Commutative Ring Theory Days 2010

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Abstracts

Von Neumann Regular and Related Elements in Commutative Rings

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[Joint work with David F. Anderson]. Let R be a commutative ring with nonzero identity. In this paper, we study the von Neumann regular elements of R, that is, $a \in R$ such that $a^2x = a$ for some $x \in R$. We also study the idempotent elements, π-regular elements (i.e., $a \in R$ such that $a^{2n}x = a^n$ for some $x \in R$ and integer $n \geq 1$), the von Neumann local elements (i.e., $a \in R$ such that either a or $1 - a$ is von Neumann regular), and the clean elements of R (i.e., elements of R that are the sum of a unit and an idempotent of R). Finally, we investigate the subgraphs of the zero-divisor graph $\Gamma(R)$ of R induced by the above elements.

Properties of associated graded rings in one-dimensional domains

Lance Bryant
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Let R be a one-dimensional Noetherian local domain such that the integral closure of R in its fraction field, denoted by V, is a DVR and a finitely generated R-module. In other words, R is an analytically irreducible Noetherian local domain. If we further assume that R and V have the same residue field, then the semigroup $S = v(R) = \{v(r) \,|\, r \in$

$R\backslash\{0\}\}$, where v is the normalized valuation of V, can provide information about R. This is especially true when R is a numerical semigroup ring, i.e., whenever $u \in S$, $t^u \in R$ where tV is the maximal ideal of V. We begin by seeing that when R is a numerical semigroup ring, whether or not the associated graded ring of R is Cohen-Macaulay or Gorenstein can be easily determined from S. Next we consider extending this in two ways. We look at arbitrary analytically irreducible Noetherian local domains and we look at associated graded rings of ideal filtrations. Although some results have been obtained in this generality, important questions remain unanswered. Lastly we consider a numerical criterion that is valid under certain conditions for determining whether or not the associated graded ring of a filtration, $\mathrm{gr}_F(R)$, is Gorenstein. In particular, this criterion is valid when F is the integral closure filtration of an ideal in R.

Finitely valuative domains

Paul-Jean Cahen
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For a pair of rings $S \subseteq T$, an element $t \in T \backslash S$ is said to be within n steps (resp., within finitely many) steps of S if there is a saturated chain of rings of length m, $S = S_0 \subset S_1 \subset \cdots \subset S_m = S[t]$, for some integer $m \leq n$ (resp., for some integer m). An integral domain R is said to be n-valuative (resp., finitely valuative) if for each nonzero element u in its quotient field, at least one of u and u^{-1} is within n steps (resp., within finitely many steps) of R. Valuative domains, presented by Tom Lucas, are thus the 1-valuative domains, and this talk presents a generalization. The integral closure of a finitely valuative domain is a Prüfer domain. An integrally closed n-valuative domain is a Bézout domain with at most $2n+1$ maximal ideals, the prime ideals contained in more than n prime ideals are totally ordered and at most n maximal ideals fail to contain every such prime.

About subsets of $\mathbf{Z}$ with simultaneous orderings

Jean-Luc Chabert
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The natural sequence $\{n\}_{n\geq 0}$ has the following nice property: whatever the integer $m \in \mathbf{Z}$ the product $\prod_{k=0}^{n-1}(m-k)$ is divisible by the product $\prod_{k=0}^{n-1}(n-k)$, that is, $n!$ divides $m!/(m-n)!$.

Analogously, consider a subset S of $\mathbf{Z}$. A sequence $\{a_n\}_{n\geq 0}$ of elements of S is said to be a *simultaneous ordering* of S when:

$$\forall x \in S \quad \prod_{k=0}^{n-1}(x-a_k) \quad \text{is divisible by} \quad \prod_{k=0}^{n-1}(a_n-a_k).$$

There are two well known examples of subsets which admit simultaneous orderings: $\{n^2 \mid n \geq 0\}$ and $\{q^n \mid n \geq 0\}$ where q is any integer ≥ 2. We are interested in finding other natural examples.

We characterize here the integer-valued polynomials f of degree 2 such that either $\{f(n) \mid n \geq 0\}$ or $\{f(n) \mid n \in \mathbf{Z}\}$ admits a simultaneous ordering.

We prove also that, for any $f \in \mathbf{Z}[X]$ and any integer $x \in \mathbf{Z}$, the orbit of x under the action of the iterates of f, that is, $\{f^n(x) \mid n \geq 0\}$ always admits a simultaneous ordering.

This is a joint work with David ADAM (Université Française du Pacifique) and Youssef FARES (LAMFA, Université de Picardie).

Abstract ideal theory, ideal systems, and semistar operations

Jesse Elliott
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We show that a generalization of quantales and prequantales provides a noncommutative and nonassociative abstract ideal theoretic setting for the theories of star operations, semistar operations, semiprime operations, ideal systems, and module systems, and conversely the latter theories motivate new results on quantales and prequantales.

Factorials in several variables

Sabine Evrard
Upjv, Amiens, France

There is no need to introduce the factorial sequence $\{n!\}_{n\geq 0}$, and its arithmetic properties are well known

- For each $k, l \in \mathbb{N}$, $k!l!$ divides $(k+l)!$.
- For any sequence $x_0, x_1, \ldots, x_n$ of integers, the product $\prod_{0\leq i<j\leq n}(x_j - x_i)$ is divisible by $1!\cdots n!$.

We can also give some properties connecting polynomials and factorials:

- For every polynomial $f \in \mathbb{Z}[X]$, with unitary content and of degree n, $d(f) = gcd\{f(k) \mid k \in \mathbb{Z}\}$ divides $n!$.
- Every integer-valued polynomial f (that is, $f(\mathbb{Z}) \subset \mathbb{Z}$), of degree n,

$$n!f(X) \in \mathbb{Z}[X].$$

Bhargava introduced the notion of factorial sequence of a subset S of a Dedekind domain D, which generalizes the usual notion of $n!$, since it has arithmetical properties similar to the classical factorials. He introduced the factorial sequence of a subset S, in a local way, thanks to the notion of v-orderings of S. On the other hand, such a sequence may be defined in a global way, thanks to the notion of integer-valued polynomial on S. In this talk, we define factorials in several variables using both, integer-valued polynomials with d indeterminates and v-orderings of subsets of D^d. We will see that these factorial sequences still generalize some arithmetical properties of the factorial sequence $n!$.

Domains having a unique Kronecker function ring

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We study the class of integrally closed domains having a unique Kronecker function ring, or equivalently, domains in which the b-operation is the only e.a.b star operation of finite type, as an approach to domains having a quite simple set of valuation overrings. We give characterizations by means of valuation overrings and integral closure of finitely generated ideals and provide new examples of such domains.

Amalgamated algebras along an ideal

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[Joint work with M. D'Anna and M. Fontana]. Let $f : A \longrightarrow B$ be a ring homomorphism and J be an ideal of B. Then we consider the following subring

$$A \bowtie^f J := \{(a, f(a) + j) : a \in A, j \in J\}$$

of $A \times B$ and call it *the amalgamation of A with B along J, with respect to f*. This construction generalizes the amalgamated duplication of a ring along an ideal, introduced and studied in [1] and [2]. Moreover, several constructions (like the rings of the type $A + XB[X]$ or $D + M$, etc.) can be seen as particular cases of the amalgamation. In this talk, I will study conditions on A, B, f and J to transfer algebraic properties from A and B to $A \bowtie^f J$ and conversely, by using also the fiber product structure of $A \bowtie^f J$. I will give a classification of all the fiber products that can arise from an amalgamation. Moreover I will study the chains of prime ideals of $A \bowtie^f J$ in order to give bounds for the dimension of $A \bowtie^f J$.

[1] M.D'Anna and M. Fontana *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl. **6** (2007), pp. 443–459.

[2] M.D'Anna and M. Fontana, *The amalgamated duplication of a ring along a multiplicative-canonical ideal*, Arkiv Mat. **45** (2007), pp. 241–252.

Semistar Noetherian domains

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We study the class of semistar Noetherian domains, defined by having the Ascending Chain Condition on the set of quasi semistar ideals. We give many different characterizations of this class, involving associated prime ideals, local Noetherianity, Delta-Noetherianity and topological Noetherianity.

Finally, we discuss how semistar Noetherian domains provide a framework in which the problem of classifying injective modules can be successfully approached.

Prüfer-like Conditions in Commutative Group Rings

Sarah Glaz
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Let R be a commutative ring, and let f be a polynomial with coefficients in R. Denote by $c(f)$, *the content* of f, the ideal of R generated by the coefficients of f. A ring R is called a *Gaussian ring* if $c(f)c(g) = c(fg)$ for any two polynomials f and g with coefficients in R. Gaussian rings were defined by Tsang in 1965, and became an active topic of investigation due to their connection to Kaplansky's conjecture, which was solved between 1997 and 2005. The focus of these investigations lied in the comparison between the Gaussian property and several related Prüfer-like ring theoretic and homological properties. Specifically the properties under consideration are:

1. R is a semihereditary ring.
2. $w.\dim R \leq 1$.
3. R is an arithmetical ring.
4. R is a Gaussian ring.
5. R is locally a Prüfer ring.
6. R is a Prüfer ring.

This talk will discuss the behavior of the six Prüfer-like properties in commutative group rings. In particular, we will consider several results and counterexamples, obtained by the speaker, to questions of ascent and descent of these properties between the ring R and the group ring RG, for an abelian group G.

Formal power series over strongly hopfian rings

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A commutative ring R is said to be strongly Hopfian if the chain of annihilators $ann(a) \subseteq ann(a^2) \subseteq ...$ stabilizes for each $a \in R$. The class of strongly Hopfian rings contains Noetherian rings, Laskerian rings, rings satisfying acc on d-annihilators and those satisfying acc on d-colons, rings satisfying accr , rings which are embeddable in a zero dimensional ring, in particular zero dimensional rings are strongly Hopfian. Recall that in [1], the authors proved that for a commutative ring R, the ring R is strongly Hopfian if and only if the ring $R[X]$ is. In this talk, we are interested in the class of strongly Hopfian rings and in the transfer of this property from a commutative ring R to the ring of the power series $R[[X]]$. We give necessary and sufficient conditions in order that $R[[X]]$ is strongly Hopfian. We provide an example of a strongly Hopfian ring R such that $R[[X]]$ is not strongly Hopfian.

[1] A. Hmaimou, A. Kaidi, E. Sanchez Campus. Generalized fitting modules and rings. Journal of Algebra. 308 (2007), 199-214.

Counting star operations on an integral domain

Evan Houston
University of North Carolina, Charlotte, USA

This talk will survey what is known about integral domains which admit only finitely many star operations. In particular, I will completely characterize integrally closed domains and Noetherian domains which admit at most two star operations.

Numerical characterizations of some integral domains

Ali Jaballah
University of Sharjah, United Arab Emirates

Let R be an integral domain and let K be its quotient field. An intermediate ring in the ring extension $R \subseteq K$ is called an overring of R. Several classes of integral domains are defined by properties satisfied by their overrings. Recently domains with only finitely many overrings have been investigated by several authors. Such integral domains have been named FO domains by Gilmer. The author gave an algorithm for computing the number of overrings of FO integrally closed domain. The main purpose of this talk is to investigate whether the number of overrings affects the properties of R. Precisely we have the following problems:

Problem 1
Given an FO integral domain R, is it possible to provide an exact list of all overrings of R?

Problem 2
Given an FO integral domain R. Does the number of overrings characterize the properties enjoyed by R?

We answer Problem 1 by providing an algorithm that produces all different overrings that are intersections of localizations of an integral domain with a tree as spectrum. The same algorithm gives a complete list of all overrings of an integrally closed domain. We then address Problem 2 by establishing a characterization of integrally closed domains in terms of the number of their overrings. We also give similar characterizations of valuation domains, Dedekind domains, integral domains with spectrums free of Y-subgraphs, and integral domains of Krull dimension 1.

Pólya fields, Pólya groups and Pólya extensions: a question of capitulation

Amandine Leriche
Université de Picardie Jules Verne, France

The embedding problem is well known: is every number field contained in a field with class number one? We consider an analogous question with a weaker conclusion: is every number field contained in a Pólya field? A Pólya field is a number field where all the products of same norm prime ideals are principal ideals. It would be explained how the Hilbert class field gives a positive answer to this question. By the way, with respect to the capitulation of these products, we introduce and study the notion of a Pólya extension.

Topology and integer-valued polynomials on subsets of valuation domains

K. Alan Loper
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Let V be a valuation domain and S a subset. We investigate the question of conditions on S such that $\mathrm{Int}(S, V)$ is a Prüfer domain. We make use of Chabert's recent results on polynomial topology.

Valuative domains

Tom Lucas
University of North Carolina, Charlotte, USA

An integral domain R with quotient field K is said to be valuative if for each nonzero element $u \in K$, at least one of the ring extensions $R \subseteq R[u]$ and $R \subseteq R[u^{-1}]$ has no proper intermediate rings. This generalizes the notion of a valuation domain. In fact, an integrally closed local domain is valuative if and only if it is a valuation domain. There do exist (both local and nonlocal) valuative domains that are not integrally closed. For example, if $F \subsetneq F[t]$ is a field extension of prime degree, then $R = F + XF[t][[X]]$ is a local valuative domain that is not integrally closed. A complete characterization of valuative domains will be given. One necessary condition for a domain to be valuative is that it have at most three maximal ideals, another is that the set of nonmaximal primes must be linearly ordered. Also, at most one maximal ideal (of a valuative domain) does not contain each nonmaximal prime. In the integrally closed case, R is valuative if and only if it is a Bézout domain with at most three maximal ideals such that at most one maximal ideal does not contain each nonmaximal prime ideal. Except for a special type of "bad valuative domain, every overring of a "good valuative domain is valuative. Moreover, the overrings of a good valuative domain can be described in a way that is similar to how one may describe the overrings of a semilocal Bézout domain. [Joint work with Paul-Jean Cahen and David Dobbs.]

Integrally closed rings in birational extensions of two-dimensional Noetherian domains

Bruce Olberding
New Mexico State University, Usa

This talk is motivated by the general goal of developing a classification framework for the integrally closed overrings of a two-dimensional Noetherian domain D. Our approach to describing such rings is via their representations as intersections of valuation rings, and from this point of view it is the non-Noetherian overrings that are the most mysterious. In this talk we seek to describe intervals of integrally closed rings between a local Noetherian domain D and a localization $D[1/f]$, where $0 \neq f \in D$. Our strongest results are when D is a regular local ring and f is a regular parameter of D. Even in this case there exists a complicated abundance of Noetherian and non-Noetherian integrally closed rings between D and $D[1/f]$.

The theoretical goal of classifying rings between D and $D[1/f]$ has applications of a Noetherian nature, and leads to what appear to be new results about the exceptional prime divisor of a normalized blow-up of a two-dimensional regular local ring.

This is joint work with Francesca Tartarone.

Parametrization of integral values of polynomials

Giulio Peruginelli
Hausdorff Research Institute for Mathematics of Bonn, Germany

A polynomial $f \in \mathbb{Q}[X]$ is called integer-valued if $f(\mathbb{Z}) \subset \mathbb{Z}$. We give a complete classification of those integer-valued polynomials $f(X)$ whose image over the integers can be parametrized by a multivariate polynomial with integer coefficients, that is, the existence of $g \in \mathbb{Z}[X_1, \ldots, X_m]$ such that $f(\mathbb{Z}) = g(\mathbb{Z}^m)$. The necessary and sufficient condition for $f(X)$ is that $f \in \mathbb{Z}[B(X)]$ for some polynomial $B(X) = sX(sX - r)/2$, where s and r are coprime odd integers and s is a prime power or it is equal to 1. In particular we obtain that 2 is the only prime factor dividing the common denominator D of the coefficients of $f(X)$ and there exists a rational $\beta = r/s$ such that

$f(X) = f(\beta - X)$. Moreover if $f(\mathbb{Z})$ is likewise parametrizable, then this can be done by a polynomial in one or two variables.

We will give also some ideas towards a generalization of this classification to number fields K, showing that the prime factors dividing D are related to the order of the roots of unity contained in K.

On intersections and composites of minimal ring extensions

Martine Picavet
Université Blaise Pascal, Aubière, France

[Joint work with G. Picavet, D.E. Dobbs and J. Shapiro]. A ring extension $A \subset B$ is **minimal** if A and B are the only A-subalgebras of B. In this case, there exists a maximal ideal M of A, called the **crucial** ideal of the extension, such that $A_P = B_P$ for each prime ideal P of A different from M. Let $A \subset B$ and $A \subset C$ be two distinct minimal (ring) extensions such that the composite $D = BC$ exists. In an earlier work [1], the two last authors examined the extensions $B \subset D$ and $C \subset D$, and, in particular, under which conditions they are minimal. In this talk, we give some complements to this study and consider the dual situation : let $B \subset D$ and $C \subset D$ be distinct minimal extensions and set $A = B \cap C$. We obtain conditions in order that $A \subset B$ and/or $A \subset C$ are minimal extensions, with a special attention to the case of integral extensions. Moreover, combining properties of intersections and composites of minimal extensions, the following result is gotten: Given two distinct rings B and C such that the composite $D = BC$ exists, and setting $A = B \cap C$, the following conditions are equivalent :

(1) $A \subset B$ and $A \subset C$ are minimal extensions with distinct crucial ideals.

(2) $B \subset D$ and $C \subset D$ are minimal extensions with crucial ideals lying over distinct maximal ideals of A.

REFERENCES

[1] D.E. Dobbs, J. Shapiro, 241–275, Composites of minimal ring extensions, 2007, 9, JP J. Algebra.

Algebraic properties of star-ideal semigroups

Andreas Reinhart
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Let R be an integral domain, $*$ a star operation on R and $\mathcal{F}_*^\bullet(R)$ the (multiplicative) semigroup of non-zero fractional $*$-ideals of R. In this talk, we focus on the semigroup structure of $\mathcal{F}_*^\bullet(R)$. Especially we provide properties on R which imply almost completeness or π-regularity of $\mathcal{F}_*^\bullet(R)$ or which force the idempotents to be trivial. Our results generalize the corresponding results in the noetherian case, proved by Dade, Taussky and Zassenhaus (1961) and Halter-Koch (2007).

Extending lenght functions to polynomial rings via algebraic entropy

Luigi Salce
Università degli Studi di Padova, Italia

[Joint work with Simone Virili]. Length functions L for categories Mod(R) of modules over arbitrary rings R, taking values in the non–negative reals plus infinity, have been introduced by Northcott and Reufel in 1965. They found all length functions over valuation domains, and Vamos found in 1968 all length functions L over commutative Noetherian rings.
I will present a recent result stating that, if L is a discrete length function on Mod(R), where R is an arbitrary ring, there is a unique discrete length function h_L on the subcategory of Mod($R[X]$) consisting of the locally $L-$finite modules, such that, for every $R-$module M of finite length:

(i) $h_L(M_f) = 0$ for every endomorphism f of M (M_f is the $R[X]-$module M with X acting on it via f).
(ii) $h_L({M^{(\mathbb{N})}}_{\beta_M}) = L(M)$ where β_M is the right Bernoulli shift on $M^{(\mathbb{N})}$.

The length function h_L coincides with the algebraic $L-$entropy ent_L, introduced by L. S. and P. Zanardo in 2009; that is, for every endomorphism g of an arbitrary $R-$module N, $h_L(N_g) = \mathrm{ent}_L(g)$. The crucial point in obtaining this result relies on the proof of the Addition Theorem for the $L-$entropy ent_L.

On Dedekind domains and Krull monoids with infinite class group

Wolfgang A. Schmid
École polytechnique, France

The arithmetic of a Dedekind domain and, more generally, a Krull monoid is determined to a large extent by the distribution of (v-)prime ideals over the classes of the ideal (divisor) class group.

In case the class group is non-trivial the domain/monoid is not factorial, and various quantities have been studied to describe the phenomena of non-uniqueness that can arise.

These quantities are finite if the number of ideal classes containing prime ideals is finite (thus, in particular, if the class group is finite); if this is not the case, they are typically infinite. Yet, they are not always infinite, or at least not all of them are infinite; and in these scenarios phenomena that are distinct from yet closely related to the 'finite case' can be observed.

We present some results, with a strong emphasize on the case that the class group is cyclic and infinite, which illustrate this.

This is joint work with A. Geroldinger, D.J. Grynkiewicz, and G.J. Schaeffer.

Zero Divisor Conditions in commutative group rings

Ryan Schwarz
University of Connecticut, Usa

Let R be a commutative ring, and let G be an abelian group. The most restrictive zero divisor condition one can impose on a ring R is to require that R be a domain. Viewed from the homological perspective, this requirement becomes a statement concerning principal ideals of R. In fact, this can be extended to the setting RG where G is torsion-free. More generally, we can speak of the following less restrictive zero divisor conditions in R:

1. R is a PF ring, i.e. every principal ideal of R is flat.

2. R is a PP ring, i.e. every principal ideal of R is projective.

3. The total ring of quotients of R, denoted $Q(R)$, is von Neumann regular.

4. The set of minimal primes of R, denoted Min R, is compact in the Zariski topology.

In this talk, we'll discuss the relationships between these conditions in R and in RG where G is torsion-free. Using these characterizations, we'll construct examples of group rings RG which exhibit some of the above properties.

Characterizations of one dimensional analytically unramified rings not necessarily residually rational

Ioana Serban
Budapest, Hungary

Research on one-dimensional local CM rings was originally motivated by questions regarding algebraic curves and their singularities. For this reason often certain additional properties - like residual rationality - were assumed.

From a geometrical point of view, residual rationality is indeed a natural assumption. However, with a more algebraic interest, it seems desirable to remove all additional assumptions and investigate the relevant properties as general as possible. In my talk I will outline a characterization of one-dimensional local CM rings of maximal and minimal length (without assuming residual rationality).

Isomorphism classes of certain Artinian Gorenstein Algebras

Giuseppe Valla
Università di Genova, Italia

Let A be the one dimensional local domain corresponding to a monomial curve in the affine space of dimension h. It is well known that the multiplicity e of A verifies the inequality $e \geq h+1$. A result of Rosales and García Sanchez says that if $h+2 \leq e \leq e \leq h+4$ and A is Gorenstein, then the defining ideal of the curve is minimally generated by $\binom{h+1}{2} - 1$ elements. We prove that the result holds true for any Gorenstein local ring of embedding codimension h and any dimension.

The result is a consequence of a more general Theorem which gives a structure for Gorenstein ideals which are called almost stretched. This refers to the property that the square of the maximal ideal is generated by two elements.

This result gives as a consequence a clean formula for the Poincare series of a stertched Gorenstein local ring, namely

$$P_A(z) = \frac{(1+z)^d}{1-hz+z^2}$$

In a more recent paper we attack the problem of classifying up to analytic isomorphism the family of almost stretched Artinian Gorenstein algebras with a given Hilbert Function. We solve the problem in the case the socle degree is large enough.

It turns out that there is an Artinian Gorenstein algebra of multiplicity 10 which has infinitely many isomorphism classes. One should remember that Artinian Algebras with multiplicity at most 6 have a finite number of isomorphism classes.

Prime and semiprime operations over one-dimensional domains

Janet C. Vassilev
University of New Mexico, Usa

Closure operations defined on a commutative ring R can be thought of of maps f_c from the set of ideals of a ring to itself satisfying the properties:

(a) $I \subseteq f_c(I)$ for all $I \subseteq R$.

(b) If $I \subseteq J$, then $f_c(I) \subseteq f_c(J)$.

(c) $f_c(f_c(I)) = f_c(I)$ for all $I \subseteq R$.

Semiprime operations are closure operations which also satisfy:

(d) $f_c(I)f_c(J) \subseteq f_c(IJ)$ for all $I, J \subseteq R$.

Prime operations are semiprime operations for which the following also holds:

(e) $f_c(bI) = bf_c(I)$ for all regular elements $b \in R$ and all ideals $I \subseteq R$.

We will classify all the semiprime and prime operations on certain one-dimensional domains and discuss the algebraic structure on the set of semiprime/prime operations over these rings. If we expand the set of ideals, to the fractional ideals over R, we will observe that some of these semiprime operations are no longer defined.

Multiplicative invariants and length functions over valuation domains

Paolo Zanardo
Università degli Studi di Padova, Italia

The notion of generalized length functions for modules over a commutative ring R was given by Northcott and Reufel in 1965. They described length functions over valuation domains. We define a multiplicative invariant as a map $\mu : \mathrm{Fin}(R) \to \Gamma$, where $(\Gamma, \cdot, \leq)$ is a partially ordered semigroup, such that $\mu(X) = \mu(Y)\mu(X/Y)$, for $Y \subseteq X$ finitely generated R-modules. We prove some results on finitely generated modules over valuation domains, that allow us to get a new description of length functions, through a universal property of multiplicative invariants.