Irredundant intersections of valuation overrings of two-dimensional Noetherian domains

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Some basic questions... Let V and W be valuation overrings of a domain H, and let R be an overring of H.

- When does $V \cap R = W \cap R \Rightarrow V = W$?
- When does $V \cap R \subseteq W \Rightarrow V \subseteq W$ or $R \subseteq W$?
- How do the structures of V and R influence $V \cap R$?

Now rephrase all these questions where appropriate with collections of valuation rings... For example, the collections might be finite, finite character, Noetherian, etc.

Probably to say something interesting (i. e. non-tautological) about these questions you need to consider specific settings...

Our setting is...

Let D be a two-dimensional Noetherian domain, and let $H \subseteq R$ be integrally closed overrings of D such that

$$H = \Big(\bigcap_{V \in \Sigma} V\Big) \cap R$$

for some Noetherian collection Σ of valuation overrings of H.

- (Today and Friday) Describe H extrinsically: Show that you may assume Σ is the unique strongly irredundant Noetherian R-representation of H... And Σ is a determined in a canonical way from H.
- (Thursday) Describe H intrinsically: What can you say the ideal-theoretic structure of H? (Note: R must be an "independent variable" in the answer.)

Given an overring R of H, Σ is an R-**representation** of H if

$$H = \left(\bigcap_{V \in \Sigma} V\right) \cap R.$$

Problem: When does H have a *unique* irredundant R-representation?

Better problem: When does H have a *unique* strongly irredundant R-representation?

Recall that the R-representation Σ is **strongly irredundant** if no member V of Σ can be replaced by a proper overring of V.

...We focus on the answer to this question when H is an overring of a two-dimensional Noetherian domain.

First we relativize the notion of a strongly irredundant representative to handle the situation:

$$H \subseteq \; ?? \subseteq R.$$

V is a **strongly irredundant** R**-representative** of H if there exists an R-representation Σ of H such that $V \in \Sigma$ and V cannot be replaced in this representation by a proper overring.

$\text{Rep}_R(H) :=$ the set of strongly irredundant R-representatives of H.

Thus $V \in \text{Rep}_R(H) \Leftrightarrow$ there exists an integrally closed overring R_1 of H such that

$$H = V \cap R_1 \cap R$$

and V is strongly irredundant in this intersection.

Note: If $R =$ quotient field of H, then $\text{Rep}_R(H) = \text{Rep}(H)$.

We restate now the theorem of Heinzer and Ohm discussed yesterday...

Theorem. (Heinzer-Ohm, 1972) If H is a domain, R is an overring of H and Σ is a finite character R-representation of H consisting of one-dimensional valuation rings:

$$H = \left(\bigcap_{V \in \Sigma} V\right) \cap R,$$

then $\mathrm{Rep}_R(H)$ is the unique irredundant finite character R-representation of H of one-dimensional valuation rings.

Note: Since the valuation rings involved are one-dimensional,

irredundant = strongly irredundant

...Allowing valuation rings of dimension > 1 complicates things...

Standing assumption: D is a two-dimensional Noetherian domain.

(Abhyankar, 1956) Only three types of valuation overrings of D:

- rational $=$ value group is isomorphic to a subgroup of $\mathbb{Q}$.
- irrational $=$ not rational but its value group is isomorphic to a subgroup of $\mathbb{R}$.
- Krull dimension 2 ($\Rightarrow V$ is discrete)

Special case: DVRs

- essential prime divisors $= \overline{D}_P$, where P is a height one prime ideal of the integral closure $\overline{D}$ of D.
- hidden prime divisors $=$ DVRs having maximal ideals contracting to a maximal ideal M of D such that the residue field of V has transcendence degree 1 over the field D/M.

Crucial fact: (Abhyankar) the residue field of a hidden prime divisor is a function field of transcendence degree 1 over the residue field of D.

Main Theorem. Suppose

- $H \subsetneq R$ are integrally closed overrings of D, and
- V is a valuation overring of H of Krull dimension 2.

Then

$$V \in \text{Rep}_R(H) \Leftrightarrow V \text{ is a member of } \underline{\text{every}}\ R\text{-representation of } H.$$

Thus: V is strongly irredundant in some intersection

$$H = V \cap R_1 \cap R$$

iff V must appear in every representation of H as an intersection of R and valuation overrings of H.

...This is special to dimension 2. It does not hold for strongly irredundant representatives of Krull dimension 1.

Define $\Sigma_1 := \{V \in \Sigma : V \text{ has dimension } 1\}$.

Corollary. (All rings are overrings of D.) Suppose that

$$H = \Big(\bigcap_{V \in \Sigma} V\Big) \cap R = \Big(\bigcap_{W \in \Gamma} W\Big) \cap R.$$

If:

- R is integrally closed (e.g. $R =$ quotient field),
- these R-representations are strongly irredundant, and
- Σ_1 and Γ_1 have finite character (e.g. Σ_1 and Γ_1 are empty),

then

$$\Sigma = \mathrm{Rep}_R(H) = \Gamma.$$

If strongly irredundant is replaced with irredundant, the corollary is false.

Example. In general

$$V \cap R = W \cap R \quad \not\Rightarrow \quad V = W,$$

even when V and W are strongly irredundant in their respective representations. (By the corollary R cannot be integrally closed.)

Let $F =$ quotient field of $A := \mathbb{Q}[X, Y]/(Y^2 - X^3 - X + 1)$.

Write $A = \mathbb{Q}[x, y]$, where $x, y \in A$ and $y^2 = x^3 + x - 1$.

Let $U = \mathbb{Q}[x]_{(x-1)}$.

Then there exist two distinct valuation rings U_1 and U_2 with quotient field F such that $U = U_1 \cap \mathbb{Q}(x) = U_2 \cap \mathbb{Q}(x)$.

Let R be the pullback of $\mathbb{Q}(x)$ to an overring of $\mathbb{Q}[X, Y]$.

Let V_i be the pullback of U_i to an overring of $\mathbb{Q}[X, Y]$.

Then $V_1 \cap R = V_2 \cap R$ but $V_1 \neq V_2$.

Outline of proof of main theorem

Suppose $V \in \text{Rep}(H)$ has Krull dimension 2, and Σ is an R-representation of H. Thus:

$$H = V \cap R_1 = (\bigcap_{W \in \Sigma} W) \cap R$$

where $R_1 \subseteq R$ is integrally closed and V cannot be replaced by a proper overring. We claim that $V \in \Sigma$.

With a good bit of work we can reduce to the case that

$$H = V \cap R = (\bigcap_{W \in \Sigma} W) \cap R.$$

We have $H = V \cap R = (\bigcap_{W \in \Sigma} W) \cap R$.

Notation: If W is a valuation overring of D, then

- $M_W :=$ maximal ideal of W.
- $P_W :=$ ht 1 prime ideal of W. (If W is a field, set $P_W = 0$.)

Then $\text{Spec}(V) = \{0, P_V, M_V\}$.

With more work: If $P_V \cap H$ is a nonmaximal prime ideal of H, then $V \in \Sigma$. (Even better, $V = H_{M_V \cap H}$.)

Thus the interesting case is: $M := P_V \cap D$ is a maximal ideal of D.

Set $B = \text{End}(M) \cap R$: $V \in \text{Rep}(H) \Rightarrow \cdots \Rightarrow B = V_{P_V} \cap R \Rightarrow H \neq B$.

So $\exists W \in \Sigma \setminus \text{Zar}(B)$. Moreover, W must have dimension 2.

Thus $V \cap B = H \subseteq W$. We show this implies $V \subseteq W$. For then, since V and W have dimension 2, $V = W \in \Sigma$.

We have established that $V \cap B = H \subseteq W$.

Key Lemma. Let A be an integrally closed overring of D, and let T and U be valuation overrings of D. Then

$$T \cap A \subseteq U \;\Rightarrow\; T \subseteq U \text{ or } T_{P_T} \cap U_{P_U} \cap A \subseteq U.$$

(Notice the lemma is trivial if U is one-dimensional.)

So $V \subseteq W$ or $B \cap W_{P_W} \subseteq W$.

However, some work shows that $B \subseteq V_{P_V} = W_{P_W}$.

Thus since $B \not\subseteq W$, we have $V \subseteq W$.

Since W has Krull dimension 2, we have $V = W \in \Sigma$.

...This proves the main theorem
(modulo the lemma).

Recall how the Key Lemma was used:

$$V \cap R \subseteq W \;\Rightarrow\; V \subseteq W \text{ or } V_{P_V} \cap W_{P_W} \cap R \subseteq W.$$

The idea of the proof of this lemma is with a good bit of work reduce to the case that $R \subseteq V_{P_V} \cap W_{P_W}$ and $\mathfrak{m} := P_V \cap D$ is a height 2 (hence maximal) ideal of D.

Then: $D/\mathfrak{m} \hookrightarrow V/P_V \cap (R + P_V)/P_V \subseteq V_{P_V}/P_V$.

Also V_{P_V}/P_V is a function field of degree 1 over $D/\mathfrak{m}$.

The interesting case turns out to be where $W \subseteq V_{P_V}$. Then:

$$V/P_V \cap (R + P_V)/P_V \subseteq W/P_V \subseteq V_{P_V}.$$

R integrally closed $\Rightarrow \cdots \Rightarrow (R + P_V)/P_V$ has quotient field V_{P_V}/P_V.

...And all of this is in the function field setting...

Lemma. Let $F|K$ be a function field of transcendence degree 1. Let A be a proper K-subalgebra of F having quotient field F, and let Σ be a collection of valuation rings containing K and having quotient field F. Suppose that there is a valuation ring U having quotient field F such that $(\bigcap_{V \in \Sigma} V) \cap A \subseteq U$. Then $U \in \Sigma$ or $A \subseteq U$.

We use this lemma in the following way:

Recall: $V/P_V \cap (R + P_V)/P_V \subseteq W/P_V \subseteq V_{P_V}$.

Also, $(R + P_V)/P_V$ has quotient field V_{P_V}/P_V.

So by the Lemma, $V \subseteq W$ or $R \subseteq W$.

...Therefore, (modulo a proof of the Lemma), we have shown:

$$V \cap R \subseteq W \quad \Rightarrow \quad V \subseteq W \text{ or } V_{P_V} \cap W_{P_W} \cap R \subseteq W.$$

Why is the lemma true?

Lemma Restated. Let $F|K$ be a function field of transcendence degree 1. Let A be a proper K-subalgebra of F having quotient field F. Then

$$(\bigcap_{V \in \Sigma} V) \cap A \subseteq U \;\Rightarrow\; U \in \Sigma \text{ or } A \subseteq U.$$

Strong Approximation Theorem. If $F|K$ is a function field of degree 1 and $\{v_1, \ldots, v_n\} \cup \{w_i : i \in I\} \cup \{v_\infty\}$ is the collection of all valuations on $F|K$, then for any integers $e_1, \ldots, e_n$ there exists $x \in F$ such that $v_k(x) = e_k$ for all i and $w_i(x) \geq 0$ for all i.

The Strong Approximation Theorem is a corollary to the Riemann-(Roch) Theorem for curves.

So it is likely very important for our approach to uniqueness that the residue fields of the valuations be function fields. (This fails if D has Krull dimension > 2 (Zariski-Samuel).)

Recall the corollary to the main theorem:

$$(\bigcap_{V\in\Sigma} V)\cap R = (\bigcap_{W\in\Gamma} W)\cap R \quad \Rightarrow \quad \Sigma = \Gamma$$

whenever

- R is integrally closed* (e.g. $R =$ quotient field),
- these R-representations are strongly irredundant*, and
- Σ_1 and Γ_1 have finite character (e.g. Σ_1 and Γ_1 are empty),

**necessary*

Example. (Thanks to Bill Heinzer and Alan Loper)
Let $H = K + Z\,K(X,Y)[Z]_{(Z)}$. Then H is a one-dimensional domain having many different strongly irredundant (Noetherian) representations.

...So the corollary does not hold in dimension 3.

Application: Let $H \subseteq R$ be integrally closed overrings of D, and let Σ be a Noetherian R-representation of H:

$$H = (\bigcap_{V \in \Sigma} V) \cap R.$$

Friday: You can replace Σ with a strongly irredundant R-representation Γ of H.

In fact, applying the Main Theorem from today, it's possible to show that $\Gamma = \text{Rep}_R(H)$.

So by the uniqueness theorem proved today $\text{Rep}_R(H)$ is the unique strongly irredundant Noetherian R-representation of H.

Thursday: We use $\text{Rep}_R(H)$ to "decompose" H as a kind of pullback...