Irredundant intersections of integrally closed overrings

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Thursday's talk: Let $H \subseteq R$ be integrally closed overrings of a two-dimensional Noetherian domain such that H has a Noetherian R-representation.

We can reduce to the case that H is quasilocal...

We saw that H is built from four components:

- an indecomposable Noetherian ring of dimension ≤ 1,
- an integrally closed Noetherian overring,
- a finite intersection of irrational valuation rings, and
- R.

The proof depended on the fact that $\text{Rep}_R(H)$ is the unique strongly irredundant Noetherian R-representation of H...

...which depended on the fact that there exists a strongly irredundant Noetherian R-representation of H.

...Today we want to settle this last detail.

Let H be a domain, and let R be an overring of H. Recall that a set Σ of valuation overrings of H is an R-**representation** of H if

$$H = \Big(\bigcap_{V \in \Sigma} V\Big) \cap R.$$

The R-representation Σ is **strongly irredundant** if no member V of Σ can be replaced by a proper overring of V.

In this talk we outline a proof of:

Theorem. If Σ is an R-representation of H that is a Noetherian subspace of the space of all valuation overrings of H, then there exists a strongly irredundant Noetherian R-representation of H.

The theorem was proved by Heinzer and Ohm (1972) in the case where every member of Σ has Krull dimension 1. Brewer and Mott (1970) proved the theorem for finite character R-representations in the case R = quotient field of H.

Corollary. If Σ is a finite character R-representation of H, then there exists a strongly irredundant finite character R-representation of H.

Consider the map $\Sigma \to \text{Spec}(H)$. Then Σ has finite character $\Leftrightarrow$ its image has finite character and the map is finite-to-one.

Thus finite character representations arise from finite character collections of prime ideals...

...But it is tricky to identify these collections of primes.

There are however some interesting applications:

- Building unusual Dedekind domains: Goldman (1964), Heitmann (1974).
- Conforming spectra: Houston-McAdam (1975)
- Slender rings and modules: L. Lady (1975)

Proposition. Here are two cases in which a domain has a finite character set of infinitely many maximal ideals.

- D is a countable domain with Jacobson radical 0.
- D is an affine K-domain, K = field (can be chosen so that each maximal ideal has residue field algebraic over K).

Proposition. Let $\{\phi_i\}_{i=1}^n$ be a subset of a transcendence basis over $\mathbb{R}$ of the ring of analytic functions on $\mathbb{R}$. Let I be an infinite compact subset of $\mathbb{R}$ and define for each $t \in I$,

$$\mathfrak{m}_t := (X_1 - \phi_1(t), X_2 - \phi_2(t), \ldots, X_n - \phi_n(t)).$$

Then $\{\mathfrak{m}_t : t \in I\}$ is an uncountable finite character set of maximal ideals in $\mathbb{R}[X_1, \ldots, X_n]$.

...e. g. $\mathfrak{m}_t = (X_1 - t, X_2 - e^t, X_3 - sin(t))$.

Returning to the original theorem, we actually prove something more general...

Let H be a domain, let R be an overring of H, and let Σ be a collection of integrally closed overrings of H such that $H = (\bigcap_{A \in \Sigma} A) \cap R$.

We say that Σ is an R**-representation** of H.

If no proper subset of Σ is an R-representation of H, then Σ is an **irredundant** R-representation of H.

If Σ is finite, then by throwing out "extra" members of Σ, you can obtain an irredundant R-representation of H.

...but this doesn't work if Σ is infinite.

Technical Problem: Given a collection Σ of integrally closed overrings of H, when does H have an irredundant R-representation consisting of integrally closed overrings of members of Σ?

Special Case: Irredundant representations of valuation overrings.

If Γ and Σ are collections of overrings of H, we say $\Gamma \leq \Sigma$ if every member of Γ is an overring of a member of Σ.

Main Theorem. Let H be a domain, let R be a proper overring of H, and let Σ be a collection of integrally closed overrings of H. If Σ is a Noetherian R-representation of H, then there exists an irredundant weakly Noetherian R-representation $\Gamma \leq \Sigma$ of H.

...The relevant topology will be introduced later...

From the proof of the Main Theorem, we derive: Domains having a Noetherian R-representation Σ of valuation overrings have also a strongly irredundant Noetherian R-representation $\Gamma \leq \Sigma$.

...This special case of the theorem was used in the previous talks.

Let H be a domain with quotient field F.

Recall: $\mathrm{Zar}(H) =$ the set of all valuation overrings of H. The **Zariski topology** is given by declaring the basic open sets to be of the form $\{V \in \mathrm{Zar}(H) : x_1, \ldots, x_n \in V\}$, where $x_1, \ldots, x_n \in F$.

Define $\mathrm{Over}(H) :=$ the set of all integrally closed overrings of H.

The **Zariski topology** on $\mathrm{Over}(H)$ has basic open sets

$$\mathcal{U}_H(x_1, \ldots, x_n) := \{R \in \mathrm{Over}(H) : x_1, \ldots, x_n \in R\},$$

where $x_1, \ldots, x_n \in F$.

Thus the Zariski topology on $\mathrm{Zar}(H)$ is the subspace topology inherited from the Zariski topology on $\mathrm{Over}(H)$.

...We will introduce a finer topology on $\mathrm{Over}(H)$.

Let I be a fractional ideal of H, and let $R \in \mathrm{Over}(H)$. Denote by $\mathrm{cl}_R(I)$ the integral closure of I in R; that is,

$$\mathrm{cl}_R(I) = \bigcap_{V \in Zar(R)} IV.$$

We define the b**-topology** on $\mathrm{Over}(H)$ to be the one induced by declaring the basic open sets to be of the form

$$\mathcal{U}_H(I,J) := \{R \in \mathrm{Over}(H) : I \subseteq \mathrm{cl}_R(J)\},$$

where I and J are finitely generated H-submodules of F.

In general: Zariski topology $\subsetneq$ b-topology...However:

Proposition. If H is a Prüfer domain, then

$$\text{Zariski topology} = b\text{-topology}.$$

Let $V \in \mathrm{Zar}(H)$. Define a valuation ring in $F(X)$ by

$$V^b := V[X]_{M[X]}.$$

If $A \in \mathrm{Over}(H)$, define

$$A^b := \bigcap_{V \in Zar(A)} V^b.$$

Then A^b is the unique minimal Kronecker function ring of A.

Proposition. The mapping $\mathrm{Over}(H) \to \mathrm{Over}(H^b) : A \mapsto A^b$ is a homeomorphism (with respect to the b-topology) onto its image.

...Thus $\mathrm{Over}(H)$ is homeomorphic to a collection of overrings of a Bézout domain.

Corollary. On $\mathrm{Zar}(H)$, Zariski topology = b-topology.

A topological space X is **Noetherian** if X satisfies the ascending chain condition for open sets.

We say $\Sigma \subseteq \text{Over}(H)$ is **weakly Noetherian** if Σ is a Noetherian subspace of $\text{Over}(H)$ in the Zariski topology.

We say Σ is **Noetherian** if Σ is a Noetherian subspace of $\text{Over}(H)$ in the b-topology.

Clearly a Noetherian collection is weakly Noetherian, since the b-topology is finer than the Zariski topology on $\text{Over}(H)$.

If H is a Prüfer domain or if $\Sigma \subseteq \text{Zar}(H)$, then *Noetherian* $=$ *weakly Noetherian.*

Proposition. If Σ is a finite character collection of integrally closed overrings of H, then Σ is a Noetherian collection.

Proposition. If H is a domain and Σ is a subspace of $\overline{\mathrm{Zar}(H)}$ such that every member of Σ has Krull dimension 1, then Σ is Noetherian if and only if Σ has finite character.

Proof: Pass to an overring of H^b, and apply known results about Prüfer domains.

Proposition. Let Σ be a Noetherian collection of integrally closed overrings of H.

(i) If Y is a flat H-submodule of F, then $Y(\bigcap_{A\in\Sigma} A) = \bigcap_{A\in\Sigma} YA$.

(ii) For every flat overring B of H, $\{\overline{BA} : A \in \Sigma\}$ is a Noetherian collection of overrings of H.

Proof: This is not so easy to prove. Idea: Pass to Over(H^b), where there are more tools since H^b is a Bézout domain. The Noetherian property is used in an essential way here.

...The mapping $\mathrm{Zar}(H) \to \mathrm{Spec}(H) : V \to M_V \cap H$ is a continuous closed mapping.

If also H is a Prüfer domain, the map is a homeomorphism.

...We need something similar for $\mathrm{Over}(H)$...

Define $\mathrm{Sat}(H) := \{\bigcup_{P \in X} P : X \subseteq \mathrm{Spec}(H)\}$.

We give $\mathrm{Sat}(H)$ the topology whose basic open sets are

$$U_H(x_1, \ldots, x_n) := \{\mathfrak{m} \in \mathrm{Sat}(H) : x_i \notin \mathfrak{m} \text{ for some } i\}.$$

For each $A \in \mathrm{Over}(H)$, define $\mathfrak{m}_A := \{x \in H : xA \neq A\} \in \mathrm{Sat}(H)$.

Proposition. If H is a Bézout domain, then the mapping

$$\mathrm{Over}(H) \to \mathrm{Sat}(H) : A \mapsto \mathfrak{m}_A$$

is a homeomorphism.

Piecing everything together, we have:

Proposition. If H is an integrally closed domain, then the mapping

$$\mathrm{Over}(H) \to \mathrm{Sat}(H^b) : A \mapsto \mathfrak{m}_{A^b}$$

is a homeomorphism of $\mathrm{Over}(H)$ onto its image in $\mathrm{Sat}(H^b)$.

Thus a Noetherian collection in $\mathrm{Over}(H)$ gives rise to a Noetherian subspace in $\mathrm{Sat}(H)$.

Returning to our original problem, we assume: H is an integrally closed domain having a Noetherian R-representation Σ.

We want to modify Σ in some fashion to produce an irredundant weakly Noetherian R-representation of H.

Define $H^{\Sigma} = \bigcap_{A \in \Sigma} A^{b}$. Then $H^{\Sigma} \cap R = H$.

Let $\mathcal{F} = \{B \in \mathrm{Over}(H^{\Sigma}) : B \cap R = H\}$. (Note: R, not F.)

...Zorn's Lemma $\Rightarrow \mathcal{F}$ contains maximal elements.

Let B be a maximal element of $\mathcal{F}$, and define $\Sigma' = \{BA^{b} : A \in \Sigma\}$.

Continuity + previous results $\Rightarrow \mathfrak{A} := \{\mathfrak{m}_A : A \in \Sigma'\}$ is a Noetherian subspace of $\mathrm{Sat}(B)$.

Next we need to "sharpen" $\mathfrak{A}$.

(This terminology comes from Gilmer's condition (#).)

The focus now shifts from Σ to the collection $\mathfrak{A}$...

Ultimately, we want to find overrings of the rings in $\{H^{b}_{H \setminus \mathfrak{m}} : \mathfrak{m} \in \mathfrak{A}\}$ that yield, when intersected with R, an irredundant R-representation.

For $\mathfrak{B} \subseteq \mathfrak{A}$, we define

$$\bigwedge_{\mathfrak{b} \in \mathfrak{B}} \mathfrak{b} := \bigcup \{P \in \mathrm{Spec}(H) : P \subseteq \bigcap_{\mathfrak{b} \in \mathfrak{B}} \mathfrak{b})\}.$$

For each $P \in \mathrm{Spec}(B)$, define

$$\mathfrak{m}(\mathfrak{A}, P) = \bigwedge \{\mathfrak{m} \in \mathfrak{A} : P \subseteq \mathfrak{m}\}.$$

Finally, define $\mathfrak{A}^{\#} = \{\mathfrak{m}(\mathfrak{A}, P) : P \in \mathrm{Spec}(H)\}$.

Lemma. $\mathfrak{A}$ Noetherian $\Rightarrow \mathfrak{A}^{\#}$ Noetherian.

Noetherian $\Rightarrow \mathfrak{A}^{\#}$ has maximal elements. Denote this set by $\mathrm{Max}(\mathfrak{A}^{\#})$.

We have obtained a certain set Max($\mathfrak{A}^\#$)... When we localize B with respect to members of this set and intersect with R, we will have an irredundant R-representation of H (and be done).

The irredundance is a consequence of the following lemma... In fact, the reason we sharpened $\mathfrak{A}$ was to arrange for precisely this situation:

Lemma. For every $\mathfrak{m} \in$ Max($\mathfrak{A}^\#$), there exists $x \in H$ such that $x \in \mathfrak{m}$ but in no other other member of Max($\mathfrak{A}^\#$).

Taking the preimage (more or less) of Max($\mathfrak{A}^\#$) under the mapping Over(H) $\to$ Sat(B) yields now a weakly Noetherian irredundant R-representation Γ of H such that $\Gamma \leq \Sigma$. $\square$

Note on proof: The reason for passing to H^b was to be able to work with unions of prime ideals in a Bèzout domain rather than integrally closed overrings of an arbitrary domain... In particular, this allowed for the sharpening construction.