

The class group of an integral domain

Cortona Lecture

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Let R be an integral domain with quotient field K .

We want to associate an abelian group G with R

$$R \longrightarrow G = \text{Cl}(R), \text{ the class group of } R.$$

If R is a Dedekind domain or a Prüfer domain, we have

$$\text{Cl}(R) = \text{Inv}(R)/\text{Prin}(R), \text{ the ideal class group of } R.$$

For any integral domain R , we define

$$\text{Pic}(R) = \text{Inv}(R)/\text{Prin}(R), \text{ the Picard group of } R.$$

If R is a Krull domain, we have

$$\text{Cl}(R) = \text{D}(R)/\text{Prin}(R), \text{ the divisor class group of } R.$$

For any integral domain R , we define

$$\text{Cl}_t(R) = \text{T}(R)/\text{Prin}(R), \text{ the (t-)class group of } R.$$

Here, $\text{T}(R) = \{ I \mid I \text{ is a t-invertible (fractional) t-ideal of } R \}$ is an abelian group under the t-multiplication $I \times J = (IJ)_t$ and $\text{Prin}(R)$ is its subgroup of principal (fractional) ideals.

We write elements of $Cl_t(R)$ as $[I]$. Thus $[I] = [J] \Leftrightarrow I = xJ$ for some $0 \neq x \in K \Leftrightarrow I \cong J$ as R -modules.

If R is either a Prüfer domain or a Krull domain, then $Cl_t(R) = Cl(R)$. So we will just write $Cl(R)$ for $Cl_t(R)$.

The Picard group

Let $\mathcal{F}(R) = \{ I \mid I \text{ is a nonzero fractional ideal of } R \}$.

$\mathcal{F}(R)$ is a commutative monoid (with identity R) under the usual multiplication of ideals.

Def. An $I \in \mathcal{F}(R)$ is **invertible** if there is a $J \in \mathcal{F}(R)$ with $IJ = R$. Necessarily $J = (R : I) = \{ x \in K \mid xI \subset R \} = I^{-1}$.

$Inv(R) = \{ I \in \mathcal{F}(R) \mid I \text{ is invertible} \}$ is an abelian group and a submonoid of $\mathcal{F}(R)$.

$Prin(R) = \{ xR \mid 0 \neq x \in K \}$ is a subgroup of $Inv(R)$.

Def. For any integral domain R , we define

$Pic(R) = Inv(R)/Prin(R)$, the **Picard group** of R .

Note that $Pic(R)$ is the group of isomorphism classes of projective ideals, or f. g. rank-one projective R -modules.

Handwritten notes:
 that's open
 not clear
 1) not given proofs - sometimes
 2) not most general results
 3) omit most important topics - nothing
 a how concepts relate to novel facts
 4) experts - sometimes modify
 5) Nagata talk

Dedekind, Prüfer, and Krull domains

Def. (1) R is a **Dedekind domain** if all nonzero (fractional) ideals of R are invertible.

(2) R a **Prüfer domain** if all nonzero f. g. (fractional) ideals of R are invertible.

(3) R is a **Bézout domain** if all f. g. (fractional) ideals of R are principal.

Facts. (1) For R Dedekind, $\text{Pic}(R) = 0 \Leftrightarrow R$ is a PID. *measure how*
(2) For R Prüfer, $\text{Pic}(R) = 0 \Leftrightarrow R$ is a Bézout domain.
(3) If R is quasilocal (or semiquasilocal), then $\text{Pic}(R) = 0$.
(4) If R is a Dedekind domain arising as the ring of integers in a finite algebraic extension of \mathbb{Q} , then $\text{Pic}(R)$ is finite.

Thm. (Claborn-1966) Let G be an abelian group. Then there is a Dedekind domain R with $\text{Pic}(R) = G$. \square

Thm. TFAE for an integral domain R (not a field).

(1) R is a Dedekind domain (all nonzero ideals of R are invertible).

(2) R is Noetherian, integrally closed, and $\dim R = 1$.

(3) R is Noetherian and R_M is a DVR for each maximal ideal M of R . \square

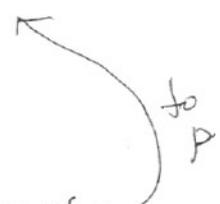
Let $X^{(1)}(R) = \{ P \mid P \text{ is a ht-one prime ideal of } R \} \subset \text{Spec}(R)$.

Def. An integral domain R (not a field) is a **Krull domain** if

- (1) R_P is a DVR for all $P \in X^{(1)}(R)$,
- (2) $R = \bigcap_{P \in X^{(1)}(R)} R_P$,
- (3) (FC) Each $0 \neq x \in R$ is a unit in R_P for almost all $P \in X^{(1)}(R)$ (equivalently, x is contained in at most a finite number of ht-one prime ideals of R).

Facts. (1) R Dedekind $\Rightarrow R$ Krull.

- (2) A Krull domain R is Dedekind $\Leftrightarrow \dim R = 1$.
- (3) R UFD $\Rightarrow R$ Krull.
- (4) R Noetherian integrally closed $\Rightarrow R$ Krull.
- (5) R Krull $\Rightarrow R_S, R[X], R[[X]]$ Krull.



Thm. (Mori, Nagata-1952-1955). The integral closure of a Noetherian integral domain is a Krull domain. \square

Thm. An integral domain R is a Krull domain if and only if

- (1) R is completely integrally closed, and
- (2) R satisfies ACC on integral divisorial ideals

(i.e., R is a **Mori domain**). \square (\Rightarrow) c.i.c. \Leftarrow harder

Def. (1) An $x \in K$ is **almost integral** over R if $rx^n \in R$ for some $0 \neq r \in R$ and all $n \geq 1$ implies $x \in R$. Then R is **completely integrally closed** (c. i. c.) if x almost integral over R implies $x \in R$.

(2) For $I \in \mathcal{F}(R)$, define $I_v = (I^{-1})^{-1}$ ($= \bigcap \{ xR \mid I \subset xR$ with $x \in K$ }). Then I is **divisorial** or a **v-ideal** if $I_v = I$.

Facts. (1) R is integrally closed $\Leftrightarrow (I : I) = R$ for all nonzero f. g. (fractional) ideals I of R .

(2) R is completely integrally closed $\Leftrightarrow (I : I) = R$ for all nonzero (fractional) ideals I of R .

(3) R is completely integrally closed $\Rightarrow R$ is integrally closed.

(4) If R is Noetherian, then R is integrally closed $\Leftrightarrow R$ is completely integrally closed.

(5) R Noetherian $\Rightarrow R$ is a Mori domain.

(6) We have $((I^{-1})^{-1})^{-1} = I^{-1}$; so $(I_v)_v = I_v$. Thus I_v is always a v -ideal. Note that an invertible ideal is divisorial.

Thm. A Noetherian integrally closed domain is a Krull domain. \square

Let $D(R) = \{ I \in \mathcal{F}(R) \mid I \text{ is divisorial} \}$.

need not be a v -ideal

$D(R)$ is a commutative monoid (with identity R) under the v -multiplication $I \times J = (IJ)_v$. We have

$$\text{Prin}(R) \subset \text{Inv}(R) \subset D(R).$$

~~Submonoid~~

Def. An $I \in D(R)$ is **v -invertible** if $(IJ)_v = R$ for some $J \in \mathcal{F}(R)$. Note that I has an inverse in $D(R) \Leftrightarrow (I : I) = R$, and $J_v = I^{-1}$. Hence I is **v -invertible** $\Leftrightarrow (I^{-1})_v = R$.

Thus if R is c. i. c. (in particular, if R is a Krull domain), then $D(R)$ is an abelian group under the v -multiplication $I \times J = (IJ)_v$ with identity R and the inverse of I is I^{-1} .

Def. For R a Krull domain, we define

$Cl(R) = D(R)/Prin(R)$, the divisor class group of R .

$Pic(R) \subset Cl(R)$, and $Cl(R) = 0 \Leftrightarrow R$ is a UFD.
not so clear *measure how far from U*

For R Dedekind, $Cl(R) = Pic(R)$ since $Inv(R) = D(R) = \mathcal{F}(R)$.

Def. For R a Krull domain, we define

$G(R) = D(R)/Inv(R)$, the local divisor class group of R . *due to A*

$$0 \longrightarrow Pic(R) \longrightarrow Cl(R) \longrightarrow G(R) \longrightarrow 0$$

Thm. (Chouinard-1981) Let G be an abelian group. Then there is a quasilocal Krull domain (R, M) with $Cl(R) = G$. Moreover, for any field K , the Krull domain R may be chosen to have the form $R = K + M$. \square

Let R be a Dedekind domain. Then each nonzero ideal of R is (uniquely) a product of prime ideals. So $\mathcal{F}(R) = Inv(R)$ is a free abelian group on $Spec(R) - \{0\} = X^{(1)}(R)$. *actually \Leftrightarrow*

Let R be a Krull domain. Then each $I \in D(R)$ may be written uniquely as $I = (P_1^{n_1} \cdots P_k^{n_k})_{\vee}$ for some $P_i \in X^{(1)}(R)$ and $n_i \in \mathbb{Z}$. This shows that a Krull domain R is a UFD \Leftrightarrow each ht-one prime ideal of R is principal $\Leftrightarrow Cl(R) = 0$.

$X^{(1)}(R)$ are divisors

Define $\text{Div}(R)$ to be the free abelian group on $X^{(1)}(R)$.

Then $D(R) \cong \text{Div}(R)$, $\text{Prin}(R) \cong \{ (n_p) \mid xR = (\prod P^{n_p})_v \text{ for some } 0 \neq x \in K \}$, and

$$\text{Cl}(R) = D(R)/\text{Prin}(R) \cong \text{Div}(R)/\text{Prin}(R).$$

$$\text{For } I = (\prod P^{n_p})_v, [I] \rightarrow (n_p) + \text{Prin}(R)$$

Star Operations

Def. A mapping $*$: $\mathcal{F}(R) \longrightarrow \mathcal{F}(R)$, $I \rightarrow I^*$ is a **star operation** on R if the following hold for all $0 \neq a \in K$, and all $I, J \in \mathcal{F}(R)$

- (1) $(a)^* = (a)$, and $(aI)^* = aI^*$,
- (2) $I \subset I^*$, and $I \subset J \Rightarrow I^* \subset J^*$,
- (3) $I^{**} = I^*$.

$$I_v = (I^{-1})^{-1} \text{ and } I_d = I \text{ define star operations on } R.$$

Def. (1) $I \in \mathcal{F}(R)$ is a ***-ideal** if $I^* = I$. Thus I is a *-ideal $\Leftrightarrow I = J^*$ for some $J \in \mathcal{F}(R)$.

(2) $I \in \mathcal{F}(R)$ is a **finite type *-ideal** if $I = J^*$ for some f. g. $J \subset I$.

We have $(IJ)^* = (I^*J)^* = (I^*J^*)^*$ for all $I, J \in \mathcal{F}(R)$.
Thus $\mathcal{F}_*(R)$ is a commutative monoid (with identity R)
with respect to the *-multiplication $I \times J = (IJ)^*$.

associative ka

The v -operation induces a new star operation on R , called the **t -operation**, by

$$I_t = \cup \{ J_v \mid 0 \neq J \subset I \text{ is f. g.} \} \text{ for } I \in \mathcal{F}(R).$$

*not in
popularity
gas bubble*

We say that $I \in \mathcal{F}(R)$ is a **t -ideal** if $I_t = I$.

Facts. (1) $I \subset I_t \subset I_v$ for all $I \in \mathcal{F}(R)$. Thus a v -ideal is also a t -ideal.

(2) $I_t = I_v$ if I is f. g.

(3) I is a t -ideal $\Leftrightarrow 0 \neq x_1, \dots, x_n \in I \Rightarrow (x_1, \dots, x_n)_v \subset I$.

(4) Any proper t -ideal is contained in a (necessarily prime) t -ideal which is maximal wrt being a proper t -ideal.

Let $t\text{-Max}(R) \subset \text{Spec}(R)$ be the set of maximal t -ideals of R .

(5) Any ht-one prime ideal is a t -ideal. *Krull domains only ones for Krull dom*

(6) $I = \bigcap_{P \in t\text{-Max}(R)} I_P$ for I a t -ideal of R . *$t\text{-Max}(R) =$*

$\mathcal{F}_t(R) = \{ I \in \mathcal{F}(R) \mid I \text{ is a } t\text{-ideal} \}$ is a commutative monoid (with identity R) under the t -multiplication $I \times J = (IJ)_t$.

Def. (1) An $I \in \mathcal{F}_t(R)$ is **t -invertible** if $(IJ)_t = R$ for some $J \in \mathcal{F}(R)$. Note that if I has an inverse J in $\mathcal{F}_t(R)$, then $J_t = I^{-1}$. Hence I is **t -invertible** $\Leftrightarrow (II^{-1})_t = R$. (We will say that an $I \in \mathcal{F}(R)$ is t -invertible if I_t is t -invertible.)

(2) Let $T(R) = \{ I \in \mathcal{F}_t(R) \mid (IJ)_t = R \text{ for some } J \in \mathcal{F}(R) \}$.
 Then $T(R)$ is an abelian group under the t -multiplication $I \times J = (IJ)_t$. $T(R)$ is the group of **t -invertible t -ideals**.

(3) Let $D_f(R) = \{ I \in \mathcal{F}(R) \mid I \text{ is a finite type } v\text{-ideal, i.e., } I = J_v \text{ for some f.g. } J \in \mathcal{F}(R) \}$. Then $D_f(R)$ is submonoid of $D(R)$ under the usual v -multiplication $I \times J = (IJ)_v$.

$(I \cdot J^{-1})_t = R$
 $R = (I_0 J_0)_t$
 $R = (J_0 v J_0 v)_t = (I_0 v I_0^{-1})_t$
 $I_t = I_v = R$
 $I_0 \in \mathcal{F}$
 $J_0 \in \mathcal{F}$

Facts. (1) $I \in T(R) \Rightarrow I \in D_f(R)$.

(2) $I \in T(R) \Rightarrow I = x_1 R \cap \dots \cap x_n R$ for some $0 \neq x_i \in K$.

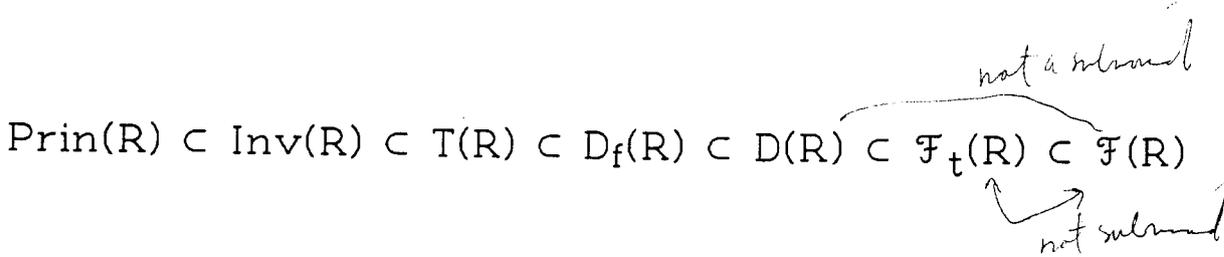
(3) TFAE for a t -ideal I of R .

(a) I is t -invertible, i.e., $(II^{-1})_t = R$.

(b) $I, I^{-1} \in D_f(R)$, and $(I : I) = R$.

(c) $I \in D_f(R)$ and I_P is principal for all $P \in t\text{-Max}(R)$.

(d) $I^{-1} \not\subseteq P$ for all $P \in t\text{-Max}(R)$.



Def. For any integral domain R , we define

Alain Broué

$\text{Cl}(R) = T(R)/\text{Prin}(R)$, the **(t) -class group** of R .

$\text{Cl}(R)$ is an abelian group and $\text{Pic}(R) \subset \text{Cl}(R)$.

Def. For any integral domain R , we define

$G(R) = Cl(R)/Pic(R)$, the local (t -)class group of R .

$$0 \longrightarrow Pic(R) \longrightarrow Cl(R) \longrightarrow G(R) \longrightarrow 0$$

Does $Cl(R) = Cl(R)$?

extended, not generalize

(1) R is a Prüfer domain.

some f.g. ideals are invertible
 $t = d$ and $T(R) = D_f(R) = Inv(R)$; so $Cl(R) = Pic(R)$.
 $Cl(R) = 0 \iff R$ is a Bézout domain.

$v = t = d$ if R is a Dedekind domain, and
 $Cl(R) = 0 \iff R$ is a PID.

(2) R is a Krull domain.

Cl + ACC non domain

$t = v$ and $T(R) = D(R) = D_f(R)$; so $Cl(R) = Cl(R)$.
 Also, $t\text{-Max}(R) = X^{(1)}(R)$.

Thm. (Kang-1989) TFAE for an integral domain R .

- (1) R is a Krull domain.
- (2) Every t -ideal is t -invertible (i.e., $T(R) = \mathcal{F}_t(R)$).
- (3) Every prime t -ideal is t -invertible.
- (4) Every nonzero prime ideal contains a t -invertible prime t -ideal.
- (5) Every proper t -ideal is a t -product of (t -invertible) prime ideals.
- (6) R_M is a UFD for each $M \in t\text{-Max}(R)$ and every minimal prime ideal is a finite type t -ideal. \square

change t to d to get Dedekind

$$\text{Prin}(R) \subseteq \text{Inv}(R) \subseteq T(R) \subseteq D_f(R) \subseteq D(R) \subseteq F_f$$

$$\underbrace{\pi_0(R)}_{\substack{= \text{Inv}(R) \\ \text{to be}}} \subseteq \text{Cl}(R) = T(R)/\text{Prin}(R), \quad G(R) = T(R)/\text{Inv}(R) = \frac{C_1}{F}$$

R Knull \Leftrightarrow c.i.c. $\frac{1}{2}$ Mori

R GCD $\quad \times R \cap yR$ primos $\quad D_f(R) = \text{Prin}(R)$

(3) R is a Mori domain (in particular, if R is Noetherian).

$$t = v \text{ and } T(R) \subset D_f(R) = D(R)$$

$$(T(R) = D(R) \Leftrightarrow R \text{ is c. i. c.} \Leftrightarrow R \text{ is a Krull domain})$$

Let R be a Mori domain. Then $R = R_1 \cap R_2$ for canonical overrings R_1 and R_2 , where R_1 is the "Krull part" of R and R_2 is the "nonKrull part" of R , and $Cl(R)$ is related to $Cl(R_1)$ and $Cl(R_2)$. This has been investigated in several papers by Barucci, Gabelli, and Roitman in the late 80's and early 90's.

(4) R is a Prüfer v -multiplication domain (PVMD).

(Recall that R is a PVMD if $D_f(R)$ is a group under v -multiplication, $\Leftrightarrow R_P$ is a valuation domain for each $P \in t\text{-Max}(R)$.)

I, I^{-1} both finite-type

Prüfer domains and Krull domains are PVMDs. If R is a PVMD, then so are R_S and $R[X]$. *finite $\cap \Rightarrow T(R)$*

\Rightarrow ~~$Cl(R) = 0$~~ R is a PVMD $\Leftrightarrow T(R) = D_f(R)$. $D_f(R) = T(R)$ PVMD

for PVMD $Cl(R) = 0 \Leftrightarrow R$ is a GCD-domain.

$D_f(R) = \text{Inv}(R)$ G-GCD domain
 $D_f(R) \neq \text{Prüfer}$ GCD

\Leftarrow always
 \Rightarrow need PVMD

(5) R is a weakly Krull domain.

(Recall that R is weakly Krull if $R = \bigcap_{P \in X(1)(R)} R_P$ and this intersection has FC.)

Any Krull domain or one-dimensional integral domain is weakly Krull.

Def. An integral domain R is **weakly factorial** if each nonzero, nonunit of R is a product of primary elements.

Thm. (DDA, Zafrullah-1990) An integral domain R is **weakly factorial** $\Leftrightarrow R$ is **weakly Krull** and $Cl(R) = 0$. \square

(6) R is completely integrally closed. *need not be Krull*

$D(R)$ is a group; thus $Cl_v(R) = D(R)/Prin(R)$ is an abelian group. However, $T(R)$ may be a proper subgroup of $D(R)$, and thus $Cl(R)$ may be a proper subgroup of $Cl_v(R)$.

Cl

Ex. Let R be a one-dimensional valuation domain with value group \mathbb{Q} . Then $Cl(R) = Pic(R) = 0$ since R is a Bézout domain (or R is one-dimensional and quasilocal). Also, $D(R)$ is a group since a one-dimensional valuation domain is completely integrally closed. However, R has nonprincipal divisorial ideals since its value group is \mathbb{Q} . Thus $Cl_v(R) \neq 0$.

*3/5 x R
1/4 v(y)
x R
y x R*

increasing bounded sequence in \mathbb{Q}

When does $Cl(R) = Pic(R)$, i.e., when is $G(R) = 0$?

Thm. Let R be an integral domain such that each maximal ideal of R is a t -ideal. Then $Cl(R) = Pic(R)$. In particular, $Cl(R) = Pic(R)$ when R is one-dimensional. *Generalized Dedekind*

Proof. Let M be a maximal ideal of R and $I \in T(R)$. If $I^{-1} \subset M$, then $R = (I^{-1})_t \subset M_t = M$, a contradiction. Thus $I^{-1} = R$; so $I \in Inv(R)$, and hence $Cl(R) = Pic(R)$.

If R is one-dimensional, then $t\text{-Max}(R) = \text{Max}(R)$. \square

back to Krull

Thm. (DDA-1978) TFAE for a Krull domain R .

- (1) $Cl(R) = Pic(R)$.
- (2) R_M is a UFD for each maximal ideal M of R .
- (3) For each $P \in X^{(1)}(R)$, $P \in Inv(R)$.
- (4) $I, J \in D(R) \Rightarrow IJ \in D(R)$. $II^{-1} \in D(R) \Rightarrow II^{-1} = R$
- (5) The intersection of any two (principal) invertible ideals is invertible. \square

Def. An integral domain R is a **generalized GCD domain** (G-GCD domain) if the intersection of two principal ideals is invertible.

(1.1) $D_f(R) = Inv(R)$ s.a. a PVMD

Thm. Let R be a PVMD. Then $Cl(R) = Pic(R) \Leftrightarrow R$ is a G-GCD domain. \square

\Rightarrow VMD $D_f(R) = T(R) \supset Inv(R) \supset Prin(R) = G$
G-GCD \perp

Thm. Let R be an integral domain. Then $Cl(R) = Pic(R) \Leftrightarrow I, J \in T(R) \Rightarrow IJ \in T(R)$. \square (\Leftarrow)

~~$\exists I^{-1} \in T(R)$ \Rightarrow I is T -ideal
 $R = (I, I^{-1}) \Rightarrow (II^{-1}) = R$
 $= II^{-1} = R \Rightarrow I$~~

Thm. (DDA-1982) Let R be an integral domain. Then $G(R_M) = 0$ (i.e., $Cl(R_M) = 0$) for all maximal ideals M of $R \Rightarrow G(R) = 0$ (i.e., $Cl(R) = Pic(R)$). \square

When is $Cl(R)$ torsion?

Def. An integral domain R is an almost GCD domain if for all $0 \neq x, y \in R$, there is an $n = n(x, y) \geq 1$ such that $x^n R \cap y^n R$ is principal.

~~replace 1 with n~~

Thm. (Storch-1967) TFAE for a Krull domain R .

- (1) R is almost factorial, i. e., $Cl(R)$ is a torsion group.
- (2) R is an almost GCD domain. $X^n R \cap Y^n R$ normal
- (3) Each subintersection of R is a localization of R . For each v is a local
- (4) Some power of each nonzero, nonunit of R is a product of primary elements.
- (5) For $P \in X^{(1)}(R)$, $P = \text{rad}(xR)$ for some $x \in R$. \square

Thm. (Zafrullah-1985) A PVMD R has $Cl(R)$ torsion $\Leftrightarrow R$ is an almost GCD domain. \square

Def. An integral domain R is **almost weakly factorial** if for each nonzero, nonunit $x \in R$, there is an $n = n(x) \geq 1$ such that x^n is a product of primary elements of R .

Thm. (DDA, Mott, Zafrullah-1992) An integral domain R is almost weakly factorial $\Leftrightarrow R$ is weakly Krull and $Cl(R)$ is a torsion group. \square

$(x^n)^{-1} = 0$

So we have for a Krull domain R

$Cl(R) = 0$: R is factorial

$G(R) = 0$: R is locally factorial

$Cl(R)$ torsion: R is almost factorial

$G(R)$ torsion: R is almost locally (or locally almost) factorial

Thm. (DDA-1978-82) TFAE for a Krull domain R .

(1) R is almost locally factorial, i. e., $G(R)$ is torsion.

(2) R is locally almost factorial, i.e., $Cl(R_M)$ is torsion for each maximal ideal M of R .

(3) Some power of each invertible ideal is a product of invertible primary ideals.

(4). For all $I, J \in \text{Inv}(R)$, $I^n \cap J^n$ is invertible for some $n = n(I, J) \geq 1$.

(5) Each $P \in X^{(1)}(R)$ is the radical of an invertible ideal.

□

Thm. (DDA-1982) Let R be an integral domain. Then $G(R_M)$ is torsion for all maximal ideals M of $R \Rightarrow G(R)$ is torsion. □

*$Cl(R_M)$ trivial
locally some power
principal \Rightarrow some power
invertible*

The early history of $Cl(R)$

1. A. Bouvier, Le groupe des classes d'un anneau int gr , 107 me Congr s National des Soci t  Savantes, Brest, Fasc. IV(1982), 85-92.
2. M. Zafrullah, A general theory of almost factoriality, Manuscripta Math. 51(1985), 29-62. (received 3/26/84)
3. A. Bouvier and M. Zafrullah, On some class groups of an integral domain, Bull. Soc. Math. Gr c. 29(1988), 46-59. (This is a revised version of "On the class group" (1984/85) by Bouvier and Zafrullah.)

Maps between class groups, I

Let $A \subset B$ be an extension of integral domains.

Then $I \in \text{Inv}(A) \Rightarrow IB \in \text{Inv}(B)$ and $x_A \rightarrow x_B$; so $I \rightarrow IB$ induces a homomorphism $\text{Pic}(A) \longrightarrow \text{Pic}(B)$ by $[I] \rightarrow [IB]$.

For $I \in T(A)$, is $IB \in T(B)$? Is IB even a t -ideal?
(The localization of a t -ideal need not be a t -ideal.)

Suppose that $A \subset B$ is a flat extension. Then for $I \in T(A)$, $I = x_1 A \cap \cdots \cap x_n A$; so $IB = x_1 B \cap \cdots \cap x_n B$ is a v -ideal (and hence a t -ideal) of B .

Lemma. Let $A \subset B$ be a flat extension of integral domains.

(1) $(IB)_v = (I_v B)_v$ for all f. g. $I \in \mathcal{F}(R)$.

(2) $(IB)_t = (I_t B)_t$ for all $I \in \mathcal{F}(R)$.

In particular, for $S \subset R$ multiplicatively closed.

$(IR_S)_v = (I_v R_S)_v$ for all f. g. $I \in \mathcal{F}(R)$ and $(IR_S)_t = (I_t R_S)_t$ for all $I \in \mathcal{F}(R)$.

Proof. (1) Let $I = (x_1, \dots, x_n)$. Then $I^{-1} = x_1^{-1}A \cap \cdots \cap x_n^{-1}A$; so $I^{-1}B = x_1^{-1}B \cap \cdots \cap x_n^{-1}B = (IB)^{-1}$ since $A \subset B$ is flat. Then $x \in I_v$ implies $xI^{-1} \subset A$; so $xI^{-1}B \subset B$, and hence $x \in (I^{-1}B)^{-1} = ((IB)^{-1})^{-1} = (IB)_v$. Thus $I_v \subset (IB)_v$, and hence $(I_v B)_v \subset (IB)_v$. The other inclusion always holds; so we have $(IB)_v = (I_v B)_v$. \square

Note that $(IB)_t = (I_t B)_t \Leftrightarrow I_t \subset (IB)_t$.

Thm. Let $A \subset B$ be a flat extension of integral domains.

(1) If $I \in T(A)$, then $IB \in T(B)$.

(2) $I \rightarrow IB$ induces a homomorphism $Cl(A) \rightarrow Cl(B)$ given by $[I] \rightarrow [IB]$.

Proof. (1) Let $I \in T(A)$; so $(II^{-1})_t = A$. Thus $((IB)(I^{-1}B))_t = (II^{-1}B)_t = ((II^{-1})_t B)_t = (AB)_t = B_t = B$; so $IB \in T(B)$.

(2) By (1), we have the map $\varphi : T(A) \rightarrow T(B)$, given by $\varphi(I) = IB$, which is a homomorphism since $\varphi(I \times J) = \varphi((IJ)_t) = (IJ)_t B = ((IJ)_t B)_t = (IJB)_t = ((IB)(JB))_t = \varphi(I) \times \varphi(J)$ for all $I, J \in T(A)$. We also have $\varphi(xA) = xB$ for all $0 \neq x \in qf(A)$. Thus we have an induced homomorphism $Cl(A) \rightarrow Cl(B)$ given by $[I] \rightarrow [IB]$. \square

The following commutative diagram has exact rows.

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \text{Pic}(A) & \longrightarrow & \text{Cl}(A) & \longrightarrow & G(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \text{Pic}(B) & \longrightarrow & \text{Cl}(B) & \longrightarrow & G(B) & \longrightarrow & 0
 \end{array}$$

two special cases

$$Cl(R) \longrightarrow Cl(R[X]), \quad [I] \longrightarrow [IR[X]]$$

$$Cl(R) \longrightarrow Cl(R_S), \quad [I] \longrightarrow [I_S]$$

generalize UFD

Thm. Let R be a Krull domain. Then $Cl(R) = Cl(R[X])$. \square

$Pic(R) \longrightarrow Pic(R[X])$ is always a split monomorphism. \checkmark
 $Cl(R) \longrightarrow Cl(R[X])$ is always a monomorphism.

Def. An integral domain R with quotient field K is **seminormal** if $x^2, x^3 \in R$ for $x \in K$ implies $x \in R$.
 (R c. i. c. $\Rightarrow R$ integrally closed $\Rightarrow R$ seminormal.
 No implication is reversible.)

Thm. (1) (Gilmer, Heitmann-1980) $Pic(R) = Pic(R[X]) \Leftrightarrow R$ is seminormal.
 (2) (Gabelli-1987) $Cl(R) = Cl(R[X]) \Leftrightarrow R$ is integrally closed. \square

An application to the $D + M$ construction

Let T be an integral domain of the form $T = K + M$, where K is a field contained in T and M is a nonzero maximal ideal of T . Let D be a proper subring of K with quotient field K and $S = D - \{0\}$. Then $R = D + M$ is a subring of T with $R_S = K + M = T$. Also, R is never a Krull domain, and M is a divisorial ideal (and hence a t -ideal) of R . (This will be generalized to general pullbacks later.)

We have $D \subset R = D + M \subset R_S = K + M = T$; all are flat extensions, so we have induced homomorphisms

$$\alpha : Cl(D) \longrightarrow Cl(D + M), [I] \longrightarrow [IR] = [I + M], \text{ and}$$

$$\beta : Cl(D + M) \longrightarrow Cl(K + M), [J] \longrightarrow [J_S]$$

Thm. (A, Rycaert-1988)

(1) $0 \longrightarrow Pic(D) \longrightarrow Pic(R) \longrightarrow Pic(T) \longrightarrow 0$ is exact.

(2) $0 \longrightarrow Cl(D) \longrightarrow Cl(R) \longrightarrow Cl(T)$ is exact.

also for Cl

(3) If T is either quasilocal or $Cl(T) = 0$, then $Pic(D) = Pic(D + M)$ and $Cl(D) = Cl(D + M)$.

Proof. (3) Suppose that T is quasilocal. Then $R_S = R_M = T$. We show that $\beta : Cl(R) \longrightarrow Cl(T), \varphi([I]) = [I_M]$, is the zero map. Since M is a t -ideal of R , we have $I^{-1} \not\subseteq M$ for all $I \in T(R)$. Thus I_M is principal in $R_M = T$; so $\beta = 0$. \square

Cor. Let D be an integral domain with $qf(D) = K$. Then

(1) $Pic(D + XK[X]) = Pic(D)$ and $Pic(D + XK[[X]]) = Pic(D)$.

(2) $Cl(D + XK[X]) = Cl(D)$ and $Cl(D + XK[[X]]) = Cl(D)$. \square

This corollary can be used to construct nonKrull domains with given class group. For example, if D is a Dedekind domain, then $D + XK[X]$ will be a two-dimensional PVMD which is neither a Noetherian domain nor a Krull domain.

Ex. In general, $\beta : Cl(R) \longrightarrow Cl(T)$ is not surjective. Let $T = \mathbb{Q}[X^2, XY, Y^2]_{(X^2, XY, Y^2)} = \mathbb{Q} + M$. Then T is a Noetherian two-dimensional local Krull domain with $Cl(T) = \mathbb{Z}/2\mathbb{Z}$. Let $R = \mathbb{Z} + M$. Then by the above theorem, we have $Cl(\mathbb{Z} + M) = Cl(\mathbb{Z}) = 0$. Thus β is not surjective.

$\varphi_S : Cl(R) \longrightarrow Cl(R_S), [I] \rightarrow [I_S]$, is a homomorphism.

Two questions

- (1) How do $Cl(R)$ and $Cl(R_S)$ compare?
- (2) What can we say about φ_S ? When is φ_S injective or surjective?

Nagata's Thm. Let R be a Krull domain and $S \subset R$ be multiplicatively closed. Then the homomorphism

$\varphi_S : Cl(R) \longrightarrow Cl(R_S)$ is surjective and $\ker \varphi_S = \langle [P] \mid P \in X^{(1)}(R) \text{ with } P \cap S \neq \emptyset \rangle$. \square

Cor. Let R be a Krull domain. Then the homomorphism φ_S is injective (and hence an isomorphism) if and only if the saturation of S is generated by principal primes. \square

Thm. Let R be a PVMD. Then $\varphi_S : Cl(R) \longrightarrow Cl(R_S)$ is surjective.

Proof. Let $J \in T(R_S) = D_f(R_S)$. Then $J = (IR_S)_v$ for some $f. g. I \in \mathcal{F}(R)$. Thus $I_v \in D_f(R) = T(R)$ and $I_v R_S = (I_v R_S)_v = (IR_S)_v = J$. Hence $T(R) \longrightarrow T(R_S)$ is surjective, and thus φ_S is surjective. \square

Ex. Let G and H be any two abelian groups. Then there is a Krull domain D with $Cl(D) = G$. Let $qf(D) = K$. Also, there is a quasilocal Krull domain $T = K + M$ with $Cl(T) = H$. Let $R = D + M$ and $S = D - \{0\}$. Then $R_S = K + M = T$, and $Cl(D + M) = Cl(D) = G$ and $Cl(R_S) = Cl(T) = H$. Note that in this case, $\varphi_S : Cl(R) \longrightarrow Cl(R_S)$ is the zero map since $R_S = R_M$.

Now we go back to: when does $\text{Cl}(R) = \text{Pic}(R)$?

\Leftrightarrow always

Does $\text{Cl}(R) = \text{Pic}(R) \Leftrightarrow \text{Cl}(R_M) = 0$ for each maximal ideal M of R ? (\Rightarrow) **No**, choose $\text{Cl}(R) = 0$ (so $\text{Cl}(R) = \text{Pic}(R)$), but $\text{Cl}(R_M) \neq 0$ in the above example.

Thm. TFAE for an integral domain R .

(1) $\text{Cl}(R_M) = 0$ for each maximal ideal M of R .

(2) $\text{Cl}(R) = \text{Pic}(R)$ and $\varphi_M : \text{Cl}(R) \longrightarrow \text{Cl}(R_M)$ is surjective for each maximal ideal M of R .

(Thus $G(R_M) = 0$ for all maximal ideals M of R $\Leftrightarrow G(R) = 0$ and each φ_M is surjective.) \square

cf. Krull

Cor. Let R be a PVMD. Then $\text{Cl}(R) = \text{Pic}(R) \Leftrightarrow R_M$ is a GCD-domain for all maximal ideals M of R .

Proof. Each φ_M is surjective when R is a PVMD. \square

Thus, in general, we can't say anything about $\text{Cl}(R)$ and $\text{Cl}(R_S)$. So we put extra conditions on R or S to try to get better results.

Thm. (AA-1988) Let $S \subset R$ be a multiplicatively closed set generated by principal primes. Then $\varphi_S : \text{Cl}(R) \longrightarrow \text{Cl}(R_S)$ is injective. Thus $\text{Pic}(R) \longrightarrow \text{Pic}(R_S)$ is also injective. \square

$Cl(R) \longrightarrow Cl(R_S)$ injective \Rightarrow $Pic(R) \longrightarrow Pic(R_S)$ injective?

$R = \mathbb{Q}[X, Y, Z]/(X^2, Y^2, Z^2)$ *for vms, not prime*

$Cl(R) \longrightarrow Cl(R_S)$ surjective \Rightarrow $Pic(R) \longrightarrow Pic(R_S)$ surjective?

$Pic(R) \longrightarrow Pic(R_S)$ injective \Rightarrow $Cl(R) \longrightarrow Cl(R_S)$ injective?

$Pic(R) \longrightarrow Pic(R_S)$ surjective \Rightarrow $Cl(R) \longrightarrow Cl(R_S)$ surjective?

DKM K+M

The case where S is generated by principal primes has been investigated by many authors.

Ex. (Gabelli, Roitman-1990) They gave two examples of an integral domain R where S is generated by principal primes, but $\varphi_S : Cl(R) \longrightarrow Cl(R_S)$ is **not** surjective.

(1) R is quasilocal and $S = \langle p \rangle$ for p a prime of R .

(2) R is a c. i. c. quasilocal domain and R_S is a Krull domain.

Def. An integral domain R is **archimedean** if $\bigcap (r^n) = \{0\}$ for all nonunits $r \in R$.

(R c. i. c., R satisfies ACCP, or $\dim R = 1 \Rightarrow R$ archimedean)

Thm. (GB-1990) Let $S \subset R$ be generated by principal primes. Then $\varphi_S : Cl(R) \longrightarrow Cl(R_S)$ is an isomorphism if

(1) R satisfies ACCP.

(2) R is archimedean and S is finitely generated.

(3) $t\text{-dim} R = 1$ (i.e., each $P \in t\text{-Max}(R)$ has ht one).

(4) $\bigcap (p^n) = \{0\}$ (equivalently, $\text{ht}(p) = 1$) for each prime $p \in S$ and $\bigcap (p_\alpha) = \{0\}$ for any infinite family $\{p_\alpha\}$ of nonassociate primes $p_\alpha \in S$. \square

Ex. Let $S = \{1, X, X^2, \dots\} \subset R[X]$. Then X is prime and $R[X]_S = R[X, X^{-1}]$. By (4) above, $\text{Cl}(R[X]) = \text{Cl}(R[X, X^{-1}])$, and $\text{Cl}(R) = \text{Cl}(R[X, X^{-1}])$ if R is integrally closed.

Thm. (Nour El Abidine-1990) Let $S \subset R$ be generated by principal primes. Then $\varphi_S : \text{Cl}(R) \longrightarrow \text{Cl}(R_S)$ is an isomorphism if R is v -coherent (i.e., $I_v \cap J_v$ is v -finite for any nonzero f. g. ideals I and J of R). \square

Many of the above results are special cases of the following theorem.

Thm. (A, Chang-2003) Let $S \subset R$ be a multiplicatively closed set generated by principal primes. Assume that if P is a prime t -ideal of R such that $P \cap S = \emptyset$, then $(PR_S)_t \neq R_S$. Then $\varphi_S : \text{Cl}(R) \longrightarrow \text{Cl}(R_S)$ is an isomorphism. \square

For an arbitrary multiplicatively closed set $S \subset R$, conditions for $\varphi_S : \text{Cl}(R) \longrightarrow \text{Cl}(R_S)$ to be surjective have also been investigated by many authors. For example, φ_S is surjective if

- (1) R is a Mori domain such that $\text{ht}P = 1$ for each $P \in t\text{-Max}(R)$ which intersects S . (GR-1990)
- (2) $\dim R = 1$. (GR-1990)
- (3) $t\text{-dim}R = 1$. (AHZ-1993)

Thm. (AHZ-1993; A, Chang-2003) Let $S \subset R$ be multiplicatively closed. Assume that for $P \in t\text{-Max}(R)$: if $P \cap S = \emptyset$, then $(PR_S)_t \neq R_S$; and if $P \cap S \neq \emptyset$, then $\text{ht}P = 1$. Then $\varphi_S : \text{Cl}(R) \longrightarrow \text{Cl}(R_S)$ is surjective. \square

$R(X) = R[X]_S$, where $S = \{ f \in R[X] \mid A_f = R \}$, and
 $R\langle X \rangle = R[X]_T$, where $T = \{ f \in R[X] \mid f \text{ is monic} \}$.

Thm. (A, Chang-2003). Let R be an integral domain.

(1) If R is integrally closed, then $Cl(R) = Cl(R\langle X \rangle)$.

(2) $0 \longrightarrow Pic(R) \longrightarrow Cl(R) \longrightarrow Cl(R(X))$ is exact.

(3) If in addition, R is integrally closed, then we have the
 SES $0 \longrightarrow Pic(R) \longrightarrow Cl(R) \longrightarrow Cl(R(X)) \longrightarrow 0$. \square

Splitting sets

Def. (1) Let $S \subset R$ be a saturated multiplicative set. The **m-complement** of S is $N = N(S) = \{ 0 \neq x \in R \mid xR \cap sR = xsR \text{ for all } s \in S \}$. Then N is a saturated multiplicatively closed set with $S \cap N = U(R)$.

(2) S is a **splitting set** of R if $SN = R - \{0\}$, i.e., if for each $0 \neq x \in R$, we have $x = st$ for some $s \in S$ and $t \in N$ (this representation is unique up to unit factors).

(3) S is an **lcm splitting set** if $sR \cap xR$ is principal for all $s \in S$ and $x \in R$.

Ex. (1) Let $S = U(R)$. Then the m-complement of S is $N = R - \{0\}$, and S is a splitting set of R .

(2) If S is a splitting set, then so is N , and $N(N(S)) = S$.

(3) Let $T = \{ p_\alpha \}$ be a set of nonassociate primes of R such that $\bigcap (p^n) = \{0\}$ for each $p \in T$ (equivalently, each $ht(p) = 1$) and $\bigcap (p_\alpha) = \{0\}$ for each infinite $\{ p_\alpha \} \subset T$. Then $S = \langle T \rangle$ is an lcm splitting set. Moreover, if R is atomic (i.e., every nonzero, nonunit of R is a finite product of irreducible elements (atoms) of R), then a saturated multiplicatively closed set is an lcm splitting set \Leftrightarrow it is generated by principal primes.

Facts. (1) $xR \cap sR = xsR \Leftrightarrow (x, s)_v = R$.

(2) S is an lcm splitting set $\Leftrightarrow R_N$ is a GCD domain.

(3) Let P be a prime t -ideal of R . Then either $P \cap S = \emptyset$ or $P \cap N = \emptyset$.

(4) Let I be a t -ideal of R . Then $I_S \cap I_N = I$.

(5) Let $s_1, \dots, s_n \in S$ and $t_1, \dots, t_n \in N$. Then $((s_1, \dots, s_n)(t_1, \dots, t_n))_v = (s_1 t_1, \dots, s_n t_n)_v$.

Thm. (AAZ-1991) Let S be a splitting set of R with m -complement N . Then $Cl(R) = Cl(R_S) \oplus Cl(R_N)$. In particular, $\varphi_S : Cl(R) \longrightarrow Cl(R_S)$ is surjective. Moreover, if S is an lcm splitting set, then φ_S is an isomorphism.

Proof. Define a homomorphism $\psi : Cl(R) \longrightarrow Cl(R_S) \oplus Cl(R_N)$ by $\psi([I]) = ([I_S], [I_N])$.

(injective) Suppose that $I_S = tR_S$ and $I_N = sR_N$ for some $s \in S$ and $t \in N$. Then $I = I_S \cap I_N = tR_S \cap sR_N = stR_S \cap stR_N = stR$; so ψ is injective.

(surjective) Let $([I], [J]) \in Cl(R_S) \oplus Cl(R_N)$. Write $I = ((t_1, \dots, t_n)_S)_v$ and $J = ((s_1, \dots, s_n)_N)_v$ for some $s_1, \dots, s_n \in S$ and $t_1, \dots, t_n \in N$. Let $L = (s_1 t_1, \dots, s_n t_n)_v = ((s_1, \dots, s_n)(t_1, \dots, t_n))_v$. Then $L \in D_f(R)$, $L_S = I$ and $L_N = J$. Hence $L \in T(R)$; so ψ is surjective.

If S is an lcm splitting set, then $Cl(R_N) = 0$ since R_N is a GCD domain. Thus φ_S is an isomorphism. \square

Note: φ_S may be an isomorphism for a multiplicative set S generated by principal primes which is not a splitting set.

Does $\text{Pic}(R) = \text{Pic}(R_S) \oplus \text{Pic}(R_N)$?

The above proof shows that $\psi|_{\text{Pic}(R)}$ is injective, but $\psi|_{\text{Pic}(R)}$ need not be surjective. For example, let $R = D[X]$ and S be the lcm splitting set generated by the prime X . Then $R_S = D[X, X^{-1}]$ and R_N is a GCD domain. If D is seminormal, but not quasinormal, then $\text{Pic}(D) = \text{Pic}(D[X]) \neq \text{Pic}(D[X, X^{-1}])$; so $\psi|_{\text{Pic}(R)}$ is not surjective.

We can also generalize splitting sets by considering multiplicative sets S such that $SN = D - P$, for P a prime $(t-)$ ideal of R . The "splitting set theory" works here because

- (1) If $I \not\subseteq P$, then $I = (I - P)$, and
- (2) If $I \subseteq P$, then we can assume $I \not\subseteq P$ by replacing I by uI for suitable $u \in I^{-1}$.

Thm. (A, Chang-2003). Let R be an integral domain, P a prime t -ideal of R , $S \subset R$ a saturated multiplicative set, and N the m -complement of S such that $SN = D - P$. Then $\text{Cl}(R) = \text{Cl}(R_S) \oplus \text{Cl}(R_N) \iff I^{-1} \not\subseteq P_S$ and $JJ^{-1} \not\subseteq P_N$ for all t -invertible ideals I and J of R_S and R_N , respectively.

In particular, this holds when either R is v -coherent, $\text{ht}P = 1$, $P = (aR : bR)$ for some $0 \neq a, b \in R$, or R is a PVMD. \square

Thm. (A, Chang-2003) Let R be an integral domain, P a prime ideal of R , $S \subset R$ a saturated multiplicative set, and N the m -complement of S such that $SN = D - P$. Assume that for each maximal ideal Q of R , either $Q \cap S = \emptyset$ or $Q \cap N = \emptyset$. Then $\text{Pic}(R) = \text{Pic}(R_S) \oplus \text{Pic}(R_N)$. \square

One can also define almost splitting sets. A saturated multiplicatively closed set $S \subset R$ is an **almost splitting set** if for each $0 \neq x \in R$, there is an $n = n(x) \geq 1$ such that $x^n = st$ for some $s \in S$ and $t \in N = N(S)$. In this case, $\text{Cl}(R)$ is torsion if and only if $\text{Cl}(R_S)$ and $\text{Cl}(R_N)$ are both torsion.

Maps between class groups, II

We have seen that if $A \subset B$ is a flat extension of integral domains, then we have an induced homomorphism $\text{Cl}(A) \longrightarrow \text{Cl}(B)$ given by $[I] \rightarrow [IB]$. What if B is not a flat A -module or A is not a subring of B ?

For any extension $A \subset B$ of integral domains, we have a map $\mathcal{F}_t(A) \longrightarrow \mathcal{F}_t(B)$ given by $I \rightarrow (IB)_t$. It seems natural to define $T(A) \longrightarrow T(B)$ by $I \rightarrow (IB)_t$. But is $(IB)_t$ t -invertible? It is easy to see that this map is a well-defined homomorphism if $(I_t B)_t = (IB)_t$ for all $I \in \mathcal{F}(R)$. However, this map may fail to be a homomorphism, and yet it induces a homomorphism $\text{Cl}(A) \longrightarrow \text{Cl}(B)$.

Ex. Let K be a field and $A = K[X, XY] \subset K[X, Y] = B$. Then $(X, XY)_t = A$ since $\text{ht}(X, XY) = 2$, but $((X, XY)B)_t = (XB)_t = XB$. Let $I = (X, XY) \subset A$. Then $(I_t B)_t = (AB)_t = B_t = B$, but $(IB)_t = (XB)_t = XB$. However, A and B are both UFDs; so $\text{Cl}(A) = \text{Cl}(B) = 0$ and $\text{Cl}(A) \longrightarrow \text{Cl}(B)$ is the zero map.

Def. An extension $A \subset B$ of integral domains is **t-linked** (or B is t-linked over A) if $(IB)^{-1} = B$ for each f. g. ideal I of A with $I^{-1} = A$ (i.e., $I_{\mathcal{V}} = A \Rightarrow (IB)_{\mathcal{V}} = B$).

In the above example, the extension $K[X, XY] \subset K[X, Y]$ is not t-linked. For $I = (X, XY)$, we have $I_{\mathcal{V}} = A$, but $(IB)_{\mathcal{V}} = XB$.

Thm. (DHLZ-1989, AHZ-1993) TFAE for an extension $A \subset B$ of integral domains.

- (1) $A \subset B$ is t-linked.
- (2) Let $I \in \mathcal{F}(A)$. If $I_t = A$, then $(IB)_t = B$.
- (3) If $Q \in \text{t-Max}(B)$ with $Q \cap A \neq 0$, then $(Q \cap A)_t \neq A$.
- (4) If $I_t \in T(A)$, then $(IB)_t = (I_t B)_t$. \square

Thm. (AHZ-1993) Let $A \subset B$ be a t-linked extension of integral domains. Then $T(A) \longrightarrow T(B)$, $I \longrightarrow (IB)_t$ induces a homomorphism $\text{Cl}(A) \longrightarrow \text{Cl}(B)$ given by $[I] \longrightarrow [(IB)_t]$. \square

Thm. (DHLZ-1989) Let R be an integral domain.

- (1) Any flat extension of R is t-linked over R . In particular, any localization of R is t-linked over R .
- (2) The complete integral closure of R in its quotient field is t-linked over R .
- (3) Any intersection of t-linked extensions of R is t-linked over R . In particular, any intersection of localizations of R is t-linked over R . \square

However, the integral closure of R need not be t-linked over R .

For Krull domains, there are (at least) two distinct ways to define homomorphisms between class groups, depending on how we consider $\text{Cl}(R)$. First, define $\varphi' : D(A) \longrightarrow D(B)$ by $I \mapsto (IB)_{\vee}$, and then define $\varphi : \text{Cl}(A) \longrightarrow \text{Cl}(B)$ by $[I] \mapsto [(IB)_{\vee}]$. This map is always well-defined; but is it a homomorphism? It is if $A \subset B$ is t-linked.

Def. An extension of Krull domains $A \subset B$ satisfies (PDE) if $\text{ht}(P \cap A) \leq 1$ for each $P \in X^{(1)}(B)$.

Note that $A \subset B$ satisfies (PDE) $\Leftrightarrow A \subset B$ is t-linked. If $A \subset B$ is either integral or flat, then it satisfies (PDE).

Let $A \subset B$ be an extension of Krull domains. Then for each $P \in X^{(1)}(A)$, there are only a finite number $Q_1, \dots, Q_n \in X^{(1)}(B)$ with $Q_k \cap A = P$. Define $\vartheta' : \text{Div}(A) \longrightarrow \text{Div}(B)$ by $(P) \mapsto a_1(Q_1) + \dots + a_n(Q_n)$, where $PB_{Q_k} = Q_k^{a_k}B_{Q_k}$. If $A \subset B$ satisfies (PDE), then $\vartheta'(\text{Prin}(A)) \subset \text{Prin}(B)$; so there is an induced isomorphism

$$\vartheta : \text{Cl}(A) = \text{Div}(A)/\text{Prin}(A) \longrightarrow \text{Div}(B)/\text{Prin}(B) = \text{Cl}(B).$$

If $A \subset B$ satisfies (PDE), then $\pi_B \varphi = \vartheta \pi_A$, where π_R is the canonical isomorphism $D(R)/\text{Prin}(R) \longrightarrow \text{Div}(R)/\text{Prin}(R)$.

$$\begin{array}{ccc} \text{Cl}(A) & \xrightarrow{\varphi} & \text{Cl}(B) \\ \pi_A \downarrow & & \downarrow \pi_B \\ \text{Cl}(A) & \xrightarrow{\vartheta} & \text{Cl}(B) \end{array}$$

Subintersections

Def. Let R be a Krull domain with quotient field K . Then for $\emptyset \neq Y \subset X^{(1)}(R)$; let $R_Y = \bigcap_{P \in Y} R_P$. Then $R \subset R_Y \subset K$ is a Krull domain, called a **subintersection** of R . Moreover, $R \subset R_Y$ is a t -linked extension.

Let B be an overring of a Krull domain R . Then B is a localization of $R \Rightarrow {}_R B$ is flat $\Rightarrow B$ is a subintersection of R . Moreover, $R_S = R_Y$, where $Y = \{ P \in X^{(1)}(R) \mid P \cap S = \emptyset \}$.

Nagata's Thm. Let R be a Krull domain and $\emptyset \neq Y \subset X^{(1)}(R)$. Then $\varphi : Cl(R) \longrightarrow Cl(R_Y)$, given by $[I] \rightarrow [(IB)_t]$, is a surjective homomorphism with $\ker \varphi = \langle [P] \mid P \in X^{(1)}(R) - Y \rangle$. \square

Thm. (AHZ-1993) Let R be a PVMD, X be the set of prime t -ideals of R , $\emptyset \neq Y \subset X$, and $R_Y = \bigcap_{P \in Y} R_P$. Then $\varphi : Cl(R) \longrightarrow Cl(R_Y)$, $[I] \rightarrow [IR_Y]$ is a surjective homomorphism. \square

Thm. (AHZ-1993) Let R be a weakly Krull domain and $\emptyset \neq Y \subset X^{(1)}(R)$. Then $R_Y = \bigcap_{P \in Y} R_P$ is a weakly Krull domain and the homomorphism $\varphi : Cl(R) \longrightarrow Cl(R_Y)$, $[I] \rightarrow [(IB)_t]$, is surjective with $\ker \varphi = \langle [P] \mid P \text{ is a } t\text{-invertible } t\text{-ideal primary to primes from } X^{(1)}(R) - Y \rangle$. \square

Pullbacks

Let T be an integral domain with nonzero maximal ideal M , residue field $K = T/M$, and $\varphi : T \longrightarrow T/M = K$ the natural projection. Let D be a proper subring of K with $\text{qf}(D) = K$. Then $R = \varphi^{-1}(D)$ is a subring of T and M is a prime divisorial (and hence a t -ideal) ideal of R . We call this a pullback of type (\square) .

$$\begin{array}{ccc}
 \varphi(D)^{-1} = R & \subset & T \\
 \downarrow & & \downarrow \varphi \\
 D & \subset & K = T/M
 \end{array}
 \quad (\square)$$

We can define a homomorphism $\alpha' : T(D) \longrightarrow T(R)$ by $\alpha'(I) = \varphi^{-1}(I)$. However, α' induces a (well-defined) homomorphism $\alpha : \text{Cl}(D) \longrightarrow \text{Cl}(R)$, $\alpha([I]) = [\varphi^{-1}(I)]$, \Leftrightarrow the induced homomorphism $\tilde{\varphi} : U(T) \longrightarrow K^*/U(D)$, given by $\tilde{\varphi}(x) = xU(D)$, is surjective. Also, $R \subset T$ is a flat extension (in fact, T is a localization of R when $\tilde{\varphi}$ is surjective); so we have a homomorphism $\beta : \text{Cl}(R) \longrightarrow \text{Cl}(T)$, given by $\beta([J]) = [JT]$. In this case, we have

Thm. (Fontana, Gabelli-1996; Khalis, Nour El Abidine-1997)
 Consider a pullback of type (\square) with $\tilde{\varphi} : U(T) \longrightarrow K^*/U(D)$ surjective. Then we have the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \text{Pic}(D) & \longrightarrow & \text{Pic}(R) & \longrightarrow & \text{Pic}(T) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \text{Cl}(D) & \xrightarrow{\alpha} & \text{Cl}(R) & \xrightarrow{\beta} & \text{Cl}(T) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & G(D) & \longrightarrow & G(R) & \longrightarrow & G(T) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Moreover, if β is surjective, then the SES

$$0 \longrightarrow \text{Cl}(D) \xrightarrow{\alpha} \text{Cl}(R) \xrightarrow{\beta} \text{Cl}(T) \longrightarrow 0 \quad \text{splits. } \square$$

When are $\tilde{\varphi}$ and β surjective?

Thm. (FG-1996) The map $\tilde{\varphi}$ is surjective if either T is semiquasilocal or $T = K + M$. \square

We have already seen that β need not be surjective in general and that β is surjective if R is a PVMD.

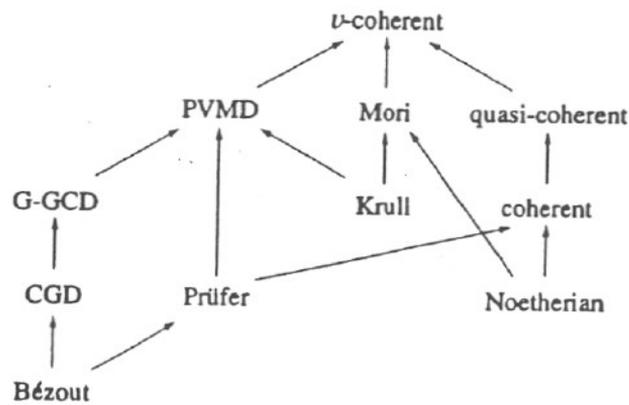
We always have $\text{Pic}(T) \subset \text{im}\beta$. In many cases (but not all) $\text{im}\beta = \text{Pic}(T)$, and in this case, $\text{Cl}(R) = \text{Cl}(D) \oplus \text{Pic}(T)$. This would be the case if either T is a Prüfer domain or if every maximal ideal of T is a t -ideal (e.g., $\dim T = 1$).

Def. An integral domain R is ν -coherent if $I_\nu \cap J_\nu$ is ν -finite for each pair of f. g. fractional ideals $I, J \in \mathcal{F}(R)$.

R is ν -coherent $\Leftrightarrow I^{-1}$ is ν -finite for each f. g. $I \in \mathcal{F}(R)$ (this is property P^* introduced by Nour El Abidine-1992).

Ex. (FG-1966)

We summarize in a diagram the principal implications among the classes of domains that we are considering:



Thm. (FG-1996) The map β is surjective if R is ν -coherent. Moreover, in this case, all the rows are split exact. \square

Thm. (D + M version) Let $T = K + M$ be an integral domain, where M is a nonzero maximal ideal of T and K is a field contained in T . Let D be a proper subring of K and $R = D + M$. Then R is a Prüfer (resp., Bézout, GCD, G-GCD, PVMD) domain if and only if $K = \text{qf}(D)$, D and T are Prüfer (resp., Bézout, GCD, G-GCD, PVMD) domains, and T_M is a valuation domain. \square

Thm. (Pullback version) Let M be a nonzero maximal ideal of an integral domain T with residue field $K = T/M$ and $\varphi : T \longrightarrow T/M = K$ the natural projection. Let D be a proper subring of K and $R = \varphi^{-1}(D)$. Then

(0) R is a Prüfer domain $\Leftrightarrow K = \text{qf}(D)$ and D and T are Prüfer domains.

(1) R is a PVMD (resp., G-GCD domain) $\Leftrightarrow K = \text{qf}(D)$, D and T are PVMDs (resp., G-GCD domains), and T_M is a valuation domain.

(2) R is a GCD domain $\Leftrightarrow K = \text{qf}(D)$, D and T are GCD domains, T_M is a valuation domain, and $\tilde{\varphi} : U(T) \longrightarrow K^*/U(D)$ is surjective.

(3) R is a Bézout domain $\Leftrightarrow K = \text{qf}(D)$, D and T are Bézout domains, and $\tilde{\varphi} : U(T) \longrightarrow K^*/U(D)$ is surjective.

Proof. (3) Suppose that R is a Bézout domain. Then R is a Prüfer domain; so $K = \text{qf}(D)$ and D and T are Prüfer domains by (0). By a result in [FG-1996], $\tilde{\varphi}$ is surjective. Thus there is a SES $0 \rightarrow \text{Cl}(D) \rightarrow \text{Cl}(R) \rightarrow \text{Cl}(T) \rightarrow 0$. Then $\text{Cl}(R) = 0$ since R is a Bézout domain, and hence $\text{Cl}(D) = \text{Cl}(T) = 0$. Thus D and T are Bézout domains.

Conversely, suppose that $K = \text{qf}(D)$, D and T are Bézout domains, and $\tilde{\varphi}$ is surjective. By (0) again, R is a Prüfer domain. There is a SES $0 \rightarrow \text{Cl}(D) \rightarrow \text{Cl}(R) \rightarrow \text{Cl}(T) \rightarrow 0$. Then $\text{Cl}(D) = \text{Cl}(T) = 0$ since D and T are Bézout domains. Thus $\text{Cl}(R) = 0$, and hence R is also a Bézout domain. \square

This result can also be obtained via “generalized” splitting sets. Let $S = U(T) \cap R \subset R$. Then the m -complement of S in R is $N = \{ x \in R \mid \varphi(x) \in U(D) \}$ and $SN = R - M \Leftrightarrow \tilde{\varphi} : U(T) \longrightarrow K^*/U(D)$ is surjective. In this case, $T = R_S$.

Thm. (A, Chang-2003) Consider a pullback of type (\square) with $\tilde{\varphi} : U(T) \longrightarrow K^*/U(D)$ surjective.

- (1) $Cl(R) = Cl(D) \oplus \text{im} \beta$.
- (2) $Cl(R) = Cl(D) \oplus Cl(T)$ if and only if $I^{-1} \not\subset M$ for all t -invertible t -ideals I of T .
- (3) If either $Cl(T) = \text{Pic}(T)$ or M is a t -ideal of T , then $Cl(R) = Cl(D) \oplus Cl(T)$. \square

Graded integral domains

$$R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots = \bigoplus_{n \in \mathbb{Z}_+} R_n \quad \text{and} \quad R = \bigoplus_{n \in \mathbb{Z}} R_n$$

$$R = D[X] \quad \text{and} \quad R = D[X, X^{-1}] \quad \text{with} \quad \deg(dX^n) = n \quad \text{for} \quad 0 \neq d \in D$$

Thm. Let R be an integral domain.

- (1) $\text{Pic}(R) = \text{Pic}(R[X])$ if and only if R is seminormal.
- (2) $Cl(R) = Cl(R[X])$ if and only if R is integrally closed.
- (3) $Cl(R) = Cl(R[X, X^{-1}])$ if and only if R is integrally closed. \square

Def. An integral domain R is said to be **quasinormal** if $\text{Pic}(R) = \text{Pic}(R[X, X^{-1}])$.

c. i. c. \Rightarrow integrally closed \Rightarrow quasinormal \Rightarrow seminormal
(no implications are reversible)

Monoid domains

Let Γ be a commutative (additive) monoid. Then the monoid ring $R[\Gamma] = \{ \sum r_\alpha X^\alpha \mid r_\alpha \in R \text{ and } \alpha \in \Gamma \}$ with $X^\alpha X^\beta = X^{\alpha+\beta}$ is an integral domain if and only if R is an integral domain and Γ is cancellative and torsionless (i.e., $nx = ny$ for $n \geq 1$ and $x, y \in \Gamma$ implies $x = y$). We call such a monoid Γ a **grading monoid**. Then Γ is a grading monoid $\Leftrightarrow \langle \Gamma \rangle = \{ \alpha - \beta \mid \alpha, \beta \in \Gamma \}$ is a torsionfree abelian group $\Leftrightarrow \Gamma$ may be totally ordered. Also, note that $U(R[\Gamma]) = \{ rX^\alpha \mid r \in U(R) \text{ and } \alpha \in U(\Gamma) = \Gamma \cap -\Gamma \}$.

For example, $R[X] = R[\mathbb{Z}_+]$, $R[X, X^{-1}] = R[\mathbb{Z}]$, and $R[X^2, X^3] = R[\{0, 2, 3, \dots\}]$. More generally, let A be a subring of $R[\{X_\alpha\}]$ generated by monomials over R . Then $A = R[\Gamma]$, where $\Gamma = \{ (n_\alpha) \mid \prod X_\alpha^{n_\alpha} \in A \}$.

Def. Let Γ be a grading monoid.

(1) Γ is **seminormal** if $2\alpha, 3\alpha \in \Gamma$ for $\alpha \in \langle \Gamma \rangle$ implies $\alpha \in \Gamma$.

(2) Γ is **integrally closed** if $n\alpha \in \Gamma$ for some $n \geq 1$ and $\alpha \in \langle \Gamma \rangle$ implies $\alpha \in \Gamma$.

(3) Γ is a **Krull monoid** if Γ is c. i. c. and satisfies ACC on divisorial ideals.

Facts. (1) $R[\Gamma]$ is seminormal $\Leftrightarrow R$ and Γ are each seminormal.

(2) $R[\Gamma]$ is integrally closed $\Leftrightarrow R$ and Γ are each integrally closed.

(3) $R[\Gamma]$ is a Krull domain $\Leftrightarrow R$ is a Krull domain, Γ is a Krull monoid, and $\langle \Gamma \rangle$ (equivalently, $U(\Gamma)$) satisfies ACC on cyclic subgroups.

Thm. (A-1990) Let R be an integral domain and Γ a grading monoid with group of units $U(\Gamma) = \Gamma \cap -\Gamma$.

(1) $\text{Pic}(R) = \text{Pic}(R[\Gamma])$ if and only if $R[\Gamma]$ is seminormal and $\text{Pic}(R) = \text{Pic}(R[U(\Gamma)])$.

(2) Suppose that $U(\Gamma) = 0$. Then $\text{Pic}(R) = \text{Pic}(R[\Gamma])$ if and only if R is seminormal and Γ is seminormal.

(3) Suppose that $U(\Gamma) \neq 0$. Then $\text{Pic}(R) = \text{Pic}(R[\Gamma])$ if and only if R is quasinormal and Γ is seminormal. \square

Cor. $\text{Pic}(R) = \text{Pic}(R[\Gamma])$ for all seminormal grading monoids Γ if and only if $\text{Pic}(R) = \text{Pic}(R[\mathbb{Z}])$. \square

Thm. (El Baghdadi, Izelgue, Kabbaj-2002) Let R be an integral domain with quotient field K and Γ a grading monoid. Then $\text{Cl}(R) = \text{Cl}(R[\Gamma])$ if and only if $R[\Gamma]$ is integrally closed and $\text{Cl}(K[\Gamma]) = 0$. \square

Facts. (1) If $R[\Gamma]$ is a Krull domain, then $\text{Cl}(R) = \text{Cl}(R[\Gamma])$ if and only if $K[\Gamma]$ is a UFD. However, we have $\text{Cl}(K[\mathbb{Q}_+]) = 0$ since $K[\mathbb{Q}_+]$ is a Bézout domain, but $K[\mathbb{Q}_+]$ is not a UFD.

(2) It is possible to have $\text{Cl}(R) \cong \text{Cl}(R[\Gamma])$ and $\text{Cl}(K[\Gamma]) \neq 0$ even when $R[\Gamma]$ is a Krull domain. For example, let R and Γ be such that $\text{Cl}(R) \cong \bigoplus_{n \in \mathbb{Z}_+} \mathbb{Z}$ and $\text{Cl}(K[\Gamma]) = \mathbb{Z}$. Thus $\text{Cl}(R[\Gamma]) \cong \text{Cl}(R) \oplus \text{Cl}(K[\Gamma]) \cong (\bigoplus_{n \in \mathbb{Z}_+} \mathbb{Z}) \oplus \mathbb{Z} \cong \bigoplus_{n \in \mathbb{Z}_+} \mathbb{Z}$.

The Krull domain case

Thm. Let R be an integral domain and G a torsionfree abelian group.

(1) (Gilmer, Parker-1974) $R[G]$ is a UFD $\Leftrightarrow R$ is a UFD and G satisfies ACC on cyclic subgroups.

(2) (Matsuda-1975) $R[G]$ is a Krull domain $\Leftrightarrow R$ is a Krull domain and G satisfies ACC on cyclic subgroups. \square

Thm. Let R be an integral domain and Γ a grading monoid.

(1) (Gilmer, Parker-1974) $R[\Gamma]$ is a UFD $\Leftrightarrow R$ is a UFD, Γ is a factorial monoid, and $U(\Gamma)$ satisfies ACC on cyclic subgroups.

(2) (Chouinard-1981) $R[\Gamma]$ is a Krull domain $\Leftrightarrow R$ is a Krull domain, Γ is a Krull monoid, and $U(\Gamma)$ satisfies ACC on cyclic subgroups. \square

Facts. (1) A torsionfree abelian group G satisfies ACC on cyclic subgroups \Leftrightarrow each element of G has type $(0, 0, 0, \dots)$.

(2) A factorial monoid Γ has the form $G \oplus F_+$, where G is a torsionfree abelian group and F is a free abelian group with the usual product order.

(3) A Krull monoid Γ has the form $G \oplus T$, where G is a torsionfree abelian group and T is a submonoid of a free abelian group F with the usual product order such that $T = \langle T \rangle \cap F_+$.

(4) Thus a UFD semigroup ring is a polynomial ring over a UFD group ring, and a Krull semigroup ring is a subring of a polynomial ring over a Krull group ring generated by monomials.

(5) Let Γ be a Krull monoid. Then $\langle \Gamma \rangle$ satisfies ACC on cyclic subgroups $\Leftrightarrow U(\Gamma)$ does.

(6) Let $R[\Gamma]$ be a Krull domain. Then $Cl(R[\Gamma]) = Cl(R) \oplus Cl(K[\Gamma])$, and $Cl(K[\Gamma])$ is independent of the field K .

Def. Let Γ be a Krull monoid. Define the **divisor class group** of Γ to be the abelian group $Cl(\Gamma)$ of divisorial fractional ideals of Γ under v -multiplication modulo its subgroup of principal fractional ideals.

Thm. (Chouinard-1981) Let $R[\Gamma]$ be a Krull domain. Then $Cl(R[\Gamma]) \cong Cl(R) \oplus Cl(\Gamma)$. \square

Let $F = \bigoplus_{\alpha} \mathbb{Z}$ be a free abelian group with the usual product order, and let each pr_{α} be the natural projection map. If $\Gamma \subset F_+$ with $\Gamma = \langle \Gamma \rangle \cap F_+$, then Γ is a Krull monoid, and if the $pr_{\alpha}|_{\langle \Gamma \rangle}$'s are distinct essential valuations on Γ , then $Cl(\Gamma) \cong F/\langle \Gamma \rangle$.

This construction can be used to show that for any abelian group G , there is a quasilocal Krull domain R with $Cl(R) = G$. A Krull domain A with $Cl(A) = G$ can be constructed to have the form $A = K[\Gamma] \subset K[\{X_{\alpha}\}]$ (for any field K) and be generated by monomials over K . Then $N = A \cap (\{X_{\alpha}\})$ is a maximal ideal of A and $A = K + N$. Let $R = A_N = K + N_N$. Then R is a quasilocal Krull domain, and $Cl(R) = G$ by Nagata's Theorem.

Def. Let Γ be a grading monoid. We define the **t-class group** of Γ to be the abelian group $\text{Cl}(\Gamma)$ of t-invertible fractional t-ideals of Γ under t-multiplication modulo its subgroup of principal fractional ideals.

Thm. (BIK-2002) Let $R[\Gamma]$ be an integrally closed domain. Then $\text{Cl}(R[\Gamma]) \cong \text{Cl}(R) \oplus \text{Cl}(\Gamma)$. \square

What if $R[\Gamma]$ is not integrally closed?

not independent $\exists K$

It is wellknown that $\text{Cl}(K[X^2, X^3]) = \text{Pic}(K[X^2, X^3]) = K$ (as an additive abelian group) for any field K . More generally, for any integral domain R with quotient field K , we have $\text{Cl}(R[X^2, X^3]) = \text{Cl}(R[X]) \oplus K$. Recall that a **numerical semigroup** Γ is an additive submonoid Γ of \mathbb{Z}_+ such that $\mathbb{Z}_+ - \Gamma$ is finite. This previous example is a special case of

Thm. (A, Chang-2004) Let R be an integral domain with quotient field K and Γ a numerical semigroup. Then $\text{Cl}(R[\Gamma]) = \text{Cl}(R[X]) \oplus \text{Pic}(K[\Gamma])$. In particular, if R is integrally closed, then $\text{Cl}(R[\Gamma]) = \text{Cl}(R) \oplus \text{Pic}(K[\Gamma])$.

Proof. Let $S = \{ X^\alpha \mid \alpha \in \Gamma \}$ and $N = D - \{0\}$. Then the natural homomorphism $\text{Cl}(R[\Gamma]) \longrightarrow \text{Cl}(R[\Gamma]_S) \oplus \text{Cl}(R[\Gamma]_N) = \text{Cl}(R[X, X^{-1}]) \oplus \text{Cl}(K[\Gamma])$, $[I] \longrightarrow ([I_S], [I_N])$, is an isomorphism.

Then $\text{Cl}(R[X, X^{-1}]) = \text{Cl}(R[X])$, and $\text{Cl}(K[\Gamma]) = \text{Pic}(K[\Gamma])$ because $K[\Gamma]$ is one-dimensional; so the result follows. If in addition, R is integrally closed, then $\text{Cl}(R[X]) = \text{Cl}(R)$. \square

Note that $\text{Pic}(K[\Gamma])$ can be computed.

Γ -graded integral domains, $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be an integral domain graded by a grading monoid Γ . Let H be the multiplicatively closed set of all nonzero homogeneous elements of R . Then R_H is a $\langle \Gamma \rangle$ -graded integral domain with $\deg(a/s) = \deg a - \deg s$; R_H is called the **homogeneous quotient field** of R . Note that $(R_H)_0$ is a field. If R is \mathbb{Z}_+ - or \mathbb{Z} -graded, then $R_H \cong (R_H)_0[X, X^{-1}]$ is a UFD. But in general, R_H is a twisted group ring over $(R_H)_0$ and is just a c. i. c. GCD-domain.

Ex. The monoid domain $R[\Gamma]$ is Γ -graded with $\deg(rX^\alpha) = \alpha$ for all $0 \neq r \in R$ and $\alpha \in \Gamma$. The homogeneous quotient field of $R[\Gamma]$ is $K[G]$, where $K = \text{qf}(R)$ and $G = \langle \Gamma \rangle$.

Note that if I is a homogeneous (fractional) ideal of R , then I^{-1} , I_v , and I_t are all homogeneous and each is contained in R_H . We can thus define $\text{HPrin}(R)$, $\text{HInv}(R)$, $\text{HT}(R)$ in the obvious way. We define the **homogeneous Picard group** and **homogeneous class group** of R to be

$$\text{HPic}(R) = \text{HInv}(R)/\text{HPrin}(R) \subset \text{Pic}(R)$$

$$\text{HCl}(R) = \text{HT}(R)/\text{HPrin}(R) \subset \text{Cl}(R)$$

Note that $\text{HCl}(R) = \text{Cl}(R) \iff$ for each integral $I \in \text{T}(R)$, $I = xJ$ for some homogeneous $J \in \text{T}(R)$ and $x \in R_H$.

Thm. Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded Krull domain. Then $HCl(R) = Cl(R)$.

Proof. Let H be the multiplicatively closed set of nonzero homogeneous elements of R . Then R_H is a UFD since R is a Krull GCD-domain; so $Cl(R_H) = 0$. Hence $\ker(Cl(R) \rightarrow Cl(R_H)) = Cl(R)$. Thus by Nagata's Theorem, $Cl(R)$ is generated by the classes of the ht-one prime ideals of R which intersect H . Such a prime ideal is necessarily homogeneous since it has ht-one. Hence $HCl(R) = Cl(R)$. \square

Ex. Let K be a field and $R = K[X^2, XY, Y^2]$. Then R is a two-dimensional Krull domain with $Cl(R) = \mathbb{Z}/2\mathbb{Z}$. We can consider R as a \mathbb{Z}_+ -graded subring of $K[X, Y]$ in the usual way with $\deg X = \deg Y = 1$; but it is better to think of R as being a $\mathbb{Z}_+ \times \mathbb{Z}_+$ -graded subring of $K[X, Y]$ with $\deg X = (1, 0)$ and $\deg Y = (0, 1)$. In this case, R has only two homogeneous ht-one prime ideals, namely $P = (X^2, XY)$ and $Q = (XY, Y^2)$. Then $(P^2)_{\vee} = (X^4, X^3Y, X^2Y^2)_{\vee} = X^2(X^2, XY, Y^2)_{\vee} = (X^2)$; $(Q^2)_{\vee} = Y^2(X^2, XY, Y^2)_{\vee} = (Y^2)$; and $(PQ)_{\vee} = (X^3Y, X^2Y^2, XY^3)_{\vee} = XY(X^2, XY, Y^2)_{\vee} = (XY)$. Thus $Cl(R) = HCl(R)$ is generated by $[P]$ and $[Q]$ with $2[P] = 2[Q] = [P] + [Q] = 0$; so $Cl(R) = \langle [P] \rangle = \mathbb{Z}/2\mathbb{Z}$.

Note that R is just the monoid domain $K[\Gamma]$, where $\Gamma = \{ (m, n) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \mid m + n \text{ is even} \}$, and $Cl(R) = \mathbb{Z}^2 / \langle \Gamma \rangle = \mathbb{Z}^2 / \langle (2, 0), (1, 1) \rangle$. Similarly, the Krull domain $R_n = K[X^n, XY, Y^n]$ has $Cl(R_n) = \mathbb{Z}/n\mathbb{Z}$ for any $n \geq 1$.

Thus $R = K[X_1^{n_1}, X_1 Y_1, Y_1^{n_1}, \dots, X_r^{n_r}, X_r Y_r, Y_r^{n_r}]$ has $Cl(R) = \mathbb{Z}/n_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/n_r\mathbb{Z}$ for any $n_1, \dots, n_r \geq 1$.

Def. Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain with homogeneous quotient field R_H .

- (1) R is **almost seminormal** if whenever $x^2, x^3 \in R$ for homogeneous $x \in R_H$ with $\deg x \neq 0$, then $x \in R$.
- (2) R is **almost normal** if whenever x is integral over R for homogeneous $x \in R_H$ with $\deg x \neq 0$, then $x \in R$.
- (3) $R_0 \subset R$ is an **inert extension** if $xy \in R_0$ for $x, y \in R$ implies that $x = ua$ and $y = u^{-1}b$ for some $u \in U(R)$ and $a, b \in R_0$.

Facts: (1) R seminormal (resp., integrally closed) $\Rightarrow R$ almost seminormal (resp., integrally closed).

(2) R is seminormal (resp., integrally closed) $\Leftrightarrow R$ is almost seminormal (resp., almost normal) and R_0 is seminormal (resp., integrally closed) in $(R_H)_0$.

(3) $R[\Gamma]$ is almost seminormal (resp., almost normal) if and only if $R[\Gamma]$ is seminormal (resp., integrally closed).

(4) $R \subset R[\Gamma]$ is always an inert extension.

(5) $R_0 \subset R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ is inert if $\Gamma \cap -\Gamma = 0$. In particular, $R_0 \subset R = \bigoplus_{n \in \mathbb{Z}_+} R_n$ is always an inert extension.

Ex. Let $R = K[X, Y, Z, W]$ with K a field. Then R is a \mathbb{Z} -graded integral domain with $\deg X = \deg Y = 1$ and $\deg Z = \deg W = -1$ and $R_0 \subset R$ is not an inert extension. Note that R is a UFD, but $R_0 = k[XZ, XW, YZ, YW]$ is not a UFD. In fact, $Cl(R_0) = \mathbb{Z}$.

The next several theorems are based on "content" results (cf. (Querre-1980), (AA-1982), (BIK-2002)). For $x = \sum x_{\alpha} \in R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$, its **content ideal** is the homogeneous ideal $C(x) = (x_{\alpha})$. Next we give a sampling of such results.

Thm. (AA-1982) Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain and H its set of nonzero homogeneous elements. Then TFAE

(1) Each integral v -ideal of R which contains a nonzero homogeneous element of R is homogeneous.

(2) $C(xy)_v = (C(x)C(y))_v$ for all $0 \neq x, y \in R$.

(3) $xR_S \cap R = xC(x)^{-1}$ for all $0 \neq x \in R$.

(4) If I is an integral v -ideal of R of finite type, then $I = xJ$ for some $x \in R_H$ and homogeneous v -ideal J of R of finite type. \square

Thm. (AA-1982) Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain.

(1) If R is integrally closed, then $C(xy)_v = (C(x)C(y))_v$ for all $0 \neq x, y \in R$.

(2) If $C(xy)_v = (C(x)C(y))_v$ for all $0 \neq x, y \in R$, then R is almost normal.

(3) Let $R_0 \subset R$ be an inert extension. Then $C(xy)_v = (C(x)C(y))_v$ for all $0 \neq x, y \in R \Leftrightarrow R$ is almost normal. \square

Thm. (AA-1982) Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain.

(1) If $\text{HPic}(R) = \text{Pic}(R)$, then R is almost seminormal. Suppose, in addition, that $R_0 \subset R$ is an inert extension.

(2) $\text{Pic}(R_0) = \text{HPic}(R)$.

(3) If R is almost normal, then $\text{Pic}(R_0) = \text{Pic}(R)$. \square

The inert hypothesis is needed in (2) and (3) above.

We now show the earlier stated result of El Baghdadi-Izelgue-Kabbaj (2002) that $Cl(R[\Gamma]) \cong Cl(R) \oplus Cl(\Gamma)$ when $R[\Gamma]$ is integrally closed.

Thm. (BIK-2002) Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain such that $R_0 \subset R$ is inert. Then $HCl(R) = Cl(R) \Leftrightarrow R$ is almost normal. In particular, if R is \mathbb{Z}_+ -graded, then $HCl(R) = Cl(R) \Leftrightarrow$ if R is almost normal. \square

Thm. (BIK-2002) Let R be an integral domain with quotient field K and Γ a grading monoid.

- (1) $HCl(R[\Gamma]) = Cl(R[\Gamma]) \Leftrightarrow R[\Gamma]$ is integrally closed.
- (2) $HCl(R[\Gamma]) = Cl(R) \oplus HCl(K[\Gamma])$. \square

Thm. (BIK-2002) Let K be a field. Then $HCl(K[\Gamma]) \cong Cl(\Gamma)$. Thus $Cl(K[\Gamma]) \cong Cl(\Gamma)$ when Γ is integrally closed.

Proof. The map is $[K[Y]] \rightarrow [Y]$. \square

Thm. (BIK-2002) Let $R[\Gamma]$ be an integrally closed domain. Then $Cl(R[\Gamma]) = Cl(R) \oplus Cl(K[\Gamma]) \cong Cl(R) \oplus Cl(\Gamma)$.

Proof. By the above results, $Cl(R[\Gamma]) = HCl(R[\Gamma]) = Cl(R) \oplus HCl(K[\Gamma]) \cong Cl(R) \oplus Cl(K[\Gamma]) \cong Cl(R) \oplus Cl(\Gamma)$. \square

Cor. (BIK-2002) Let R be an integral domain and G a nonzero torsionfree abelian group. Then $Cl(R) = Cl(R[G]) \Leftrightarrow R$ is integrally closed. \square

Homogeneous splitting sets

Def. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain and H the multiplicative set of nonzero homogeneous elements of R

(1) A multiplicative subset $S \subset R$ is **homogeneous** if $S \subset H$.

(2) For a homogeneous multiplicative set S , its **homogeneous complement** is $N_h(S) = \{ x \in H \mid (d, s)_v = R \text{ for all } s \in S \}$.

(3) A saturated homogeneous multiplicative set S is a **homogeneous splitting set** if $H = SN_h(S)$.

Ex. If S is a splitting set of R , then $S^* = S \cap H$ is a homogeneous splitting set with $N_h(S^*) = N_h(S) = N(S) \cap H$.

Thm. (A, Chang-2005) Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, and let S be a homogeneous splitting set of R with homogeneous complement $N = N_h(S)$. Then $HCl(R) = HCl(R_S) \oplus HCl(R_N)$. \square

Cor. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, and let S be a homogeneous splitting set of R with homogeneous complement $N = N_h(S)$.

(1) If R is a Krull domain, then $Cl(R) = Cl(R_S) \oplus Cl(R_N)$.

(2) If $R_0 \subset R$ is an inert extension and R is almost normal, then $Cl(R) = Cl(R_S) \oplus Cl(R_N)$. \square

We recover the BIK result.

Cor. Let $R[\Gamma]$ be an integrally closed domain. Then $\text{Cl}(R[\Gamma]) = \text{Cl}(R[\langle \Gamma \rangle]) \oplus \text{Cl}(K[\Gamma])$.

Proof. Let S be the saturation of $R - \{0\}$ in $R[\Gamma]$. Then $N = N_h(S) = \{uX^\alpha \mid u \in U(R) \text{ and } \alpha \in \Gamma\}$. So S is a homogeneous splitting set. Note that $R[\Gamma]_S = K[\Gamma]$ and $R[\Gamma]_N = R[\langle \Gamma \rangle]$. Thus $\text{Cl}(R[\Gamma]) = \text{Cl}(R[\Gamma]_N) \oplus \text{Cl}(R[\Gamma]_S) = \text{Cl}(R[\langle \Gamma \rangle]) \oplus \text{Cl}(K[\Gamma])$. Note that $\text{Cl}(R[\langle \Gamma \rangle]) = \text{Cl}(R)$ since $R[\Gamma]$ is integrally closed. \square

Cor. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, and let S be a splitting set of R with $S^* = S \cap H$. If either R is a Krull domain, or $R_0 \subset R$ is an inert extension and R is almost normal, then $\text{Cl}(R_{S^*}) = \text{Cl}(R_S)$. \square

The $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots = \bigoplus_{n \in \mathbb{Z}_+} R_n$ case

Thm. (A-1982) Let $R = \bigoplus_{n \in \mathbb{Z}_+} R_n$ be a graded integral domain. Then $\text{Pic}(R_0) = \text{Pic}(R) \Leftrightarrow R$ is almost seminormal. In particular, $\text{Pic}(R_0) = \text{Pic}(R)$ when R is seminormal. \square

Very rarely does $\text{Cl}(R_0) = \text{Cl}(R)$ even when R is a Krull domain. We may have $\text{Cl}(R_0) = 0$ and $\text{Cl}(R) \neq 0$. Is $\text{Cl}(R_0) \longrightarrow \text{Cl}(R)$ even a homomorphism?

Let $R = \bigoplus_{n \in \mathbb{Z}_+} R_n$ be a Krull domain. The map $\text{Cl}(R_0) \longrightarrow \text{Cl}(R)$, given by $[I] \rightarrow [(IR)_\vee]$, is always defined, and is injective since $(IR)_\vee = I \oplus \cdots$ is homogeneous. Is it a homomorphism?

The $A + XB[X]$ and $D + XD_S[X]$ constructions

Let $A \subset B$ be an extension of integral domains. Then $R = A + XB[X] = \{ f(X) \in B[X] \mid f(0) \in A \} \subset B[X]$ is a graded subring of $B[X]$. This is called the " $A + XB[X]$ " construction. It has been very useful for constructing examples.

Facts. (1) $A + XB[X]$ is almost seminormal \Leftrightarrow B is seminormal.

(2) $A + XB[X]$ is almost normal \Leftrightarrow B is integrally closed.

(3) $A \subset A + XB[X]$ is always an inert extension.

(4) If B is a flat A -module, then $A \subset A + XB[X]$ is a flat extension; so the mapping $\varphi : Cl(A) \rightarrow Cl(A + XB[X])$, $[I] \rightarrow [IR]$, is an injective homomorphism.

(5) $\varphi : Cl(A) \rightarrow Cl(A + XB[X])$, $[I] \rightarrow [(IR)_t]$, is always injective. Is it a homomorphism?

Thm. (A, El Baghdadi, Kabbaj-1999-2002) Let $R = A + XB[X]$.

(1) $Pic(A) = Pic(R)$ if and only if B is seminormal.

(2) $Cl(R) = HCl(R)$ if and only if B is integrally closed.

Thus if $Cl(A) = Cl(R)$, then B must be integrally closed.

(3) If B is an integrally closed flat overring of A , then $Cl(A) = Cl(R)$.

(4) If B is integrally closed and $qf(A) \subset B$, then $Cl(A) = Cl(R)$.

(5) If A is integrally closed and $B = A[\{X_\alpha\}]$, then $Cl(A) = Cl(R)$. \square

Ex. In (3) above, it is not enough to just assume that B is an integrally closed flat A -module. Let $A = \mathbb{Z} \subset \mathbb{Z}[i] = B$; so $R = \mathbb{Z} + X\mathbb{Z}[i][X]$. Then $\mathbb{Z}[i]$ is integrally closed and is a flat \mathbb{Z} -module (but not an overring). Then $\text{Cl}(A) = 0$, and it may be shown that $\text{Cl}(R) = \mathbb{Z}/2\mathbb{Z} \oplus (\bigoplus_{n \in \mathbb{Z}_+} \mathbb{Z})$.

Thm. (ABK-2002) Let $R = A + XB[X]$ and $S = A - \{0\}$.

- (1) Suppose that $\text{qf}(A) \subset B$. Then $\text{Cl}(A) = \text{HCl}(R)$.
- (2) Suppose that B is a flat overring of A . Then $\text{Cl}(A) = \text{HCl}(R)$.
- (3) Suppose that $A \subset B$ is inert, $\text{Cl}(B_S) = 0$, and $\text{qf}(A) \cap B = A$. Then $\text{HCl}(R) = \text{Cl}(B)$. \square

A special case of the $A + XB[X]$ construction is when $A = D$ is an integral domain, $S \subset D$ is multiplicatively closed, and $B = D_S$; the " $D + XD_S[X]$ " construction. In this case, $D \subset D_S$ is always a flat overring; so we have an injective homomorphism $\text{Cl}(D) \longrightarrow \text{Cl}(D + XD_S[X])$, $[I] \longrightarrow [I + XID_S[X]]$.

Cor. (ABK-1999-2002) Let D be an integral domain and $S \subset D$ multiplicatively closed.

- (1) $\text{Pic}(D) = \text{Pic}(D + XD_S[X]) \iff D_S$ is seminormal.
- (2) $\text{Cl}(D) = \text{Cl}(D + XD_S[X]) \iff D_S$ is integrally closed.

In particular, $\text{Pic}(D) = \text{Pic}(D + XD_S[X])$ when D is seminormal and $\text{Cl}(D) = \text{Cl}(D + XD_S[X])$ when D is integrally closed. \square

Thm. (A, Chang-2003) Let S be a splitting multiplicative subset of an integral domain D with m -complement N . Then $\text{Cl}(D + XD_S[X]) = \text{Cl}(D_S[X]) \oplus \text{Cl}(D_N)$ and $\text{Pic}(D + XD_S[X]) = \text{Pic}(D_S[X]) \oplus \text{Cl}(D_N)$.

Proof. Let $D^{(S)} = D + XD_S[X]$. If $S \neq U(D)$ is a splitting set of D , then $ST = D^{(S)} - XD_S[X]$, where T is the m -complement of S in $D^{(S)}$ (and $N = T \cap D$) and $\text{Cl}(D^{(S)}_T) = \text{Cl}(D_N)$. Thus $\text{Cl}(D^{(S)}) = \text{Cl}(D^{(S)}_S) \oplus \text{Cl}(D^{(S)}_T) = \text{Cl}(D_S[X]) \oplus \text{Cl}(D_N)$. \square

Generalizations

There are usually many star operations other than the d -, t -, and v -operations defined on an integral domain R . (However, for R a Dedekind domain, $v = d$; so all star operations on R are the same.) Recently the **w-operation** has received considerable attention, where

$$I_w = \{ x \in K \mid xJ \subset I \text{ for some f. g. ideal } J \text{ with } J^{-1} = D \}.$$

Let $\mathcal{S}(R)$ be the set of all star operations on an integral domain R . Then $\mathcal{S}(R)$ may be partially ordered by $*_1 \leq *_2 \Leftrightarrow I^{*1} \subset I^{*2}$ for all $I \in \mathcal{F}(R)$. Moreover, $\mathcal{S}(R)$ is then a complete lattice with least element d and greatest element v , i.e., $d \leq * \leq v$ for all $* \in \mathcal{S}(R)$.

For example, let R be an affine domain with $\dim R \geq 2$ (e. g., if $R = k[X_1, \dots, X_n]$ for a field k with $n \geq 2$).

Then $|\mathcal{S}(R)| = 2^{|R|}$.

Def. (1) A star operation $*$ on R has **finite type** if $I^* = \bigcup \{J^* \mid 0 \neq J \subset I \text{ is f. g.}\}$ for all $I \in \mathcal{F}(R)$. Note that the d - and t -operations always have finite type, and the v -operation has finite type $\Leftrightarrow v = t$.

(2) Let $*$ be a star operation on R . Then $*$ defines a finite type star operation $*_s$ on R by

$$I^*_s = \bigcup \{J^* \mid 0 \neq J \subset I \text{ is f. g.}\}.$$

Note that $v_s = t$ and that $d \leq * \leq t$ for any finite type star operation $*$ on R .

Let $*$ be a star operation on an integral domain R . We define $\text{Inv}_*(R) = \{I \in \mathcal{F}_*(R) \mid (IJ)^* = R \text{ for some } J \in \mathcal{F}(R)\}$ to be the abelian group of $*$ -invertible $*$ -ideals of R under the usual $*$ -multiplication $I \times J = (IJ)^*$.

For star operations $*_1 \leq *_2$, we have $\mathcal{F}_{*_2}(R) \subset \mathcal{F}_{*_1}(R)$ and $\text{Inv}_{*_1}(R) \subset \text{Inv}_{*_2}(R)$. Thus $\text{Inv}_d(R) = \text{Inv}(R) \subset T(R) = \text{Inv}_t(R)$.

Def. We define the $*$ -class (local $*$ -class) group of R by

$$\text{Cl}_*(R) = \text{Inv}_*(R) / \text{Prin}(R)$$

$$G_*(R) = \text{Inv}_*(R) / \text{Inv}(R) = \text{Cl}_*(R) / \text{Pic}(R)$$

We have $\text{Pic}(R) = \text{Cl}_d(R)$ and $\text{Cl}(R) = \text{Cl}_t(R)$, and

$\text{Cl}_{*1}(R) \subset \text{Cl}_{*2}(R)$ if $*_1 \leq *_2$. In particular,

$$\text{Pic}(R) \subset \text{Cl}_*(R) \subset \text{Cl}_v(R) \quad \text{and}$$

$$\text{Pic}(R) \subset \text{Cl}_*(R) \subset \text{Cl}_t(R) \quad \text{if } * \text{ has finite type.}$$

Def. Let $A \subset B$ be an extension of integral domains with star operations $*_A$ and $*_B$, respectively. We say that $*_A$ and $*_B$ are **compatiable** if $(IB)^{*}_B = (I^*_A B)^{*}_B$ for all $I \in \mathcal{F}(A)$. We then have an induced homomorphism

$$\varphi : \text{Cl}_{*_A}(A) \longrightarrow \text{Cl}_{*_B}(B) \quad \text{given by } \varphi([I]) = [(IB)^{*}_B].$$

*d by *_B compatible*

Ex. (1) Let R be an integral domain with quotient field K and $*$ a star operation on $R[X]$. Then $*$ induces a star operation $*$ on R by $I^* = (IR[X])^* \cap K$ for $I \in \mathcal{F}(R)$, which satisfies $(IR[X])^* = (I^*R[X])^*$ for all $I \in \mathcal{F}(R)$. Thus the two star operations are compatiable; so there is an induced homomorphism $\varphi : \text{Cl}_*(R) \longrightarrow \text{Cl}_*(R[X])$ given by $[I] \rightarrow [IR[X]]$. Clearly φ is injective, and φ is surjective if R is integrally closed.

(2) Let $T = K + M$, D be a subring of K , and $R = D + M$. A star operation $*$ on R induces a star operation $*$ on D by $I^* = (I + M)^* \cap \text{qf}(D)$ for $I \in \mathcal{F}(D)$. Then the two star operations are compatiable; in fact, $(I + M)^* = I^* + M$ for all $I \in \mathcal{F}(D)$. So we have an induced homomorphism $\varphi : \text{Cl}_*(D) \longrightarrow \text{Cl}_*(D + M)$ given by $[I] \rightarrow [I + M]$. Then φ is always injective, and is an isomorphism if T is quasilocal.

*-splitting sets

Let R be an integral domain and $S \subset R$ a saturated multiplicatively closed set. Then a finite type star operation $*$ on R induces a finite type star operation $*_S$ on R_S by defining $(J)^{*S} = (J \cap R)^* R_S$ for all nonzero integral ideals J of R_S , and then extending to fractional ideals, which satisfies $(IR_S)^{*S} = I^* R_S$ for all $I \in \mathcal{F}(R)$. Thus $*$ and $*_S$ are compatible, and the localization of a $*$ -ideal is a $*_S$ -ideal. Note that $d_S = d$, but we only have $t_S \leq t$ (the localization of a t -ideal need not be a t -ideal). In fact, t_S is the t -operation on $R_S \Leftrightarrow (I_t)R_S = (IR_S)_t$ for all $I \in \mathcal{F}(R)$. However, $t_S = t$ when S is a splitting set.

Def. Let R be an integral domain, $S \subset R$ a saturated multiplicatively closed set, and $*$ a finite type star operation on R .

(1) The $*$ -complement of S is $N_*(S) = \{ 0 \neq x \in R \mid (x, s)^* = R \text{ for all } s \in S \} \subset N(S)$. Then $N_*(S)$ is a saturated multiplicatively closed subset of R with $S \cap N_*(S) = U(R)$.

(2) S is a $*$ -splitting set if $SN_*(S) = R - \{0\}$. Thus a t -splitting set is just a splitting set.

Facts. (1) Let $* \leq *'$ be finite type star operations on R . If S is a $*$ -splitting set, then S is a $*'$ -splitting set. In particular, a $*$ -splitting set is a splitting set.

(2) A splitting set S is a $*$ -splitting set $\Leftrightarrow N_*(S) = N(S)$
 $\Leftrightarrow P \cap S = \emptyset$ or $P \cap N = \emptyset$ for all prime $*$ -ideals P of R .

(3) Any splitting set generated by principal primes is a $*$ -splitting set for any finite type star operation $*$.

Thm. (A, Chang, J. Park-2005) Let $*$ be a finite type star operation on an integral domain R , P a prime $*$ -ideal of R , S a saturated multiplicatively closed subset of R , $N = \{ 0 \neq x \in R \mid (s, t)^* = R \text{ for all } s \in S \}$ its $*$ -complement, and $*_S$ (resp., $*_N$) the star operation on R_S (resp., R_N) induced by $*$. Suppose that $SN = R - P$. Then $Cl_*(D) = Cl_{*_S}(R_S) \oplus Cl_{*_N}(R_N)$. Thus the natural homomorphism $Cl_*(R) \longrightarrow Cl_{*_S}(R_S)$ is surjective. In particular, the above holds when S is a $*$ -splitting set. \square

If S is a splitting set, then the t -operation on R induces the t -operation on R_S , i.e., $t_S = t$. Also, a splitting set S is a d -splitting set \Leftrightarrow either $M \cap S = \emptyset$ or $M \cap N = \emptyset$ for each maximal ideal M of R , the d -operation on R induces the d -operation on R_S , and $Cl_d(\) = Pic(\)$. Thus

Cor. Let S be a splitting multiplicative subset of an integral domain R and $N = \{ 0 \neq x \in R \mid (s, t)_v = R \text{ for all } s \in S \}$.

(1) $Cl(D) = Cl(R_S) \oplus Cl(R_N)$.

(2) Suppose that either $M \cap S = \emptyset$ or $M \cap N = \emptyset$ for each maximal ideal M of R . Then $Pic(D) = Pic(R_S) \oplus Pic(R_N)$. \square

We can also define induced star operations on pullbacks.

Let T be an integral domain with nonzero maximal ideal M , residue field $K = T/M$, and $\varphi : T \longrightarrow T/M = K$ the natural projection. Let D be a subring of K with $\text{qf}(D) = K$. Then $R = \varphi^{-1}(D)$ is a subring of T and M is a prime divisorial (and hence a t -ideal) ideal of R . We call this a pullback of type (\square) .

A star operation $*$ on R induces star operations $*_{\varphi}$ on D and $(*)_T$ on T by

$$I^{*_{\varphi}} = \bigcap \{ y\varphi((\varphi^{-1}(y^{-1}I))^*) \mid y \in K, I \subset yD \} \text{ for } I \in \mathcal{F}(D)$$

$$I^{(*)_T} = I^* \cap (T : (T : I)) \text{ for } I \in \mathcal{F}(T) (\subset \mathcal{F}(R)).$$

If $* = t$, then $*_{\varphi} = t$, but $(*)_T$ need not be t .

Thm. (Fontana, M. Park-2005) Given a pullback of type (\square) with $\tilde{\varphi} : U(T) \longrightarrow K^*/U(D)$ surjective and $*$ a finite type star operation on R , there is a split exact sequence

$$0 \longrightarrow \text{Cl}_{*_{\varphi}}(D) \xrightarrow{\alpha} \text{Cl}_*(R) \xrightarrow{\beta} \text{Cl}_{(*)_T}(T) \longrightarrow 0.$$

Some Problems

1. Investigate $\text{Cl}(R[X])$.

Is $\text{Cl}(R) \longrightarrow \text{Cl}(R[X])$ a split monomorphism?

Describe $\text{Cl}(R[X])/\text{Cl}(R)$.

2. Investigate $\text{Cl}(R[\Gamma])$ for arbitrary grading monoids.
3. Investigate the map $\text{Cl}(R_0) \longrightarrow \text{Cl}(R)$ when R is a graded domain.
4. Investigate $\text{Cl}(R[[X]])$.

Note that $\text{Pic}(R) = \text{Pic}(R[[X]])$ for any integral domain R . However, R is a UFD does not imply that $R[[X]]$ is a UFD. So even for Krull domains, we may have $\text{Cl}(R) \neq \text{Cl}(R[[X]])$.

5. Investigate $\text{Cl}_*(R)$ for arbitrary star operations.

The class group of an integral domain

D. F. Anderson - Cortona - 6/4/2006

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