Representations of integrally closed domains as intersections of valuation overrings

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Motivation

(Krull) If H is an integrally closed domain, then there exists a collection Σ of valuation overrings of H such that

$$H = \bigcap_{V \in \Sigma} V.$$

We say that Σ is a **representation** of H.

Problem: Determine "nice" representations of H, and determine which domains have nice representations.

For example, when does there exist a representation Σ of H such that no proper subset of Σ is a representation of H?

In other words, when does H have an irredundant representation?

And when does H have a *unique* irredundant representation?

More general idea...Let R be an overring of H.

Σ is an R**-representation** of H if

$$H = (\bigcap_{V \in \Sigma} V) \cap R.$$

Problem: If R is an integrally closed overring of H, describe the integrally closed rings between H and R:

$$H \subseteq \ ?? \subseteq R.$$

This is equivalent to asking: What are the R-representations of H?

When does there exist an irredundant R-representation of H (i. e. no member of Σ can be omitted)?

And when does H have a *unique* irredundant R-representation?

Special case:

$$\mathbb{Z}[X] \subseteq \; ?? \subseteq \mathbb{Q}[X]$$

Problem: Describe the $\mathbb{Q}[X]$-representations of integrally closed overrings of $\mathbb{Z}[X]$.

Warning: $\mathbb{Z}[X] \subseteq \text{Int}(\mathbb{Z}) \subseteq \mathbb{Q}[X]$

This problem recently has been solved by Loper and Tartarone.

…Deep paper; lots of cases; key polynomials (Maclane, 1936).

We will consider (with less success!) the more general problem:

$$D \subseteq \; ?? \subseteq R,$$

where $D =$ Noetherian domain of Krull dimension 2, and $R =$ integrally closed overring of D.

Outline

- **Monday**: Background, basic results
- **Tuesday**: Uniqueness of irredundant representations of overrings of two-dimensional Noetherian domains
- **Thursday**: Overrings of two-dimensional Noetherian domains representable by Noetherian spaces of valuation rings
- **Friday**: When can a domain that is an intersection of integrally closed overrings be expressed as an irredundant intersection of overrings of these overrings? (No assumption on base domain.)

These talks are taken from 3 papers, roughly corresponding to the topics on Tuesday, Thursday and Friday. Rather than try to give an overview of the papers, each day outlines a single illustrative theorem from one of these papers.

Historically important example

H is a **Krull domain** iff H is a finite character intersection of DVRs.

Recall: A collection Σ of overrings of H has **finite character** if every nonzero element of H is a unit in all but at most finitely many members of Σ.

Theorem: A Krull domain has a unique irredundant finite character representation of DVRs.

Krull domains motivate most of the research done on irredundance and uniqueness. Can DVRs be replaced with one-dimensional valuation rings? (Brewer, Gilmer, Heinzer, Mott, Ohm, Ribenboim,...1968-1972)

Uniqueness need not hold for irredundant finite character representations...

Let D be a local Noetherian UFD of Krull dimension 2, and let f be an irreducible element of D.

Then $D = D_{(f)} \cap (\bigcap_{g \neq f} D_{(g)})$, where g ranges over the irreducible elements that are not associate with f.

On the other hand, there exists a valuation overring $V \subsetneq D_{(f)}$.

Thus $D = V \cap (\bigcap_{g \neq f} D_{(g)})$, and this gives rise to another irredundant finite character representation of D.

So even irredundant finite character representations of Krull domains do not have to be unique.

...But in this case uniqueness fails for an uninteresting reason (which we will circumvent)...

A valuation overring V is a **strongly irredundant representative** of H if there exists an integrally closed overring R of H such that

- $H = V \cap R$, and
- V cannot be replaced in this intersection by a proper overring.

Triviality: If V has Krull dimension 1, then V is a strongly irredundant representative of H iff $H = V \cap R$ and $H \neq R$.

Define $\text{Rep}(H) =$ the set of strongly irredundant representatives of H.

Warning: $\text{Rep}(H)$ can be empty.

When does a domain have a strongly irredundant representation?
When is such a representation unique?

Note: An irredundant representation that is not strongly irredundant can never be unique among irredundant representations.

Prüfer domains

In addition to providing examples, the Prüfer case is important because representations of integrally closed domains correspond to representations of Kronecker function rings (which are Prüfer domains)...So some proofs pass to Prüfer domains.

An integral domain H is a **Prüfer domain** if every valuation overring of H is a localization of H.

...Thus Prüfer domains are a kind of **limit case**.

Theorem: (Gilmer-Heinzer, 1968) If H is a Prüfer domain and H has an irredundant representation, then this representation is the unique (strongly) irredundant representation of H.

Thus for Prüfer domains: irredundant representations are unique.

There exist Prüfer domains that do not have an irredundant representation...

...Even worse, there exist Prüfer domains that do not have **any** irredundant representatives.

These occur in nature:

Let $K =$ non-algebraically closed field of characteristic 0.

Let $D = K[X_1, \ldots, K_n]$.

Let $H = \bigcap$ all valuation overrings of D having residue field K.

Then H is a Prüfer domain...This is because K is not algebraically closed (Dress, Gilmer, Roquette, Loper, Bröcker-Schulting, Rush).

$H = \bigcap$ all valuation overrings of $K[X_1, \ldots, X_n]$ having residue field K.

Theorem. (O., 2007) If $n > 1$, then Rep(H) is empty.

It is enough to show that no maximal ideal is the radical of a finitely generated ideal (Gilmer-Heinzer, 1972).

In fact, for this ring H every proper nonzero finitely generated ideal has infinitely many minimal prime ideals.

...This is true because given finite data about a valuation on a function field, you can replace it with all sorts of other valuations behaving the same way with respect to this data. (Zariski; Seidenberg; Kuhlmann-Prestel; Bröcker-Schülting; Kuhlmann)

Conclusion: An integrally closed domain may fail in a strong way to have an irredundant representation...

However, there exist interesting positive cases...

Theorem. (Heinzer-Ohm, 1972) If H is a domain, R is an overring of H and Σ is a finite character R-representation of H consisting of one-dimensional valuation rings:

$$H = (\bigcap_{V \in \Sigma} V) \cap R,$$

then there exists an irredundant finite character R-representation Γ of one-dimensional valuation rings.

...In fact, the only choice for Γ is $\Gamma = \mathrm{Rep}_R(H)$.

So irredundant finite character R-representations consisting of one-dimensional valuation rings are unique.

The previous theorem follows from something more general:

Theorem. (Heinzer-Ohm, 1972) If H is a domain, R is an overring of H and Σ is a finite character R-representation of H consisting of one-dimensional valuation overrings, then every one-dimensional irredundant representative of H is a member of Σ.

Proof: $H = V \cap (\bigcap_{U \in ??} U) \cap R = (\bigcap_{W \in \Sigma} W) \cap R$.

Choose $x \in ((\bigcap_{U \in ??} U) \cap R) \setminus V$.

$\Rightarrow$ $H = V \cap H[x] = W_1 \cap \cdots \cap W_n \cap H[x]$ (need finite character!).

$\Rightarrow$ $H = V \cap (W_2 \cap \cdots \cap W_n \cap H[x]) = W_1 \cap (W_2 \cap \cdots \cap W_n \cap H[x])$.

$\Rightarrow$ W.O.L.O.G.: $H = V \cap R = W \cap R$.

We have $H = V \cap R = W \cap R$.

Case 1: V has rational value group. Then V is a localization of H (Heinzer-Ohm, 1972, *Noetherian intersections*...) $\Rightarrow$ $V = W$.

Case 2: V has irrational (real) value group (V has dimension 1).

Suppose $\frac{a}{b} \in W \setminus V$ with $a, b \in H$. So $v(b) > v(a) \geq 0$.

Choose $y \in R \setminus W$. Then $\exists n > 0$ such that $-v(y^n) > v(a)$.

> **Lemma.** (Heinzer-Ohm, 1972) If V is real but not discrete (*e.g. our case*), then H contains elements of arbitrarily small v-value.

$\Rightarrow$ $\exists h \in H$ such that $0 \leq v(a) < -v(hy^n) < v(b)$

$\Rightarrow$ $bhy^n \in H, ahy^n \notin H \Rightarrow \ldots \Rightarrow$ Contradiction. $\square$

In another direction...

Theorem (Brewer-Mott, 1970) Let H be a domain. If there exists a finite character representation of H, then there exists an irredundant finite character representation of H.

We will later (Friday) prove this for R-representations...

...In doing so, we loosen the finite character requirement to something topological, namely that Σ is a Noetherian space.

Strategy: Pass to the Kronecker function ring, where you can use properties of Prüfer domains. Solve the problem there, then pass it back to the base domain.

...This works because the space of valuation overrings of H is homeomorphic to that of H^b (Dobbs-Fontana, 1986), so Noetherian spaces transfer back and forth.

On Tuesday and Thursday we will focus mainly on overrings of two-dimensional Noetherian domains.

...Why restrict to such rings?

In general the overrings of a Noetherian domain of dimension > 1 are mysterious, and it seems a hard problem, even in dimension 2, to try to classify these overrings.

However, as is often the case in Commutative Algebra, dimension 2 seems to be special in many ways...So the problem is maybe more tractable here.

In any case, the problem of overrings is often unavoidable even when studying properties of Noetherian domains: One encounters non-Noetherian overrings in applications such as those that involve the affineness of open sets of projective schemes, rings of invariants, local uniformization and resolution of singularities.

Thus it could be of interest to classify the sorts of overrings that can arise from Noetherian domains.

The results to be outlined, or at least their proofs, seem to depend heavily on the fact that the base domain is Noetherian and two-dimensional... Certain basic properties seem crucial...

Proposition. Let H be an integrally closed overring of a two-dimensional Noetherian domain D, and suppose that P is a nonzero prime ideal of H. Then:

(i) H has Krull dimension ≤ 2. (well-known)

(ii) H/P is a Noetherian domain.

(iii) If P is not a maximal ideal of H, then H_P is a DVR.

(iv) If P is an invertible ideal of H, then H_P is a valuation domain.

(ii), (iii) and (iv) involve passing to a f.g. D-subalgebra of H.

From the standpoint of non-Noetherian commutative ring theory, statements (ii) and (iii) are very strong...Even so, the ring H can be a very long ways from being Noetherian (e. g. consider Int($\mathbb{Z}$)).

(Abhyankar, 1956) Only three types of valuation overrings of a two-dimensional Noetherian domain D:

- rational $=$ value group is isomorphic to a subgroup of $\mathbb{Q}$.
- irrational $=$ not rational but its value group is isomorphic to a subgroup of $\mathbb{R}$; the value group is free of rational rank 2.
- Krull dimension 2 ($\Rightarrow V$ is discrete).

Everything that is permissible within these parameters happens...

For example, every nonzero subgroup of $\mathbb{Q}$ can arise as a value group of a rational valuation on $K(X, Y)$.

Similarly, every rational rank 2 free subgroup of $\mathbb{R}$ arises as a value group of an irrational valuation on $K(X, Y)$.

Define $\Sigma_1 := \{V \in \Sigma : V \text{ has dimension } 1\}$.

Tuesday's talk. (All rings are overrings of a 2-dimensional Noetherian domain.) Suppose that

$$H = \Big(\bigcap_{V \in \Sigma} V\Big) \cap R = \Big(\bigcap_{W \in \Gamma} W\Big) \cap R.$$

If:

- R is integrally closed (e.g. $R =$ quotient field),
- these R-representations are strongly irredundant, and
- Σ_1 and Γ_1 have finite character (e.g. Σ_1 and Γ_1 are empty),

then $\Sigma = \Gamma$.

The assumptions strongly irredundant and R is integrally closed are necessary.

Summary

Given two strongly irredundant R-representations Σ and Γ of a domain:

$$(\bigcap_{V \in \Sigma} V) \cap R = (\bigcap_{W \in \Gamma} W) \cap R.$$

Then $\Sigma = \Gamma$ in the cases:

(1) Σ and Γ have finite character and consist of one-dimensional valuation rings. (But $\Sigma \neq \Gamma$ can happen without the one-dimensional assumption.)

(2) H is an overring of a two-dimensional Noetherian domain, R is integrally closed and every member of Σ and Γ has Krull dimension 2.

(3) Same context as (2), but Σ and Γ have members of Krull dimension ≥ 1 and Σ_1 and Γ_1 have finite character.

If H is an overring of 3-dimensional Noetherian domain, then (2) and (3) can fail.