Critical exponents for semilinear equations of mixed elliptic-hyperbolic and degenerate types

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Abstract

For semilinear Gellerstedt equations with Tricomi, Goursat or Dirichlet boundary conditions we prove Pohozaev type identities and derive non existence results that exploit an invariance of the linear part with respect to certain non homogeneous dilations. A critical exponent phenomenon of power type in the nonlinearity is exhibited in these mixed elliptic hyperbolic or degenerate settings where the power is one less than the critical exponent in a relevant Sobolev imbedding.

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1 Introduction

In the theory of semilinear elliptic equations, starting from the classical model of the Dirichlet problem for a semilinear Laplace equation

\[
\begin{cases}
-\Delta u = f(u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]  

(1.1)

phenomena of critical growth in the nonlinearity \( f(u) \) are well known. The critical growth manifests itself in results of nonexistence of the form (cf. Pohozaev [21]): the problem (1.1) has no nontrivial sufficiently regular solution in a bounded region \( \Omega \subset \mathbb{R}^n \) with \( n \geq 3 \) if \( \Omega \) is starshaped with respect to some interior point (say the origin) and if \( f(u) = |u|^p \) with \( p \geq 2^* - 1 = (n + 2)/(n - 2) \).

On the other hand, if the nonlinearity has subcritical growth at infinity, generically one does have nontrivial solutions to the problem (1.1).

The key ingredients in the proof of such nonexistence results are maximum principles for the solutions to the associated linear problem and the so-called Pohozaev identities for the nonlinear problem. These identities result from an energy integral method of finding a suitable vector field \( D = \sum_{j=1}^{n} a_j(x) \partial_{x_j} \) for which multiplying the differential equation in (1.1) by \( Du \) and applying the divergence theorem yields an identity between volume and surface integrals whose signs (with the aid of the maximum principle) are incompatible with nontrivial solutions (the solution must vanish identically). A good choice for the vector field is the infinitesimal generator of some coordinate invariance in the solutions to the linear homogeneous equation (homogeneous dilations in the case of the Laplacian) and the starshaped condition on \( \Omega \) can be viewed as a strong form of pseudoconvexity of \( \Omega \) with respect to the flow generated by \( D \). The same mechanism is responsible for the critical exponent phenomenon in Sobolev imbeddings in starshaped domains such as \( W^{1,2} (\Omega) \subset L^p (\Omega) \) for \( 1 \leq p \leq 2^* = 2n/(n - 2) \) with compactness of the inclusion if \( 1 \leq p < 2^* \). Such a loss of compactness due to the action of a noncompact group like \( \mathbb{R}^+ \) and means to overcome it is the source of much recent work having origins in the seminal paper of Brezis and Nirenberg [3]. For dimension 2, the critical growth in (1.1) is not of power type but rather exponential and related to imbeddings into Orlicz spaces (cf. [4] and references therein).
The aim of the present paper is to exhibit a critical exponent phenomenon of power type for suitable boundary value problems in the plane for semilinear equations of either mixed elliptic-hyperbolic or degenerate type. In particular, we consider problems of the form

\[
\begin{aligned}
Lu &= f(u) \quad \text{in } \Omega \\
0 &= 0 \quad \text{on } \Gamma \subseteq \partial \Omega,
\end{aligned}
\]  

(1.2)

where \( \Omega \) is a bounded, open, simply connected subset of \( \mathbb{R}^2 \) with piecewise \( C^1 \) boundary \( \partial \Omega \), \( f \in C^0(\mathbb{R}) \), and \( L \) is the Gellerstedt operator [12]

\[
L = -y^m \partial_x^2 - \partial_y^2,
\]  

(1.3)

with \( m > 0 \). In the case of \( m = 2k + 1 \) an odd integer, \( L \) is of mixed elliptic-hyperbolic type and degenerates on \( y = 0 \), the so-called sonic line due to its importance in the theory of two dimensional transonic fluid flow (cf. [9], [19]). In particular, when \( m = 1 \) one has the classical Tricomi operator, where our sign convention on \( L \) is chosen so that \( L \) is positive definite on the elliptic half space \( y > 0 \). In the case \( m = 2k \) an even integer, \( L \) is a degenerate elliptic operator of Grushin type in the half space \( y > 0 \).

As for boundary conditions, we will consider classical choices in three distinct settings. In the mixed type case with \( m = 2k + 1 \), we consider the Tricomi problem [27] in which \( \Omega \) has non empty intersection with both the elliptic and hyperbolic half spaces and the boundary is given by an elliptic arc \( \sigma \) and two characteristics \( AC \) and \( BC \) for \( L \) in the hyperbolic region (cf. section 3 for a precise description). In this case, the boundary condition in (1.2) is placed on the proper subset \( \Gamma = AC \cup \sigma \) and is known to yield results on well posedness for the linear problem and has its importance in the problem of transonic flow in nozzles (cf. [9]). In the degenerate hyperbolic setting with \( m = 2k + 1 \) we will consider the Goursat problem in \( \Omega^- \) a characteristic triangle where the boundary condition is placed on \( \Gamma = AC \cup AB \) the union of a characteristic and the parabolic segment \( AB \). In the degenerate elliptic setting, we consider the Dirichlet problem in a domain \( \Omega^+ \) in the elliptic half space whose boundary partially coincides with the parabolic segment \( AB \) so that \( \Gamma = \partial \Omega^+ = \sigma \cup AB \).
Our main results are the following. All of the operators defined in (1.3) come equipped with a coordinate invariance in the solutions to the linear homogeneous equation with respect to a certain non homogeneous dilation. The infinitesimal generator of the associated non homogeneous dilation operator is a vector field with linear coefficients which generates a natural notion of being *starshaped with respect to the flow of D* as is sketched in section 2. Moreover, this non homogeneous dilational invariance is related to a critical exponent phenomena with respect to the imbedding of weighted $L^2$ Sobolev spaces (naturally associated to the operator L) into $L^p$ where the value of the critical exponent $2^*(m) = 2N/(N - 2)$ where $N = (m + 4)/2$ plays the role of a homogeneous dimension for the operators L (cf. [5], [11], [7]). Using D as the auxiliary vector field D in an energy integral argument yields Pohozaev type identities for solutions to the semilinear boundary value problems in the various settings as shown in section 3 (cf. Theorems 3.1, 3.4 and 3.5). These Pohozaev identities, calibrated to the non homogeneous dilational invariance, are then applied to the semilinear problems (1.2) with $f(u) = |u|^\alpha$ in the various settings yielding non existence results of the form (cf. Theorems 4.2, 4.6, and 4.7): for regular solutions in domains starshaped with respect to the flow of D there are no nontrivial solutions if $\alpha > \alpha^*(m)$ where $\alpha^*(m) = (m + 8)/8 = 2^*(m) - 1$. That is, there is critical exponent phenomenon of power type where the critical power is one less than the critical exponent in the relevant Sobolev imbedding. The proof in the degenerate elliptic setting follows along classical lines, but there is more work to do in the mixed type and degenerate hyperbolic cases since the boundary conditions, which place no restriction on part of the hyperbolic boundary, do not eliminate all of the usual boundary integrals. However, by exploiting a Hardy-Sobolev inequality one is able to show that the sum of the boundary terms has the right sign and yields our nonexistence result which is delicate since an almost sharp value of the constant in the Hardy-Sobolev inequality is needed. Various extensions of the results presented here are possible; section 5 sketches a few such avenues.

We conclude this introduction with a few remarks. First, to the best of our knowledge, we present the first such Pohozaev type results in a mixed type setting. Moreover, our results in the degenerate settings can be viewed as natural consequences of the mixed type study. Our degenerate elliptic results can be compared with existing results of [26] for Grushin type operators with interior degeneration (whereas ours degenerate at the boundary ) or with [22] for cases of an isotropic
degeneration (whereas ours is anisotropic; see also [11]) and our degenerate hyperbolic results can be compared to those in [17] for Cauchy problems in half spaces. Third, one should ask whether there are generically solutions to the problem for subcritical exponents. In the degenerate elliptic case the answer is yes. In the mixed type case, the answer is less complete. For the Tricomi operator, the authors have established a maximum principle for generalized solutions that is compatible with an \( L^2 \) based solvability theory for the linear problem in normal domains [14] which is then used to prove the existence of non trivial generalized solutions to certain subcritical problems [15]. These generalized solutions belong to the weighted Sobolev space naturally associated to the operator as recalled in section 5. Similar results should hold for the general Gellerstedt operator under suitable technical conditions. In order to complete the picture, it would be interesting to examine problems with critical exponents and to know whether the non existence results in the supercritical case extend to a suitable class of weak solutions. We introduce the notion of a strong solution in Definition 5.1 for which there are no nontrivial solutions in the supercritical case. Closing the gap between such strong and generalized solutions awaits better results on regularity for the linear problem as described in section 5.

2 Non homogeneous dilations and D-starshaped domains

The Pohozaev identities and nonexistence results of this paper exploit an invariance of the operators under consideration with respect to certain non homogeneous dilations. The infinitesimal generators of these transformations will be used in the Pohozaev type scheme to prove the nonexistence results in domains which are starshaped with respect to the flow of the generator. Moreover, the critical growth phenomenon that results from these non homogeneous dilations can be connected to a critical exponent phenomenon in the question of imbedding certain Sobolev spaces with weights into Lebesgue spaces. The purpose of this section is to present these considerations as preparation for the main results which follow.

For \( \alpha, \beta > 0 \) we will consider the one parameter family of non homogeneous dilations

\[
\Phi_\lambda(x, y) = (\lambda^{-\alpha}x, \lambda^{-\beta}y), \quad \lambda > 0
\]  

(2.1)
which determines a one parameter family of non homogeneous dilation operators \( \Psi_\lambda u = u \circ \Phi_\lambda = u_\lambda \) whose infinitesimal generator

\[
\left[ \frac{d}{d\lambda} u_\lambda \right]_{\lambda=1} = Du = -\alpha x \partial_x - \beta y \partial_y
\] (2.2)

is the vector field \( D \). This vector field has linear coefficients and hence will determine a flow \( F_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) that is defined for each \( t \in \mathbb{R} \) where we recall that \( (x(t), y(t)) = F_t(x_0, y_0) \) is the unique integral curve of \( D \) that passes through \( (x_0, y_0) \) in time \( t = 0 \). In fact one integrates easily to find \( (x(t), y(t)) = (x_0 e^{-\alpha t}, y_0 e^{-\beta t}) \) and that \( (0,0) = F_{t+\infty}(x_0, y_0) = \lim_{t \to +\infty} F_t(x_0, y_0) \) for each initial point \( (x_0, y_0) \in \mathbb{R}^2 \). Hence \( \mathbb{R}^2 \) is “starshaped” with respect to the origin where one uses the flow of \( D \) in place of the segment \( \{ (\lambda x, \lambda y) : 0 \leq \lambda \leq 1 \} \) in the usual notion of starshapedness with respect to the origin. The usual notion is recovered in the homogeneous case \( \alpha = \beta \). Moreover, when using the flow of \( D \) the only point which can serve as the center of the star is the origin; an obvious translation allows for other choices.

**Definition 2.1** Let \( \alpha, \beta > 0 \). An open set \( \Omega \subseteq \mathbb{R}^2 \) is said to be starshaped with respect to the flow of \( D = -\alpha x \partial_x - \beta y \partial_y \) (or more simply \( D \)-starshaped) if for each \( (x_0, y_0) \in \Omega \) one has \( F_t(x_0, y_0) \subset \Omega \) for each \( t \in [0, +\infty] \).

As in the usual notion of starshapedness, bounded starshaped domains will have starlike boundaries.

**Lemma 2.2.** Let \( \Omega \) be an open region with piecewise \( C^1 \) boundary \( \partial \Omega \). If \( \Omega \) is starshaped with respect to the flow of \( D = -\alpha x \partial_x - \beta y \partial_y \) with \( \alpha, \beta > 0 \) then \( \partial \Omega \) is starlike in the sense that

\[
(\alpha x, \beta y) \cdot n(x, y) \geq 0
\] (2.3)

at each regular point \( (x, y) \in \partial \Omega \) where \( n(x, y) \) is the exterior unit normal to \( \partial \Omega \) at the point \( (x, y) \).

**Proof:** Assume that \( (x_0, y_0) \in \partial \Omega \) is a regular point. Then there exists \( \delta > 0 \) such that the boundary neighborhood \( \Gamma_\delta = B_\delta(x_0, y_0) \cap \partial \Omega \) is a \( C^1 \) curve. Consider the cone \( K_\delta \) with respect to the \( D \) flow determined by \( \Gamma_\delta \); that is, consider

\[
K_\delta = \{ F_t(x, y) : (x, y) \in \Gamma_\delta, \ t \in [0, +\infty] \}.
\]

Since \( \Omega \) is \( D \)-starshaped one has \( K_\delta \subset \Omega \).
Now applying the divergence theorem to the vector field \( V = (\alpha x, \beta y) \) in the cone \( K_\delta \) gives
\[
\int_{\partial K_\delta} (\alpha x, \beta y) \cdot n \, ds = (\alpha + \beta)|K_\delta|
\]
but the field \( V \) is tangential to the lateral boundary arcs of \( K_\delta \) and hence
\[
\frac{1}{|\Gamma_\delta|} \int_{\Gamma_\delta} (\alpha x, \beta y) \cdot n \, ds = \frac{1}{|\Gamma_\delta|} (\alpha + \beta)|K_\delta| > 0.
\]
Using the continuity of the integrand above and passing to the limit as \( \delta \to 0 \) yields the inequality \( (\alpha x_0, \beta y_0) \cdot n(x_0, y_0) \geq 0 \) as claimed.

**Definition 2.3** A boundary arc \( \Gamma \subset \partial \Omega \) will be called \( D \)-starlike if the condition (2.3) holds on \( \Gamma \) which is equivalent to the condition
\[
\beta dx - \alpha dy \geq 0 \quad \text{on} \quad \Gamma
\]
where \( \partial \Omega \) is given the positive orientation of leaving the interior of \( \Omega \) on the left.

We now turn to the behavior of Gellerstedt type operators with respect to certain non homogeneous dilations. One easily verifies the following invariance properties. In the mixed elliptic-hyperbolic case, if \( u \) is a regular solution of \( Lu = 0 \) in \( \mathbb{R}^2 \) where
\[
L = -y^{2k+1} \partial_x^2 - \partial_y^2, \quad k \in \mathbb{N}
\]
then so is \( u_\lambda = u \circ \Phi_\lambda \) where
\[
\Phi_\lambda(x, y) = (\lambda^{-(2k+3)}x, \lambda^{-2}y), \quad \lambda > 0.
\]
This invariance also generates a conservation law associated to the equation \( Lu = 0 \) which is investigated in [16]. Similarly, in the degenerate elliptic case, if \( u \) is a regular solution of \( Lu = 0 \) in \( \mathbb{R}^+_y = \{(x, y) \in \mathbb{R} : y > 0\} \) where
\[
L = -y^m \partial_x^2 - \partial_y^2, \quad m \in \mathbb{R}^+
\]
then so is \( u_\lambda = u \circ \Phi_\lambda \) where
\[
\Phi_\lambda(x, y) = (\lambda^{-(m+2)}x, \lambda^{-2}y), \quad \lambda > 0.
\]
Moreover, the same affirmations hold true with \( \lambda \in (0, 1] \) for domains \( \Omega \) which are starshaped with respect to the flow of the relevant vector field.

In addition, there are certain weighted Sobolev spaces that are associated to equations of Gellerstedt type \( Lu = f \) with \( L \) given by (2.5) or (2.7). These spaces are the natural candidates in which to search for an \( L^2 \) based theory of existence of weak or variational solutions. In the mixed type case (2.5), the norm is given by (cf. [14])

\[
||u||^2_{W^{1,2,2k+1} (\Omega)} = \int_{\Omega} (|y|^{2k+1}u_x^2 + u_y^2 + u^2) \, dxdy = ||\nabla_k u||^2_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)}
\]

where we have introduced the weighted gradient \( \nabla_k u = (|y|^{(2k+1)/2}u_x, u_y) \).

With respect to these weighted Sobolev spaces, one finds a critical exponent phenomena of Sobolev-Gagliardo-Nirenberg type. As in the classical case, the value of the critical exponents can be captured by the relevant scaling.

**Proposition 2.4** Let \( k \in \mathbb{N} \) be given. Assume that there exists a constant \( C > 0 \) independent of \( u \) such that

\[
||u||_{L^p(\mathbb{R}^2)} \leq C||\nabla_k u||_{L^2(\mathbb{R}^2)}, \quad u \in C_0^\infty (\mathbb{R}^2).
\]

Then necessarily \( p = 2^*(k) = (4k + 10)/(2k + 1) \).

**Proof:** The argument is classical, it is enough to observe that by scaling \( u_\lambda(x, y) = u(\lambda^{-(2k+3)}x, \lambda^{-2}y) \) one finds

\[
||u_\lambda||_{L^p} = \lambda^{(2k+5)/p}||u||_{L^p} \quad \text{and} \quad ||\nabla_k u_\lambda||_{L^2} = \lambda^{(2k+1)/2}||\nabla_k u_\lambda||_{L^2}.
\]

Then inserting \( u = u_\lambda \) into (2.10) yields

\[
||u||_{L^p} \leq C\lambda^{(2k+1)/2-(2k+5)/p}||\nabla_k u||_{L^2}
\]

and considering the limiting cases as \( \lambda \to 0 \) and \( \lambda \to +\infty \) one must necessarily have \( (2k + 1)/2 = (2k + 5)/p \).

A similar argument, using only \( \lambda \to 0 \), shows that if \( \Omega \) starshaped with respect to the flow of \( D = -(2k + 3)x \partial_x - 2y \partial_y \) and if one has an embedding of the form \( W^{1,2}_{2k+1,0} (\Omega) \hookrightarrow L^p(\Omega) \) then necessarily \( p \leq 2^*(k) \). This critical exponent satisfies \( 2^*(k) = 2N/(N - 2) \) for \( N = (2k + 5)/2 \) which
is the so-called homogeneous dimension of $\mathbb{R}^2$ when equipped with the relevant metric structure naturally associated to the operator $L$ (cf. [5]). That such imbeddings exist, are continuous, and compact for $p < 2^*(k)$ follows from the estimates of Franchi-Lanconelli [6] in domains which are not necessarily $D$-starshaped (see also [7]). Similar considerations hold for any $m > 0$.

3 Pohozaev type identities

In this section we derive Pohozaev type identities for semilinear problems that seek to exploit the invariance of the linear part with respect to the relevant non homogeneous dilation as explained in the previous section. This is accomplished by using the infinitesimal generator of the dilation as the multiplier in an energy integral argument.

We first consider the semilinear Tricomi problem for the Gellerstedt equation

$$
\begin{aligned}
Lu &= -y^{2k+1}u_{xx} - u_{yy} = f(u) \quad \text{in } \Omega \\
u &= 0 \quad \text{on } AC \cup \sigma
\end{aligned}
$$

where $\Omega$ is a Tricomi domain for the Gellerstedt operator $L$. That is, $\Omega$ is an open, bounded, and simply connected set with piecewise $C^1$ boundary $\partial \Omega = \sigma \cup AC \cup BC$ formed by the elliptic arc $\sigma \subset \{(x,y) \in \mathbb{R}^2 : y > 0\}$ whose endpoints lie on the $x$-axis in $A = (2x_0,0)$ and $B = (0,0)$ with $x_0 < 0$ and the characteristics $AB$ and $BC$ of $L$ that descend from $A$ and $B$ respectively and meet in the point $C$. In particular, the domains depend on $k \geq 0$ which characterizes the operator $L$ and $x_0$ which determines the “parabolic diameter” $2|x_0| = |AB|$. One has explicit formulas for the characteristics

$$AC := \{(x,y) \in \mathbb{R}^2 \mid y \leq 0; \ (2k+3)(x - 2x_0) = 2(-y)^{(2k+3)/2}\}$$

$$BC := \{(x,y) \in \mathbb{R}^2 \mid y \leq 0; \ (2k+3)x = -2(-y)^{(2k+3)/2}\}$$

and the point $C$ which is given by $C = (x_0, y_C)$ with $y_C = -[(2k+3)x_0/2]^{2/(2k+3)}$. The exterior unit normals on the characteristics are thus given by

$$n_{AC} = (1 - y^{2k+1})^{-1/2} \left(-1, -(-y)^{(2k+1)/2}\right)$$
\[ n_{BC} = (1 - y^{2k+1})^{-1/2} \left(1, (-y)^{(2k+1)/2}\right) , \]  

where \( y \in [y_C, 0] \) can be used to parameterize the characteristics.

**Theorem 3.1.** Let \( \Omega \) be a Tricomi domain for the Gellerstedt operator \( L \) with \( k \geq 0 \) whose boundary \( \partial \Omega \) is piecewise \( C^1 \) with external unit normal field \( n \). If \( u \in C^2(\overline{\Omega}) \) is a solution of problem (3.1) then

\[
(2k+5) \int_{\Omega} F(u) \, dx dy - \frac{2k+1}{2} \int_{\Omega} uf(u) \, dx dy = \frac{1}{2} \left( \int_{BC \cup \sigma} \omega_1 \, ds + \int_{BC} \omega_2 \, ds \right) \tag{3.6}
\]

where \( F \) the primitive of \( f \in C^0(\mathbb{R}) \) such that \( F(0) = 0 \), \( \omega_1 \) and \( \omega_2 \) are defined by

\[
\omega_1 := [2Du(-y^{2k+1}u_x, -u_y) + (y^{2k+1}u_x^2 + u_y^2)(a, b)] \cdot n \tag{3.7}
\]

\[
\omega_2 := [-2F(u)(a, b) - (2k+1)u(-y^{2k+1}u_x, -u_y)] \cdot n \tag{3.8}
\]

and \( D \) is the vector field

\[
D = a \partial_x + b \partial_y = -(2k+3)x \partial_x - 2y \partial_y. \tag{3.9}
\]

The argument used is classical, but for completeness we will give the proof below in order to illustrate the effect of having placed the homogeneous Dirichlet condition on a proper subset of the boundary. In particular, we note that the first boundary integral in (3.6) is a “classical term” but it is calculated on the proper subset \( BC \cup \sigma \) of \( \partial \Omega \) because the integrand defined by (3.7) vanishes on \( AC \). The second boundary term is “new” and reflects the fact that \( u \) need not vanish on \( BC \).

**Proof of Theorem 3.1.** One obtains (3.6) by multiplying the equation \( Lu = f(u) \) by \( Du \) and integrating over the domain where the regularity of the solution allows for utilizing the classical divergence theorem. As a first step, one finds the identity

\[
\int_{\Omega} DuLu \, dx dy = \frac{2k+1}{2} \int_{\Omega} (y^{2k+1}u_x^2 + u_y^2) \, dx dy \tag{3.10}
\]

\[
+ \frac{1}{2} \int_{BC \cup \sigma} [2Du(-y^{2k+1}u_x, -u_y) + (y^{2k+1}u_x^2 + u_y^2)(a, b)] \cdot n \, ds
\]

Indeed, using the fact that \( Lu = \text{div}(\mathcal{X}u) \) where \( \mathcal{X}u = (-y^{2k+1}u_x, -u_y) \) and applying the divergence theorem one finds
\[
\int\Omega DuLu\,dxdy = \int\Omega Du\mathcal{X}u \cdot \mathbf{n}\,ds - \int\Omega \nabla (Du) \cdot \mathcal{X}u\,dxdy
\]

and then splitting the last integrand into two suitable pieces and applying once more the divergence theorem gives

\[
\int\Omega DuLu\,dxdy = \frac{2k+1}{2} \int\Omega (y^{2k+1}u_x^2 + u_y^2)\,dxdy + \frac{1}{2} \int_{\partial\Omega} [2Du\mathcal{X}u + (y^{2k+1}u_x^2 + u_y^2)(a, b)] \cdot \mathbf{n}\,ds,
\]

which is the identity (3.6) if the contribution of the line integral on \(AC\) is zero. Denoting by

\[
\partial_{\pm} u := u_y \pm (-y)^{(2k+1)/2}u_x,
\]

which give essentially the directional derivatives along the characteristics, one has that \(u|_{AC} = 0\) implies \(\partial_- u|_{AC} = 0\) and that

\[
(y^{2k+1}u_x^2 + u_y^2)|_{AC} = (\partial_+ u)(\partial_- u)|_{AC} \equiv 0.
\]

Next by (3.4), one has

\[
(-y^{2k+1}u_x, -u_y) \cdot \mathbf{n}|_{AC} = \left[-y^{2k+1}/(1 - y^{2k+1})\right]^{1/2} \partial_- u|_{AC} \equiv 0
\]

and hence (3.10) follows. On the other hand, a straightforward computation, using \(u|_{AC\cup\sigma} = 0\) and \(F(0) = 0\) gives

\[
\int\Omega Du f(u)\,dxdy = (2k + 5) \int\Omega F(u)\,dxdy + \int_{BC} F(u)(a, b) \cdot \mathbf{n}\,ds
\]

(3.12)

Furthermore, since \(u|_{AC\cup\sigma} = 0\) and \(Lu = f(u)\) on \(\Omega\), one obtains

\[
\int\Omega uf(u)\,dxdy = \int\Omega uLu\,dxdy = \int_{BC} u(-y^{2k+1}u_x, -u_y) \cdot \mathbf{n}\,ds + \int\Omega (y^{2k+1}u_x^2 + u_y^2)\,dxdy,
\]

(3.13)

Since \(\int\Omega DuLu\,dxdy = \int\Omega Du f(u)\,dxdy\), the identities (3.13), (3.12) and (3.10) give the result.

If \(k = 0\) we get the following result in the special case of Tricomi operator \(T\) for the boundary value problem.

11
\[
\begin{cases}
Tu = -yu_{xx} - u_{yy} = f(u) & \text{in } \Omega \\
u = 0 & \text{on } AC \cup \sigma
\end{cases}
\] (3.14)

**Corollary 3.2.** Let \( \Omega \) be a Tricomi domain for the Tricomi operator \( T \). If \( u \in C^2(\Omega) \) is a solution of problem (3.14) then

\[
5 \int_{\Omega} F(u) \, dx \, dy - \frac{1}{2} \int_{\Omega} uf(u) \, dx \, dy = \frac{1}{2} \left[ \int_{BC \cup \sigma} \omega_1 \, ds + \int_{BC} \omega_2 \, ds \right]
\] (3.15)

where \( F \) a primitive of \( f \in C^0(\mathbb{R}) \) such that \( F(0) = 0 \) and \( \omega_1 \) and \( \omega_2 \) are defined by

\[
\omega_1 := [2(-3xu_x - 2yu_y)(-yu_x, -u_y) + (yu^2_x + u^2_y)(-3x, -2y)] \cdot n
\]

\[
\omega_2 := [-2F(u)(-3x, -2y) - u(-yu_x, -u_y)] \cdot n
\]

**Remark 3.3.** In the case of the Tricomi operator, the appropriate vector field is given by \( D = -3x\partial_x - 2y\partial_y \) which generates the \((3,2)\)-dilation under which solutions to the linear homogeneous problem are invariant. This invariance has been used in the research for fundamental solutions by Barros-Neto and Gelfand [2].

We now consider the degenerate hyperbolic Goursat problem for the semilinear Gellerstedt equation

\[
\begin{cases}
Lu = -y^{2k+1}u_{xx} - u_{yy} = f(u) & \text{in } \Omega^- \\
u = 0 & \text{on } AC \cup AB
\end{cases}
\] (3.16)

in the characteristic triangle \( \Omega^- = ABC \) whose boundary \( \partial \Omega^- = AC \cup BC \cup AB \) is formed by the two characteristics plus the parabolic segment \( AB = \{(x, y) \in \mathbb{R}^2 \mid y = 0, 2x_0 \leq x \leq 0\} \). Again we assume \( f \in C^0(\mathbb{R}) \) with primitive \( F \) such that \( F(0) = 0 \).
Theorem 3.4 Let \( u \in C^2(\overline{\Omega^-}) \) be a solution of the problem (3.16). Then it satisfies

\[
(2k + 5) \int_{\Omega^-} F(u) \, dx \, dy - \frac{2k + 1}{2} \int_{\Omega^-} uf(u) \, dx \, dy = \\
\frac{1}{2} \int_{\partial \Omega^-} |2Du - (2k + 1)u| (-y^{2k+1}u_x, -u_y) \cdot n \, ds
\]

(3.17)

Proof. Following exactly the proof Theorem 3.1 with \( \Omega^- \) in the place of \( \Omega \) one finds the identity

\[
(2k + 5) \int_{\Omega^-} F(u) \, dx \, dy - \frac{2k + 1}{2} \int_{\Omega^-} uf(u) \, dx \, dy = \frac{1}{2} \int_{\partial \Omega^-} (\omega_1 + \omega_2) \, ds
\]

(3.18)

with \( \omega_1 \) and \( \omega_2 \) given by (3.7) and (3.8). One has \( \omega_1 = \omega_2 = 0 \) on \( AC \) as shown in the proof of Theorem 3.1. Moreover \( \omega_1 = \omega_2 = 0 \) on \( AB \) as well since \( u \equiv 0 \) on \( AB \), the flow of \( D \) is everywhere tangential to \( AB \) and \( F(0) = 0 \). Hence the boundary integral in (3.18) is nontrivial only on \( BC \). Finally, the flow of \( D \) is also everywhere tangential to the characteristic \( BC \) so that the formulas (3.7) and (3.8) for \( \omega_1 \) and \( \omega_2 \) simplify to give the integrand on the right in (3.17).

Finally, we consider the degenerate elliptic Dirichlet problem for the semilinear Gellerstedt equation in a domain whose boundary partially coincides with the parabolic line \( AB \):

\[
\begin{cases}
Lu = -y^m u_{xx} - u_{yy} = f(u) & \text{in } \Omega^+ \\
u = 0 & \text{on } AB \cup \sigma
\end{cases}
\]

(3.19)

where \( m > 0 \), \( \partial \Omega^+ = AB \cup \sigma \) with external unit normal field \( n \), \( f \in C^0(\mathbb{R}) \) with primitive \( F \) such that \( F(0) = 0 \), and \( AC, \sigma \) are as before.

In this case, the suitable auxiliary vector field \( D \) is given by \( D = -(m + 2)x \partial_x - 2y \partial_y \) which gives the Pohozaev identity generated by the \((m + 2,2)\)-dilation invariance of the operator.

Theorem 3.5 Suppose to have a solution \( u \in C^2(\overline{\Omega^+}) \) of problem (3.19). Then it satisfies

\[
(m + 4) \int_{\Omega^+} F(u) \, dx \, dy - \frac{m}{2} \int_{\Omega^+} uf(u) \, dx \, dy = \\
\frac{1}{2} \int_{\sigma} [2Du(-y^m u_x, -u_y) + (y^m u_x^2 + u_y^2)(-(m + 2)x, -2y)] \cdot n \, ds
\]

(3.20)
**Proof.** Following the same ideas used in the proof of Theorem 3.1 one finds

\[
\int_{\Omega^+} uLu \, dx \, dy = \int_{\Omega^+} \left( y^m u_x^2 + u_y^2 \right) \, dx \, dy \tag{3.21}
\]

and

\[
\int_{\Omega^+} DuLu \, dx \, dy = \frac{m}{2} \int_{\Omega^+} y^m u_x^2 + u_y^2 \, dx \, dy \tag{3.22}
\]

\[
+ \frac{m}{2} \int_{\partial \Omega^+} \left[ 2Du(-y^m u_x, -u_y) + (y^m u_x^2 + u_y^2)(-(m + 2)x, -2y) \right] \cdot n \, ds.
\]

Since \( F(0) = 0 \) and \( u_{|\partial \Omega^+} \equiv 0 \) it follows that

\[
\int_{\Omega^+} Du f(u) \, dx \, dy = (m + 4) \int_{\Omega^+} F(u) \, dx \, dy. \tag{3.23}
\]

Combining (3.21), (3.22) and (3.23) with

\[
\int_{\Omega^+} DuLu \, dx \, dy = \int_{\Omega^+} Du f(u) \, dx \, dy
\]

yields

\[
(m + 4) \int_{\Omega^+} F(u) \, dx \, dy - \frac{m}{2} \int_{\Omega^+} uf(u) \, dx \, dy = \frac{1}{2} \int_{\partial \Omega^+} \left[ 2Du(-y^m u_x, -u_y) + (y^m u_x^2 + u_y^2)(-(m + 2)x, -2y) \right] \cdot n \, ds.
\]

However, the contribution of the line integral vanishes on \( AB \), where \( y = 0 \), since \( n = (0, -1), ds = dx \), and \( Du = -(m + 2)xu_x - 2yu_y \equiv 0 \) as \( u \equiv 0 \) on \( AB \) implies that \( u_x \equiv 0 \) as well, hence the result.

**4 Non existence results**

In this section we use the Pohozaev identities of section 3 to prove results on the nonexistence of nontrivial regular solutions to the semilinear problems (3.1), (3.16) and (3.19). In particular, a critical exponent phenomenon of power type in the nonlinearity will be exhibited by calibrating the Pohozaev identity to the relevant non homogeneous dilation in domains which are suitably starshaped as explained in section 2. In so doing, we show that the classical approach of Pohozaev
for elliptic equations has its analog in these degenerate elliptic and mixed elliptic-hyperbolic settings.

We begin with the mixed elliptic-hyperbolic problem (3.1) in a Tricomi domain $\Omega$ which by translation in $x$ was normalized to have the parabolic boundary point $B$ at the origin. In this section we will also assume $\Omega$ to be starshaped with respect to the flow of $D = -(2k+3)x\partial_x - 2y\partial_y$. This flow is everywhere tangential to the boundary arc $BC$ with the normalization $B = (0,0)$. In particular, we record the following observations as a remark.

**Remark 4.1.** Let $\Omega$ be a Tricomi domain for the Gellerstedt operator with $B = (0,0)$. Then

a. $\Omega$ is starshaped with respect to the flow of $D = -(2k+3)x\partial_x - 2y\partial_y$ in the sense of Definition 2.1 if and only if this is true for the positive part $\Omega^+ = \{(x,y) \in \Omega : y > 0\}$.

b. The boundary $\partial \Omega$ is $D$-starlike in the sense of Definition 2.4 if and only if this is true for the elliptic arc $\sigma$.

That is, with the choice of $B$ as the singular point of $D$, the negative part of $\Omega$ is always $D$-starshaped and the hyperbolic boundary arcs are always $D$-starlike where we note explicitly that

$$((2k+3)x, 2y) \cdot n_{BC} = 0 \quad \text{and} \quad ((2k+3)x, 2y) \cdot n_{AC} = -3x_0 > 0 \quad (4.1)$$

as easily follows from (3.2) – (3.5).

**Theorem 4.2** Let $\Omega$ be a Tricomi domain for the Gellerstedt operator with $B = (0,0)$ which is starshaped with respect to the $D = -(2k+3)x\partial_x - 2y\partial_y$ flow with $k \geq 0$. Let $u \in C^2(\Omega)$ be a regular solution to

$$\begin{cases} 
Lu = -y^{2k+1}u_{xx} - u_{yy} = |u|^\alpha & \text{in } \Omega \\
u = 0 & \text{on } AC \cup \sigma
\end{cases} \quad (4.2)$$

with $\alpha > \frac{2k+9}{2k+1}$. Then $u \equiv 0$.

The proof of this Theorem will depend strongly on the following Hardy-Sobolev inequality, which will be used to control the sign of the boundary integrals on the characteristic $BC$ on which
no boundary condition has been placed. The inequality is a weighted Sobolev inequality in which the weights belong to the class

$$W := \{ w : [\alpha, \beta] \to \mathbb{R} \mid w \text{ is measurable, positive, and finite a.e.} \}$$

and applies to classes of absolutely continuous functions, namely to

$$\mathcal{AC}_L(\alpha, \beta) = \{ \varphi : [\alpha, \beta] \to \mathbb{R} \mid \varphi|_J \in \mathcal{AC}(J), \forall J = [\gamma, \delta] \subset (\alpha, \beta) \text{ and } \lim_{x \to \alpha^+} \varphi(x) = 0 \}$$

and the proof can be found for instance in [18], Theorem 1.14.

**Lemma 4.3.** Let $1 < p \leq q < +\infty$ and $v, w \in W$ be given. Then

$$\left[ \int_{\alpha}^{\beta} |\varphi(x)|^q w(x) \, dx \right]^\frac{1}{q} \leq C_L \left[ \int_{\alpha}^{\beta} |\varphi'(x)|^p v(x) \, dx \right]^\frac{1}{p}$$

for every $\varphi \in \mathcal{AC}_L(\alpha, \beta)$ if and only if

$$M_L = \sup_{\alpha < x < \beta} G_L(x) < +\infty \quad (4.3)$$

where $G_L(x) := \left( \int_{x}^{\beta} w(t) \, dt \right)^\frac{1}{2} \left( \int_{\alpha}^{x} v^{1-p'}(t) \, dt \right)^\frac{1}{p'}$. Moreover the best constant $C_L$ satisfies

$$M_L \leq C_L \leq r(p, q) M_L$$

where $r(p, q) := \left( 1 + \frac{q}{p'} \right)^\frac{1}{2} \left( 1 + \frac{p'}{q} \right)^\frac{1}{p}$. 

**Proof of Theorem 4.2.**

**Step 1: Apply the maximum principle**

Since we are working with regular solutions of problem (4.2) we can apply the maximum principle of [1] to say that $u \geq 0$ and hence $f(u) = |u|^\alpha = |u|^\alpha$, therefore $F(u) = u^{\alpha+1}/(\alpha + 1)$.

**Step 2: Apply the Pohozaev identity**

Using the notation of Theorem 3.1, the Pohozaev identity (3.6) yields

$$\frac{-\alpha(2k + 1) + 2k + 9}{2(\alpha + 1)} \int_{\Omega} u^{\alpha+1} \, dx \, dy = \frac{1}{2} \left[ \int_{\sigma} \omega_1 \, ds + \int_{BC} (\omega_1 + \omega_2) \, ds \right]$$
and hence, since $u^{n+1} \geq 0$, the result will follow if the boundary integrals can be shown to be non-negative.

**Step 3:** Verify $\int_\sigma \omega_1 \, ds \geq 0$.

From formula (3.7) one has

$$\int_\sigma \omega_1 \, ds = \int_\sigma \left[ 2 Du \left( y^{2k+1} u_x, -u_y \right) + \left( y^{2k+1} u_x^2 + u_y^2 \right) (a, b) \right] \cdot \mathbf{n}_\sigma \, ds$$

$$= \int_\sigma V_1 \, dy - V_2 \, dx$$

where we have defined $\mathbf{V} = (V_1, V_2) \in C^1(\overline{\Omega}, \mathbb{R}^2)$ by

$$V_1 = 2au_x + 2bu_y \left( y^{2k+1} u_x \right) + \left( y^{2k+1} u_x^2 + u_y^2 \right) a$$

$$V_2 = (2au_x + 2bu_y)(-u_y) + \left( y^{2k+1} u_x^2 + u_y^2 \right) b.$$

Straightforward computation, which takes into account the boundary condition $(u_x \, dx + u_y \, dy)_{\partial \sigma} = 0$ yields

$$\int_\sigma \omega_1 \, ds = \int_\sigma \left( y^{2k+1} u_x^2 + u_y^2 \right) \left( bdx - ady \right) + \left( -2by^{2k+1} u_x + 2au_y \right) \left( u_x dx + u_y dy \right)$$

$$= \int_\sigma \left( y^{2k+1} u_x^2 + u_y^2 \right) \left( bdx - ady \right) \geq 0, \quad (4.4)$$

where the non-negativity comes from the fact that $\sigma$ must be $D$-starlike since $\Omega$ is assumed to be $D$-starshaped (cf. Lemma 2.2).

**Step 4:** Rewrite the integral $\int_{BC} (\omega_1 + \omega_2) \, ds$

By parameterizing the curve $BC$ with

$$\Gamma(t) = (-2(-t)^{2k+3}/(2k + 3), t), \quad t \in [y_C, 0] \quad (4.5)$$

one finds that

$$\int_{BC} (\omega_1 + \omega_2) \, ds = \int_{y_C}^{0} \left[ 4(-t)^{2k+3} \varphi'(t)^2 - \frac{(2k + 1)^2}{4} (-t)^{2k+1} \varphi^2(t) \right] dt \quad (4.6)$$

where we have defined

$$\varphi(t) = u(\Gamma(t)) \in C^2([y_C, 0]) \cap C^1([y_C, 0]). \quad (4.7)$$
Indeed, since the flow of $D = a \partial_x + b \partial_y = -(2k+3)x \partial_x - 2y \partial_y$ is everywhere tangential to $BC$, the formulas (3.7) and (3.8) for $\omega_1$ and $\omega_2$ simplify as in the proof of Theorem 3.4 to give

$$
\int_{BC} (\omega_1 + \omega_2) \, ds = \int_{BC} [2Du - (2k+1)u](y^{2k+1}u_x, -u_y) \cdot n \, ds.
$$

(4.8)

Recalling that $BC$ is given by $(2k+3)x + 2(-y)^{(2k+3)/2} = 0$ one finds that on $BC$

$$
ds = \sqrt{1 - y^{2k+1}}dy \quad \text{and} \quad Du = -2(-y)\partial_y u,
$$

(4.9)

where $\partial_+ = \partial_x + (-y)^{(2k+1)/2}\partial_y$ was introduced in (3.11). Using (4.9) and the expression (3.5) for $n_{BC}$ shows that (4.8) becomes

$$
\int_{BC} (\omega_1 + \omega_2) \, ds = \int_{BC} \left[4(-y)^{2k+3} (\partial_+ u)^2 - (2k+1)(-y)^{2k+1}u \partial_+ u\right] \, dy.
$$

(4.10)

Next one notes that $\varphi$ defined by (4.7) satisfies

$$
\varphi'(t) = \partial_+ u(\Gamma(t)), \quad t \in [y_C, 0]
$$

from which (4.10) becomes

$$
\int_{BC} (\omega_1 + \omega_2) \, ds = \int_{y_C}^{0} \left[4(-t)^{2k+3} \varphi'(t) - (2k+1)(-t)^{2k+1} \varphi'(t)\right] \, dt.
$$

(4.11)

Finally, rewriting the second term on the right of (4.11) by using

$$
(-t)^{2k+1} \varphi'(t) \, dt = \frac{1}{2} \frac{d}{dt} \left[(-t)^{(2k+1)/2} \varphi^2\right] + \frac{2k+1}{4} (-t)^{(2k-1)/2} \varphi^2(t)
$$

(4.12)

where $\varphi(y_C) = 0 = \varphi(0)$ since $u \in C^0(\overline{\Omega})$ vanishes on $AC \cup \sigma$. Inserting (4.12) into (4.11) yields (4.6) as claimed.

**Step 5: Apply the Hardy-Sobolev inequality**

It remains only to show that the boundary integral given by (4.6) is non-negative which happens exactly when

$$
\left[\int_{y_C}^{0} (-t)^{2k+1} \varphi^2(t) \, dt\right]^{1/2} \leq \frac{4}{2k+1} \left[\int_{y_C}^{0} (-t)^{2k+3} \varphi'(t)^2 \, dt\right]^{1/2},
$$

(4.13)

which requires a Hardy-type inequality with constant $C \leq 4/(2k+1)$. One applies Lemma 4.3 on the interval $(\alpha, \beta) = (y_C, 0)$ with weights $v(t) := (-t)^{2k+1}$ and $w(t) := (-t)^{2k+1}$ and exponents $p = q = 2$ to the function $\varphi$ which belongs to $AC_L(y_C, 0)$ by (4.6). One finds that
\[ M_L = \sup_{y < x < 0} G_L = \frac{2}{2k+1} < +\infty \text{ and } r(2, 2) = 2 \]

for which the best constant satisfies the upper bound

\[ C_L \leq \frac{4}{2k+1}, \quad (4.14) \]

from which (4.13) follows completing the proof.

We note that result depends on the inequality (4.13) for which an almost sharp value of the best constant is needed (4.14).

**Remark 4.4.** In Theorem 4.2 the polynomial growth \( \alpha = (2k + 9)/(2k + 1) \) is critical for the semilinear Tricomi problem in the sense that for \( \alpha \) greater than this quantity there are no nontrivial solutions to the problem in general. This critical exponent is exactly \( 2^*(k) - 1 \) where \( 2^*(k) \) is the critical Sobolev exponent of immersion discussed in section 2.

These considerations give support the suggestion (cf. Lemma 3.10 of [15]) of the existence of a critical exponent \( \alpha = 9 \) for the semilinear Tricomi problem which corresponds to the critical exponent of embedding of the weighted Sobolev space \( W^1_{AC,1} \) in \( L^p \). The following corollary to Theorem 4.2 is given in this spirit.

**Corollary 4.5.** Let \( \Omega \) be a Tricomi domain for the Tricomi operator which is starshaped with respect to the \( D = (-3x\partial_x - 2y\partial_y) \)-flow. Let \( u \in C^2(\Omega) \) be a solution to

\[
\begin{align*}
Tu &= -yu_{xx} - u_{yy} = |u|^{\alpha} \quad &\text{in } \Omega \\
\alpha &> 9. \quad &\text{Then } u \equiv 0.
\end{align*}
\]

One finds the same critical exponents for the degenerate hyperbolic Goursat problem in which the domains are automatically \( D \) starshaped as noted in Remark 4.1.

**Theorem 4.6.** Let \( \Omega^- \) be a characteristic triangle for the Gellerstedt operator. Let \( u \in C^2(\Omega^-) \) be
a solution to

\[
\begin{cases}
Lu = -y^{2k+1}u_{xx} - u_{yy} = |u|^\alpha & \text{in } \Omega^- \\
u = 0 & \text{on } AC \cup AB
\end{cases}
\]

(4.16)

with \( \alpha > (2k + 9)/(2k + 1) \). Then \( u \equiv 0 \).

**Proof:** Following the scheme of the proof in the mixed type case, one applies the maximum principle of [1] valid for the degenerate hyperbolic operator \( L \) in the characteristic triangle to find \( u \geq 0 \). Then applying the Pohozaev identity (3.17) of Theorem 3.4 one finds

\[
-\alpha(2k + 1) + 2k + 9 \int_{\Omega^-} u^{\alpha+1} \, dx \, dy = \frac{1}{2} \int_{BC} \omega \, ds
\]

(4.17)

where \( \omega \) is the sum \( \omega_1 + \omega_2 \) which simplifies to the expression in (4.8). The left hand member of (4.17) being non positive it suffices to know that the boundary integral is non negative, which is the content of Steps 4 and 5 in the proof of Theorem 4.2.

We now turn to the degenerate elliptic problem for the Gellerstedt operator for which there are analogous results. One again finds a critical polynomial growth whose exponent is \( \alpha^*(m) = 2^*(m) - 1 \) where \( 2^*(m) = (2m + 8)/m \) is the critical exponent for the immersion of the weighted Sobolev space with norm

\[
\left[ \int_{\Omega^+} (|y|^m u_x^2 + u_y^2) \, dx \, dy \right]^{1/2}
\]

which is the correct space in which to have an \( L^2 \) based theory for the linear problem. One notices that for the limiting case \( m = 0 \) (the Laplace operator) there is no critical polynomial growth and that for \( m = 1 \) (the Tricomi operator as a degenerate elliptic operator in \( \Omega^+ \)) one recovers the critical growth \( \alpha^*(1) = 9 \).

**Theorem 4.7.** Let \( \Omega^+ \) be a domain as in Theorem 3.5 which is assumed to be starshaped with respect to the \( D = (-m+2)x\partial_x - 2y\partial_y \)-flow. Let \( u \in C^2(\overline{\Omega}) \) be a regular solution to

\[
\begin{cases}
Lu = -y^m u_{xx} - u_{yy} = |u|^\alpha & \text{in } \Omega \\
u = 0 & \text{on } AB \cup \sigma
\end{cases}
\]

(4.18)
with $\alpha > \frac{m+8}{m}$. Then $u \equiv 0$.

**Proof.** By applying the maximum principle and the Pohozaev identity (3.20) of Theorem 3.5 one finds

\[
\frac{2m + 8 - m(\alpha + 1)}{2(\alpha + 1)} \int_{\Omega^+} u^{\alpha+1} \, dx \, dy = \frac{1}{2} \int_{\Sigma} \omega \, ds \tag{4.19}
\]

where

\[
\omega = [2Du(-y^m u_x, -u_y) + (y^m u_x^2 + u_y^2)(a, b)] \cdot n_x \tag{4.20}
\]

with $Du = au_x + bu_y = -(m + 2)xu_x - 2yu_y$ and $u \geq 0$ in $\Omega^+$. The left hand member of (4.19) is non-positive for $\alpha > \alpha^*(m) = (m + 8)/8$ and hence it suffices to show that $\int_{\Sigma} \omega \, ds \geq 0$. One proceeds exactly as in Step 3 of the proof of Theorem 4.2 to find

\[
\int_{\Sigma} \omega \, ds = \int_{\Sigma} (u_y^2 + y^m u_x^2) (bdx - ady) \geq 0
\]

in analogy to (4.4) where the non-negativity again comes from the starshaped hypothesis on $\Omega^+$ with respect to the flow of $D$.

**5 Concluding remarks.**

In this section we collect in a less formal way some observations concerning various generalizations that are possible for the results in section 4.

First of all, it is clear that Theorems 4.2, 4.6 and 4.7 are merely model results; the choice of the nonlinearity $f(u) = |u|^\alpha$ was made to put clearly into focus the critical exponent phenomena. In particular, a choice like $f(u) = u|u|^\alpha - 1$ works perfectly well and without the need of the maximum principle for and hence one can show the non existence of nontrivial regular solutions in cases of “critical growth” such as

\[
f(u) = u|u|^{\alpha^*-1} + \lambda u, \quad \lambda < 0 \tag{5.1}
\]

where $\alpha^* = 2^* - 1$ is the critical exponent. Moreover, for any $f \in C^0(\mathbb{R})$ such that $f(0) = 0$ there will be no nontrivial non negative regular solutions provided
\[(2k + 1)F(u) - \frac{2k + 1}{2}uf(u) \leq 0, \quad \text{for} \quad u \geq 0 \] (5.2)
in the mixed type case or

\[(m + 4)F(u) - \frac{m}{2}uf(u) \leq 0, \quad \text{for} \quad u \geq 0 \] (5.3)
in the degenerate elliptic case.

Secondly, it is also clear that the mixed type and degenerate hyperbolic results presented here extend to the full class of operators considered by Gellerstedt [12]; namely to the class

\[L = -\text{sign}(y)|y|^m \partial_x^2 - \partial_y^2\]

with \(m > 0\) which has the \((m + 2, 2)\) non homogeneous dilational invariance. We have chosen to focus on the case for which the mixed type operator has smooth coefficients but this is not essential; it is enough when applying the divergence theorem to split the given domain along the parabolic segment. The resulting Pohozaev identity will have a form similar to (3.6) and the critical power has the expected value \(\alpha^*(m) = (m + 8)/8\).

Somewhat less clear is the role of the regularity \(u \in C^2(\Omega)\) or \(u \in C^2(\overline{\Omega})\) assumed throughout. In the mixed type setting for example, a more careful reading of the regularity needs for the Pohozaev identity shows that for Theorem 3.1 it suffices to assume

\[u_y, xu, yu_y, yu_x^2 \in C^1(\overline{\Omega})\] (5.4)
in order to apply the classical divergence theorem for \(C^1(\overline{\Omega})\) vector fields and to assume

\[yu, xu \in C^2(\Omega)\] (5.5)
in order to exchange the order of certain derivatives in the calculation. For the nonexistence result, one needs only

\[u \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{A, B\}) \cap C^0(\overline{\Omega})\] (5.6)
in order to apply the maximum principle of [1]. Furthermore, either of the hypothesis (5.4) or (5.6) will ensure the applicability of the Hardy-Sobolev inequality of Lemma 4.3. Perhaps it is worth noting that the classical regularity of (5.6) is the best one expects in general for solutions to linear Tricomi problem for $T$ in which often one verifies a lack of continuity in the derivative $u_x$ at the parabolic boundary point $B$. In any case, nonexistence of nontrivial solutions will continue to hold for solutions satisfying the above regularity (or being uniform limits of such regular functions). Moreover, the same will be true for a suitable class of *quasi-regular solutions* in which one relaxes the request of the classical divergence theorem to vector fields continuous on the closure, $C^1$ in the interior with the integral of the divergence interpreted as an improper integral, if need be.

As for weak solutions in the mixed type case, the authors have established existence of nontrivial solutions for the case of the Tricomi operator for classes of nonlinearities with subcritical growth [15] in spaces of *generalized solutions* $W^{1}_{AC\cup\sigma,1}(\Omega)$ which is the closure of

$$C^2_{AC\cup\sigma}(\overline{\Omega}) = \{ u \in C^2(\overline{\Omega}) : u_{AC\cup\sigma} = 0 \}$$

the norm $W^{1,2}_1(\Omega)$ defined in (2.9). In particular, in so-called *normal domains* in which the boundary meets the parabolic segment $AB$ in an orthogonal way, one has a maximum principle for such generalized solutions which is compatible with an $L^2$ based existence theory for the linear problem [14] and allows for a treatment of various semilinear but subcritical problems by the method of upper and lower solutions or the use of fixed point theory [15]. Similar results should hold for general Gellerstedt operators under suitable technical conditions. One should then ask if the nonexistence results here can be extended to this class of weak solutions.

One such approach would be to try to pass from a Pohozaev type identity along a regular approximating sequence to a suitably strong limit in $W^{1}_{AC\cup\sigma,1}(\Omega)$. We give the following definition to this end.

**Definition 5.1.** We call $u \in W^{1}_{AC\cup\sigma,1}(\Omega)$ a strong solution to the semilinear Tricomi problem (3.14) if

$$u \in L^{2\alpha}(\Omega), \quad (5.7)$$

there exists $\{ u_j \} \subset C^2_{AC\cup\sigma}(\overline{\Omega})$ such that
\[ \| Tu_j - u_j^\alpha \|_{L^2(\Omega)} + \| u_j - u \|_{W^{\alpha+1}_{AC,\sigma;1}(\Omega)} \to 0, \ j \to +\infty \] (5.8)

and

\[ \| u_j - u \|_{L^{\alpha+1}(\Omega)} \to 0, \ j \to +\infty. \] (5.9)

With this definition one easily checks that there are no nontrivial strong solutions to (4.15) with \( \alpha > 9 \). Indeed, the hypothesis (5.7) ensures that \( |u|^\alpha \in L^2(\Omega) \) and hence by the maximum principle [14] one has \( u \geq 0 \) almost everywhere. This hypothesis can be eliminated in examples like \( f(u) = u|u|^{\alpha-1} \) as noted previously. Next one considers the integral identity

\[ \int_{\Omega} Du_j(Tu_j - u_j^\alpha) \, dx \, dy = \int_{\Omega} Du_j R_j \, dx \, dy \]

where \( R_j = Tu_j - u_j^\alpha \). Performing the same energy integral argument that gives Theorem 3.1 one finds here

\[ \frac{9 - \alpha}{2(\alpha - 1)} \int_{\Omega} u_j^{\alpha+1} \, dx \, dy = \frac{1}{2} \left[ \int_{\sigma} \omega_1^{(j)} \, ds + \int_{BC\cup\sigma} \omega_2^{(j)} \, ds \right] - \frac{1}{2} \int_{\Omega} [2R_j Du_j - u_j R_j] \, dx \, dy \] (5.10)

where the boundary integrals have the same form as those in (3.7) and (3.8) and are non-negative. By the convergence \( u_j \to u \) in \( W^1_{AC,\sigma;1}(\Omega) \) in (5.8) one has the convergence of \( u_j, Du_j \) in \( L^2(\Omega) \). The rest of the hypothesis (5.8) then ensures that the area integrals on the right of (5.10) tend to zero. Exploiting the hypothesis (5.9) then gives the nonexistence claim.

We conclude with some remarks concerning Definition 5.1. In the notion of generalized solution one weakens (5.8) by asking only that \( Tu_j \) converges in the norm on the dual space \( W^{-1}_{BC,\sigma;1}(\Omega) \) to \( W^1_{AC,\sigma;1}(\Omega) \); it is this convergence which comes for free from the linear solvability theory (cf. [14]). However, for certain special domains (which are not normal) existence and uniqueness of strong solutions for the linear Tricomi problem is known [24] and hence one has (5.8). We suspect that the same is true for normal domains as well; that is, a “weak equals strong” result for solutions to a relevant first order system (cf. [10], [13], [20], and [23]). The request of the \( L^{\alpha+1} \) convergence in (5.9) is potentially more troublesome since for critical or supercritical \( \alpha \) one loses the imbedding of \( W^1_{AC,\sigma;1}(\Omega) \) in \( L^{\alpha+1} \). A completely satisfactory answer would benefit from results on higher
order regularity for the linear mixed type problems which to date are not available. Even in the degenerate elliptic case, higher order regularity results seem to be known only in special cases of boundary geometry (cf. [25], [8] and references therein).

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References


