JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 44, 661–675 (1973)

On the Calculation of an Integral*

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Submitted by Gian-Carlo Rota

Received September 7, 1972

1. INTRODUCTION

We devote our attention to the following integral

$$I_n(g, \omega) = \int_{R^n} dq_1 \cdots dq_n \exp \left\{ -\frac{1}{2} \sum_{i \neq j}^{1,n} \frac{g^2}{(q_i - q_j)^2} - \frac{1}{2} \sum_{i=1}^{n} \omega^2 q_i^2 \right\}, \tag{1.1}$$

and we show that

$$I_n(g, \omega) = I_n(0, \omega) \exp \left\{ -\omega g \frac{n(n-1)}{2} \right\}. \tag{1.2}$$

The above integral does not seem to be known except in the case $n = 2$. The technique for the computation of (1.1) is quite interesting and relies on some results on the spectrum of the operator

$$\hat{H} = \frac{-\hbar^2}{2} \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \sum_{i \neq j}^{1,n} \frac{g^2}{(x_i - x_j)^2} + \frac{1}{2} \sum_{i=1}^{n} \omega^2 x_i^2, \tag{1.3}$$

where $\hbar^2$, g^2, ω^2 are positive constants.

This operator has been studied in Refs. [1] and [2], where it has been shown to have a discrete spectrum, which has also been exactly described in *closed form.*

* Research partially supported by the Consiglio Nazionale delle Ricerche (G.N.F.M.).

The knowledge of the spectrum of H allows to compute the quantity

$$Z = \mathrm{Tr}\exp(-\beta H) = \sum_{\alpha} \exp(-\beta\epsilon_\alpha) = f(\omega, g, \hbar, \beta). \tag{1.4}$$

A *formal* (see below) limit as $\hbar \to 0$ gives the result

$$\begin{aligned}\lim_{\hbar\to 0}(2\pi\hbar)^n \,\mathrm{Tr}\exp(-\beta H) &= \int dp_1 \cdots dp_n\, dq_1 \cdots dq_n \\ &\quad \cdot \exp\left[-\beta\left(\frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{g^2}{2}\sum_{i\neq j}^{1,n}\frac{1}{(q_i-q_j)^2} + \frac{\omega^2}{2}\sum_{i=1}^{n} q_i^2\right)\right] \\ &= \left(\left(\frac{2\pi}{\beta}\right)^{1/2}\right)^n \cdot I_n(\beta^{1/2}g, \beta^{1/2}\omega).\end{aligned} \tag{1.5}$$

The above formula is based on the mechanical interpretation of (1.3); it is, in fact, the Hamiltonian of a quantum system of particles in one dimension. Therefore, (1.4) describes the statistical mechanics of the system and the limit $\hbar \to 0$ should describe the statistical mechanics of the corresponding classical system; according to the correspondence rule the first equality in (1.5) follows [3].

Since the problem of computing (1.4) from the explicity known energy levels ϵ_α is easy, the real mathematical problems that have to be solved are the ones connected with a rigorous proof of the limit relation (1.5). It is the purpose of this paper to provide such a proof thus providing also a rigorous proof of (1.2).

This paper is self-contained. In Section 2 we summarize into a more mathematical language the known results on the spectrum of H. In Section 3 we find different representation for the trace (1.4) in terms of a Wiener integral using the Feynman–Kac formula. Using this representation we show, in Section 4, the limit relation (1.5), i.e., that the $\hbar \to 0$ limit is, indeed, the classical limit.

In Section 5 we discuss a conjecture, so far unproved, on the system of Hamiltonian equations

$$\frac{d^2 q_i}{dt^2} = g^2 \sum_{j=1, j\neq i}^{n} \frac{1}{(q_i - q_j)^3} - \omega^2 q_i\,, \tag{1.6}$$

which implies (1.2).

In Section 6 we list a number of integrals which follow from the knowledge of the one in (1.1).

2. The Operator H

We shall regard the formal operator (1.3) as a densely defined operator on $L_2(R^n)$.

Decompose the space $L_2(R^n)$ into $n!$ orthogonal subspaces labeled by the permutations of n objects

$$L_2(R^n) = \bigoplus_P L_2{}^P(R^n). \tag{2.1}$$

The space $L_2{}^P(R^n)$ is the space of the functions $f \in L_2$ such that

$$f(x_1, ..., x_n) = 0 \qquad \text{unless} \qquad x_{P_1} > \cdots > x_{P_n} \tag{2.2}$$

where $(P_1, ..., P_n)$ is the set into which $(1, ..., n)$ is permuted by P.

If e is the identity (as a permutation) let $\mathscr{D}^e_{\lambda,\xi} \subset L_2{}^e(R^n)$ be the subspace of the functions of the form

$$f(x_1, ..., x_n) = \left\{ \prod_{i=1}^n \prod_{j=i+1}^n (x_i - x_j)^\lambda \right\} P(x_1 \cdots x_n) \exp \left\{ -\xi \sum_{i=1}^n x_i{}^2 \right\} \tag{2.3}$$

if $x_1 > x_2 > \cdots > x_n$ and $f = 0$ otherwise; here $P(x_1, ..., x_n)$ is a symmetric polynomial of arbitrary degree. The spaces $\mathscr{D}^P_{\lambda,\xi}$ are similarly defined for all the permutations.

On $\bigoplus_P \mathscr{D}^P_{\lambda,\xi}$ we define the operator $H_{\lambda,\xi}$ as

$$(H_{\lambda,\xi} f)(x_1, ..., x_n) = -\frac{\hbar^2}{2} \sum_{i=1}^n \frac{\partial^2 f(x_1 \cdots x_n)}{\partial x_i{}^2} + \left[\frac{g^2}{2} \sum_{i \neq j}^{1,n} \frac{1}{(x_i - x_j)^2} + \frac{\omega^2}{2} \sum_{i=1}^n x_i{}^2 \right] f(x_1 \cdots x_n), \tag{2.4}$$

if $x_i \neq x_j$, $i \neq j$ and otherwise

$$(H_{\lambda,\xi} f)(x_1, ..., x_n) = 0.$$

Lemma. *If $\lambda > \frac{3}{2}$ the operator $H_{\lambda,\xi}$ is symmetric on $\mathscr{D}_{\lambda,\xi}$ and*

$$H_{\lambda,\xi} \mathscr{D}^P_{\lambda,\xi} \subset L_2{}^P(R^n).$$

The condition $\lambda > \frac{3}{2}$ is necessary in order to be sure that f is in the domain of the Laplace operator (notice the singularity of f when $x_i = x_j$). We leave the easy proof to the reader.

Next one needs the following theorem which is due to Calogero and Sutherland [1, 2]. Let

$$\bar{\lambda} = \frac{1}{2} \left(1 + \left(1 + \frac{4g^2}{\hbar^2} \right)^{1/2} \right), \qquad \bar{\xi} = \frac{\omega}{2\hbar}. \tag{2.5}$$

Then

THEOREM *If $\bar{\lambda} > 3$, the operators $H_{\bar{\lambda},\xi}$ are essentially self-adjoint on the domain $\bigcup_P \mathscr{D}^P_{\bar{\lambda},\xi}$ and*

(1) *$H_{\bar{\lambda},\xi}$ has a discrete spectrum*

(2) *the eigenvalues of $H_{\bar{\lambda},\xi}$ are of the form*

$$\epsilon_{k_1,\dots,k_n} = \omega\hbar \sum_{i=1}^{n} (k_i + \tfrac{1}{2}) + \omega\hbar\lambda \frac{n(n-1)}{2}, \qquad k_i = 0, 1, \dots \tag{2.6}$$

The multiplicity of each eigenvalue is given by $n!$ times the number of sets $(k_1, \dots, k_n)$ of nonnegative integers (two sets are regarded as equal if they differ by a permutation, e.g. (0, 0, 0, 1, 3, 18) = (18, 0, 1, 3, 0, 0)).

When $\bar{\lambda} > 3$ we shall call the self-adjoint extension of $H_{\bar{\lambda},\xi}$ simply H.

COROLLARY. *The operator* $\exp - \beta H$ *is a trace class operator and*

$$\operatorname{Tr} \exp(-\beta H) = n! \exp\left\{-\beta \frac{\lambda\omega\hbar}{2} n(n-1)\right\} \sum_{k_1,\dots,k_n}^{*} \exp\left\{-\beta\omega\hbar \sum_{i=1}^{n} (k_i + \tfrac{1}{2})\right\}. \tag{2.7}$$

Furthermore,

$$\begin{aligned}\lim_{\hbar\to 0} (2\pi\hbar)^n \operatorname{Tr} \exp(-\beta H) &= \exp\left\{-\beta\omega g \frac{n(n-1)}{2}\right\} \left(\frac{2\pi}{\beta\omega}\right)^n \\ &\equiv \exp\left\{-\beta\omega g \frac{n(n-1)}{2}\right\} \int \exp\left\{-\frac{\beta}{2} \sum_{i=1}^{n} p_i^2 - \frac{\beta}{2}\omega^2 \sum_{i=1}^{n} q_i^2\right\} \\ &\quad \times dp_1 \cdots dp_n \, dq_1 \cdots dq_n .\end{aligned} \tag{2.8}$$

The * in (2.7) remembers that the sum is over the set $(k_1, \dots, k_n)$ of n nonnegative integers. The corollary is a simple consequence of (2.6), and its proof is left to the reader; the theorem is proved in Appendix.

3. THE OPERATOR $\exp - \beta H$

We now show that the kernel associated with $\exp(-\beta H)$ can be expressed by means of a Wiener integral through the Feynman–Kac formula [4].

Let Θ be the space of the Brownian sample paths (θ from x to y, i.e., θ is a function from $[0, \beta]$ to R^n such that $\theta(0) = x$, $\theta(\beta) = y$). On Θ we consider the Wiener measure (with fixed initial and final points) generated by gaussian transition probability such that [4]

$$\int_{\Theta} P_{xy}(d\theta) = \frac{\exp\{-(x-y)^2/2\beta\hbar^2\}}{(2\pi\beta\hbar^2)^{1/2}} . \tag{3.1}$$

We shall need only the following properties of the measure $P_{xy}(d\theta)$ [4]:

(1) P_{xy} is concentrated on the continuous trajectories from x to y.

(2) $$\int dy \int_{\theta \not\subset [x-\delta, x+\delta]} P_{xy}(d\theta) \leqslant 2 \exp\left\{-\frac{\delta^2}{2\beta\hbar^2}\right\} . \tag{3.2}$$

(3) $\int dy \int P_{xy}(d\theta) \equiv 1.$

Using the above definitions the Feynman–Kac formula for the kernel $K_\beta(x_1 \cdots x_n, y_1 \cdots y_n)$ of the $\exp(-\beta\hat{H})$, where $\hat{H}$ is the formal operator (1.3), is given by

$$\begin{aligned} &K_\beta(x_1 \cdots x_n, y_1 \cdots y_n) \\ &\quad = \int_{\Theta^n} \left(\prod_{i=1}^n P_{x_i y_i}(d\theta_i)\right) \\ &\qquad \times \exp\left\{-\frac{g^2}{2}\sum_{i\neq j} \int_0^\beta \frac{d\tau}{(\theta_i(\tau)-\theta_j(\tau))^2} - \frac{\omega^2}{2}\sum_{i=1}^n \int_0^\beta \theta_i(\tau)^2\, d\tau\right\} . \end{aligned} \tag{3.3}$$

The above integral certainly exists (since the integrand is measurable and $\leqslant 1$); furthermore, it is easy to check that K_β defines a semigroup on $L_p(R^n)$, $1 \leqslant p \leqslant \infty$, through the formula

$$(K_\beta f)(x_1 \cdots x_n) = \int dy_1 \cdots dy_n\, K_\beta(x_1 \cdots x_n, y_1 \cdots y_n) f(y_1 \cdots y_n) \tag{3.4}$$

and $\| K_\beta \|_1 \leqslant 1$, $\| K_\beta \|_\infty \leqslant 1$; hence,

$$\| K_\beta \|_p \leqslant 1, \qquad 1 \leqslant p \leqslant \infty. \tag{3.5}$$

We shall now show that K_β is the kernel of the operator $\exp(-\beta H)$. Of course, the problem we have to solve is wether the generator (if existing) of the semigroup associated with K_β coincides with H; in fact, it might correspond to a different self-adjoint operator associate to the formal operator $\hat{H}$.

In this section, from now on, we restrict ourselves to the case $n = 2$ since it already contains all the difficulties and the extension to the general case is trivial. The results of the computations that follow are summarized in the two lemmas below and in the theorem at the end of the section.

Put (see (3.3))

$$V(\theta_1, \theta_2) = \frac{g^2}{2} \int_0^\beta \frac{d\tau}{(\theta_1(\tau) - \theta_2(\tau))^2} + \frac{\omega^2}{2} \int_0^\beta (\theta_1(\tau)^2 + \theta_2(\tau)^2)\, d\tau. \tag{3.5}$$

We prove the following two lemmas.

LEMMA 1. *If $\bar\lambda > \frac{3}{2}$ then for all $f \in \mathscr{D}_{\bar\lambda,\bar\xi}$*

$$\lim_{\beta\to 0} \int dy_1\, dy_2 \int P_{x_1 y_2}(d\theta_1)\, P_{x_2 y_2}(d\theta_2) \frac{\exp[-V(\theta_1, \theta_2)] - 1}{\beta} f(y_1 y_2)$$
$$= -\left(\frac{g^2}{2} \frac{1}{(x_1 - x_2)^2} + \frac{\omega^2}{2} (x_1{}^2 + x_2{}^2)\right) f(x_1 x_2) \tag{3.6}$$

almost everywhere in (x_1, x_2).

LEMMA 2. *If $\bar\lambda > 3$ then for all $f \in \mathscr{D}_{\bar\lambda,\bar\xi}$ one can find a function $g \in L_2(R^2)$ such that the double integral in the limit relation* (3.6) *has a modulus not exceeding $g(x_1, x_2)$ for all $0 < \beta < 1$. In other words the convergence of the limit* (3.6) *is not only almost everywhere convergence but it is also a dominated convergence.*

Clearly the above two lemmas imply that the limit (3.6) takes place also in the L_2 norm.

Proof of Lemma 1. Suppose $x_1 \neq x_2$, say $x_1 > x_2$. Assume

$$\beta^{1/3} < \tfrac{1}{2} \mid x_1 - x_2 \mid$$

then the integral in (3.6) can be written

$$\int_{\theta_1 \subset S_{x_1}(\beta^{1/3}), \theta_2 \subset S_{x_2}(\beta^{1/3})} P_{x_1 y_1}(d\theta_1)\, P_{x_2 y_2}(d\theta_2)\, dy_1\, dy_2 \frac{\exp[-V(\theta_1 \theta_2)] - 1}{\beta} f(y_1 y_2)$$
$$+ \int_{\theta_1 \not\subset S_{x_1}(\beta^{1/3}), \theta_2 \subset S_{x_2}(\beta^{1/3})} (\text{same}) + \int_{\theta_1 \subset S_{x_1}(\beta^{1/3}), \theta_2 \not\subset S_{x_2}(\beta^{1/3})} (\text{same})$$
$$+ \int_{\theta_1 \not\subset S_{x_1}(\beta^{1/3}), \theta_2 \not\subset S_{x_2}(\beta^{1/3})} (\text{same}), \tag{3.7}$$

where $S_{x_1}(\beta^{1/3})$ is the interval $[x_1 - \beta^{1/3}, x_1 + \beta^{1/3}]$.

Since $V \geqslant 0$ and the function $f \in \mathscr{D}_{\bar\lambda,\bar\xi}$ has a maximum

$$M = \max_{y_1 y_2} \mid f(y_1, y_2) \mid < \infty$$

we can use (3.2) to deduce that the sum of the last three integrals in (3.7) does not exceed

$$3M\frac{2}{\beta}\cdot 2\exp\left[-\frac{(\beta)^{2/3}}{2\beta\hbar^2}\right] = 12M\frac{\exp[-1/(2\hbar^2\beta^{1/3})]}{\beta} \xrightarrow[\beta\to\infty]{} 0; \tag{3.8}$$

hence, only the first integral survives in the limit $\beta\to 0$. Since $x_1 \neq x_2$ it is easy to deduce the statement of the lemma from the continuity of $f(y_1y_2)$ and from the fact that, uniformly in $\theta_1, \theta_2 \subset S_{x_{1,2}}(\beta^{1/3})$, one has

$$\lim_{\beta\to 0}\frac{\exp[-V(\theta_1,\theta_2)]-1}{\beta} = -\left(\frac{g^2}{2}\frac{1}{(x_1-x_2)^2}+\frac{\omega^2}{2}(x_1{}^2+x_2{}^2)\right). \tag{3.9}$$

Proof of Lemma 2. Since $f\in\mathscr{D}_{\bar\lambda,\xi}$ we can write

$$f(x_1, x_2) = |x_1-x_2|^{\bar\lambda}\phi(x_1x_2)\exp\left[-\frac{\omega}{2\hbar}(x_1{}^2+x_2{}^2)\right], \tag{3.10}$$

where ϕ is a polynomial in each sector of $L_2(R^2)$.

We apply the Schwartz inequality to the integral in (3.6) assuming $dy_1\,dy_2P_{x_1y_1}(d\theta_1)\,P_{x_2y_2}(d\theta_2)$ as measure and

$$\phi(y_1y_2)\exp[-(\omega/4\hbar)(y_1{}^2+y_2{}^2)]$$

and

$$\frac{\exp[-V(\theta_1,\theta_2)]-1}{\beta}|y_1-y_2|^{\bar\lambda}\exp[-(\omega/4\hbar)(y_1{}^2+y_2{}^2)]$$

as function to be integrated; calling $I_\beta(x_1, x_2)$ the integral in (3.6) we, therefore, find

$$\begin{aligned}
I_\beta(x_1,x_2)^2 &\\
&\leqslant \left[\iint |\phi(y_1y_2)|^2\exp\left[-\frac{\omega}{2\hbar}(y_2{}^2+y_2{}^2)\right]dy_1\,dy_2\,P_{x_1y_1}(d\theta_1)\,P_{x_2y_2}(d\theta_2)\right]\\
&\quad\cdot\left[\iint\exp\left[-\frac{\omega}{2\hbar}(y_1{}^2+y_2{}^2)\right]\left(\frac{\exp[-V(\theta_1,\theta_2)]-1}{\beta}\right)^2|y_1-y_2|^{2\bar\lambda}\right.\\
&\quad\left.\times P_{x_1y_1}(d\theta_1)\,P_{x_2y_2}(d\theta_2)\,dy_1\,dy_2\right]\\
&\leqslant\tilde M\left[\iint\exp\left[-\frac{\omega}{4\hbar}(y_1{}^2+y_2{}^2)\right]dy_1\,dy_2\,P_{x_1y_1}(d\theta_1)\,P_{x_2y_2}(d\theta_2)\right]\cdot I,
\end{aligned} \tag{3.11}$$

where I is the integral in the second square bracket and

$$\tilde M = \max_{y_1y_2}|\phi(y_1y_2)|^2\exp\left[-\frac{\omega}{4\hbar}(y_1{}^2+y_2{}^2)\right].$$

The integral which multiplies I in (3.11) is a Gaussian integral and can be performed exactly giving

$$\frac{1}{(1+(\omega\beta\hbar/2))^{1/2}}\exp\left\{-(x_1^2+x_2^2)\frac{\omega}{8\hbar(1+(\omega\beta\hbar/2))}\right\} \leqslant \frac{\exp\left\{-(x_1^2+x_2^2)\frac{\omega}{8\hbar(1+(\omega\hbar/2))}\right\}}{(1+(\omega\hbar/2))^{1/2}}, \tag{3.12}$$

since $\beta < 1$.

Hence, all that remains to show is that I is uniformly bounded for $0<\beta<1$.

Proceeding as in (3.7), and (3.8) we find

$$I \leqslant \int_{\theta_1\subset S_{x_1}(\beta^{1/3}/4),\theta_2\subset S_{x_2}(\beta^{1/3}/4)} P_{x_1y_1}(d\theta_1)\, P_{x_2y_2}(d\theta_2) \times \left(\frac{\exp[-V(\theta_1\theta_2)]-1}{\beta}\right)^2 |y_1-y_2|^{2\bar\lambda} \cdot \exp\left[-\frac{\omega}{2\hbar}(y_1^2+y_2^2)\right] dy_1\, dy_2 + 3M_1\frac{4}{\beta^2}\exp\left[-\frac{1}{32\hbar^2\beta^{1/3}}\right], \tag{3.13}$$

where

$$M_1 = \max_{y_1y_2} |y_1-y_2|^{2\bar\lambda}\exp\left[-\frac{\omega}{2\hbar}(y_1^2+y_2^2)\right].$$

Consider next only the first term I_1 in the left side of (3.13), the other being manifestly bounded in β.

We distinguish two cases $|x_1-x_2| \leqslant \beta^{1/3}$ and $|x_1-x_2| > \beta^{1/3}$; in the first case, if $\bar\lambda > 3$

$$I_1 \leqslant \frac{4}{\beta^2}(2\beta^{1/3})^{2\bar\lambda}\left(\max_{y_1y_2}\left\{\phi^2(y_1y_2)\exp\left[-\frac{\omega}{2\hbar}(y_1^2+y_2^2)\right]\right\}\right) < \bar M < \infty, \tag{3.14}$$

where we have used the fact that the measure $dy_1\, dy_2 P_{x_1y_1}(d\theta_1)\, P_{x_2y_2}(d\theta_2)$ is normalized upon integration of θ_1, θ_2 and y_1, y_2.

In the second case when $|x_1-x_2| > \beta^{1/3}$

$$I_1 \leqslant \int_{\theta_1\subset S_{x_1}(\beta^{1/3}/4),\theta_2\subset S_{x_2}(\beta^{1/3}/4)} dy_1\, dy_2\, P_{x_1y_1}(d\theta_1)\, P_{x_2y_2}(d\theta_2) \cdot \left[\frac{\exp\left\{-\frac{\beta g^2}{(|x_1-x_2|-(\beta^{1/3}/2))^2}-\beta\omega^2\left(x_1^2+x_2^2+\frac{\beta^{2/3}}{2}\right)\right\}-1}{\beta} \cdot \left(|x_1-x_2|+\frac{\beta^{1/3}}{2}\right)^{2\bar\lambda} \times \exp\left\{-\frac{\omega}{2\hbar}\left(\min_\pm\left(x_1\pm\frac{\beta^{1/3}}{4}\right)^2+\min_\pm\left(x_2\pm\frac{\beta^{1/3}}{4}\right)^2\right)\right\}\right]; \tag{3.15}$$

hence, if $M_0(\beta) =$ (maximum of the integrand in (3.15) over (x_1, x_2), $|x_1 - x_2| > \beta^{1/3}$), we find, using again the normalization properties of the measure,

$$I_1 \leqslant M_0(\beta) \leqslant M_0 \tag{3.16}$$

since a straightforward calculation shows that $M_0(\beta)$ is uniformly bounded for $0 < \beta < 1$ if $\bar{\lambda} > 3$. Therefore, Lemma 2 is proved by combining (3.14) and (3.16).

From the above two lemmas we deduce the following theorem.

THEOREM. *If* $\bar{\lambda} > 3$ *the semigroup generated by* K_β *is strongly continuous and coincides with the semigroup* $\exp(-\beta H)$.

Proof. In fact if $f \in \mathscr{D}_{\bar{\lambda},\xi}$ (we consider the case $n = 2$ for simplicity),

$$\begin{aligned}&\frac{(K_\beta f)(x_1x_2) - f(x_1x_2)}{\beta}\\ &\quad= \frac{\int dy_1\, dy_2\, P_{x_1y_1}(d\theta_1)\, P_{x_2y_2}(d\theta_2) \exp[-V(\theta_1\theta_2)]\, f(y_1y_2) - f(x_1x_2)}{\beta}.\end{aligned} \tag{3.17}$$

Using the normalization of the measure,

$$\begin{aligned}&\int dy_1\, dy_2\, P_{x_1y_1}(d\theta_1)\, P_{x_2y_2}(d\theta_2)\, \frac{\exp[-V(\theta_1\theta_2)] - 1}{\beta} f(y_1y_2)\\ &\qquad + \int dy_1\, dy_2\, P_{x_1y_1}(d\theta_1)\, P_{x_2y_2}(d\theta_2) \frac{f(y_1y_2) - f(x_1x_2)}{\beta}.\end{aligned} \tag{3.18}$$

The preceding lemmas imply, as we have seen, that the first integral in (3.18) converges in $L_2(R^n)$ to

$$-\left(\frac{g^2}{(x_1 - x_2)^2} + \frac{\omega^2}{2}(x_1^2 + x_2^2)\right) f(x_1x_2). \tag{3.19}$$

Furthermore, since $\bar{\lambda} > 3 > \frac{3}{2}$, f is in the domain of the Laplacian, and, therefore,

$$\begin{aligned}&\int dy_1\, dy_2\, P_{x_1y_1}(d\theta_1)\, P_{x_2y_2}(d\theta_2) \frac{f(y_1y_2) - f(x_1x_2)}{\beta}\\ &\quad= \int dy_1\, dy_2\, \frac{\exp[-((x_1 - y_1)^2/2\beta\hbar^2)] \exp[-((x_2 - y_2)^2/2\beta\hbar^2)]}{2\pi\beta\hbar^2}\\ &\qquad \times \frac{f(y_1y_2) - f(x_1x_2)}{\beta}\end{aligned}$$

tends, in $L_2(R^n)$, to

$$-\frac{\hbar^2}{2}\left(\frac{\partial^2 f}{\partial x_1}+\frac{\partial^2 f}{\partial x_2}\right)(x_1 x_2).$$

Therefore, the operator K_β is strongly differentiable on the dense set $\mathscr{D}_{\bar\lambda,\xi}$, and its derivates on this domain coincides with $-H$. Since $\| K_\beta \| \leqslant 1$ (see (3.5)) it follows that K_β has a generator and its generator is just $-H$: $K_\beta = \exp(-\beta H)$.

4. The Limit $\hbar \to 0$

The kernel K_β defined in (3.3) is clearly such that

$$K_\beta(x_1, ..., x_n, y_1, ..., y_n) \leqslant K_\beta^{(0)}(x_1, ..., x_n, y_1, ..., y_n), \tag{4.1}$$

where $K^{(0)}$ is the kernel obtained by setting $g = 0$ in (3.3). Clearly $K^{(0)}$, by the Feynman–Kac formula for the harmonic oscillator, is the kernel of the semigroup generated by the operator

$$-\frac{\hbar^2}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}+\frac{\omega^2}{2}\sum_{i=1}^{n} x_i^2. \tag{4.2}$$

This kernel is well known. We need its expression only when $y_1 \cdots y_n = x_1 \cdots x_n$:

$$\begin{aligned} K_\beta^{(0)}&(x_1 \cdots x_n, x_1 \cdots x_n) \\ &= \prod_{i=1}^{n}\sqrt{\frac{\omega\hbar}{\pi(1-\exp(-2\beta\omega\hbar))}} \\ &\quad\cdot \exp\left(-\frac{\beta\omega\hbar}{2}\right)\exp\left\{-\frac{\omega}{\hbar}\left(\frac{1-\exp(-\beta\omega\hbar)}{1+\exp(-2\beta\omega\hbar)}\right)x_i^2\right\}. \end{aligned} \tag{4.3}$$

It is easy to deduce from (4.1), (4.3), and the fact that $K_\beta^{(0)}$ is a trace class operator (for $\beta > 0$) that K_β is a trace class operator and

$$(2\pi\hbar)^n \operatorname{Tr} K_\beta = (2\pi\hbar)^n \operatorname{Tr}\exp(-\beta H) = \int dx_1 \cdots dx_n\, K_\beta(x_1 \cdots x_n, x_1 \cdots x_n). \tag{4.4}$$

Notice (4.3) implies

$$\begin{aligned} (2\pi\hbar)^n & K_\beta^{(0)}(x_1 \cdots x_n, x_1 \cdots x_n) \\ &\leqslant c_1(\beta,\omega)\exp[-c_2(\beta,\omega)(x_1^2+x_2^2+\cdots+x_n^2)], \end{aligned} \tag{4.5}$$

with $C_1(\beta, \omega) < \infty$, $C_2(\beta, \omega) < \infty$, and $\hbar$ independent. Furthermore, if $x_1 \neq x_2$ a computation identical to the one used to prove Lemma 1 in Section 3 gives (using also (3.1))

$$\lim_{\hbar\to 0}(2\pi\hbar)^2 K_\beta(x_1 \cdots x_n, y_2 \cdots y_n) = \frac{1}{((\beta/2\pi)^{1/2})^n} \exp\left\{-\frac{\beta g^2}{2}\sum_{i\neq j}\frac{1}{(x_i - x_j)^2} - \frac{\beta\omega^2}{2}\sum_{i=1}^n x_i^2\right\}, \tag{4.6}$$

hence, since the convergence in (4.6) is dominated by the right side of (4.5), we deduce

$$\lim_{\hbar\to 0}(2\pi\hbar)^n \operatorname{Tr}\exp(-\beta H) = \left(\left(\frac{2\pi}{\beta}\right)^{1/2}\right)^n \cdot \int dx_1 \cdots dx_n \exp\left\{-\frac{\beta g^2}{2}\sum_{i\neq j}\frac{1}{(x_i - x_j)^2} - \frac{\beta\omega^2}{2}\sum_{i=1}^n x_i^2\right\}, \tag{4.7}$$

which is our main result and implies (1.2) as shown in Section 2.

5. A Conjecture

Consider for $q_1 > q_2 > \cdots > q_n$ the classical Hamiltonian

$$\tilde{H}(p, q) = \frac{1}{2}\sum_{i=1}^n p_i^2 + \frac{1}{2}\sum_{i=1}^n \omega^2 q_i^2 + \frac{1}{2}\sum_{i\neq j} g^2 \frac{1}{(q_i - q_j)^2}, \tag{5.1}$$

and assume that there exists a canonical transformation

$$(p_1 \cdots p_n, q_1 \cdots q_n) \to (\eta_1 \cdots \eta_n, \varphi_1 \cdots \varphi_n)$$

such that $\eta_i \geqslant 0$,

$$\hat{H}(\eta, \varphi) = \omega \sum_{k=1}^n k\eta_k + \frac{\omega g}{2} n(n-1), \tag{5.2}$$

and the variables φ_i are "angles" (i.e., $\varphi_i = \varphi_i + 2\pi$).

Then, since the canonical transformations leave the Lebesgue measure invariant,

$$\int_{q_1>q_2>\cdots>q_n} dp_1 \cdots dp_n\, dq_1 \cdots dq_n \exp[-\beta\tilde{H}(p, q)] = \int d\eta_1 \cdots d\eta_n\, d\varphi_1 \cdots d\varphi_n \exp\left\{-\beta\omega\sum_{k=1}^n k\eta_k - \frac{\beta\omega g}{2} n(n-1)\right\} = \exp\left\{-\frac{\omega g}{2} n(n-1)\right\} \frac{1}{n!}\left(\frac{2\pi}{\beta\omega}\right)^n, \tag{5.3}$$

which is the right result.

The conjecture of the existence of a canonical transformation changing (5.1) into (5.2) is verified, so far, only for $n = 2, 3$.

6. Other Integrals

Using Laplace transforms and the knowledge of

$$I_n(\beta^{1/2}g, \beta^{1/2}\omega) = \int \left(\exp\left\{-\frac{\beta g^2}{2}\sum_{i\neq j}\frac{1}{(x_i - x_j)^2} - \frac{\beta\omega^2}{2}\sum_{i=1}^{n} x_i^2\right\}\right) dx_1 \cdots dx_n\,, \tag{6.1}$$

and putting

$$\begin{aligned} V(x_1 \cdots x_n) &= \frac{g^2}{2}\sum_{i\neq j}\frac{1}{(x_i - x_j)^2} + \frac{\omega^2}{2}\sum_{i=1}^{n} x_i^2 \\ h(p_1 \cdots p_n\,, x_1 \cdots x_n) &= \tfrac{1}{2}\sum_{i=1}^{n} p_i^2 + V(x_1 \cdots x_n), \end{aligned} \tag{6.2}$$

one can compute the following integrals

$$\int dx_1 \cdots dx_n \frac{\exp[-\beta V(x_1 \cdots x_n)]}{(V(x_1 \cdots x_n)^s}\,, \tag{6.3}$$

$$\int dx_1 \cdots dx_n\,, dp_1 \cdots dp_n \frac{\exp[-\beta h(x_1 \cdots x_n\,, p_1 \cdots p_n)]}{(h(p_1 \cdots p_n\,, x_1 \cdots x_n))^s}\,. \tag{6.4}$$

For instance,

$$\int_{R^n} dx_1 \cdots dx_n \frac{1}{(V(x_1 \cdots x_n))^s} = \left(\frac{\pi}{2}\right)^{-n/2} \frac{1}{\omega^n} \frac{\Gamma(s - n/2)}{\Gamma(s)} \left(\omega g \frac{n(n-1)}{2}\right)^{(n/2)-s} \tag{6.5}$$

is valid for $\omega > 0$, $g > 0$, $s > n/2$. Also,

$$\begin{aligned} &\int dx_1 \cdots dx_n \frac{\exp[-\beta V(x_1 \cdots x_n)]}{(V(x_1 \cdots x_n))^s} \\ &= \left(\frac{2}{\pi\omega}\right)^n \gamma^{-(2s+n-2)/4} \beta^{s-(n/2)} \exp\left[-\frac{\gamma}{2}\right] W_{(-2s-n+2)/4,(-2s+n)/4} \end{aligned} \tag{6.6}$$

with $\gamma = \beta\omega g(n(n-1)/2)$, $s \geqslant 0$, $\beta > 0$, $\omega > 0$, $g > 0$ and $W_{\mu,\nu}$ are the Whittaker functions.

7. Concluding Remarks

It is natural to ask whether one really has to pass through the calculation of Tr $\exp(-\beta H)$ and the limit as $\hbar \to 0$ to reach the results of the previous sections.

It seems to us that a simply way of obtaining the results in question would be intimately related both to a simpler way of deducing the spectrum of H (which differs only by a constant from that of the case $g = 0$) and to a proof of the conjecture discussed in Section 5. The extreme simplicity of the spectrum of H makes one believe that there might be a simple group theoretic structure behind the theory of the operator H.

Appendix

Assume $q_1 > q_2 > \cdots > q_n$ and put

$$\psi(q_1 \cdots q_n) = Z^{\lambda} P(q_1 \cdots q_n) \exp\left\{-\frac{\omega}{2\hbar} \sum_{i=1}^{n} q_i^2\right\}, \tag{A.1}$$

where

$$Z = \prod_{i=1}^{n} \prod_{j=i+1}^{n} (q_i - q_j). \tag{A.2}$$

Then a straightforward calculation gives

$$\begin{aligned}(H\psi)(q_1 \cdots q_n) &= n\omega\hbar\left(\frac{1}{2} + \lambda\frac{n-1}{2}\right)\psi(q_1 \cdots q_n) + Z^{\lambda} \exp\left\{-\frac{\omega}{2\hbar}\sum_{k=1}^{n} q_k^2\right\} \\ &\times \left\{-\frac{\hbar^2}{2}\sum_{k=1}^{n}\frac{\partial^2 P}{\partial q_k^2} + \omega\hbar \sum_{k=1}^{n} q_k \frac{\partial P}{\partial q_k} - \frac{\lambda\hbar^2}{2}\right. \\ &\left. \cdot \sum_{h\neq k}\left(\frac{(\partial P/\partial q_h) - (\partial P/\partial q_k)}{q_h - q_k}\right)\right\}. \end{aligned} \tag{A.3}$$

We choose

$$P_{k_1 \cdots k_n}(q_1 \cdots q_n) = S\left(\prod_{i=1}^{n} H_{k_i}\left(q_i\left(\frac{\omega}{\hbar}\right)^{1/2}\right)\right),$$

where $H_k(\xi)$ is a Hermite polynomial and S denotes symmetrization and denote $\psi_{k_1 \cdots k_n}$ the corresponding element of $\mathscr{D}^{e}_{\lambda,\xi}$ (see (A.1)).

Using the equation

$$H_k'' - 2\xi H_k = -2kH_k$$

and assuming $q_1 > q_2 > \cdots > q_n$, we find

$$(H\psi_{k_1\cdots k_n})(q_1 \cdots q_n)$$
$$= \epsilon_{k_1\cdots k_n}\psi_{k_1\cdots k_n}(q_1 \cdots q_n)$$
$$- \frac{\lambda\hbar^2}{2} Z^\lambda \exp\left\{-\frac{\omega}{2\hbar}\sum_{i=1}^n q_i^2\right\} \sum_{h\neq k} \frac{1}{q_h - q_k}\left(\frac{\partial P_{k_1\cdots k_n}}{\partial q_h} - \frac{\partial P_{k_1\cdots k_n}}{\partial q_k}\right), \tag{A.4}$$

where

$$\epsilon_{k_1\cdots k_n} = \omega\hbar \sum_{i=1}^n \left(k_i + \frac{1}{2}\right) + \frac{\lambda\omega\hbar}{2} n(n-1).$$

Hence, if we put

$$\tilde{P}_{k_1\cdots k_n}(q_1 \cdots q_n) = P_{k_1\cdots k_n}(q_1 \cdots q_n) + \sum_{\substack{h_1\cdots h_n \\ \sum_{i=1}^n h_i < \sum_{i=1}^n k_i}} c_{h_1\cdots h_n} P_{h_1\cdots h_n}(q_1 \cdots q_n) \tag{A.5}$$

and using the fact that

$$\epsilon_{k_1\cdots k_n} - \epsilon_{h_1\cdots h_n} \geqslant \omega\hbar > 0$$

and the fact that

$$\sum_{h\neq k} \frac{1}{q_h - q_k}\left(\frac{\partial \tilde{P}_{k_1\cdots k_n}}{\partial q_h} - \frac{\partial \tilde{P}_{k_1\cdots k_n}}{\partial q_k}\right) \tag{A.6}$$

is a polynomial of degree at most equal to $\sum_{i=1}^n k_i - 2$, it is easy to see that the coefficients $c_{k_1\cdots k_n}$ can be recursively determined through the equation obtained by equating to zero (A.6). The $\tilde{P}_{k_1\cdots k_n}$ obtained in this way are eigenvectors of H if $\lambda > \frac{3}{2}$.

It is also clear that the polynomials $P_{k_1\cdots k_n}$ can be expressed as finite linear combinations of the new polynomials $\tilde{P}_{k_1\cdots k_n}$. Hence, the completeness of the Hermite polynomials can be easily used to deduce the completeness in $L_2^e(R^n)$ of the functions $\Psi_{k_1\cdots k_n}(q_1 \cdots q_n)$. The above constructions can be repeated in the other sectors $L_2^P(R^n)$.

The above remarks simultaneously prove the essential self-adjointens of H on $\bigcup_P \mathscr{D}_{\lambda,\xi}^P$ for $\lambda > \frac{3}{2}$ and the remaining statements of Theorem 1.

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