Physica **65** (1973) 73–88 © *North-Holland Publishing Co.*

SURFACE TENSION IN THE TWO-DIMENSIONAL ISING MODEL

D. B. ABRAHAM

Department of Theoretical Chemistry, Oxford University, Oxford, England

G. GALLAVOTTI

Instituto di Mathematico dell'Università di Roma, Italia

and

A. MARTIN-LÖF

Department of Mathematics, Royal Institute of Technology, Stockholm, Sverige

Received 25 July 1972

Synopsis

We examine several definitions of surface tension between oppositely magnetised phases in the two-dimensional Ising model. With nearest-neighbour attractive interactions, we prove that at low temperatures (large β) all the definitions agree with that of Onsager.

1. *Introduction.* For the two-dimensional Ising model with nearest-neighbour interactions, Onsager[2]) defined and calculated exactly a surface tension. This was later shown by Fisher and Ferdinand[5]) to be related to the incremental free energy of a lattice with a vertical ladder of perturbed horizontal bonds. Unfortunately the generalization of this approach to the situation with non-nearest neighbour interactions is not straightforward; neither is it obviously related to the more general definition based on a detailed discussion of the phase-separation phenomenon; this is described in ref. 1. Later in this paper we shall evaluate the surface tension according to yet another definition, referred to in ref. 1 as grand canonical. We shall enlarge upon these definitions in subsequent sections; at this point we only wish to stress that considerably less detailed information is required to evaluate the surface tension for these *macroscopic* definitions. Therefore it is desirable to explore their equivalence, if any, to the microscopic definition in terms of phase separation; this is the aim of the present paper.

In the following sections, we review the results of refs. 1 and 3 and outline the definitions of surface tension to which we alluded above. We then prove that, for the two-dimensional lattice with nearest-neighbour interactions, the Onsager definition coincides with the microscopic definition in ref. 1 for the region in β where the latter has been shown to make sense. Finally, we evaluate the grand-canonical surface tension, and reflect on its equality with the Onsager value.

2. *Notation and previous results.* Let Ω be a lattice with N columns and $(H + 2)$ rows. The opposite ends of each row are joined giving a cylinder with base length N. At each site of Ω there is located a classical spin σ_i with values $\sigma_i = \pm 1$. The Ising model with nearest-neighbour interactions of strength J is then specified by assigning an energy $E(X)$ to each spin configuration X as follows:

$$E(X) = J \times (\text{total number of bonds in } \Omega) - 2J \times (\text{number of bonds with opposite spins at their extremes}). \tag{2.1}$$

We shall be interested in constrained situations where the spins on the bases have specified values; in particular $M^{(x,y)}(\Omega)$ denotes the set of configurations X for which spins on the upper base have value x while those on the lower base have value y; $x, y = \pm 1$. A set of lines is associated with each configuration by drawing a unit segment symmetrically perpendicular to the midpoint of any bond which has opposite spins at its extremes. One realises that the set of lines thus obtained on the dual lattice can be split, because of the selected boundary conditions, into several disjoint closed contours $\gamma_1, \ldots, \gamma_n$, which are self-avoiding in a sense made precise in ref. 1. Formula (2.1) can be written immediately in terms of the lengths $|\gamma_j|$ of the contours $\gamma_j, j = 1, \ldots, n$, associated with a given spin configuration X:

$$E(x) = Jp\,|\Omega| - 2J \sum_1^n |\gamma_j|, \tag{2.2}$$

where $|\Omega|\,p$ is the total number of bonds in the lattice, $|\Omega|$ being its area.

There are important restrictions on the contour configurations $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ appropriate for the ensemble $M^{(x,y)}(\Omega)$: if $x = y$ there must be an even number of contours which wind round the cylinder, whereas if $x = -y$ there must be an odd number of such contours. With these stipulations, there is a 1 : 1 correspondence between sets of contours Γ on the dual lattice and the spin configurations $X \in M^{(x,y)}(\Omega)$: we shall refer to X and $\Gamma(X)$ in an interchangeable way.

In ref. 1, the following results were obtained [we take $J = -\frac{1}{2}$ in (2.2)]:

2.1. Phase separation. Let $M^{+-}(\Omega, m)$ be the set of configurations in $M^{+-}(\Omega)$ with fixed magnetization $|\Omega|\,m$, where

$$m = \alpha m^* + (1 - \alpha)(-m^*) = (2\alpha - 1)\,m^*, \qquad 0 < \alpha < 1, \tag{2.3}$$

m^* being the spontaneous magnetization. Let the subset $\tilde{M}_0^{+-}(\Omega, m)$ be defined by the additional restrictions:

1) There is one and only one contour λ which winds round the cylinder. Its length satisfies the restriction

$$|\lambda| \leqslant N(1 + G\beta^{-1}), \tag{2.4}$$

for a suitable $G > 0$.

2) Area fluctuations: let Ω_λ be the region above λ; then

$$\begin{aligned} &||\Omega_\lambda| - \alpha\,|\Omega|| \leqslant k(\beta)\,|\Omega|^p, \\ &||\Omega - \Omega_\lambda| - (1 - \alpha)\,|\Omega|| \leqslant k(\beta)\,|\Omega|^p, \qquad \tfrac{3}{4} \leqslant p < 1. \end{aligned} \tag{2.5}$$

$k(\beta) = c_1\,\mathrm{e}^{-c_2\beta}$ for suitable constants $c_1, c_2 > 0$.

3) Magnetization fluctuations: let $m^\pm$ be the average magnetizations above and below the contour. Then

$$\begin{aligned} &|m^+\,|\Omega_\lambda| - m^*\alpha\,|\Omega|| \leqslant k(\beta)\,|\Omega|^p, \\ &|m^-\,|\Omega - \Omega_\lambda| + m^*(1 - \alpha)\,|\Omega|| \leqslant k(\beta)\,|\Omega|^p. \end{aligned} \tag{2.6}$$

4) Excluding λ, the contours satisfy the length restriction

$$|\gamma| \leqslant c_0 \log|\Omega|, \tag{2.7}$$

with a few exceptions; the total length of these contours does not exceed $N\beta^{-1}$.

For β sufficiently large, it was proved in ref. 1 that

$$\lim_{N\to\infty} \frac{Z(\tilde{M}_0^{+-}(\Omega, m), \beta)}{Z(M^{+-}(\Omega, m), \beta)} = 1, \tag{2.8}$$

where $Z(M, \beta)$ is the partition function for the ensemble M. This equation shows that we may regard the lattice as two seas of spins having well defined but opposite magnetizations separated by a fairly well defined interface.

2.2. Surface tension. The surface tension τ associated with the interface described above is defined by

$$\tau = \lim_{N\to\infty} \frac{1}{N} \log \frac{Z(M^{+-}(\Omega, m), \beta)}{Z(M^{++}(\Omega, m^*), \beta)}. \tag{2.9}$$

For β sufficiently large, this limit was shown in (1) to exist and to be independent of m when $H = [N^\delta]$, $\delta > 1$. It is then independent of δ as well. Formula (2.9) was analysed by successively restricting the ensembles M in the partition functions. Thus we have

$$Z(M^{+-}(\Omega, m), \beta) \approx \sum_{\lambda} e^{-\beta|\lambda|} Z(M_0^{++}(\Omega_\lambda), \beta)\, Z(M_0^{--}(\Omega - \Omega_\lambda), \beta), \tag{2.10}$$

where the ensembles $M_0^{\pm\pm}(\Omega)$ have no magnetization restriction, and no contour which will wind round the cylinder.

The sum over λ is restricted by the bound

$$0 \leqslant |\Omega_\lambda| - \alpha|\Omega| < N, \tag{2.11}$$

as well as by (2.4). Analogously, the partition function $Z(M^{++}(\Omega, m^*), \beta)$ in (2.9) may also be replaced by $Z(M_0^{++}(\Omega), \beta)$. The symbol $\approx$ means that the two sides of (2.10) are identical up to a factor which behaves like $\exp(No(N))$.

2.3. Virial expansion. Given $n(\geq 1)$ closed self-avoiding contours, not necessarily disjoint, nor even different, none of which winds round the cylinder Ω one can define a function $\varphi^{T}(\gamma_1 \cdots \gamma_n)$ on the sets of contours $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ such that

1) $\varphi^{T}(\Gamma) = 0$ if Γ is disconnected, that is, if Γ can be partitioned into two or more subsets such that every γ in one is compatible with every γ in the other.

2) $\varphi^{T}(\Gamma)$ is translationally invariant; it is not lattice dependent unless Γ winds round the cylinder.

3) We have the bounds

$$\sum_{\Gamma \ni x} |\varphi^{T}(\Gamma)| \leqslant e^{-c\beta}, \qquad c > 0, \tag{2.12}$$

$$\sum_{\substack{\Gamma \ni x \\ \Gamma \ni y}} |\varphi^{T}(\Gamma)| \leqslant F e^{-2\beta} (4\, e^{-\frac{1}{2}\beta})^{|x-y|^{\frac{1}{2}}}, \qquad F > 0, \tag{2.13}$$

where $\Gamma \ni x$ implies that $x \in \Omega$ is contained in some $\gamma_i \in \Gamma$.

4) The functions $\varphi^{T}(\Gamma)$ are such that the following virial expansion is valid:

$$Z(M_0^{++}(\theta), \beta) = \exp \sum_{\Gamma \subset \theta} \varphi^{T}(\Gamma), \tag{2.14}$$

for any subregion $\theta \subset \Gamma$. Then

$$Z(M^{+-}(\Omega, m), \beta) \approx Z(M_0^{++}(\Omega), \beta) \sum_{\lambda} e^{-\beta|\lambda| - \mu_\Omega(\lambda)}, \tag{2.15}$$

where

$$\mu_\Omega(\lambda) = \sum_{\substack{\Gamma \text{ i } \lambda \\ \Gamma \subset \Omega}} \varphi^{T}(\Gamma). \tag{2.16}$$

The symbol $\Gamma \mathrm{i} \lambda$ means Γ intercepts the contour λ. Finally, by using (2.12), it follows that

$$\tau = \lim_{N\to\infty} \frac{1}{N} \log \sum_{\lambda} (\mathrm{e}^{-\beta|\lambda| - \mu(\lambda)}), \tag{2.17}$$

where the sum satisfies (2.11) and (2.4), and $\mu(\lambda)$ is defined by

$$\mu(\lambda) = \sum_{\Gamma \mathrm{i} \lambda} \varphi^{\mathrm{T}}(\Gamma), \tag{2.18}$$

independent of the height of Ω, and thus of δ.

3. *The Onsager definition.* Let Ω be a cylinder with base N and height H. Suppose N is even and H is odd. Let the bases of the cylinder be joined by additional bonds so that Ω becomes a torus. We define columns of spins on the torus to be lines of spins parallel to the original cylinder axis. Rows are defined *mutatis mutandis.* The sites on the torus are labelled by the ordered pair (r, s); r and s are the row and column indices, respectively. The spins have an interaction energy similar to (2.1), but the interaction in rows and columns is not necessarily the same.

$$E = \sum_{r=1}^{H} \sum_{s=1}^{N} [J(v)\, \sigma_{rs}\, \sigma_{r+1\,s} + J(h)\, \sigma_{rs}\sigma_{rs+1}]. \tag{3.1}$$

We shall consider the case $J(v) = J(h) = J > 0$ which describes the antiferromagnet. The associated partition function is denoted by $Z(\Omega, \beta, a)$.

There are two related energy assignments which are of interest. Let every spin be reversed on alternant columns. Since N is even, we obtain a lattice with ferromagnetic interactions within rows, but antiferromagnetic interaction within columns. Such a scheme was considered by Onsager[2]). Let the associated partition function be denoted by $Z(\Omega, \beta, fa)$; then evidently

$$Z(\Omega, \beta, fa) \equiv Z(\Omega, \beta, a). \tag{3.2}$$

Suppose now a further spin reversal is applied to every other row on the lattice. Then the lattice obtained thereby has $J = J(h) < 0$ and $J = J(v) < 0$ for all rows of vertical bonds, except one which has bond strengths $-J$, because H is odd. Let the associated partition function be $Z^{\mathrm{X}}(\Omega, \beta, f)$. Then we have

$$Z^{\mathrm{X}}(\Omega, \beta, f) \equiv Z(\Omega, \beta, fa). \tag{3.3}$$

This relationship was first pointed out by Fisher and Ferdinand[5]).

The Onsager definition of the surface tension τ is

$$\tau = \lim_{N\to\infty} \frac{1}{N} \log \frac{Z(\Omega,\beta,fa)}{Z(\Omega,\beta,f)}, \tag{3.4}$$

where $Z(\Omega,\beta,f)$ is the partition function for the purely ferromagnetic case with $J(h) = J(v) = J < 0$. The limit is taken so that H is the least odd integer greater than N^δ for $\delta > 1$. We now establish the equivalence of (3.4) and (2.17) using a contour technique.

4. *Equivalence of definitions.* In order to discuss $Z(\Omega,\beta,a)$ introduced in the previous section, we shall represent a spin configuration X in Ω by a set of contours constructed in a different way: draw a unit line segment perpendicular to each bond of Ω which has *equal* spins at its extremes[4]). Clearly the set of lines on the dual lattice obtained in this way splits into a set of closed, disjoint, self-avoiding contours $\gamma_1, \ldots, \gamma_n$, and the energy of the corresponding configuration X can be written

$$E(X) = -pJ|\Omega| + 2J\sum|\gamma_i|, \tag{4.1}$$

by analogy with (2.2). The set of contours is subject to a restriction: there must be an odd number of contours which wind round the torus in the row direction. Such sets of contours are termed compatible. The correspondence between them and the set of spin configurations is 1:2. If $J = +\frac{1}{2}$ the partition function $Z(\Omega,\beta,a)$ is given by

$$Z(\Omega,\beta,a) = 2 \sum_{\substack{\Gamma\subset\Omega \\ \Gamma \text{ allowed}}}' \exp\Big(-\beta\sum_j|\gamma_j|\Big). \tag{4.2}$$

In the treatment of $Z(\Omega,\beta,f)$ we use the same contour construction as in ref. 1 and section 2 of this paper. The energy is given by

$$E(X) = pJ|\Omega| - 2J\sum|\gamma_i|, \tag{4.3}$$

but, because of the toroidal boundary conditions, the correspondence between allowed configurations of spins and the contours is 2:1. There must be an even number of contours which wind round the cylinder. If $J = -\frac{1}{2}$, then

$$Z(\Omega,\beta,f) = 2 \sum_{\substack{\Gamma\subset\Omega \\ \Gamma \text{ allowed}}} \exp\Big(-\beta\sum_j|\gamma_j|\Big). \tag{4.4}$$

By the following an argument similar to that used in Peierls's proof of the occurrence of a phase transition for β sufficiently large[10]) it is easy to see that the contribution to $Z(\Omega, \beta, a)$ from configurations with more than one contour winding round Ω and to $Z(\Omega, \beta, f)$ from configurations having any such contour at all is negligible. Consider $Z(\Omega, \beta, f)$, *e.g.*, and call the contribution from the configurations without such "long" contours $Z_0(\Omega, \beta, f)$. Then

$$\begin{aligned} 1 &- Z_0(\Omega,\beta,f)/Z(\Omega,\beta,f) \\ &= \Big[\sum_{\lambda_1,\lambda_2} e^{-\beta(|\lambda_1|+|\lambda_2|)} \sum_{\gamma_1\ldots\gamma_n\subset\Omega\setminus\lambda_1\cup\lambda_2} \exp\Big(-\beta\sum_i|\gamma_i|\Big)\Big] \\ &\times \Big[\sum_{\gamma_1\ldots\gamma_n\subset\Omega} \exp\Big(-\beta\sum_j|\gamma_i|\Big)\Big]^{-1}, \end{aligned} \tag{4.5}$$

where λ_1 and λ_2 both wind round Ω in the same direction. The last sum in the numerator is contained in that of the denominator, so their ratio is ≤ 1 and:

$$\begin{aligned} \text{eq. (4.5)} &\leqslant \sum_{\lambda_1,\lambda_2} e^{-\beta(|\lambda_1|+|\lambda_2|)} \leqslant \Big(\sum_{p\in\Omega}\sum_{\substack{\lambda\ni p\\ |\lambda|\geqslant N}} e^{-\beta|\lambda|}\Big)^2 \leqslant \Big[NH\sum_{l=N}^{\infty}(3\,e^{-\beta})^l\Big]^2 \\ &= [NH/(1-3\,e^{-\beta})]^2\,(3\,e^{-\beta})^{2N}, \end{aligned} \tag{4.6}$$

using the estimate 3^l for the number of contours of length l which pass through a given point p. For β large $1 - Z_0/Z$ hence goes exponentially to zero as $N \to \infty$. The contribution $Z_0(\Omega, \beta, a)$ to $Z(\Omega, \beta, a)$ from configurations with only one "long" contour is estimated similarly. We have thus the following relations:

$$Z(\Omega,\beta,a) \approx Z_0(\Omega,\beta,a) = 2\sum_{\lambda} e^{-\beta|\lambda|} \sum_{\gamma_1\ldots\gamma_n\subset\Omega\setminus\lambda}^{0} \exp\Big(-\beta\sum_i|\gamma_i|\Big), \tag{4.7}$$

$$Z(\Omega,\beta,f) \approx Z_0(\Omega,\beta,f) = 2\sum_{\gamma_1\ldots\gamma_n\subset\Omega}^{0} \exp\Big(-\beta\sum_i|\gamma_i|\Big), \tag{4.8}$$

where λ, but none of $\gamma_1, \ldots, \gamma_n$ winds round Ω and no γ_i intercepts λ.

On the torus Ω a similar virial expansion as on a cylinder is valid in the ensemble of contours not winding round Ω, so that we have:

$$Z_0(\Omega,\beta,a) = 2\sum_{\lambda} e^{-\beta|\lambda|} \exp \sum_{\Gamma\subset\Omega\setminus\lambda} \varphi^{\mathrm{T}}(\Gamma), \tag{4.9}$$

$$Z_0(\Omega,\beta,f) = 2\exp\sum_{\Gamma\subset\Omega}\varphi^{\mathrm{T}}(\Gamma). \tag{4.10}$$

[The $\varphi^{\mathrm{T}}(\Gamma)$ appropriate for a torus are defined also for other configurations than those on a cylinder, but for a Γ that can be located both on a torus and on a cylinder $\varphi^{\mathrm{T}}(\Gamma)$ is the same in both cases, see ref. 1]. We obtain as in (2.15):

$$\frac{Z(\Omega,\beta,a)}{Z(\Omega,\beta,f)} \approx \frac{Z_0(\Omega,\beta,a)}{Z_0(\Omega,\beta,f)} = \sum_{\lambda} e^{-\beta|\lambda| - \tilde{\mu}_\Omega(\lambda)}, \tag{4.11}$$

where

$$\tilde{\mu}_\Omega(\lambda) = \sum_{\substack{\Gamma\subset\Omega \\ \Gamma \mid \lambda}} \varphi^{\mathrm{T}}(\Gamma). \tag{4.12}$$

(Here we use the fact that $\varphi^{\mathrm{T}}(\Gamma) = 0$ if Γ is "disconnected".) In (4.11) we can impose the length restriction (2.4) without affecting τ. This follows from the estimate $|\tilde{\mu}_\Omega(\lambda)| \leqslant |\lambda|\, \mathrm{e}^{-\beta c}$ implied by (2.12), which allows us to conclude that

$$\begin{aligned}
&\Big(\sum_{|\lambda| \geqslant N(1+G\beta^{-1})} \mathrm{e}^{-\beta|\lambda| - \tilde{\mu}_\Omega(\lambda)}\Big)\Big(\sum_{\lambda} \mathrm{e}^{-\beta|\lambda| - \tilde{\mu}_\Omega(\lambda)}\Big)^{-1} \\
&\leqslant \sum_{|\lambda| \geqslant N(1+G\beta^{-1})} \exp\left(-\beta\,|\lambda| + 2\,|\lambda|\,\mathrm{e}^{-\beta c} + N\beta\right) \\
&\leqslant NH\,\mathrm{e}^{N\beta} \sum_{l=N(1+G\beta^{-1})}^{\infty} [3\exp(-\beta + 2\,\mathrm{e}^{-\beta c})]^l \\
&\leqslant \mathrm{const}\; NH\,\{3^{1+G\beta^{-1}} \exp[-G + (1 + G\beta^{-1})\,2\,\mathrm{e}^{-\beta c}]\}^N,
\end{aligned} \tag{4.13}$$

which goes exponentially to zero as $N \to \infty$ for all large enough β if G is not too small. [G in (2.4) was chosen in ref. 1 to fulfil this requirement.]

Both in (4.11) and (2.17) we can restrict the summation to one representative from each congruence class under translation along the cylinder axis without affecting τ. The difference between $\tilde{\mu}_\Omega(\lambda)$ in (4.12) and $\mu_\Omega(\lambda)$ in (2.18) is of the order at most $\mathrm{e}^{-\alpha N^{\frac{1}{2}}}$ for some $\alpha > 0$, because by (2.13) this is a bound on the contribution from the configurations which appear in one of the sums but not the other. To see this let λ_1 and λ_2 be congruent to λ at distance $\frac{1}{2}N$ from it ($N < H$). Then both $\mu_\Omega(\lambda)$ and $\tilde{\mu}_\Omega(\lambda)$ differ from the corresponding sum with Γ restricted not to intercept $\lambda_1 \cup \lambda_2$ by at most $\sum_{\Gamma \mid \lambda,\, \mid \lambda_1 \cup \lambda_2} |\varphi^{\mathrm{T}}(\Gamma)|$, which is bounded as indicated using (2.13). Hence (3.4) and (2.9) are identical for sufficiently large values of β.

5. *Grand canonical surface tension.* In ref. 1, a grand-canonical definition of the surface tension was proposed. This can be formulated as follows: let Ω be a cylinder with base N and height $(H + 2) = [N^\delta]$, $\delta > 1$. Using the notation of section 2, consider the grand-canonical partition functions $Z(M^{++}(\Omega), \beta)$ and $Z(M^{+-}(\Omega), \beta)$ and define

$$\tau'(\beta) = \lim_{N\to\infty} \frac{1}{N} \log\left(\frac{Z(M^{+-}(\Omega), \beta)}{Z(M^{++}(\Omega), \beta)}\right), \tag{5.1}$$

where the limit is to be taken with $H = [N^\delta]$, $\delta > 1$ as in (3.4). The physical arguments given in ref. 1 and reviewed in section 2 of this paper lead one to conclude that $\tau'(\beta)$ is perhaps another viable definition of surface tension. It has an expression analogous to (4.9) in terms of the contour expansion. We are unable to demonstrate in general the coincidence of (2.9) and (5.1). Neither is the existence of (5.1) proved directly. However, for nearest-neighbour interactions we can compute $\tau'(\beta)$ exactly. Its value is obtained in the appendix:

$$\tau'(\beta) = -\beta - \log \tanh(\tfrac{1}{2}\beta), \tag{5.2}$$

which is the same as Onsager's value[2]). In the computation it is necessary to stipulate that $H = [N^{\delta}]$, $\delta > 1$. The physical meaning of such a condition is quite simple: the ensemble $M^{+-}(\Omega)$ is the union of the ensembles of configurations with given average magnetization m. The configurations with $m > m^*$ have vanishing probability in the limit, but those for which $m \in [-m^*, m^*]$ have comparable probabilities. If $m \in (-m^*, m^*)$ then a typical configuration $X \in M^{+-}(\Omega, m)$ splits into two oppositely magnetised phases separated by a line λ as in section 2. If $m = (2\alpha - 1)\,m^*$, then λ is located at a mean distance αH from the upper base, and has a width of the order of N [see (2.4)]. Hence if $H = [N^{\delta}]$ this line can be made arbitrarily distant from the bases of Ω for $0 < \alpha < 1$ merely by choosing N large enough; thus the effect of interference between the interface and the bases is negligible. This is not so for those "few" configurations with $m = \pm m^*$. Thus it is not clear from this point of view why $\tau'(\beta)$ and $\tau(\beta)$ should be equal, and we are unable to control the necessary estimates to prove this equality in general by direct considerations.

Suppose a spin reversal is applied to s adjacent rows of spins, beginning at the lower base. We then have a ferromagnetic lattice with + spins on each base and a row of reversed vertical bonds at a height s; the incremental free energy is independent of s. We can introduce a contour description as follows: on the line of reversed bonds, draw a unit perpendicular segment on the dual lattice wherever there are equal spins at the extremes. For the other bonds we adhere to the usual rule. With (+, +) bases, there must be an odd number of contours which wind round the cylinder. With free ends, the surface tension is zero (as may be seen along the lines indicated in appendix A). τ' can hence also be defined as the limit of the incremental free energy for this configuration divided by N.

Let a column X of reversed horizontal bonds of length l be inserted into a large lattice Λ. Let the thermodynamic limit $|\Lambda| \to \infty$ be taken so that $d(X, \partial\Lambda) \to \infty$. Then the incremental free energy $F^X(l)$ exists in this limit, and may be seen to be independent of the boundary conditions on Ω [8]). For this system Fisher and Ferdinand[5]) defined a surface tension τ'' by

$$\tau'' = \lim_{l \to \infty} [F^X(l)/l], \tag{5.3}$$

which they evaluated, obtaining the Onsager value. The relation of their result to the grand-canonical definition may be understood readily by appealing to the transfer-matrix formalism (see appendix A). Finally we mention the interesting relation (known, at least, to Fisher and Ferdinand[5,9]) between the grand-canonical surface tension and the inverse spin–spin correlation length for the high temperature region. Let contours be drawn for the column of reversed horizontal bonds in the Fisher–Ferdinand approach according to the following rules: 1) If there are identical spins on opposite ends of a reversed bond, let a unit segment be drawn in a symmetrical way perpendicular to the mid point of the bond. 2) For the remaining bonds lines are drawn in the same way on the dual lattice if and only if the associated spins are opposite in sign.

In this way one obtains a set of contours with the restriction that at any point on the dual lattice 0, 2 or 4 bonds may meet, except at the end points of the row of reversed bonds from which either one or three bonds may emanate. Thus the ends of one of the long contours are tied down. The energy associated with the contour is given once again by (2.2). On the other hand the well-known tanh K (high temperature) expansion for the pair correlation functions[9]) leads to an exactly similar contour description, except that the edge weight should be tanh K rather than e^{-2K}. Thus the surface tension is obtained from the high-temperature inverse correlation length by merely interchanging K and K^* [see (A.2)].

Acknowledgement. We thank Professor M.E. Fisher for a clarifying discussion.

APPENDIX A

1. *Grand-canonical surface tension.* By using the methods reviewed in ref. 6, the surface tension $\tau'(\beta)$ defined in (5.1) can be expressed in terms of the matrices V_1, V_2 for transfer parallel to the cylinder axis, which are given by

$$\begin{aligned} V_1 &= (2 \sinh 2K_1)^{\frac{1}{2}N} \exp\left(-K_1^* \sum_1^N \sigma_j^z\right), \\ V_2 &= \exp\left(K_2 \sum_1^N \sigma_j^x \sigma_{j+1}^x\right), \end{aligned} \tag{A.1}$$

where

$$K_2 = \beta J(h), \qquad K_1 = \beta J(v), \qquad \tanh K_1 = e^{-2K_1^*}. \tag{A.2}$$

Let $|\pm\rangle$ be eigenstates of the $\{\sigma_j^x\}$ defined by

$$\sigma_j^x |\pm\rangle = \pm |\pm\rangle \qquad \text{for} \qquad j = 1, \ldots, N. \tag{A.3}$$

Then if Ω is a cylinder with base N and height H,

$$\frac{Z(M^{+-}(\Omega), \beta)}{Z(M^{++}(\Omega), \beta)} = \frac{\langle + | V^{H-1} | - \rangle}{\langle + | V^{H-1} | + \rangle}, \tag{A.4}$$

where

$$V = V_2^{\frac{1}{2}} V_1 V_2^{\frac{1}{2}}. \tag{A.5}$$

As shown in refs. 6 and 7, V may be written in the diagonal form

$$V = (2 \sinh 2K_1)^{\frac{1}{2}N} (Q_+ V_+ + Q_- V_-), \tag{A.6}$$

where

$$V(\pm) = \exp\left[-\tfrac{1}{2} \sum_{q \in J_\pm} \gamma(q) \, (2G_q^+ G_q - 1)\right], \tag{A.7}$$

and

$$J_{+/-} = \left\{\frac{(2r-1)\,\pi}{N} \Big/ \frac{2r\pi}{N}; \qquad r = 1, \ldots, N\right\}. \tag{A.8}$$

The G_q^+ are Fermi operators given by

$$G_q^+ = \cos\theta_q F_q^+ - \mathrm{i} \sin\theta_q F_{-q}, \tag{A.9}$$

where

$$F_q^+ = N^{-\frac{1}{2}} \sum_1^N \mathrm{e}^{\mathrm{i}qr} \prod_1^{r-1} (-\sigma_j^z) \, \tfrac{1}{2} (\sigma_r^x + \mathrm{i}\sigma_r^y). \tag{A.10}$$

The transformation angle θ is specified by

$$2\theta = \delta^*(\omega) - \omega + \pi, \tag{A.11}$$

where

$$\sinh\gamma \cos\delta^* = \sinh 2K_2 \cosh 2K_1^* - \cosh 2K_2 \sinh 2K_2^* \cos\omega,$$

$$\sinh\gamma \sin\delta^* = \sinh 2K_1^* \sin\omega, \tag{A.12}$$

with

$$\cosh\gamma = \cosh 2K_1^* \cosh 2K_2 - \sinh 2K_1^* \sinh 2K_2 \cos\omega, \tag{A.13}$$

the positive real root being taken for real ω. The functions γ and δ^* were defined by Onsager[2]). Finally, the operators $Q_\pm$ are the projectors

$$Q_\pm = \tfrac{1}{2}(1 \pm (-1)^\sigma), \tag{A.14}$$

with

$$(-1)^\sigma = \prod_1^N (-\sigma_j^z). \tag{A.15}$$

When $K_1^* < K_2$ (low-temperature region), the spectrum of (A.5) has an asymptotically doubly-degenerate ground state; thus

$$V|\Phi_\pm\rangle = \Lambda_\pm |\Phi_\pm\rangle, \tag{A.16}$$

where

$$(-1)^\sigma |\Phi_\pm\rangle = \pm |\Phi_\pm\rangle \tag{A.17}$$

and

$$\frac{\Lambda_-}{\Lambda_+} = 1 - A(N)\,\mathrm{e}^{-N\varkappa}, \qquad \varkappa = 2\,(K_1 - K_2^*), \tag{A.18}$$

as shown in appendix B, because $\gamma(\omega)$ is periodic with nearest singularities to the real axis at

$$\omega = \pm \mathrm{i}\varkappa + 2n\pi, \qquad n = 0, \pm 1, \dots .$$

Let $|\Phi_\pm\rangle_0$ be defined by

$$|\Phi_\pm\rangle_0 = \lim_{K_2 \to \infty} |\Phi_\pm\rangle. \tag{A.19}$$

Then because

$$(-1)^\sigma\,|\pm\rangle = |\mp\rangle, \tag{A.20}$$

(we consider N even), one has

$$|\Phi_\pm\rangle_0 = \pm 2^{-\frac{1}{2}}\,(|+\rangle \pm |-\rangle), \tag{A.21}$$

bearing in mind (A.17). Now

$$|\Phi_\pm\rangle = K(\pm) \prod_{q \in J_\pm} (\cos\theta_q + \mathrm{i}\sin\theta_q F_{-q}^+ F_q^+)\,|0\rangle, \tag{A.22}$$

where

$$K(+/-) = 1/(F_0^+ + F_0), \tag{A.23}$$

and $|0\rangle$ is the "vacuum" for the F_q, *i.e.*, $F_q\,|0\rangle = 0$ for all q. It now follows directly from the spectral decomposition of V in (A.4) and the above remarks that

$$\frac{Z\,(M^{+-}(\Omega), \beta)}{Z\,(M^{++}(\Omega), \beta)} = \frac{1 - \mu\,(\Lambda_-/\Lambda_+)^{H-1} + \mathcal{O}(\mathrm{e}^{-2\gamma(0)H})}{1 + \mu\,(\Lambda_-/\Lambda_+)^{H-1} + \mathcal{O}(\mathrm{e}^{-2\gamma(0)H})}, \tag{A.24}$$

where [see (A.11), (A.19) and (A.22)]

$$\mu = \prod_{q \in J_-} \cos\left[\tfrac{1}{2}\delta^*(a)\right] \Big/ \prod_{q \in J_+} \cos\left[\tfrac{1}{2}\delta^*(a)\right]. \tag{A.25}$$

We have

$$\mu = (1 + B(N)\, \mathrm{e}^{-\varkappa N}), \tag{A.26}$$

where $B(N)$ has a power dependence on N, because (see appendix B) $\cos\left[\frac{1}{2}\delta^*(\omega)\right]$ has period 2π and nearest singularities to the real ω axis at

$$\omega = \pm \mathrm{i}\varkappa + 2n\pi, \qquad n = 0, \pm 1, \ldots . \tag{A.27}$$

For large N eq. (A.24) behaves as

$$\frac{Z(M^{+-}(\Omega), \beta)}{Z(M^{++}(\Omega), \beta)} \simeq \tfrac{1}{2}(AH - B)\, \mathrm{e}^{-\varkappa N} + \mathcal{O}(\mathrm{e}^{-2\gamma(0)H}). \tag{A.28}$$

If the limit as $N \to \infty$ in (5.1) is taken over the sequence of cylinders with $H = [N^{\delta}]$, $\delta > 1$, then the second term in (A.28) vanishes, giving

$$\tau' = -2(K_1 - K_2^*), \tag{A.29a}$$

in agreement with the Onsager result. For $J(v) = J(h) = \frac{1}{2}$, this gives

$$\tau' = -2\beta - \log \tanh(\tfrac{1}{2}\beta), \tag{A.29b}$$

as in formula (5.2).

2. *Incremental free energies.* If there is a vertical ladder length r of reversed horizontal bonds on the torus Ω of section 5, then

$$\lim_{H \to \infty} [Z^X(\Omega, \beta)/Z(\Omega, \beta)] = \Lambda_+^{-r} \langle \Phi_+ |\, V_{(-)}^r \,| \Phi_+ \rangle. \tag{A.30}$$

Using the remarks of section 5, and a result of ref. 7, one makes a spectral decomposition of $V(-)^r$, obtaining immediately

$$\tau^X = -\gamma(0), \tag{A.31}$$

in agreement with (A.28) (on interchange of K_1 and K_2). This answer is essentially independent of the way the limit on the torus is performed, and was first obtained by Fisher and Ferdinand[5]) using a pfaffian method.

APPENDIX B

In this appendix we estimate the behaviour of Λ_-/Λ_+ and μ for large N. From (A.7) and (A.25) we have

$$\Lambda_-/\Lambda_+ = \exp \tfrac{1}{2} \sum_{j=1}^{2N} (-1)^j \gamma(\pi j/N), \tag{B.1}$$

and

$$\mu = \exp \tfrac{1}{2} \sum_{j=1}^{2N} (-1)^j \log\left[1 + \cos \delta^*(\pi j/N)\right]. \tag{B.2}$$

From (A.13) the following expressions for δ^* and γ as functions of z can be derived:

$$\begin{aligned} e^{i\delta^*} &= \left(\frac{b}{a}\right)^{\frac{1}{2}} \left(\frac{(z-a)(z-b^{-1})}{(z-a^{-1})(z-b)}\right)^{\frac{1}{2}}, \\ \tanh\left(\frac{\gamma}{2}\right) &= \left(\frac{(z-a)(z-a^{-1})}{(z-b)(z-b^{-1})}\right)^{\frac{1}{2}}, \end{aligned} \tag{B.3}$$

where $a = \tanh K_1^* \coth K_2$, $b = \tanh K_1^* \tanh K_2$ and the branches are such that $e^{i\delta^*} = 1$, $\tanh(\frac{1}{2}\gamma) \geq 0$ for $z = -1$. We thus see that γ has branch points at $z = a^{\pm 1}$, $b^{\pm 1}$. For low temperatures ($K_1^* < K_2$, $0 < b < a < 1$) it can be shown that $\cos \delta^* > -1$, so that $\log(1 + \cos \delta^*)$ has these branch points as well. Thus we are led to investigate sums of the form

$$S_N(g) = \sum_{j=1}^{2N} (-1)^j g(\exp(\pi i j/N)), \tag{B.4}$$

where $g(z)$ satisfies the conditions

(i) $g(z) = g(1/z)$, (B.5)

(ii) $g(z)$ is analytic in the annulus D given by

$$a < |z| < a^{-1} \tag{B.6}$$

and, in fact, in the z plane cut on (b, a) and (a^{-1}, b^{-1}).

We introduce the sequence of functions

$$F_N(z) = z^N/(z^N - 1)(z^N + 1), \tag{B.7}$$

which have simple poles at the points

$$z_j = \exp(\pi i j/N), \qquad j = 1, \ldots, 2N, \tag{B.8}$$

with residues given by

$$R(z_j) = (-1)^j \, z_j/2N. \tag{B.9}$$

Let $C_\pm$ be simple closed curves around the origin in the domains $D \cap \{z \gtrless 1\}$. Then from (B.4), $S_N(g)$ is given by

$$2NS_N(g) = \frac{1}{2\pi i} \left(\oint_{C_+} - \oint_{C^-} \right) F_N(z) \, g(z) \frac{dz}{z}, \tag{B.10}$$

which simplifies using (B.5) to

$$NS_N(g) = -\frac{1}{2\pi i} \oint_{C_-} F_N(z) \, g(z) \frac{dz}{z}, \tag{B.11}$$

or

$$NS_N(g) = \frac{1}{2\pi i} \int_b^a f_N(x) \, \Delta \, [g(x)] \frac{dx}{x}, \tag{B.12}$$

where Δ $[g(x)]$ is the jump in $g(z)$ across the cut.

The behaviour of (B.1) and (B.2) may be estimated by examining γ and $\log(1 + \cos \delta^*)$ near $z = a$; from (B.3) we have

$$\Delta \, [\gamma(x)] \approx (a - x)^{\frac{1}{2}}, \qquad \Delta \, [\log(1 + \cos \delta^*)(x)] \approx \text{constant}. \tag{B.13}$$

Thus (B.1) and (B.2) give

$$\mu \, (\Lambda_-/\Lambda_+)^H = [A(N) \, H - B(N)] \, a^N + \mathcal{O}(a^{2N}),$$

where $A(N)$ and $B(N)$ have a power-law dependence on N, and $A(N)\,H - B(N)$ does not vanish when $H = [N^\delta]$, $\delta > 1$.

REFERENCES

1) Gallavotti, G. and Martin-Löf, A., Commun. math. Phys. **25** (1972) 87; see also Abraham, D.B., Gallavotti, G. and Martin-Löf, A., Nuovo Cimento **4** (1971) 143.
2) Onsager, L., Phys. Rev. **65** (1944) 117.
3) Minlos, R.A. and Sinai, J.G., Transactions of the Moscow Math. Soc. **19** (1968) 121; see also Math. USSR Sbornik **2** (1967) 335.
4) Dobrushin, R.L., Funct. Anal. Appl. **2** (1967) 302.

5) Fisher, M.E. and Ferdinand, A.E., Phys. Rev. Letters **19** (1967) 169.
6) Schultz, T.D., Mattis, D.C. and Lieb, E.H., Rev. mod. Phys. **36** (1964) 856.
7) Abraham, D.B., Studies in Applied Math. **51** (1972) 179.
8) Martin-Löf, A., Commun. math. Phys. **24** (1972) 253.
9) Fisher, M.E., Phys. Rev. **162** (1967) 480.
10) Peierls, R.E., Proc. Cambridge Phil. Soc. **32** (1936) 47.
Dobrushin, R.L., Theoretical Prob. and Applications **10** (1965) 209; Griffiths, R.B., Phys. Rev. **136** (1964) 1437.