Overrings of two-dimensional Noetherian domains representable by Noetherian spaces of valuation rings

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Standing assumption: D is a two-dimensional Noetherian domain.

Let H be an overring of D.

$\text{Zar}(H) :=$ the set of valuation overrings of H.

The **Zariski topology** on $\text{Zar}(H)$ is defined by declaring the basic open sets to be those of the form:

$$U(x_1, \ldots, x_n) := \{V \in \text{Zar}(H) : x_1, \ldots, x_n \in V\},$$

where $x_1, \ldots, x_n$ are in the quotient field of H.

A topological space is **Noetherian** if the open sets satisfy the ascending chain condition.

Finite character subset of $\text{Zar}(H) \Rightarrow$ Noetherian

...but the converse is not true.

Notation: If W is a valuation overring of D, then

- $M_W :=$ maximal ideal of W.
- $P_W :=$ ht 1 prime ideal of W. (If W is a field, set $P_W = 0$.)

For $\Sigma \subseteq \text{Zar}(D)$, define $\Sigma^* = \{V_{P_V} : V \in \Sigma\}$.

Proposition. Σ Noetherian $\Rightarrow$ Σ^* has finite character.

A partial converse is true...

Proposition. Suppose that $\Sigma \subseteq \text{Zar}(D)$, Σ^* has finite character and there are only finitely many essential prime divisors of D in Σ^*. Then Σ is a Noetherian subspace of $\text{Zar}(D)$.

...In particular: Σ^* finite $\Rightarrow$ Σ Noetherian.

...Hence Noetherian $\not\Leftarrow$ finite character.

Any domain that admits a Noetherian R-representation admits a strongly irredundant Noetherian R-representation (more on Friday).

Thus the uniqueness theorem discussed on Tuesday implies:

Theorem. If Σ is a Noetherian space, then $\mathrm{Rep}_R(H)$ is a Noetherian R-representation of H and it is the unique strongly irredundant R-representation of H.

The theorem is false if you assume only that Σ^* has finite character.

To see this, it is enough to find $\Sigma \subseteq \mathrm{Zar}(D)$ such that

- $\Sigma^* \subseteq \{D_\mathfrak{p} : \mathrm{ht}(\mathfrak{p}) = 1\}$ (a finite character set).
- $H := \bigcap_{V \in \Sigma} V$ is a Prüfer overring of D for which every f. g. ideal is contained in infinitely many (i. e. 2) maximal ideals (Gilmer-Heinzer, 1968).

The last condition $\Rightarrow$ $\mathrm{Rep}_R(H)$ is empty.

Let $(D, \mathfrak{m})$ be a local ring that is the localization of a two-dimensional affine K-domain, where K is non-algebraically closed of char. 0.

Let Σ be the set of all two-dimensional valuation overrings of D centered on $\mathfrak{m}$ with residue field K such that $V \subseteq D_{\mathfrak{p}}$ for some height 1 prime ideal $\mathfrak{p}$ of D.

Claim: $H := \bigcap_{V \in \Sigma} V$ is a Prüfer overring (✓) such that every f. g. ideal of H is contained in infinitely many maximal ideals of H.

Let $I = (x_1, \dots, x_n)H$, and define $B = D[x_1, \dots, x_n]$. We want to replace B with a regular domain...

$\exists$ smooth projective K-variety Y of $F|K$ and a birational morphism $Y \to \text{Spec}(B)$ (Resolution of Singularities).

Hence $\Sigma \to Y \to \text{Spec}(B)$.

We have $\Sigma \to Y \to \mathrm{Spec}(B)$

Since K is not algebraically closed, all the K-rational points on Y are contained in an open affine subvariety $\mathrm{Spec}(C)$ of Y (Bröcker-Schülting, 86).

Thus $x_1, \ldots, x_n \in C \subseteq H$ and C is regular.

Now $I \subsetneq H \Rightarrow \exists V \in \Sigma$ such that $x_1, \ldots, x_n \in \mathfrak{n} := M_V \cap C$.

Hence $K = C_{\mathfrak{n}}/\mathfrak{n}C_{\mathfrak{n}}$ (because $V/M_V = K$).

Since $C_{\mathfrak{n}}$ is regular, there exist infinitely many prime ideals $\mathfrak{p}$ of C such that $C_{\mathfrak{n}}/\mathfrak{p}C_{\mathfrak{n}}$ is a DVR.

...Only finitely many of these can be centered on $\mathfrak{m}$.

Choose infinitely many, say $\mathfrak{p}_1, \mathfrak{p}_2, \ldots$, that are not.

Then $x_1, \ldots, x_n \in$ center on H of $V_i := C_{\mathfrak{n}} + \mathfrak{p}_i C_{\mathfrak{p}_i} \in \Sigma$. $\square$

What overrings of D have a Noetherian R-representation? Clearly Krull overrings have a Noetherian representation.

Theorem (Heinzer, 1969). Every Krull overring of D is a Noetherian domain.

Example. Let R be a Krull overring of D, and let $V_1, \ldots, V_n$ be valuation overrings of D. Then $H := V_1 \cap \cdots \cap V_n \cap R$ has a finite character (hence Noetherian) representation.

Example. (J. Ohm, 1966) Let p be a prime, and define v on $\mathbb{Q}(X)$ by

$$v(X) = \pi \quad \text{and} \quad v(p) = 1.$$

Let V be the valuation ring corresponding to v. Then

$$H := V \cap \mathbb{Q}[X]$$

is an irredundant intersection of an irrational valuation ring and a PID. Moreover, $M_V \cap H = \sqrt{pH}$. (More general: see Loper-Tartarone.)

Example. (Abhyankar-Eakin-Heinzer, 1972; Loper-Tartarone) Let p be a prime. Let $V_1, V_2, \ldots, V_n$ be DVR overrings of $\mathbb{Z}_p[X]$ such that $V_i \cap \mathbb{Q} = \mathbb{Z}_p$ for each i. Let $H := V_1 \cap V_2 \cap \ldots \cap V_n \cap \mathbb{Q}[X]$. Then D is a Dedekind domain provided the residue field of each V_i is algebraic over the field of p elements. Otherwise, D is a two-dimensional Noetherian domain... Every f.g. abelian group arises as a class group of a Dedekind domain H with $\mathbb{Z}[X] \subseteq H \subseteq \mathbb{Q}[X]$ (Eakin-Heinzer, 1973).

Example. Let U be a hidden prime divisor of D, and let $\Sigma = \{V \in \text{Zar}(D) : V \subseteq U\}$. Then $H := \bigcap_{V \in \Sigma} V$ is a one-dimensional quasilocal domain with maximal ideal M_U. The ring H is clearly not a valuation domain nor a Noetherian domain. However, Σ is a Noetherian representation of H.

Example. If $D = K[X, Y]$, where K is a countable field, then there exists a finite character collection Σ of two-dimensional valuation overrings of D such that for $H := \bigcap_{V \in \Sigma} V$, each V is a localization at a maximal ideal of H. However, H is not a Prüfer domain.

Main Example.

Let $B =$ an integrally closed Noetherian overring of D of Krull dimension 2.

Let J be a height 1 radical ideal of B such that $J \cap D$ is a height 2 ideal of D and B/J is indecomposable.

Then the integral closure H of $D + J$ is a quasilocal overring that has a Noetherian representation.

If also B is a finitely generated D-algebra, then H has a finite character representation.

...This example follows from the Main Theorem given later.

A consequence of the Main Theorem is that this is the only way that non-Noetherian, non-valuation, quasilocal examples which can be represented without irrational valuation rings can arise.

Notation. Suppose that $H \subseteq R$ are overrings of D. For a nonzero ideal I of H, we define

$$R(I) = \{r \in R : rI \subseteq I\}.$$

Then I is an ideal of the overring $R(I)$ of H with:

$$R(I) = \mathrm{End}(I) \cap R \quad \text{and} \quad H \subseteq R(I) \subseteq R.$$

Proposition. If $V \in \mathrm{Zar}(H) \setminus \mathrm{Zar}(R(I))$, then V has Krull dimension 2 and I is contained in every nonzero prime ideal of V.

...Thus $H \subseteq R(I)$ is a very "tight" containment with $H \subseteq R(I) \subseteq H^* :=$ c.i.c. of H.

We will use $R(I)$ to decompose Noetherian R-representations...

It is possible to reduce to the quasilocal case and give for this case an intrinsic description (i. e. no reference to Σ):

Theorem (Main Classification–“coarse” version). Let $H \subsetneq R$ be integrally closed overrings of D. Suppose that H is a quasilocal domain with maximal ideal M.

Then H has a Noetherian (finite character) R-representation **iff**

- H is a valuation domain, or
- $R(M)/M$ is a Noetherian ring (finitely generated H/M-algebra) and

$$R(M) = A \cap B \cap R,$$

where $B =$ Noetherian integrally closed overring of H, and

  - $A =$ quotient field of H, or
  - $A =$ finite intersection of irrational valuation rings.

...$\exists$ “finer” theorems that distinguish between the 2 possibilities for A.

Outline of proof

Define $X^1_R(H) = \{P \in \text{Spec}(H) : H_P \text{ is a DVR and } R \not\subseteq H_P\}$.

Lemma. If there exists a Noetherian R-representation of H, then $B := \bigcap_{V \in X^1_R(H)} V$ is an integrally closed Noetherian domain.

(Proof: Show that $X^1_R(H)$ has finite character; use Heinzer's theorem.)

Next step: Carefully calculate $\text{End}(M)$ to obtain: $R(M) = B \cap A \cap R$, where A is a finite intersection of one-dimensional valuation rings with residue fields algebraic over H/M... A useful observation is:

Lemma. Let S be an integrally closed domain, and let N be a maximal ideal of S. If N is a prime ideal of $\text{End}(N)$, then there exists an integrally closed overring T of S with $S \subseteq T \subseteq \text{End}(N)$ such that N is a nonmaximal prime ideal of T.

This is a consequence of Zariski's Main Theorem (Peskine's version).

So far: $R(M) = B \cap A \cap R$, where $B =$ integrally closed Noetherian domain and $A =$ quotient field or $A =$ finite intersection of one-dimensional valuation rings.

Lemma (Heinzer-Ohm, 1972). A rational valuation overring of a domain S that is an irredundant representative of S is a localization.

...Hence A is a finite intersection of irrational valuation rings.

Finally, with more work, using the fact that H has a Noetherian R-representation, along with the fact that H is an overring of a two-dimensional Noetherian domain, one obtains that $R(M)/M$ is a finite subdirect product of Noetherian rings, hence Noetherian.

Furthermore, if H has a finite character R-representation, then one can, with work, pass to a finite direct product of function fields over H/M... Finite character $\Rightarrow$ finitely many points at infinity for the curve that is the image of H in the product of these function fields.

Lemma. Let $F|K$ be a function field of degree 1, and let A be a K-subalgebra of F. TFAE:

(i) A is a finitely generated K-algebra.

(ii) There are at most finitely many valuation rings $V \in \text{Zar}(F|K)$ that do not contain A.

(iii) There exist valuation rings $V_1, \ldots, V_n \in \text{Zar}(F|K)$ not containing A such that $V_1 \cap \cdots \cap V_n \cap A \subseteq \overline{K}$.

...The equivalence of (i) and (ii) is due to Alamelu, 1978.

Thus...if H has a finite character R-representation, $R(M)/M$ is a finitely generated H/M-algebra.

What about the converse? That is, suppose that $R(M)/M$ is a Noetherian ring and

$$R(M) = A \cap B \cap R,$$

where $B =$ integrally closed Noetherian overring and $A =$ field or $A =$ finite intersection of irrational valuation overrings.

...Hence $R(M)$ has a finite character R-representation.

Lemma. Let A be an integrally closed domain, and let N be a maximal ideal of A. Then $\text{End}(N)$ is an integrally closed domain and $\text{End}(N)/N$ is a reduced indecomposable ring.

Thus $R(M)/M$ is indecomposable. Also, since $R(M)/M$ is Noetherian, there exist finitely many prime ideals of $R(M)$ that are minimal over M.

Now we analyze $\Sigma := \mathrm{Zar}(H) \setminus \mathrm{Zar}(R(M))$.

Clearly $H = (\bigcap_{V \in \Sigma} V) \cap R(M)$.

Using that $R(M)/M$ is an indecomposable Noetherian ring, along with Zariski's Main Theorem, one eventually obtains that Σ^* is finite.

Hence Σ is a Noetherian $R(M)$-representation of H.

Since $H = (\bigcap_{V \in \Sigma} V) \cap R(M) = (\bigcap_{V \in \Sigma} V) \cap A \cap B \cap R$, it follows that H has a Noetherian R-representation.

If also $R(M)/M$ is a finitely generated H/M-algebra, then, viewing $R(M)/M$ as contained in a finite product of function fields, the image of $R(M)/M$ in each of these fields has finitely many points at infinity.

This is used to show: Σ is a finite set.

...Hence H has a finite character R-representation. $\square$

Aside: Suppose that (H, M) is a quasilocal overring of D.

Let R be an integrally closed overring of H.

If $R(M)/M$ has finitely many minimal prime ideals, then

$$\text{Zar}(H) \setminus \text{Zar}(R(M))$$

is the unique irredundant $R(M)$-representation of H.

(Depends on technical lemmas that are behind Tuesday's talk.)

...Thus $H \subseteq R(M)$ is a very tight containment.

Special Case: $R =$ quotient field of H.

If M is a maximal ideal of H, set $\text{Rep}(M) = \{V \in \text{Rep}(H) : MV \neq V\}$.

Theorem. Let (H, M) be a quasilocal overring of D of Krull dimension 2 such that H has a Noetherian representation, but H is neither a Noetherian domain nor a valuation domain. Then:

(i) $\text{Rep}(M)$ contains only irrational valuation rings $\Leftrightarrow$ H is completely integrally closed.

(ii) $\text{Rep}(M)$ contains only two-dimensional valuation rings $\Leftrightarrow$ $\text{End}(M)$ is a Noetherian domain.

(iii) $\text{Rep}(M)$ contains both irrational and two-dimensional valuation rings and valuation rings $\Leftrightarrow$ $\text{End}(M) \neq H$ and $\text{End}(M)$ is not a Noetherian domain.

In fact, if H has a Noetherian representation, then H is Noetherian; a valuation domain; or satisfies exactly one of (i), (ii) or (iii).