ and y in the support of χ_D, the function $\hat{z}_{\varepsilon,Q}$ belongs to $H^1_{\mathcal{D}}(\Omega_\varepsilon)$. We also point out that, by our choice of $\hat{\Gamma}_D$, the support of χ_D intersects $d(\partial D\tilde{S} - Q_0)$ only on $\{y_1 = d\}$.

We prove next that the $\hat{z}_{\varepsilon,Q}$'s are good approximate solutions to (M_ε) for suitable conditions on Q.

Proposition 3.12 *Let μ_0 be the constant appearing in Proposition 2.5. Then there exists another constant $C_\Omega > 0$ (independent of ε) such that, for $C_\Omega \le d \le \frac{1}{\varepsilon C_\Omega}$ and for $Dd < \frac{1}{C_\Omega} \frac{\mu_0}{\varepsilon}$, the functions $\hat{z}_{\varepsilon,Q}$ satisfy*

$$\|I'_\varepsilon(\hat{z}_{\varepsilon,Q})\| \le C \left(\varepsilon^2 + \varepsilon e^{-d(1+o(1))} + e^{-\frac{(p+1)d}{2}(1+o(1))} + e^{-\frac{3d}{2}(1+o(1))} \right), \tag{68}$$

for some fixed $C, K > 0$ and for ε sufficiently small.

Proof. Using the coordinates y, let us write $\hat{z}_{\varepsilon,Q} = \overline{z}_{\varepsilon,Q} + \check{z}_{\varepsilon,Q}$ (see (31)), where

$$\check{z}_{\varepsilon,Q} = \chi_{\mu_0}(\varepsilon y) \left[(\chi_D(y) - 1) U(y) - \chi_D(y) \Xi_d(y) + \varepsilon (1 - \chi_0(y_1 - d)) \tilde{w}_Q(y) \right]. \tag{69}$$

With this notation, testing the gradient of I_ε at $\hat{z}_{\varepsilon,Q}$ on any function $v \in H^1_{\mathcal{D}}(\Omega_\varepsilon)$ we can write that

$$\begin{aligned} I'_\varepsilon(\hat{z}_{\varepsilon,Q})[v] &= \int_{\Omega_\varepsilon} \left(\nabla_{g_y} \hat{z}_{\varepsilon,Q} \nabla_{g_y} v + \hat{z}_{\varepsilon,Q} v \right) dy - \int_{\Omega_\varepsilon} \hat{z}^p_{\varepsilon,Q} v \, dy \\ &= \int_{\Omega_\varepsilon} \left(\nabla_{g_y} \overline{z}_{\varepsilon,Q} \nabla_{g_y} v + \overline{z}_{\varepsilon,Q} v \right) dy - \int_{\Omega_\varepsilon} \overline{z}^p_{\varepsilon,Q} v \, dy \\ &+ \int_{\Omega_\varepsilon} \left(\nabla_{g_y} \check{z}_{\varepsilon,Q} \nabla_{g_y} v + \check{z}_{\varepsilon,Q} v \right) dy + \int_{\Omega_\varepsilon} (\overline{z}^p_{\varepsilon,Q} - \hat{z}^p_{\varepsilon,Q}) v \, dy \\ &= I'_\varepsilon(\overline{z}_{\varepsilon,Q})[v] + A_1 + A_2, \end{aligned} \tag{70}$$

where

$$A_1 = \int_{\Omega_\varepsilon} \left(\nabla_{g_y} \check{z}_{\varepsilon,Q} \nabla_{g_y} v + \check{z}_{\varepsilon,Q} v \right) dy; \qquad A_2 = \int_{\Omega_\varepsilon} (\overline{z}^p_{\varepsilon,Q} - \hat{z}^p_{\varepsilon,Q}) v \, dy.$$

By Proposition 2.7 (see (32)) we have that $I'_\varepsilon(\overline{z}_{\varepsilon,Q})[v]$ is of order at most ε^2, so we only need to estimate A_1 and A_2 in the last line of (70).

To estimate A_1 we divide further $\check{z}_{\varepsilon,Q}$ into the three parts $\check{z}_{\varepsilon,Q,1}, \check{z}_{\varepsilon,Q,2}$ and $\check{z}_{\varepsilon,Q,3}$

$$\check{z}_{\varepsilon,Q,1} = \chi_{\mu_0}(\varepsilon y)(\chi_D(y) - 1) U(y); \qquad \check{z}_{\varepsilon,Q,2} = \chi_{\mu_0}(\varepsilon y) \chi_D(y) \Xi_d(y);$$

$$\check{z}_{\varepsilon,Q,3} = \chi_{\mu_0}(\varepsilon y) \varepsilon (1 - \chi_0(y_1 - d)) \tilde{w}_Q(y),$$

and write $A_1 = A_{1,1} + A_{1,2} + A_{1,3}$, where

$$A_{1,i} = \int_{\Omega_\varepsilon} \left(\nabla_{g_y} \check{z}_{\varepsilon,Q,i} \nabla_{g_y} v + \check{z}_{\varepsilon,Q,i} v \right) dy; \qquad i = 1, 2, 3.$$

Since $\chi_D(y)$ is identically equal to 1 for $|y| \leq \frac{dD}{16}$ and since $(1 - \chi_0(y_1 - d)) = 0$ for $y_1 \leq d - 1$, from (4) and (30) one finds

$$|A_{1,1}| \leq e^{-\frac{dD}{16}(1+o(1))} \|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}; \qquad |A_{1,3}| \leq C\varepsilon (1 + |d|^K) e^{-|d|} \|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}. \tag{71}$$

To control $A_{1,2}$ we write that

$$\begin{aligned} A_{1,2} &= \int_{\Omega_\varepsilon} \left(\nabla_{g_y} \check{z}_{\varepsilon,Q,2} \nabla_{g_y} v + \check{z}_{\varepsilon,Q,2} v \right) dy = \int_{\Omega_\varepsilon} \left(g^{ij}_y \partial_i \check{z}_{\varepsilon,Q,2} \partial_j v + \check{z}_{\varepsilon,Q,2} v \right) dy \\ &= \int_{\Omega_\varepsilon} (\nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q,2} \nabla_{\mathbb{R}^n} v + \check{z}_{\varepsilon,Q,2} v) \, dy + \int_{\Omega_\varepsilon} \left(g^{ij}_y - \delta^{ij} \right) \partial_i \check{z}_{\varepsilon,Q,2} \partial_j v \, dy. \end{aligned}$$

From (26) (and the subsequent comments) we have that $|(g_y)^{ij} - \delta_{ij}| \leq C\varepsilon |y|$, and hence

$$\left| A_{1,2} - \int_{\Omega_\varepsilon} (\nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q,2} \nabla_{\mathbb{R}^n} v + \check{z}_{\varepsilon,Q,2} v) \, dy \right| \leq C\varepsilon \left(\int_{\Omega_\varepsilon} |y|^2 |\nabla \check{z}_{\varepsilon,Q,2}|^2 dy \right)^{\frac{1}{2}} \|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}.$$

Since $\check{z}_{\varepsilon,Q,2}$ is supported in $\left\{ |y| \leq \frac{dD}{8} \right\}$, we obtain from the above formula and (59) that

$$\left| A_{1,2} - \int_{\Omega_\varepsilon} (\nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q,2} \nabla_{\mathbb{R}^n} v + \check{z}_{\varepsilon,Q,2} v) \, dy \right| \leq C\varepsilon d D e^{-d(1+o(1))} \|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}.$$

Next, since Ξ_d satisfies (57), we have

$$\int_{\Omega_\varepsilon} (\nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q,2} \nabla_{\mathbb{R}^n} v + \check{z}_{\varepsilon,Q,2} v) \, dy = \int_{\Omega_\varepsilon} (\nabla_{\mathbb{R}^n} (\Xi_d (\chi_{\mu_0}(\varepsilon y) \chi_D - 1)) \nabla_{\mathbb{R}^n} v + (\Xi_d (\chi_{\mu_0}(\varepsilon y) \chi_D - 1)) v) \, dy. \tag{72}$$

Since also $Dd < \frac{1}{C_\Omega}\frac{\mu_0}{\varepsilon}$, the function $\chi_{\mu_0}(\varepsilon y)\chi_D - 1$ is identically zero for $|y| \leq \frac{dD}{16}$ (if C_Ω is sufficiently large), so from (60), (61) and the Hölder inequality one finds (also for D large)

$$\left|\int_{\Omega_\varepsilon} (\nabla_{\mathbb{R}^n}(\Xi_d(\chi_{\mu_0}(\varepsilon y)\chi_D - 1))\nabla_{\mathbb{R}^n} v + (\Xi_d(\chi_{\mu_0}(\varepsilon y)\chi_D - 1))v)\, dy\right| \leq e^{-\frac{dD}{16}(1+o(1))}\|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}. \tag{73}$$

The last three formulas imply

$$|A_{1,2}| \leq \left(\varepsilon dDe^{-d(1+o(1))} + e^{-\frac{dD}{16}(1+o(1))}\right)\|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}.$$

From (71) and the latter formula it follows that

$$|A_1| \leq C\left(\varepsilon dDe^{-d(1+o(1))} + e^{-\frac{dD}{16}(1+o(1))} + \varepsilon(1+|d|^K)e^{-|d|}\right)\|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}. \tag{74}$$

We can now turn to the estimate of A_2: one has the inequalities

$$|\overline{z}^p_{\varepsilon,Q} - \hat{z}^p_{\varepsilon,Q}| \leq \begin{cases} C\overline{z}^{p-1}_{\varepsilon,Q}|\check{z}_{\varepsilon,Q}| & \text{for } \check{z}_{\varepsilon,Q} \in \left(0, \frac{1}{2}\overline{z}_{\varepsilon,Q}\right); \\ C|\overline{z}_{\varepsilon,Q}|^{p-1}|\check{z}_{\varepsilon,Q}| + C|\check{z}_{\varepsilon,Q}|^p & \text{otherwise,} \end{cases} \tag{75}$$

for a fixed constant C depending only on p. First of all we notice the following: by (4) and (30) there exists a (small) constant $c_{K,n}$ such that

$$\overline{z}_{\varepsilon,Q}(y) \geq \frac{7}{8}\frac{e^{-|y|}}{1+|y|^{\frac{n-1}{2}}}; \qquad \text{for } |y| \leq \frac{1}{\varepsilon^{c_{K,n}}}.$$

We divide next Ω_ε into the two regions

$$\mathcal{B}_1 = \left\{|y| < \min\left\{\frac{d}{2}, \frac{1}{\varepsilon^{c_{K,n}}}\right\}\right\}; \qquad \mathcal{B}_2 = \Omega_\varepsilon \setminus \mathcal{B}_1.$$

For $y \in \mathcal{B}_1$ we have that $\chi_{\mu_0}(\varepsilon y) \equiv 1$, $\chi_D(y) \equiv 1$, $\chi_0(y_1 - d) \equiv 1$, and hence $\check{z}_{\varepsilon,Q}(y) \equiv -\Xi_d(y)$. By (60) one has also that $|\check{z}_{\varepsilon,Q}(y)| = |\Xi_d(y)| \leq e^{-\frac{3}{2}d+o(d)} < \frac{1}{2}\overline{z}_{\varepsilon,Q}$ for $y \in \mathcal{B}_1$. This fact, (75) and the Hölder inequality yield

$$\begin{aligned} \int_{\mathcal{B}_1} |\overline{z}^p_{\varepsilon,Q} - \hat{z}^p_{\varepsilon,Q}||v|\, dy &\leq C\int_{\mathcal{B}_1} \overline{z}^{p-1}_{\varepsilon,Q}|\check{z}_{\varepsilon,Q}||v|\, dy \leq C\left(\int_{\mathcal{B}_1} e^{-\frac{3(p+1)}{2}d+o(d)}dy\right)^{\frac{1}{p+1}}\|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)} \\ &\leq Ce^{-\frac{3}{2}d(1+o(1))}\|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}. \end{aligned} \tag{76}$$

On the other hand, in $\mathcal{B}_2$ we have that $|\overline{z}_{\varepsilon,Q}| < C\left(e^{-\frac{d}{2}+o(d)} + e^{-\frac{1+o(1)}{\varepsilon^{c_{K,n}}}}\right)$ and that $|\check{z}_{\varepsilon,Q}| \leq e^{-d+o(d)}$, therefore (75) and the Hölder inequality imply again

$$\int_{\mathcal{B}_2} |\overline{z}^p_{\varepsilon,Q} - \hat{z}^p_{\varepsilon,Q}||v|dy \leq C\left[\left(e^{-\frac{(p-1)d}{2}+o(d)} + e^{-\frac{p-1+o(1)}{\varepsilon^{c_{K,n}}}}\right)e^{-d+o(d)} + e^{-pd+o(d)}\right]\|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}.$$

The last formula and (76) provide

$$|A_2| \leq C\left(e^{-\frac{(p-1)d}{2}+o(d)} + e^{-\frac{p-1+o(1)}{\varepsilon^{c_{K,n}}}} + e^{-\frac{d}{2}+o(d)}\right)e^{-d+o(d)}\|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}. \tag{77}$$

Finally, we obtain the conclusion from (70), (32), (74) and (77). ■

We have next a related estimate when we vary the parameters in the definition of $\hat{z}_{\varepsilon,Q}$, in the spirit of condition *ii)* in Subsection 2.1.

Proposition 3.13 *There exists a constant $C_\Omega > 0$ (independent of ε) such that, for $C_\Omega \le d \le \frac{1}{\varepsilon C_\Omega}$ and for $Dd < \frac{1}{C_\Omega}\frac{\mu_0}{\varepsilon}$, the functions $\hat{z}_{\varepsilon,Q}$ satisfy*

$$\|I''_\varepsilon(\hat{z}_{\varepsilon,Q})[q]\| \le C\left(\varepsilon^2 + \varepsilon e^{-d(1+o(1))} + e^{-\frac{(p+1)d}{2}(1+o(1))} + e^{-\frac{3d}{2}(1+o(1))}\right)\|q\|, \tag{78}$$

for some fixed $C, K > 0$ and for ε sufficiently small. In the above formula q represents a vector in $H^1_{\mathcal{D}}(\Omega_\varepsilon)$ which is tangent to the manifold of the $\hat{z}_{\varepsilon,Q}$'s (when Q varies).

PROOF. Since the arguments are quite similar to those in the proof of Proposition 3.12, we will be rather sketchy. Using (25) and the first line in (70), for any given test function $v \in H^1_{\mathcal{D}}(\Omega_\varepsilon)$ we can write that

$$I'_\varepsilon(\hat{z}_{\varepsilon,Q})[v] = \sum_{i,j}\int_{\mathbb{R}^n_+}\left((g^y)^{ij}\partial_i\hat{z}_{\varepsilon,Q}\partial_j v + \hat{z}_{\varepsilon,Q}v\right)dy - \int_{\mathbb{R}^n_+}\hat{z}^p_{\varepsilon,Q}v\,dy.$$

We want to differentiate next with respect to the parameter Q, taking first a variation q_T of the point Q for which d stays fixed, namely we take the tangential derivative to the level set of the distance d to the interface. In the above formula the dependence on Q is in the metric coefficients $(g^y)^{ij}$ and in the function $\tilde{w}_Q$ appearing in the expression of $\hat{z}_{\varepsilon,Q}$, see (67). Therefore we obtain

$$\begin{aligned}\frac{\partial}{\partial Q_T}I'_\varepsilon(\hat{z}_{\varepsilon,Q})[v] &= I''_\varepsilon(\hat{z}_{\varepsilon,Q})\left[\frac{\hat{z}_{\varepsilon,Q}}{\partial Q_T}, v\right] = \sum_{i,j}\int_{\mathbb{R}^n_+}\frac{\partial(g^y)^{ij}}{\partial Q_T}\partial_i\hat{z}_{\varepsilon,Q_T}\partial_j v\,dy \\ &+ \sum_{i,j}\int_{\mathbb{R}^n_+}\left((g^y)^{ij}\partial_i\frac{\partial\hat{z}_{\varepsilon,Q}}{\partial Q_T}\partial_j v + \frac{\partial\hat{z}_{\varepsilon,Q}}{\partial Q_T}v\right)dy - p\int_{\mathbb{R}^n_+}\hat{z}^{p-1}_{\varepsilon,Q}\frac{\partial\hat{z}_{\varepsilon,Q}}{\partial Q_T}v\,dy.\end{aligned} \tag{79}$$

From Remark 2.6 (b) we have that $\frac{\partial(g^y)^{ij}}{\partial Q}$ is of order $\varepsilon^2|y|$, while computing the expression of $\frac{\partial\hat{z}_{\varepsilon,Q}}{\partial Q_T}$ we obtain $\frac{\partial\hat{z}_{\varepsilon,Q}}{\partial Q_T} = \varepsilon\chi_{\mu_0}(\varepsilon y)\chi_0(y_1-d)\frac{\partial\tilde{w}_Q(y)}{\partial Q_T}$. Since the dependence of $\tilde{w}_Q$ on A_Q, $\tilde{A}_Q$ is linear (see (29)) and since the derivatives of the latter quantities with respect to Q are of order ε (by the arguments in Remark 2.6) we find that $\frac{\partial\hat{z}_{\varepsilon,Q}}{\partial Q_T}(y) = O\left(\varepsilon^2(1+|y|^K)e^{-|y|}\right)$. Reasoning as in the proof of Proposition 3.12 we then have

$$\left\|\frac{\partial}{\partial Q_T}I'_\varepsilon(\hat{z}_{\varepsilon,Q})[v]\right\| \le C\varepsilon^2\|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)} \qquad \text{for every } v \in H^1_{\mathcal{D}}(\Omega_\varepsilon). \tag{80}$$

On the other hand, when we take a variation q_d of Q along the gradient of d, similarly to (79) we get

$$\begin{aligned}\frac{\partial}{\partial Q_d}I'_\varepsilon(\hat{z}_{\varepsilon,Q})[v] &= I''_\varepsilon(\hat{z}_{\varepsilon,Q})\left[\frac{\hat{z}_{\varepsilon,Q}}{\partial Q_d}, v\right] = \sum_{i,j}\int_{\mathbb{R}^n_+}\frac{\partial(g^y)^{ij}}{\partial Q_d}\partial_i\hat{z}_{\varepsilon,Q_d}\partial_j v\,dy \\ &+ \sum_{i,j}\int_{\mathbb{R}^n_+}\left((g^y)^{ij}\partial_i\frac{\partial\hat{z}_{\varepsilon,Q}}{\partial Q_d}\partial_j v + \frac{\partial\hat{z}_{\varepsilon,Q}}{\partial Q_d}v\right)dy - p\int_{\mathbb{R}^n_+}\hat{z}^{p-1}_{\varepsilon,Q}\frac{\partial\hat{z}_{\varepsilon,Q}}{\partial Q_d}v\,dy.\end{aligned} \tag{81}$$

Concerning the derivatives of $(g^y)^{ij}$ with respect to Q_d we can argue exactly as for Q_T, to find

$$\left|\sum_{i,j}\int_{\mathbb{R}^n_+}\frac{\partial(g^y)^{ij}}{\partial Q_d}\partial_i\hat{z}_{\varepsilon,Q_d}\partial_j v\,dy\right| \le C\varepsilon^2\|v\|.$$

However, $\frac{\partial\hat{z}_{\varepsilon,Q}}{\partial Q_d}$ has a more involved expression compared to the previous case. Assuming that the cutoff function $\chi_D(y)$ is defined as $\overline{\chi}_D\left(\frac{y}{d}\right)$ for some fixed $\overline{\chi}_D$, we obtain

$$\begin{aligned}\frac{\partial\hat{z}_{\varepsilon,Q}}{\partial Q_d} &= -\chi_{\mu_0}\chi_D\frac{\partial\Xi_d}{\partial d} + \frac{1}{d^2}\chi_{\mu_0}\Xi_d\,y\cdot\nabla\overline{\chi}_D\left(\frac{y}{d}\right) + \varepsilon\chi_{\mu_0}\tilde{w}_Q\frac{\partial\chi_0(y_1-d)}{\partial d} \\ &+ \varepsilon\chi_{\mu_0}(\varepsilon y)\chi_0(y_1-d)\frac{\partial\tilde{w}_Q(y)}{\partial Q_d}.\end{aligned} \tag{82}$$

The last two terms in the right hand side are easily seen to give contributions to (81) of order at most $\varepsilon e^{-d(1+o(1))}\|v\|$ and $\varepsilon^2 e^{-d(1+o(1))}\|v\|$ respectively. Concerning the second one, we can use the fact that $\nabla\chi_D$ is supported in $|y| \geq \frac{Dd}{16}$, together with (60), (61) to see that the contribution of this term is at most of order $e^{-\frac{dD}{16}(1+o(1))}\|v\|$.

We can then focus on the first term in the right hand side of (82), and consider the quantity

$$-\sum_{i,j}\int_{\mathbb{R}^n_+}\left((g^y)^{ij}\partial_i\left(\chi_{\mu_0}\chi_D\frac{\partial\Xi_d}{\partial d}\right)\partial_j v + \chi_{\mu_0}\chi_D\frac{\partial\Xi_d}{\partial d}v\right)dy + p\int_{\mathbb{R}^n_+}\hat{z}^{p-1}_{\varepsilon,Q}\chi_{\mu_0}\chi_D\frac{\partial\Xi_d}{\partial d}v\,dy. \tag{83}$$

First of all, by (26) and (64), if we substitute the coefficients $(g^y)^{ij}$ with the Kronecker symbols we find a difference of order $\varepsilon e^{-d(1+o(1))}$. Next, since Ξ_d satisfies $-\Delta\Xi_d + \Xi_d = 0$, when we differentiate with respect to d we get the same equation for $\frac{\partial\Xi_d}{\partial d}$, so reasoning as for (72), (73) (and using the fact that $v \in H^1_{\mathcal{D}}(\Omega_\varepsilon)$ together with (65), (66)) we find

$$\left|\int_{\mathbb{R}^n_+}\left(\nabla\left(\chi_{\mu_0}\chi_D\frac{\partial\Xi_d}{\partial d}\right)\cdot\nabla v + \chi_{\mu_0}\chi_D\frac{\partial\Xi_d}{\partial d}v\right)dy\right| \leq Ce^{-\frac{dD}{16}(1+o(1))}\|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}.$$

For the last term in (83) one can use (64), (65) and the exponential decay of $\hat{z}_{\varepsilon,Q}$ (reasoning with arguments similar to those for (77)) to find that it is of order $e^{-d(1+o(1))}\left(e^{-\frac{d}{2}} + e^{-\frac{(p-1)d}{2}} + o(\varepsilon^2)\right)\|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}$. All the above comments yield that

$$\left\|\frac{\partial}{\partial Q_d}I'_\varepsilon(\hat{z}_{\varepsilon,Q})[v]\right\| \leq C\left(\varepsilon^2 + \varepsilon e^{-d(1+o(1))} + e^{-\frac{(p+1)d}{2}(1+o(1))} + e^{-\frac{3d}{2}(1+o(1))}\right)\|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}. \tag{84}$$

From (80) and (84) we finally obtain the desired conclusion. ■

4 Proof of Theorem 1.1

We can now prove our main theorem. We first derive an accurate expansion of the energy of approximate solutions, and then obtain the existence result using the abstract theory in Subsection 2.1.

4.1 Energy expansions for the approximate solutions $\hat{z}_{\varepsilon,Q}$

Here we expand $I_\varepsilon(\hat{z}_{\varepsilon,Q})$ in terms of Q and ε, where $\hat{z}_{\varepsilon,Q}$ is the function defined in (67).

Proposition 4.1 *If $\tilde{C}_0$ and $\tilde{C}_1$ are the constants in Proposition 2.5 and if $d = d(Q)$ is as in Subsection 2.2, then we have the following expansion*

$$I_\varepsilon(\hat{z}_{\varepsilon,Q}) = \tilde{C}_0 - \tilde{C}_1\varepsilon H(\varepsilon Q) + e^{-2d(1+o(1))} + O(\varepsilon^2),$$

as $\varepsilon \to 0$ and $d \to +\infty$.

Proof. As in the proof of Proposition 3.12, let us write $\hat{z}_{\varepsilon,Q} = \overline{z}_{\varepsilon,Q} + \check{z}_{\varepsilon,Q}$, see (31) and (69). Then, using the coordinates y introduced in Subsection 2.2 we find that

$$\begin{aligned} I_\varepsilon(\hat{z}_{\varepsilon,Q}) &= I_\varepsilon(\overline{z}_{\varepsilon,Q}) + \int_{\Omega_\varepsilon}\left(\nabla_{g_y}\overline{z}_{\varepsilon,Q}\nabla_{g_y}\check{z}_{\varepsilon,Q} + \overline{z}_{\varepsilon,Q}\check{z}_{\varepsilon,Q}\right)dy + \frac{1}{2}\int_{\Omega_\varepsilon}\left(|\nabla_{g_y}\check{z}_{\varepsilon,Q}|^2 + \check{z}^2_{\varepsilon,Q}\right)dy \\ &+ \frac{1}{p+1}\int_{\Omega_\varepsilon}\left(|\overline{z}_{\varepsilon,Q}|^{p+1} - |\hat{z}_{\varepsilon,Q}|^{p+1}\right)dy. \end{aligned} \tag{85}$$

Using (26) (and the subsequent comments) we have that

$$\begin{aligned} &\left|\int_{\Omega_\varepsilon}\left(\nabla_{g_y}\overline{z}_{\varepsilon,Q}\nabla_{g_y}\check{z}_{\varepsilon,Q} + \overline{z}_{\varepsilon,Q}\check{z}_{\varepsilon,Q}\right)dy - \int_{\mathbb{R}^n_+}\left(\nabla_{\mathbb{R}^n}\overline{z}_{\varepsilon,Q}\nabla_{\mathbb{R}^n}\check{z}_{\varepsilon,Q} + \overline{z}_{\varepsilon,Q}\check{z}_{\varepsilon,Q}\right)dy\right| \\ &\leq C\varepsilon\int_{\mathbb{R}^n_+}|y|\,|\nabla_{\mathbb{R}^n}\overline{z}_{\varepsilon,Q}|\,|\nabla_{\mathbb{R}^n}\check{z}_{\varepsilon,Q}|dy; \end{aligned} \tag{86}$$

$$\left| \int_{\Omega_\varepsilon} \left(|\nabla_{g_y} \check{z}_{\varepsilon,Q}|^2 + \check{z}_{\varepsilon,Q}^2 \right) dy - \int_{\mathbb{R}^n_+} \left(|\nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q}|^2 + \check{z}_{\varepsilon,Q}^2 \right) dy \right| \leq C\varepsilon \int_{\mathbb{R}^n_+} |y| |\nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q}|^2 dy. \tag{87}$$

Concerning (86), we can divide the domain of integration into $B_{\frac{d}{2}}(0)$ and its complement and use (4), (30), (59), (60), (61) to find

$$C\varepsilon \int_{\Omega_\varepsilon} |y| \, |\nabla_{\mathbb{R}^n} \overline{z}_{\varepsilon,Q}| \, |\nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q}| dy \leq C\varepsilon e^{-\frac{3}{2} d(1+o(1))}.$$

For (87), the same estimates yield

$$C\varepsilon \int_{\Omega_\varepsilon} |y| |\nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q}|^2 dy \leq C\varepsilon e^{-2d(1+o(1))}.$$

The last two formulas, (85), (86) and (87) imply

$$\begin{aligned} I_\varepsilon(\hat{z}_{\varepsilon,Q}) &= I_\varepsilon(\overline{z}_{\varepsilon,Q}) + \int_{\mathbb{R}^n_+} \left(\nabla_{\mathbb{R}^n} \overline{z}_{\varepsilon,Q} \nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q} + \overline{z}_{\varepsilon,Q} \check{z}_{\varepsilon,Q} \right) dy + \frac{1}{2} \int_{\mathbb{R}^n_+} \left(|\nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q}|^2 + \check{z}_{\varepsilon,Q}^2 \right) dy \\ &+ \frac{1}{p+1} \int_{\Omega_\varepsilon} \left(|\overline{z}_{\varepsilon,Q}|^{p+1} - |\hat{z}_{\varepsilon,Q}|^{p+1} \right) dy + O\left(\varepsilon e^{-\frac{3}{2} d(1+o(1))} \right). \end{aligned} \tag{88}$$

Using the same notation as in the proof of Proposition 3.12, we write $\check{z}_{\varepsilon,Q} = \check{z}_{\varepsilon,Q,1} + \check{z}_{\varepsilon,Q,2} + \check{z}_{\varepsilon,Q,3}$. Formulas (4) and (30) imply

$$\left| \int_{\mathbb{R}^n_+} \left(\nabla_{\mathbb{R}^n} \overline{z}_{\varepsilon,Q} \nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q,1} + \overline{z}_{\varepsilon,Q} \check{z}_{\varepsilon,Q,1} \right) dy \right| \leq C e^{-\frac{dD}{16}(1+o(1))};$$

$$\left| \int_{\mathbb{R}^n_+} \left(\nabla_{\mathbb{R}^n} \overline{z}_{\varepsilon,Q} \nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q,3} + \overline{z}_{\varepsilon,Q} \check{z}_{\varepsilon,Q,3} \right) dy \right| \leq C\varepsilon e^{-2d(1+o(1))},$$

from which we deduce that

$$\begin{aligned} \int_{\mathbb{R}^n_+} \left(\nabla_{\mathbb{R}^n} \overline{z}_{\varepsilon,Q} \nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q} + \overline{z}_{\varepsilon,Q} \check{z}_{\varepsilon,Q} \right) dy &= \int_{\mathbb{R}^n_+} \left(\nabla_{\mathbb{R}^n} \overline{z}_{\varepsilon,Q} \nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q,2} + \overline{z}_{\varepsilon,Q} \check{z}_{\varepsilon,Q,2} \right) dy \\ &+ O\left(e^{-\frac{dD}{16}(1+o(1))} + \varepsilon e^{-2d(1+o(1))} \right). \end{aligned}$$

Similar estimates also yield

$$\int_{\mathbb{R}^n_+} \left(|\nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q}|^2 + \check{z}_{\varepsilon,Q}^2 \right) dy = \int_{\mathbb{R}^n_+} \left(|\nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q,2}|^2 + \check{z}_{\varepsilon,Q,2}^2 \right) dy + O\left(e^{-\frac{dD}{16} - d(1+o(1))} + \varepsilon e^{-2d(1+o(1))} \right).$$

From a straightforward computation one finds that for any function v

$$\nabla \check{z}_{\varepsilon,Q,2} \nabla v + \check{z}_{\varepsilon,Q,2} v = \nabla \Xi_d \nabla(\chi_{\mu_0}(\varepsilon \cdot) \chi_D(\cdot) v) + \Xi_d \chi_{\mu_0}(\varepsilon \cdot) \chi_D(\cdot) v + \nabla(\chi_{\mu_0}(\varepsilon \cdot) \chi_D(\cdot)) \left(\Xi_d \nabla v - v \nabla \Xi_d \right).$$

Applying this relation for $v = \overline{z}_{\varepsilon,Q}$ and $v = \check{z}_{\varepsilon,Q}$ respectively, and using (4), (30), (59), (60) and (61) we find that

$$\begin{aligned} &\int_{\mathbb{R}^n_+} \left(\nabla_{\mathbb{R}^n} \overline{z}_{\varepsilon,Q} \nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q} + \overline{z}_{\varepsilon,Q} \check{z}_{\varepsilon,Q} \right) dy \\ = &\int_{\mathbb{R}^n_+} \left(\nabla_{\mathbb{R}^n} (\chi_{\mu_0}(\varepsilon \cdot) \chi_D(\cdot) \overline{z}_{\varepsilon,Q}) \nabla_{\mathbb{R}^n} \Xi_d + \chi_{\mu_0}(\varepsilon \cdot) \chi_D(\cdot) \overline{z}_{\varepsilon,Q} \Xi_d \right) dy + O\left(e^{-\frac{dD}{16}(1+o(1))} \right); \end{aligned}$$

$$\int_{\mathbb{R}^n_+} \left(|\nabla_{\mathbb{R}^n} \check{z}_{\varepsilon,Q}|^2 + \check{z}_{\varepsilon,Q}^2 \right) dy = \int_{\mathbb{R}^n_+} \left(|\nabla_{\mathbb{R}^n} (\chi_{\mu_0}(\varepsilon \cdot) \chi_D(\cdot) \Xi_d)|^2 + (\chi_{\mu_0}(\varepsilon \cdot) \chi_D(\cdot) \Xi_d)^2 \right) dy + O\left(e^{-\frac{dD}{16}(1+o(1))} \right).$$

Using now the fact that, by our construction, the function $\chi_{\mu_0}(\varepsilon\cdot)\chi_D(\cdot)\hat{z}_{\varepsilon,Q} = \chi_{\mu_0}(\varepsilon\cdot)\chi_D(\cdot)(\overline{z}_{\varepsilon,Q} + \check{z}_{\varepsilon,Q})$ vanishes on $d(\hat{\Gamma}_D - Q_0)$, from (57) we obtain

$$\begin{aligned} & \int_{\mathbb{R}^n_+} (\nabla_{\mathbb{R}^n}(\chi_{\mu_0}(\varepsilon\cdot)\chi_D(\cdot)\overline{z}_{\varepsilon,Q})\nabla_{\mathbb{R}^n}\Xi_d + \chi_{\mu_0}(\varepsilon\cdot)\chi_D(\cdot)\overline{z}_{\varepsilon,Q}\Xi_d)\,dy \\ + & \frac{1}{2}\int_{\mathbb{R}^n_+} \left(|\nabla_{\mathbb{R}^n}(\chi_{\mu_0}(\varepsilon\cdot)\chi_D(\cdot)\Xi_d)|^2 + (\chi_{\mu_0}(\varepsilon\cdot)\chi_D(\cdot)\Xi_d)^2\right) dy \\ = & \frac{1}{2}\int_{\mathbb{R}^n_+} (\nabla_{\mathbb{R}^n}(\chi_{\mu_0}(\varepsilon\cdot)\chi_D(\cdot)\overline{z}_{\varepsilon,Q})\nabla_{\mathbb{R}^n}\Xi_d + \chi_{\mu_0}(\varepsilon\cdot)\chi_D(\cdot)\overline{z}_{\varepsilon,Q}\Xi_d)\,dy. \end{aligned}$$

From (88) and the last eight formulas we find

$$\begin{aligned} I_\varepsilon(\hat{z}_{\varepsilon,Q}) &= I_\varepsilon(\overline{z}_{\varepsilon,Q}) + \frac{1}{2}\int_{\mathbb{R}^n_+} (\nabla_{\mathbb{R}^n}\overline{z}_{\varepsilon,Q}\nabla_{\mathbb{R}^n}\check{z}_{\varepsilon,Q} + \overline{z}_{\varepsilon,Q}\check{z}_{\varepsilon,Q})\,dy \\ &+ \frac{1}{p+1}\int_{\Omega_\varepsilon} \left(|\overline{z}_{\varepsilon,Q}|^{p+1} - |\hat{z}_{\varepsilon,Q}|^{p+1}\right) dy + O\left(e^{-\frac{dD}{16}(1+o(1))} + \varepsilon e^{-\frac{3}{2}d(1+o(1))}\right). \end{aligned}$$

From (4), (30), (32), (59) we then obtain

$$\begin{aligned} I_\varepsilon(\hat{z}_{\varepsilon,Q}) &= I_\varepsilon(\overline{z}_{\varepsilon,Q}) + \frac{1}{2}\int_{\Omega_\varepsilon} \overline{z}^p_{\varepsilon,Q}\check{z}_{\varepsilon,Q}dy \\ &+ \frac{1}{p+1}\int_{\Omega_\varepsilon} \left(|\overline{z}_{\varepsilon,Q}|^{p+1} - |\hat{z}_{\varepsilon,Q}|^{p+1}\right) dy + O\left(e^{-\frac{dD}{16}(1+o(1))} + \varepsilon e^{-\frac{3}{2}d(1+o(1))} + \varepsilon^2 e^{-d(1+o(1))}\right). \end{aligned} \tag{89}$$

Using a Taylor expansion we can write that

$$|\overline{z}_{\varepsilon,Q}|^{p+1} - |\hat{z}_{\varepsilon,Q}|^{p+1} = \begin{cases} -(p+1)\overline{z}^p_{\varepsilon,Q}\check{z}_{\varepsilon,Q} + O\left(\overline{z}^{p-1}_{\varepsilon,Q}\check{z}^2_{\varepsilon,Q}\right) & \text{for } \check{z}_{\varepsilon,Q} \in \left(0, \frac{1}{2}\overline{z}_{\varepsilon,Q}\right); \\ O\left(|\overline{z}_{\varepsilon,Q}|^p|\check{z}_{\varepsilon,Q}| + |\check{z}_{\varepsilon,Q}|^{p+1}\right) & \text{otherwise.} \end{cases} \tag{90}$$

As for the estimate of A_2 in (77), we divide the domain into the two regions $\mathcal{B}_1, \mathcal{B}_2$, and deduce that

$$\begin{aligned} & \frac{1}{p+1}\int_{\Omega_\varepsilon} \left(|\overline{z}_{\varepsilon,Q}|^{p+1} - |\hat{z}_{\varepsilon,Q}|^{p+1}\right) dy = -\int_{\Omega_\varepsilon} \overline{z}^p_{\varepsilon,Q}\check{z}_{\varepsilon,Q}dy \\ & +O\left(e^{-3d(1+o(1))} + e^{-\frac{p+3}{2}d(1+o(1))} + e^{-d(1+o(1))}e^{-\frac{1}{\varepsilon^{c_{K,n}}}}\right). \end{aligned}$$

Therefore using (89) the energy becomes

$$\begin{aligned} I_\varepsilon(\hat{z}_{\varepsilon,Q}) &= I_\varepsilon(\overline{z}_{\varepsilon,Q}) - \frac{1}{2}\int_{\Omega_\varepsilon} \overline{z}^p_{\varepsilon,Q}\check{z}_{\varepsilon,Q}dy \\ &+ O\left(e^{-3d(1+o(1))} + e^{-\frac{p+3}{2}d(1+o(1))} + \varepsilon e^{-\frac{3}{2}d(1+o(1))} + \varepsilon^2 e^{-d(1+o(1))}\right). \end{aligned}$$

From (60), the expression of $\check{z}_{\varepsilon,Q}$ and estimates in the same spirit as above one finds that

$$\int_{\Omega_\varepsilon} \overline{z}^p_{\varepsilon,Q}\check{z}_{\varepsilon,Q}dy = -e^{-2d(1+o(1))},$$

and hence from Proposition 2.7 we finally find

$$\begin{aligned} I_\varepsilon(\hat{z}_{\varepsilon,Q}) &= \tilde{C}_0 - \tilde{C}_1\varepsilon H(\varepsilon Q) + O(\varepsilon^2) + e^{-2d(1+o(1))} \\ &+ O\left(e^{-3d(1+o(1))} + e^{-\frac{p+3}{2}d(1+o(1))} + \varepsilon e^{-\frac{3}{2}d(1+o(1))} + \varepsilon^2 e^{-d(1+o(1))}\right). \end{aligned} \tag{91}$$

The conclusion follows from the Schwartz inequality. ■

We have a related result concerning the derivative of the energy with respect to Q: again, we will be rather quick in the proof since the arguments are quite similar to the previous ones.

Proposition 4.2 *If $\tilde{C}_0$ and $\tilde{C}_1$ are the constants in Proposition 2.5 and if $d = d(Q)$ is as in Subsection 2.2, then in the same notation as in the previous section we have the following expansion*

$$\frac{\partial}{\partial Q_T} I_\varepsilon(\hat{z}_{\varepsilon,Q}) = -\tilde{C}_1 \varepsilon^2 \nabla_T H(\varepsilon Q) + o(\varepsilon^2); \tag{92}$$

$$\frac{\partial}{\partial Q_d} I_\varepsilon(\hat{z}_{\varepsilon,Q}) = -\tilde{C}_1 \varepsilon^2 \nabla_d H(\varepsilon Q) - e^{-2d(1+o(1))} + o(\varepsilon^2), \tag{93}$$

as $\varepsilon \to 0$ and $d \to +\infty$.

PROOF. After some elementary calculations, recalling the definition of $\overline{z}_{\varepsilon,Q}$ in (31), we can write

$$\begin{aligned} I'_\varepsilon(\hat{z}_{\varepsilon,Q}) \left[\frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q} \right] &= \frac{\partial}{\partial Q} I_\varepsilon(\overline{z}_{\varepsilon,Q}) + \int_{\Omega_\varepsilon} \left(\nabla_{g_y} \overline{z}_{\varepsilon,Q} \nabla_{g_y} \frac{\partial \check{z}_{\varepsilon,Q}}{\partial Q} + \overline{z}_{\varepsilon,Q} \frac{\partial \check{z}_{\varepsilon,Q}}{\partial Q} \right) dy - \int_{\Omega_\varepsilon} \overline{z}^p_{\varepsilon,Q} \frac{\partial \check{z}_{\varepsilon,Q}}{\partial Q} \, dy \\ &+ \int_{\Omega_\varepsilon} \left(\nabla_{g_y} \check{z}_{\varepsilon,Q} \nabla_{g_y} \frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q} + \check{z}_{\varepsilon,Q} \frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q} \right) dy + \int_{\Omega_\varepsilon} (\overline{z}^p_{\varepsilon,Q} - \hat{z}^p_{\varepsilon,Q}) \frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q} \, dy, \end{aligned} \tag{94}$$

where $\check{z}_{\varepsilon,Q} = \hat{z}_{\varepsilon,Q} - \overline{z}_{\varepsilon,Q}$ was defined in (69). The first term on the right hand side is estimated in Proposition 2.7. The next two, integrating by parts and using Proposition 2.7, can be estimated in terms of a quantity like

$$C \varepsilon^2 \int_{\Omega_\varepsilon} (1 + |y|^K) \left| \frac{\partial \check{z}_{\varepsilon,Q}}{\partial Q} \right|.$$

From the same arguments as in the proof of Proposition 3.13 one deduces that the latter integral is of order $\varepsilon^2 e^{-2d(1+o(1))}$. To control the first integral in the last line of (94) we can reason as for the estimate of $A_{1,2}$ in the proof of Proposition 3.12 to see that this is of order $e^{-d(1+o(1))}(\varepsilon + e^{-d}) \left\| \frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q} \right\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)}$. From the proof of Proposition 3.13 one can deduce that $\left\| \frac{\partial \hat{z}_{\varepsilon,Q}}{\partial Q} \right\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)} \leq C \left(\varepsilon^2 + e^{-d(1+o(1))} \right)$, and hence the integral under interest is controlled by $o(\varepsilon^2) + e^{-3d(1+o(1))}$.

Finally, the last term in (94) can be estimated using a Taylor expansion as for the term A_2 in the proof of Proposition 3.12, and up to higher order is given by

$$p \int_{\mathbb{R}^n_+} U(y)^{p-1} \check{z}_{\varepsilon,Q} \nabla U(y) \cdot q \, dy,$$

where q stands either for the variation of Q in the coordinates y. If q preserves d, the latter integral gives a negligible contribution, and we find (92). If instead q is directed toward the gradient of d the above estimates (and in particular (60)) allow to deduce (93). ■

4.2 Finite-dimensional reduction and study of the constrained functional

We apply now the abstract setting developed in Subsection 2.1. First of all, we state the following two lemmas.

Lemma 4.3 *If C_Ω is as in the previous section and if we choose*

$$Z_\varepsilon = \left\{ \hat{z}_{\varepsilon,Q} \ : \ C_\Omega < d < \frac{1}{C_\Omega \varepsilon} \right\}; \qquad \mathcal{H} = H^1_{\mathcal{D}}(\Omega_\varepsilon),$$

then properties i), iii) and iv) in Subsection 2.1 holds true, with $\alpha = \min\{1, p-1\}$.

PROOF. The proof of *i)* and *iii)* is immediate: the value of α in particular is determined by the standard properties of Nemitski operators. Property *iv)* can be easily deduced from the (well known) fact that the kernel of the linearization of (1) in the half space is spanned by $\frac{\partial U}{\partial x_1}, \dots, \frac{\partial U}{\partial x_{n-1}}$, as proved in [41], and from some localization arguments which can be found in Subsections 4.2, 9.2 and 9.3 of [2]. ■

The results in the next lemma could have been collected with the previous one (with some small modification): however, since in the second part of the paper, [18], we are going to need the way it is stated, it was convenient to leave the two of them separated.

Lemma 4.4 *For any small positive constant δ, if we take*

$$Z_\varepsilon = \left\{ \hat{z}_{\varepsilon,Q} \;:\; (1-\delta)|\log\varepsilon| < d < \frac{1}{C_\Omega \varepsilon} \right\}; \qquad \mathcal{H} = H^1_{\mathcal{D}}(\Omega_\varepsilon),$$

then also property ii) in Subsection 2.1 holds true, with $f(\varepsilon) = \varepsilon^{\min\left\{2-2\delta, \frac{p+1}{2}(1-2\delta)\right\}}$.

PROOF. This lemma simply follows from Propositions 3.12 and 3.13. ■

As a corollary of the above two lemmas we can apply Propositions 2.1 and 2.3, so we expand next the reduced functional and its gradient on the natural constraint $\tilde{Z}_\varepsilon$.

Proposition 4.5 *With the choice of Z_ε in Lemma 4.4, if ω_ε is given by Proposition 2.1, then we have*

$$\mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) := I_\varepsilon(\hat{z}_{\varepsilon,Q} + \omega_\varepsilon(\hat{z}_{\varepsilon,Q})) = \tilde{C}_0 - \tilde{C}_1 \varepsilon H(\varepsilon Q) + e^{-2d(1+o(1))} + O(\varepsilon^2); \tag{95}$$

$$\frac{\partial}{\partial Q_T}\mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) = -\tilde{C}_1 \varepsilon^2 \nabla_T H(\varepsilon Q) + o(\varepsilon^2); \tag{96}$$

$$\frac{\partial}{\partial Q_d}\mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) = -\tilde{C}_1 \varepsilon^2 \nabla_d H(\varepsilon Q) - e^{-2d(1+o(1))} + o(\varepsilon^2), \tag{97}$$

as $\varepsilon \to 0$, where $\tilde{C}_0$ and $\tilde{C}_1$ are as in Proposition 4.1, and where Q_T, Q_d are as in the proof of Proposition 3.13.

PROOF. By Propositions 2.1 and 3.12 we have that

$$\|\omega_\varepsilon(\hat{z}_{\varepsilon,Q})\| \leq C_1 \|I'_\varepsilon(\hat{z}_{\varepsilon,Q})\| \leq C\left(\varepsilon^2 + \varepsilon e^{-d(1+o(1))} + e^{-\frac{(p+1)d}{2}(1+o(1))} + e^{-\frac{3d}{2}(1+o(1))}\right).$$

From the regularity of I_ε and Proposition 4.1 we then have

$$\begin{aligned} I_\varepsilon(\hat{z}_{\varepsilon,Q} + \omega_\varepsilon(\hat{z}_{\varepsilon,Q})) &= I_\varepsilon(\hat{z}_{\varepsilon,Q}) + I'_\varepsilon(\hat{z}_{\varepsilon,Q})[\omega_\varepsilon(\hat{z}_{\varepsilon,Q}))] + O\left(\|\omega_\varepsilon(\hat{z}_{\varepsilon,Q})\|^2\right) \\ &= \tilde{C}_0 - \tilde{C}_1 \varepsilon H(\varepsilon Q) + e^{-2d(1+o(1))} + O(\varepsilon^2) \\ &+ O\left(\varepsilon^{4-4\delta} + \varepsilon^{(p+1)(1-2\delta)}\right). \end{aligned}$$

This immediately gives (95), since $p > 1$ and since δ is small.

The remaining two estimates are also rather immediate for $p \geq 2$: in fact in this case property *iii)* in Subsection 2.1 holds true for $\alpha = 1$, so we also have $\|\partial_Q \omega_\varepsilon\| \leq C f(\varepsilon)$ by the last statement in Proposition 2.1. This, together with the Lipschitzianity of I'_ε implies that

$$\begin{aligned} \frac{\partial}{\partial Q}\mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) &= I'_\varepsilon(\hat{z}_{\varepsilon,Q} + \omega_\varepsilon)\left[\partial_Q \hat{z}_{\varepsilon,Q} + \partial_Q \omega_\varepsilon\right] = \frac{\partial}{\partial Q} I_\varepsilon(\hat{z}_{\varepsilon,Q}) \\ &+ I''_\varepsilon(\hat{z}_{\varepsilon,Q})\left[\omega_\varepsilon, \partial_Q \hat{z}_{\varepsilon,Q}\right] + I''_\varepsilon(\hat{z}_{\varepsilon,Q})\left[\omega_\varepsilon, \partial_Q \omega_\varepsilon\right] + \|\omega_\varepsilon\|^{1+\alpha}\left(\|\partial_Q \hat{z}_{\varepsilon,Q}\| + \|\partial_Q \omega_\varepsilon\|\right) \\ &= \frac{\partial}{\partial Q} I_\varepsilon(\hat{z}_{\varepsilon,Q}) + O(f(\varepsilon)^2) = \frac{\partial}{\partial Q} I_\varepsilon(\hat{z}_{\varepsilon,Q}) + O\left(\varepsilon^{4-4\delta} + \varepsilon^{(p+1)(1-2\delta)}\right), \end{aligned} \tag{98}$$

since $\alpha = 1$. The last two estimates then follow from Proposition 4.2.

For the case $1 < p < 2$, we can arrive to the end of the second line in (98), but since $\alpha = p - 1 \in (0, 1)$ we cannot conclude anymore, and we need to use a slightly more technical argument, which requires the improved estimate mentioned in Remark 2.2. We only give a brief sketch, and refer the reader for example to Subsection 4.2 in [4], where a similar issue is treated in more detail. The main point is to show that the function ω_ε satisfies suitable pointwise estimates in some regions of the domain. The auxiliary equation in (7), after testing on any smooth function and integrating by parts yields

$$-\Delta(\hat{z}_{\varepsilon,Q} + \omega_\varepsilon) + \hat{z}_{\varepsilon,Q} + \omega_\varepsilon - (\hat{z}_{\varepsilon,Q} + \omega_\varepsilon)^p = \overline{\alpha} \cdot \nabla_Q(-\Delta \hat{z}_{\varepsilon,Q} + \hat{z}_{\varepsilon,Q}) \qquad \text{in } \Omega_\varepsilon, \tag{99}$$

where $\overline{\alpha}$ is a vector of $\mathbb{R}^{n-1}$ with norm of order at most $\|I'_\varepsilon(\hat{z}_{\varepsilon,Q} + \omega_\varepsilon)\|$, which in turn is of order $f(\varepsilon)$.

Using the coordinates y in Subsection 2.2 and restricting our attention to $d(\mathcal{O} - Q_0) \cup (\mathbb{R}^n_+ \setminus B_{\frac{dD}{8}}(0))$, see (60) and (67), we have here pointwise estimates on the right-hand side, by (60) and (61). This control, together with a bootstrap argument for (99) allows to prove that in $d(\mathcal{O} - Q_0) \cup (\mathbb{R}^n_+ \setminus B_{\frac{dD}{8}}(0))$ the function ω_ε is pointwise of order $f(\varepsilon)(U(y) + |\Xi_d|)$. On the other hand, in the complement of this set, still by (99) (used in its weak form), the H^1 norm of ω_ε is controlled by $\overline{\alpha}(\|\overline{z}_{\varepsilon,Q}\|_{H^1((\mathbb{R}^n_+ \setminus d(\mathcal{O}-Q_0)) \cap B_{\frac{dD}{8}}(0))} + \|\Xi_d\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)})$. Hence, given any test function $v \in H^1_{\mathcal{D}}(\Omega_\varepsilon)$, if we use Taylor expansions in the regions where we have pointwise estimates (as for (75)) and the Hölder-Sobolev inequalities elsewhere we find that

$$\left| \int_{\Omega_\varepsilon} \left[(\hat{z}_{\varepsilon,Q} + \omega_\varepsilon)^{p-1} - \omega_\varepsilon^{p-1} \right] \partial_Q \hat{z}_{\varepsilon,Q} v dx \right| \leq \varepsilon^{1+\tilde{\delta}_p} \|v\|_{H^1_{\mathcal{D}}(\Omega_\varepsilon)} \qquad \text{as } \varepsilon \to 0,$$

where $\tilde{\delta}_p$ is a positive constant depending only on p. Remark 2.2 then implies that $\|\partial_Q \omega_\varepsilon\| \leq \varepsilon^{1+\tilde{\delta}_p}$ for ε small. Finally, a Taylor expansion of the functional I_ε gives

$$\begin{aligned} \frac{\partial}{\partial Q} \mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) &= I'_\varepsilon(\hat{z}_{\varepsilon,Q} + \omega_\varepsilon)\left[\partial_Q \hat{z}_{\varepsilon,Q} + \partial_Q \omega_\varepsilon\right] = \frac{\partial}{\partial Q} I_\varepsilon(\hat{z}_{\varepsilon,Q}) \\ &+ O\left(\|I'_\varepsilon(\hat{z}_{\varepsilon,Q} + \omega_\varepsilon) - I'_\varepsilon(\hat{z}_{\varepsilon,Q})\|\right) + O\left(\|I'_\varepsilon(\hat{z}_{\varepsilon,Q} + \omega_\varepsilon)\| \, \|\partial_Q \omega_\varepsilon\|\right) \\ &= \frac{\partial}{\partial Q} I_\varepsilon(\hat{z}_{\varepsilon,Q}) + O(f(\varepsilon)^2), \end{aligned} \tag{100}$$

provided we choose δ small compared to $\tilde{\delta}_p$. This concludes the proof. ■

Proof of Theorem 1.1 We use degree theory and the previous expansions: first of all, since $\overline{Q}$ is non-degenerate for $H|_{\mathcal{I}_\Omega}$, we can find a small neighborhood $\mathcal{V}$ of $\overline{Q}$ in $\mathcal{I}_\Omega$ such that $\nabla H|_{\mathcal{I}_\Omega} \neq 0$ on $\partial \mathcal{V}$ and such that in some set of coordinates

$$\deg(\nabla H|_{\mathcal{I}_\Omega}, \mathcal{V}, 0) \neq 0.$$

Then, if δ is as in Lemma 4.4, we choose $0 < \beta < \frac{\delta}{2}$, and consider the set

$$\mathcal{Y} = \left\{ (d, Q) \; : \; d \in ((1-\beta)|\log \varepsilon|, (1+\beta)|\log \varepsilon|), \varepsilon Q \in \mathcal{V} \right\}.$$

Since $\nabla H|_{\mathcal{I}_\Omega}(Q)$ corresponds to $\nabla_T H(\varepsilon Q)$ in the scaled domain Ω_ϵ, by using (96) and our choice of $\mathcal{V}$ we know that, as $\varepsilon \to 0$

$$\nabla_{Q_T} \mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) = -\tilde{C}_1 \varepsilon^2 \nabla_T H(\varepsilon Q) + o(\varepsilon^2) \neq 0 \qquad \text{on } \frac{1}{\varepsilon} \partial \mathcal{V}. \tag{101}$$

On the other hand, by (97) we also have

$$\nabla_{Q_d} \mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) = -\varepsilon^{2(1-\beta)(1+o(1))} \qquad \text{for } d = (1-\beta)|\log \varepsilon|, \tag{102}$$

and

$$\nabla_{Q_d} \mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) = -\tilde{C}_1 \varepsilon^2 \nabla_d H(\varepsilon Q) + o(\varepsilon^2), \qquad \text{for } d = (1+\beta)|\log \varepsilon|. \tag{103}$$

Since we are assuming that the gradient of H points toward $\partial_{\mathcal{D}}\Omega$ near the interface $\mathcal{I}_\Omega$, $\nabla_d H(\varepsilon Q)$ is negative and therefore the two d-derivatives in the last two formulas have opposite signs. It follows from the product formula for the degree and (101)-(103) that

$$\deg(\nabla \mathbf{I}_\varepsilon, \mathcal{Y}, 0) = -\deg(\nabla H|_{\mathcal{I}_\Omega}, \mathcal{V}, 0) \neq 0,$$

which proves the existence of a critical point for $\mathbf{I}_\varepsilon$ in $\mathcal{Y}$. Since we can choose $\mathcal{V}$ and β arbitrarily small, the solution has the asymptotic behavior required by the theorem (and more precisely by Remark 1.2 (b): the uniqueness of the global maximum follows from the asymptotics of the solution and standard elliptic regularity estimates). ■

To prove also the assertion in Remark 1.2 (a), using (95) in the case of local maximum is it easy to construct an open set of Z_ε where the maximum of $\mathbf{I}_\varepsilon$ at the interior is strictly larger than the maximum at the boundary. On the other hand, when we have a local minimum, one can construct a mountain-pass path connecting the two points parameterized by $\left(\frac{1}{\varepsilon}\overline{Q}, (1-\beta)|\log\varepsilon|\right)$ and $\left(\frac{1}{\varepsilon}\overline{Q}, (1+\beta)|\log\varepsilon|\right)$. Using a suitably truncated pseudo-gradient flow, one can prove that the evolution of the path remains inside $\frac{1}{\varepsilon}\mathcal{V} \times ((1-\beta)|\log\varepsilon|, (1+\beta)|\log\varepsilon|)$, and still find a critical point of $\mathbf{I}_\varepsilon$.

Remark 4.6 *(a) Even without assuming* $d > (1-\delta)|\log\varepsilon|$ *(as required by Lemma 4.4) but only that* d *is large, Propositions 3.12 and 4.1 yield*

$$\mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) = \tilde{C}_0 - \tilde{C}_1 \varepsilon H(\varepsilon Q) + O(\varepsilon^2) + e^{-2d(1+o(1))}. \tag{104}$$

Moreover, the second last equality in (98) *with Propositions 3.12 and 4.2 give*

$$\frac{\partial}{\partial Q_T}\mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) = -\tilde{C}_1 \varepsilon^2 \nabla_T H(\varepsilon Q) + o(\varepsilon^2) + e^{-(p+1)d(1+o(1))} + e^{-3d(1+o(1))}; \tag{105}$$

$$\frac{\partial}{\partial Q_d}\mathbf{I}_\varepsilon(\hat{z}_{\varepsilon,Q}) = -\tilde{C}_1 \varepsilon^2 \nabla_d H(\varepsilon Q) - e^{-2d(1+o(1))} + o(\varepsilon^2) + e^{-(p+1)d(1+o(1))} + e^{-3d(1+o(1))}. \tag{106}$$

(b) In [18] we will show that, under suitable geometric assumptions on Ω *and* $\mathcal{I}_\Omega$*, the ground state solution* u_ε *of* (M_ε) *belongs to the set* $Z_{\varepsilon,\tilde{c}}$ *(see the notation in Proposition 2.3), where*

$$Z_\varepsilon = \{\hat{z}_{\varepsilon,Q} \;:\; d_{1,\varepsilon} \leq d \leq d_{2,\varepsilon}, Q \in \mathcal{V}\},$$

where $d_{1,\varepsilon} \to +\infty$, $\varepsilon d_{2,\varepsilon} \to 0$ *as* $\varepsilon \to 0$*, and where* $\mathcal{V}$ *is a neighborhood of a point* $\overline{Q}$ *which realizes* $\max_{Q \in \mathcal{I}_{\Omega_\varepsilon}} H$*. Using the previous observation (a) we will prove that* $u_\varepsilon = \hat{z}_{\varepsilon,Q} + \omega(\hat{z}_{\varepsilon,Q})$*, where the distance of* Q *from* $\mathcal{I}_{\Omega_\varepsilon}$ *is of order* $\varepsilon|\log\varepsilon|$*, as in Theorem 1.1, see Remark 1.2* (b).

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