JOURNAL OF MATHEMATICAL PHYSICS VOLUME 11, NUMBER 1 JANUARY 1969

Absence of Phase Transitions in Hard-Core One-Dimensional Systems with Long-Range Interactions

G. GALLAVOTTI
Rockefeller University, New York City, New York 10021

AND

S. MIRACLE-SOLE
Institut des Hautes Etudes Scientifiques, 91 *Bures-sur-Yvette, France*

(Received 22 November 1968)

We prove the impossibility of phase transitions for a class of infinite-range potentials extending recent analogous results. We prove also a cluster property for the equilibrium state $\bar{\gamma}$ and apply some collateral results to describe, in the case of finite-range interactions, the state $\bar{\gamma}$ in terms of a family of density distributions, and to verify a general variational property of $\bar{\gamma}$.

1. INTRODUCTION

It has been conjectured that even in one-dimensional systems a phase transition can occur if the range of the potential in infinite (in the sense that the first moment of the potential diverges) and the potential is attractive.[1] [*Note added in proof:* A proof of this statement has been given by F. J. Dyson in Commun. Math. Phys. **12,** 91 (1969).]

On the contrary, if the potential has infinite range but the first moment is convergent, it is commonly accepted that, at least if the potential is sufficiently regular, no phase transitions occur in one dimension[2]; in this case one can also conjecture that the pressure has some analyticity properties with respect to the chemical potential and the temperature, and also that the correlation functions have some cluster property.

Most of these conjectures have been proved to be true by Ruelle in the case of lattice gases[3]; using his technique, it has been possible to obtain similar results in the continuous hard-core case.[4] In fact, the possibility of phase transitions has been excluded for the class of continuous bounded-pair potentials, bounded in absolute value by a decreasing function $\varphi(x)$ such that

$$\int_a^{+\infty} \varphi(x)x\,dx < +\infty, \tag{1}$$

and this extends considerably the well-known results of Van Hove.[2]

In this paper we present an extension of these results to more general unbounded and not necessarily continuous potentials involving two or more bodies; we also prove a cluster property for the correlation functions and, using collateral results, we give a description of the equilibrium state in the case of finite-range forces in terms of a family of density distribution, and we verify an extremum property of the equilibrium state.

2. DESCRIPTION OF THE SYSTEM; RESULTS

Let us consider a one-dimensional system of hard rods. A configuration will be represented by the set X of the points of the real axis R occupied by the centers of the rods. If $a > 0$ is the length of the rods, then X is restricted to verify the condition $|x - x'| \geq a$ if $x, x' \in X$, $x \neq x'$.

We say that a sequence $\{X_\alpha\}$ of configurations tends as $\alpha \to \infty$ to the limit configuration X_0 if, for every bounded interval (a, b) such that $a \notin X_0$, $b \notin X_0$, the configuration $X_\alpha \cap (a, b)$ tends point by point to $X_0 \cap (a, b)$.

With this definition of convergence the set K of all the configurations (finite or not) becomes a compact space. It will also be useful to consider the compact subsets K_+, $K_- \subset K$ built up, respectively, with the configurations of K contained in $R_- = (-\infty, 0]$ and in $R_+ = [0, +\infty)$. The symbols $C(K)$, $C(K_+)$, and $C(K_-)$ will denote the continuous functions, respectively, over K, K_+, and K_-; $C_{[0,b)}$ $(b > 0)$ will denote the set of continuous functions in $C(K_+)$, with the property that $f(Y) = f(Y \cap [0, b))$ for all $Y \in K_+$. It can be shown that the set $\bigcup_{b>0} C_{[0,b)}$ is dense in $C(K_+)$ [in the sup norm on $C(K_+)$].

Suppose the rods interact through symmetric translationally invariant many-body potentials $\Phi^{(k)}(x_1, \cdots, x_k)$ and consider these as a function Φ on the configurations $X \in K$ defined as $\Phi(X) = \Phi^{(k)}(x_1, \cdots, x_k)$ if $X = \{x_1, \cdots, x_k\}$ $0 < k < \infty$ and $\Phi(X) = 0$ if $k = 0, +\infty$; the one-particle potential $\Phi^{(1)}$ has to be interpreted as minus the chemical potential, so the

[1] M. E. Fisher, Physics 3, 255(1967); M. Kac, "Mathematical Mechanism of Phase Transitions" (to be published).
[2] L. Van Hove, Physica **16,** 137 (1950).
[3] D. Ruelle, Commun. Math. Phys. **9,** 267 (1968).
[4] G. Gallavotti, S. Miracle-Sole, and D. Ruelle, Phys. Letters **26A,** 350 (1968).

energy of a configuration is

$$U(X) = \sum_{S \subset X} \Phi(S). \tag{2}$$

We want to allow the potential Φ to be of the form

$$\Phi = \tilde{\Phi}_0 + \tilde{\Phi}, \tag{3}$$

where $\tilde{\Phi}_0$ is a nonnegative measurable finite-range pair potential (not necessarily bounded), while $\tilde{\Phi}$ is supposed to belong to the real space $\mathfrak{B}$ defined as the closure in the norm (4) below of the set $\mathfrak{B}_0$ of the real finite-range bounded measurable potentials with the "continuity" property that if $0 \leq h \leq k-1$ and $\bar{x}_1, \cdots, \bar{x}_h$ are fixed, then the function $\Phi^{(k)}(x_1, \cdots, x_h, x_{h+1}, \cdots, x_k)$ is continuous in the variables $x_1, \cdots, x_h$ at the point $\bar{x}_1, \cdots, \bar{x}_h$ for almost all $x_{h+1}, \cdots, x_k$ (which is a further restriction on $\tilde{\Phi}^{(k)}$ only for $k \geq 3$). The norm with respect to which the closure has to be taken is

$$\|\tilde{\Phi}\| = \sup_{X \in P} \sum_{\substack{T \subset X \\ T \cap R_+ \neq \varnothing \neq T \cap R_-}} |\Phi(T)|. \tag{4}$$

One can convince himself that the condition that $\tilde{\Phi}$ should be in some closure of finite-range potentials can be interpreted as a condition of decrease at large distances and the fact that this closure has to be taken with respect to the norm (4) is a condition similar to that of having a finite first moment. In fact, one can see that the requirement that $\Phi \in \mathfrak{B}$ is in general less restrictive than the requirement that $\tilde{\Phi}$ verifies simultaneously a condition of type (1) and a decreasing condition in the sense usually found in the literature.[5] The "continuity" condition imposed on the potentials is probably unnecessary and simple measurability should be sufficient. In view of technical difficulties we shall not deal with this point.

The thermodynamics of the system is described by the partition function (grand canonical ensemble):

$$Z_{(b_1,b_2)}(\Phi) = \int_{X \subset (b_1,b_2)} e^{-U(X)}\, dX, \tag{5}$$

where

$$\int \cdot \, dX = \sum_{n \geq 0} \int \cdot \frac{dx_1 \cdots dx_n}{n!}$$

and the inverse temperature factor β has been included in the interaction energy.

The main result of this paper is the theorem below, which will be proved in Sec. 4, using the preliminary lemmas of Sec. 3, following the scheme used in Ref. 3 to prove the analogous results in the lattice case.

[5] M. E. Fisher, Arch. Ratl. Mech. Anal. 17, 377 (1964).

Theorem: Let the interaction potential Φ be given by $\Phi = \tilde{\Phi}_0 + \tilde{\Phi}$, where $\tilde{\Phi}_0$ is a nonnegative finite-range pair potential and $\tilde{\Phi} \in \mathfrak{B}$. Then:

(i) The function

$$P(\Phi) = \lim_{|b_2 - b_1| \to \infty} |b_2 - b_1|^{-1}\, lg Z_{(b_1,b_2)}(\Phi)$$

has the property that, given $\tilde{\Phi}_1, \cdots, \tilde{\Phi}_n \in \mathfrak{B}$, then $P(\sum_{i=0}^n \lambda_i \tilde{\Phi}_i)$ is continuously differentiable in $\lambda_0 \in [0, \infty)$ and $\lambda_1, \cdots, \lambda_n \in (-\infty, +\infty)$.

(ii) If $\bar{\gamma}$ denotes the probability measure (on the set K of all the configurations) which represents the equilibrium state relative to the potential Φ, and if $A_1, A_2, \cdots, A_n \in C(K)$ and τ_x denotes the translation operator on $C(K)$ [defined as $(\tau_x A)(Y) = A(Y + x)$ for $x \in (-\infty, +\infty)$, $A \in C(K)$], then $\bar{\gamma}$ is translationally invariant and

$$\lim_{\substack{|x_i - x_j| \to \infty \\ i \neq j}} \bar{\gamma}(\tau_{x_1} A_i \cdot \tau_{x_2} A_2 \cdot \cdots \cdot \tau_{x_n} A_n) = \bar{\gamma}(A\ ) \cdots \bar{\gamma}(A_n). \tag{6}$$

This theorem excludes the possibility not only of first-order phase transitions, but also those of higher order in the sense that (i) and (ii) imply not only that the density is continuous as a function of the temperature and the chemical potential, but also that the equilibrium state is unique (given the interaction), and that all the correlation functions are continuous with respect to the interaction potentials and do not exhibit long-range order.

3. STUDY OF A SEMI-INFINITE SYSTEM

In this section the symbols m, n, r, and k will denote nonnegative integers and the symbol τ_x will mean the translation operator on the set of configurations $X \in K$ defined by $\tau_x X = X + x$.

Given Φ as discussed in the preceding section, we define a family of operators which map $C(K_+)$ into itself linearly and continuously with respect to the sup norm with which we suppose $C(K_+)$ to be equipped. These operators are defined for all $x \geq 0$ as

$$(\mathfrak{L}_x f)(Y) = \int_{X \subset [0,x)} e^{-U(X \mid \tau_x Y)} f(X \cup \tau_x Y)\, dX, \quad \text{for } Y \in K_+, \tag{7}$$

where $U(R \mid W)$ is defined for R and W in K as

$$\begin{aligned} U(R \mid W) &= \sum_{\substack{\varnothing \neq S \subset R \\ T \subset W}} \Phi(S \cup T) \\ &= \sum_{\substack{\varnothing \neq S \subset R \\ \varnothing \neq T \subset W}} \Phi(S \cup T) + \sum_{\varnothing \neq S \subset R} \Phi(S) \\ &= I(R \mid W) + U(R), \end{aligned} \tag{8}$$

and where $\Phi(S \cup T)$ has to be taken $+\infty$ if $S \cup T \notin K$. One verifies, as a consequence of the assumed continuity properties of Φ, that $\mathfrak{L}_x$ maps $C(K_+)$ into itself and $\|\mathfrak{L}_x\| \leq e^{\|\tilde{\Phi}\|} Z_{[0,x)}$.

The adjoint operators to $\mathfrak{L}_x$ are operators on the space $C(K_+)^* = \{\text{space of bounded measures on } K_+\} = \{\text{dual of } C(K_+)\}$:

$$(\mathfrak{L}_x^*\mu)(dX\, d\tau_x Y) = e^{-U(X \mid \tau_x Y)}\, dX \mu(dY), \qquad X \subset [0, x), \quad Y \in K_+. \quad (9)$$

A simple calculation based on (7) shows that

$$\mathfrak{L}_x \mathfrak{L}_y = \mathfrak{L}_{x+y}, \quad \mathfrak{L}_x^* \mathfrak{L}_y^* = \mathfrak{L}_{x+y}^*, \quad x, y \geq 0. \quad (10)$$

Let us call $\bar{\gamma}_+$ the probability measure on K_+ which describes the equilibrium state of a semi-infinite system (contained in R_+); then, formally, one expects that $\mathfrak{L}_x^* \bar{\gamma}_+ = e^{xP} \bar{\gamma}_+$, where P is the thermodynamic pressure and also (if there are no phase transitions) that $\bar{\gamma}_+$ is unique. Two steps towards the proof of such a property are lemmas 1 and 2:

Lemma 1: There exists a probability measure $\nu \in C(K_+)^*$ and $\lambda \geq 1$ such that

$$\mathfrak{L}_a^* \nu = \lambda \nu, \quad (11)$$

where a denotes the hard rod length.

In fact, let E be the set of probability measures on K_+; then, since $\mathfrak{L}_a^* \mu(1) \geq 1$ if $\mu \in E$, the mapping of E into itself defined by $\mu \to [\mathfrak{L}_a^* \mu(1)]^{-1}\, \mathfrak{L}_a^* \mu$ is unambiguously defined and weakly continuous. The set E being convex and weakly compact, the Schauder–Tychonoff theorem[6] applies to give a fixed point $\nu \in E$ which verifies (11).

Lemma 2: There exists $C_0 > 0$ such that, for all integers $n \geq 0$,

$$C_0 e^{-\|\tilde{\Phi}\|} \leq \lambda^n Z_{[0,na)}^{-1} \leq e^{\|\tilde{\Phi}\|}; \quad (12)$$

hence it follows that $\lambda = e^{aP}$, where P is the thermodynamic pressure.

We have, if 1 denotes the function of $C(K_+)$ identically equal to unity,

$$(\mathfrak{L}_{na} 1)(Y) \leq \int_{X \subset [0,na)} e^{-U(X)} e^{\|\tilde{\Phi}\|}\, dX = e^{\|\tilde{\Phi}\|} Z_{[0,na)}. \quad (13)$$

We have also, if ra is greater than the range of $\tilde{\Phi}_0$,

$$(\mathfrak{L}_{na} 1)(Y) \geq \int_{X \subset [0,na)} e^{-U(X)} e^{-\|\tilde{\Phi}\|}\, dX \bar{f}_r(Y) = \bar{f}_r(Y) e^{-\|\tilde{\Phi}\|} Z_{[0,na)}, \quad (14)$$

where the function $\bar{f}_r(Y)$ is defined to be 1 if $Y \subset [2ra, +\infty)$ and zero otherwise. This function is a Borel function, so we can integrate (13) and (14) with respect to ν and obtain (12) with $C_0 = \nu(\bar{f}_r)$. To prove that $C_0 > 0$ let m be an integer, $m > 2r$; then

$$(\lambda^{-m} \mathfrak{L}_{ma} \bar{f}_r)(Y) = \lambda^{-m} \int_{[0,ma)} e^{-U(X \mid \tau_{ma} Y)} \bar{f}_r(X)\, dX \geq \lambda^{-m} \int_{[ra,2ra)} e^{-U(X)}\, dX e^{-\|\tilde{\Phi}\|}. \quad (15)$$

Hence,

$$\nu(\bar{f}_r) = \nu(\lambda^{-m} \mathfrak{L}_{ma} \bar{f}_r) \geq \lambda^{-m} e^{-\|\tilde{\Phi}\|} Z_{[ra,2ra)} > 0.$$

We now want to show that ν is the unique solution of the eigenvalue problem $\mathfrak{L}_a^* \nu = e^{aP} \nu$. We observe that the "matrix elements" of $\mathfrak{L}_a^*$ are all nonnegative, so it is tempting to try to obtain unicity on the same lines of the proof of the Frobenius theorem for finite matrices.[7] The key for that theorem is the study of the adjoint eigenvalue problem, which in our case would be $\mathfrak{L}_a h = \lambda h$; this problem is solved by means of the lemmas below.

Lemma 3: If $f \in C(K_+)$ and $\nu(|f|) = 0$, then $\|f\| = 0$. Hence, if $\{f_n\}$ is a conditionally compact sequence of elements of $C(K_+)$, from the limit $\lim \nu(|f_n|) = 0$ as $n \to \infty$, we can deduce that $\lim \|f_n\| = 0$ as $n \to \infty$.

Let $f \geq 0$ and $\nu(f) = 0$. Suppose $f \not\equiv 0$; thus there exists $\bar{Y} \in K_+$ and $f(\bar{Y}) > 0$. Given $\epsilon < \frac{1}{2} f(\bar{Y})$, one can find, because of the continuity of f and the nature of the topology on K_+, a $k > 0$ and a nonempty open set $G \subset K_+$ such that

$$Y \in G \leftrightarrow Y \cap [0, ka) \in G,$$
$$Y \in G \Rightarrow |f(Y) - f(\bar{Y})| < \epsilon.$$

If χ_G is the characteristic function of G, we have

$$\nu(f) \geq \nu(f\chi_G) \geq \epsilon \nu(\chi_G). \quad (16)$$

To prove that $\nu(\chi_G) > 0$, let n be an integer greater than $(k + r)$ where r is chosen such that ra is greater than the range of $\tilde{\Phi}_0$. Then

$$\lambda^{-n} \mathfrak{L}_{na} \chi_G(Y) = \lambda^{-n} \int_{X \subset [0,na)} e^{-U(X \mid \tau_{na} Y)} \chi_G(X)\, dX \geq \lambda^{-n} e^{-\|\tilde{\Phi}\|} \int_{X \subset [0,ka)} e^{-U(X)} \chi_G(X)\, dX > 0,$$

[6] N. Dunford and J. Schwartz, *Linear Operators* (Interscience Publishers, Inc., N.Y., 1958), Vol. I, Chap. V, Sec. 10, item 5.

[7] D. Ruelle, *Statistical Mechanics. Rigorous Results* (W. A. Benjamin, Inc., N.Y., 1969), Chap. IV, Sec. 6, item 3.

so that

$$\nu(\chi_G) = \nu(\lambda^{-n}\mathfrak{L}_{na}\chi_G) \geq \lambda^{-n}e^{-\|\tilde\Phi\|}\int_{[0,ka)} e^{-U(X)}\chi_G(X)\,dX > 0.$$

Lemma 4: If r is so chosen that ra is greater than the range of $\tilde\Phi_0$ and if $g \in C_{[0,ma)}$, there exists $A > 0$ such that

$$\mathfrak{L}_{na}g(Y)/\mathfrak{L}_{na}g(Y') \leq A, \quad \text{for all} \quad Y, Y' \in K_+ \text{ and } n > m + 2r. \quad (17)$$

In fact, if $n > m + 2r$, using Lemma 2, we have

$$\frac{\mathfrak{L}_{na}g(Y)}{\mathfrak{L}_{na}g(Y')} = \left(\int_{X\subset[0,na)} dXe^{-U(X)}g(X)e^{-I(X|\tau_{na}Y)}\right)(Y\to Y')^{-1}$$
$$= \left(\int_{X_1\subset[0,ma)} e^{-U(X_1)}g(X_1)\,dX_1\int_{X_2\subset[ma,na)} e^{-U(X_2)}\,dX_2 \exp[-I(X_1\,|\,X_2\cup\tau_{na}Y)] \times e^{-I(X_2|\tau_{na}Y)}\right)(Y\to Y')^{-1}$$
$$\leq \sup_{X_1\subset[0,ma)}\left(\int_{X_2\subset[ma,na)} dX_2e^{-U(X_2)}\exp[-I(X_1\,|\,X_2\cup\tau_{na}Y)] \times e^{-I(X_2|\tau_{na}Y)}\right)(Y\to Y')^{-1}$$
$$\leq \frac{e^{2\|\tilde\Phi\|}\int_{X_2\subset[ma,na)} dX_2e^{-U(X_2)}}{e^{-2\|\tilde\Phi\|}\int_{X_2\subset[(m+r)a,(n-r)a)} dX_2e^{-U(X_2)}} \leq e^{4\|\tilde\Phi\|}\frac{Z_{[0,(n-m)a)}}{Z_{[0,(n-m-2r)a)}} \leq \frac{e^{6\|\tilde\Phi\|}}{C_0}\lambda^{2r},$$

where $Y\to Y'$ means the same term as in the numerator with Y replaced by Y'.

Lemma 5: If $f\in C_{[0,ma)}$ and $\nu(f) = 0$, then, if r denotes an integer such that ra is greater than the range of Φ_0, we have

$$\nu(|\lambda^{-n}\mathfrak{L}_{na}f|) \leq (1 - A^{-1})\nu(|f|), \quad n > m + 2r. \quad (18)$$

In fact, if f' is any positive function in $C_{[0,ma)}$, then, using the preceding lemma, we find, for $n > m + 2r$,

$$\lambda^{-n}\mathfrak{L}_{na}f' \geq \inf \lambda^{-n}\mathfrak{L}_{na}f' \geq A^{-1}\sup\lambda^{-n}\mathfrak{L}_{na}f' \geq A^{-1}\nu(\lambda^{-n}\mathfrak{L}_{na}f') = A^{-1}\nu(f'). \quad (19)$$

Now let $f\in C_{[0,ma)}$ $\nu(f) = 0$ and let f_+, f_- be, respectively, the positive and negative parts of f [i.e., $f_+ = \sup(f, 0), f_- = \sup(-f, 0)$]; then, using $\nu(f_+) = \nu(f_-)$, $\nu(|f|) = \nu(f_+) + \nu(f_-)$, and Eq. (19), we find

$$\nu(|\lambda^{-n}\mathfrak{L}_{na}f|)$$
$$= \nu(|\lambda^{-n}\mathfrak{L}_{na}f_+ - \lambda^{-n}\mathfrak{L}_{na}f_-|)$$
$$= (\nu|\lambda^{-n}\mathfrak{L}_{na}f_+ - A^{-1}\nu(f_+)) - (\lambda^{-n}\mathfrak{L}_{na}f_- - A^{-1}\nu(f_-))|$$
$$\leq \nu(\lambda^{-n}\mathfrak{L}_{na}f_+ - A^{-1}\nu(f_+)) + \nu(\lambda^{-n}\mathfrak{L}_{na}f_- - A^{-1}\nu(f_-))$$
$$= (1 - A^{-1})\nu(|f|).$$

Lemma 6: If $f\in C(K_+)$, then, given a positive integer N, there exists an integer $m(N)$ such that all the functions $\lambda^{-n}\mathfrak{L}_{na}f$ with $n \geq m(N)$ can be approximated within $1/N!$ by functions $f_n \in C_{[0,m(N)a)}$; i.e., for $n > m(N)$, there exists $f_n \in C_{[0,m(N)a)}$ and

$$\|\lambda^{-n}\mathfrak{L}_{na}f - f_n\| < 1/N!. \quad (20)$$

Since $\|\lambda^{-n}\mathfrak{L}_{na}\| \leq e^{2\|\tilde\Phi\|}/C_0$ (Lemma 2), it is sufficient to prove Lemma 6 for $f\in C_{[0,ma)}$. Let $f\in C_{[0,ma)}$; then, if $n > m$,

$$\lambda^{-n}\mathfrak{L}_{na}f(Y) = \lambda^{-n}\int_{[0,na)} e^{-U(X)}e^{-I(X|\tau_{na}Y)}f(X)\,dX.$$

Now denote Φ_k the potential obtained from Φ by replacing $\tilde\Phi$ with a potential $\tilde\Phi_k \in \mathcal{B}$ with range ka, and such that $\|\tilde\Phi - \tilde\Phi_k\| \to 0$ as $k\to\infty$ and $\|\tilde\Phi_k\| \leq \|\tilde\Phi\|$. If r denotes, as everywhere in the paper, an integer such that ra is greater than the range of $\tilde\Phi_0$, let m (N) be an integer greater than $2r$ and m and such that

$$\|\tilde\Phi - \tilde\Phi_{m(N)}\| < \frac{1}{N!}C_0e^{-2\|\tilde\Phi\|}\|f\|^{-1}.$$

Then for $n > m(N)$ define

$$f_n(Y) = \lambda^{-n}\int_{[0,na)} e^{-U(X)}f(X)\,dX \times \exp[-I(X\,|\,\tau_{na}(Y\cap[0, m(N)a))].$$

We now have $f_n \in C_{[0,m(N)a)}$ and

$$|\lambda^{-n}\mathfrak{L}_{na}f(Y) - f_n(Y)|$$
$$\leq \lambda^{-n}\|f\|\int_{[0,na)} e^{-U(X)}\,dX \times |\exp[-I(X\,|\,\tau_{na}(Y\cap[0, m(N)a))] - e^{-I(X|\tau_{na}Y)}|$$
$$\leq \lambda^{-n}\|f\|\int_{[0,na)} e^{-U(X)}\,dXe^{\|\tilde\Phi\|}\|\tilde\Phi - \tilde\Phi_{m(N)}\|$$
$$\leq \lambda^{-n}Z_{[0,na)}\|f\|\,e^{\|\tilde\Phi\|}\|\tilde\Phi - \tilde\Phi_{m(N)}\| \leq \frac{1}{N!}.$$

Lemma 7: If $f\in C(K_+)$ and $\nu(f) = 0$, then

$$\lim_{n\to\infty}\nu(|\lambda^{-n}\mathfrak{L}_{na}f|) = 0. \quad (21)$$

Let $m(N)$ be the number defined in the preceding lemma and k be an arbitrary integer. Let

$$n > 2\sum_{i=1}^{k}(m(N+i)+2r).$$

One has $n - m(N+k) - 2r > m(N+k) + 2r$, so using Lemma 6, we can approximate within $1/(N+k)!$ the function $\lambda^{-(n-m(N+k)-2r)}\mathfrak{L}_{(n-m(N+k)-2r)a}f$ with a function $\bar{f}_k \in C_{[0,m(N+k)a)}$; then, using Lemma 5 and the bound $\|\lambda^{-s}\mathfrak{L}_{sa}\| \le e^{2\|\tilde{\Phi}\|}/C_0$ (consequence of Lemma 2), we have

$$\begin{aligned}
\nu(|\lambda^{-n}\mathfrak{L}_{na}f|) &= \nu(|\lambda^{-(m(N+k)+2r)}\mathfrak{L}_{(m(N+k)+2r)a}\lambda^{-(n-m(N+k)-2r)} \\
&\quad\times \mathfrak{L}_{(n-m(N+k)-2r)a}f|) \\
&\le \frac{e^{2\|\tilde{\Phi}\|}}{C_0}\frac{1}{(N+k)!} + (1-A^{-1})\nu(|\bar{f}_k|) \\
&\le \frac{1}{(N+k)!}\left(\frac{e^{2\|\tilde{\Phi}\|}}{C_0} + (1-A^{-1})\right) \\
&\quad + (1-A^{-1})\nu(|\lambda^{-(n-m(N+k)-2r)}\mathfrak{L}_{(n-m(N+k)-2r)a}f|);
\end{aligned}$$

and since $n - m(N+k) - 2r > 2\sum_{i=1}^{k-1}(m(N+i) + 2k)$, we can iterate the above procedure and find

$$\nu(|\lambda^{-n}\mathfrak{L}_{na}f|) \le 1\left(\frac{e^{2\|\tilde{\Phi}\|}}{C_0} + (1-A^{-1})\right)\sum_{i=0}^{k-1}\frac{(1-A^{-1})^i}{(N+k-i)!} + (1-A^{-1})^k\frac{e^{2\|\tilde{\Phi}\|}}{C_0}\|f\|.$$

This proves the lemma since N and k were arbitrary.

Lemma 8: If $f \in C(K_+)$, the following limit holds uniformly in n:

$$\lim_{Y'\to Y}\lambda^{-n}\mathfrak{L}_{na}f(Y') = \lambda^{-n}\mathfrak{L}_{na}f(Y). \tag{22}$$

Since $\|\lambda^{-n}\mathfrak{L}_{na}\| \le e^{2\|\tilde{\Phi}\|}C_0^{-1}$, it will be sufficient to prove this lemma in the case $f \in C_{[0,ma)}$. Suppose $f \in C_{[0,ma)}$ and $n > m$; let Φ_k, $\tilde{\Phi}_k$ be the potentials introduced in the proof of Lemma 6. For fixed $k > r$ (where r is such that ra is greater than the range of $\tilde{\Phi}_0$), we have

$$\begin{aligned}
&|\lambda^{-n}\mathfrak{L}_{na}f(Y) - \lambda^{-n}\mathfrak{L}_{na}f(Y')| \\
&\le \|f\|\,\lambda^{-n}\int_{[0,na)} e^{-U(X)}\,dX \times |e^{-I(X\,|\,\tau_{na}Y)} - e^{-I(X\,|\,\tau_{na}Y')}| \\
&\le \lambda^{-n}\|f\|\int_{[0,na)} e^{-U(X)}\,dX \\
&\quad\times|\exp[-I_\Phi(X\,|\,\tau_{na}Y)] - \exp[-I_{\Phi_k}(X\,|\,\tau_{na}Y)]| \\
&\quad+ \lambda^{-n}\|f\|\int_{[0,na)} e^{-U(X)}\,dX \\
&\quad\times|\exp[-I_\Phi(X\,|\,\tau_{na}Y')] - \exp[-I_{\Phi_k}(X\,|\,\tau_{na}Y')]| \\
&\quad+ \lambda^{-n}\|f\|\int_{[0,na)} dX e^{-U(X)} \\
&\quad\times|\exp[-I_{\Phi_k}(X\,|\,\tau_{na}Y)] - \exp[-I_{\Phi_k}(X\,|\,\tau_{na}Y')]| \\
&\le 2\|f\|\,\lambda^{-n}Z_{[0,na)}e^{\|\tilde{\Phi}\|}\,\|\tilde{\Phi} - \tilde{\Phi}_k\| \\
&\quad+ \lambda^{-(n-k)}Z_{[0,(n-k)a)}e^{\|\tilde{\Phi}\|}\,\|f\| \\
&\quad\times\lambda^{-k}\int_{[(n-k)a,na)}|\exp[-I_{\Phi_k}(X\,|\,\tau_{na}Y)] \\
&\quad- \exp[-I_{\Phi_k}(X\,|\,\tau_{na}Y')]|\,e^{-U(X)}\,dX,
\end{aligned}$$

when $\lambda^{-(n-k)}Z_{[0,(n-k)a)}$ has to be taken equal to one if $n - k \le 0$; now using Lemma 2, the chain of inequalities ends as

$$\begin{aligned}
&\le 2\|f\|\frac{e^{2\|\tilde{\Phi}\|}}{C_0}\|\tilde{\Phi} - \tilde{\Phi}_k\| \\
&\quad+ \|f\|\frac{e^{2\|\tilde{\Phi}\|}}{C_0}\lambda^{-k}\int_{[0,ka)}|\exp[-I_{\Phi_k}(X\,|\,\tau_{ka}Y)] \\
&\quad- \exp[-I_{\Phi_k}(X\,|\,\tau_{ka}Y')]|\,e^{-U(X)}\,dX,
\end{aligned} \tag{22'}$$

and this proves the lemma because the function of Y, Y' appearing in the last inequality tends to zero as $Y' \to Y$ (consequence of the continuity properties of the potentials) and because $\lim\|\tilde{\Phi} - \tilde{\Phi}_k\| = 0$.

Remark: This lemma implies that the set of functions $\{\lambda^{-n}\mathfrak{L}_{na}f\}$ is equicontinuous and (since $\|\lambda^{-n}\mathfrak{L}_{na}\| \le e^{2\|\tilde{\Phi}\|}C_0^{-1}$) norm bounded, so it is conditionally compact[8]; in particular, there exists a subsequence $\{n_i\}$ and a function $h \in C(K_+)$ (depending on f and $\{n_i\}$) such that

$$\lim_{i\to\infty}\|\lambda^{-n_i}\mathfrak{L}_{n_ia}f - h\| = 0. \tag{23}$$

Lemma 9: There exists $h \in C(K_+)$, $h \ge 0$, $\nu(h) = 1$, and

(i) $$\lambda^{-1}\mathfrak{L}_a h = h, \tag{24}$$

(ii) $$\lim_{n\to\infty}\|\lambda^{-n}\mathfrak{L}_{na}f - \nu(f)h\| = 0,\quad f \in C(K_+), \tag{25}$$

(iii) $$\lim_{n\to\infty}\lambda^{-n}\mathfrak{L}^*_{na}\mu - \mu(h)\nu = 0,\quad \mu \in C(K_+)^*, \tag{26}$$

where the limit holds in the weak sense.

In fact, consider the function $g = 1 - \lambda^{-1}\mathfrak{L}_a 1$; we have $\nu(g) = 0$. Thus, using Lemmas 7 and 3, we find

$$\lim_{n\to\infty}\|\lambda^{-n}\mathfrak{L}_{na}(1 - \lambda^{-1}\mathfrak{L}_a 1)\| = 0, \tag{27}$$

and then (27) and (23) imply $\|h - \lambda^{-1}\mathfrak{L}_a h\| = 0$. Since clearly $\nu(h) = 1$, $h \ge 0$, (i) is proved.

[8] N. Dunford and J. Schwartz, *Linear Operators* (Interscience Publishers, Inc., N.Y., 1958), Vol. I, Chap. IV, Sec. 6, item 7.

To prove (ii) consider a function $f \in C(K_+)$ and define $\tilde{g} \in C(K_+)$ as $\tilde{g} = f - \nu(f)h$; clearly $\nu(\tilde{g}) = 0$. Thus, using Lemmas 7 and 3,

$$0 = \lim_{n\to\infty} \|\lambda^{-n}\mathfrak{L}_{na}g\| = \lim_{n\to\infty} \|\lambda^{-n}\mathfrak{L}_{na}f - \nu(f)h\|.$$

Part (iii) is simply the dual statement of (ii).

Remark 1: (ii) implies that h is the unique solution of the equation $e^{+aP}h = \mathfrak{L}_a h$. Since the commutativity of the operators $\mathfrak{L}_x$ implies that also $\mathfrak{L}_x h$ is a solution of the same equation, we must have $\mathfrak{L}_x h = \lambda(x)h$, and (10) implies $\lambda(x)\lambda(y) = \lambda(x + y)$ $x, y \geq 0$; we have also that $\lambda(x)$ is a finite-valued continuous function of x because $(\mathfrak{L}_x f)(Y)$, as is easily verified, is a finite-valued continuous function of x at fixed Y. Thus[9] $\lambda(x) = e^{xP}$.

An analogous argument holds for ν which turns out to be the unique solution of the equation $\mathfrak{L}_x^* \nu = e^{xP}\nu$.

Remark 2: one can drop in (ii) and (iii) the condition that n is an integer. This is a consequence of the following inequality, holding for n integer and $0 < x < a$:

$$\begin{aligned}\|e^{-naP}\mathfrak{L}_{na}e^{-xP}\mathfrak{L}_x f - \nu(f)h\| &= \|e^{-xP}\mathfrak{L}_x(\mathfrak{L}_{na}e^{-anP}f - \nu(f)h)\| \\ &\leq e^{\|\tilde{\Phi}\|}Z_{[0,a)}\,\|\mathfrak{L}_{na}e^{-naP}f - \nu(f)\,h\|.\end{aligned}$$

4. PROOF OF THEOREM 1

To prove differentiability of $P(\Phi)$ it is necessary to study the dependence of ν and h on Φ. Let $\tilde{\Phi}'$ be either $\tilde{\Phi}_0$ or an arbitrary element in $\mathcal{B}$. Let us consider the potential $\Phi + z\tilde{\Phi}'$. Let us also emphasize the dependence of $\mathfrak{L}_\alpha$, h, ν, and λ on Φ by writing $\mathfrak{L}_{a,\Phi}$, h_Φ, ν_Φ, and λ_Φ. From the proof of Lemma 8 and from the continuity of $\lambda_{\Phi+z\tilde{\Phi}'}$ in z (which follows[9] from the convexity properties of λ_Φ as a function of Φ), it follows easily that the limit

$$\lim_{Y'\to Y} \lambda_{\Phi+z\tilde{\Phi}'}^{-n}\mathfrak{L}_{na,\Phi+z\tilde{\Phi}'}f(Y') = \lambda_{\Phi+z\tilde{\Phi}'}^{-n}\mathfrak{L}_{na,\Phi+z\tilde{\Phi}'}f(Y) \tag{28}$$

holds uniformly not only in n, but also in z for z in a bounded interval. This implies that (Lemma 9) the limit

$$\lim_{Y'\to Y} \nu_{\Phi+z\tilde{\Phi}'}(f)h_{\Phi+z\tilde{\Phi}'}(Y') = \nu_{\Phi+z\tilde{\Phi}'}(f)h_{\Phi+z\tilde{\Phi}'}(Y) \tag{29}$$

holds uniformly for z in a bounded interval. From this equicontinuity property it follows that if $\{z_n\}$ is a sequence $z_n \xrightarrow[n\to\infty]{} z_0$, there exists a subsequence $\{n_i\}$ such that the limits

$$\lim_{i\to\infty} \nu_{\Phi+z_{n_i}\tilde{\Phi}'}(f), \tag{30}$$

$$\lim_{i\to\infty} \nu_{\Phi+z_{n_i}\tilde{\Phi}'}(f)h_{\Phi+z_{n_i}\tilde{\Phi}'} \tag{31}$$

exist (the second in the norm sense). Since $\mathfrak{L}_{\Phi+z\tilde{\Phi}'}$ and $\lambda_{\Phi+z\tilde{\Phi}'}$ depend continuously (in the operator norm sense) on z, the limit (31) has to be an eigenvalue of $\mathfrak{L}_{\Phi+z_0\tilde{\Phi}'}$ and so it must be proportional to $h_{\Phi+z_0\tilde{\Phi}'}$. From this it follows that

$$\lim_{z\to z_0} \nu_{\Phi+z\tilde{\Phi}'}(f) = \nu_{\Phi+z_0\tilde{\Phi}'}(f), \quad f \in C(K_+), \tag{32}$$

$$\lim_{z\to z_0} h_{\Phi+z\tilde{\Phi}'} = h_{\Phi+z_0\tilde{\Phi}'}, \tag{33}$$

i.e., $h_{\Phi+z\tilde{\Phi}'}$, $\nu_{\Phi+z\tilde{\Phi}'}$ are, respectively, norm and weakly continuous in z.

Now observe that the operator defined by

$$\left(\frac{\partial\mathfrak{L}_{a,\Phi}}{\partial\tilde{\Phi}'}f\right)(Y) = -\int_{X\subset[0,a)} e^{-U_\Phi(X\,|\,\tau_a Y)} \times U_{\tilde{\Phi}'}(X \mid \tau_a Y)f(x \cup \tau_a Y)\,dX \tag{34}$$

is such that

$$\left\|\frac{\mathfrak{L}_{a,\Phi+z\tilde{\Phi}'} - \mathfrak{L}_{a,\Phi}}{z} - \frac{\partial\mathfrak{L}_{a,\Phi}}{\partial\tilde{\Phi}'}\right\| \xrightarrow[z\to 0]{} 0. \tag{35}$$

So, using the just-proved continuity properties of ν, h, and λ and the identity

$$\frac{\lambda_{\Phi+z\tilde{\Phi}'} - \lambda_\Phi}{z} = \nu_\Phi\left(\frac{\mathfrak{L}_{a,\Phi+z\tilde{\Phi}'} - \mathfrak{L}_{a,\Phi}}{z}\,\frac{h_{\Phi+z\tilde{\Phi}'}}{\nu_\Phi(h_{\Phi+z\tilde{\Phi}'})}\right), \tag{36}$$

we find that

$$\lim_{z\to 0} \frac{\lambda_{\Phi+z\tilde{\Phi}'} - \lambda_\Phi}{z} = \nu_\Phi\left(\frac{\partial\mathfrak{L}_{a,\Phi}}{\partial\tilde{\Phi}'}\,h_\Phi\right), \tag{37}$$

which proves that

$$\frac{d}{dz}\,\lambda_{\Phi+z\tilde{\Phi}'} = \nu_{\Phi+z\tilde{\Phi}'}\left(\frac{\partial\mathfrak{L}_{a,\Phi+z\tilde{\Phi}'}}{\partial z}\,h_{\Phi+z\tilde{\Phi}'}\right)$$

and also that this derivative is a continuous function of z (since $\nu_{\Phi+z\tilde{\Phi}'}$ is weakly continuous in z and $\partial\mathfrak{L}_{a,\Phi+z\tilde{\Phi}'}/\partial z$ and $h_{\Phi+z\tilde{\Phi}'}$ are norm continuous in z). Now part (i) of Theorem 1 follows easily.

To prove the cluster property we use a procedure essentially contained in Ref. 2 and used there to prove some ergodicity properties of the equilibrium state in lattice systems.

We have first to construct the state $\bar{\gamma}$ which corresponds to the equilibrium state of the system when it occupies all R.

[9] N. Dunford and J. Schwartz, *Linear Operators* (Interscience Publishers, Inc., N.Y., 1958), Vol. I, Chap. VIII, Sec. 1, item 2.

In the remainder of this section the letters b and b' with or without indices will denote finite real numbers. Since the sets $C_{(b,b')} \subset C(K)$ of functions $A \in C(K)$ such that $A(X) = A(X \cap (b, b'))$ are dense in $C(K)$, it is sufficient, in order to determine the probability measure $\bar{\gamma}$ on K, to find $\bar{\gamma}(A)$ for all $A \in C_{(b,b')}$ with $b < b'$ arbitrary.

We remark that if $A \in C_{(b,b')}$, then $\tau_b A$ (translate of A by a length b) can be identified with an element $(\tau_b A)_+ \in C(K_+)$ defined for $X \in K_+$ as

$$(\tau_b A)_+(X) = (\tau_b A)(X \cap (0, b' - b)) = A(X \cap (0, b' - b) + b). \tag{38}$$

With this notation and denoting $\bar{\gamma}_{(c,d)}$ the normalized Gibbs measure on (c, d), i.e., the measure

$$\bar{\gamma}_{(c,d)}(dX) = [e^{-U(X)}\, dX/Z_{(c,d)}], \quad X \subset (c, d),$$

the value of $\bar{\gamma}(A)$ can be defined as

$$\bar{\gamma}(A) = \lim_{\substack{y\to+\infty\\ y'\to+\infty}} \bar{\gamma}_{(-y,y')}(A) = \lim_{\substack{y\to+\infty\\ y'\to+\infty}} \bar{\gamma}_{(0,y+y')}(\tau_y A) = \lim_{\substack{y_1\to+\infty\\ y_2\to+\infty}} \bar{\gamma}_{(0,y_1+y_2)}(\tau_{y_1}(\tau_b A)_+), \tag{39}$$

provided the limits exist.

To prove the existence of the limit (39), consider a function $f \in C_{(0,ma)} \subset C(K_+)$ and $n > m$. Then one easily verifies, using definitions (7) and (9) of $\mathfrak{L}_a$ and $\mathfrak{L}_a^*$, that

$$\bar{\gamma}_{(0,na)}(f) = \frac{(e^{-naP}\mathfrak{L}^*_{na}\delta_\varnothing)(f)}{(e^{-naP}\mathfrak{L}^*_{na}\delta_\varnothing)(1)}, \tag{40}$$

where $\delta_\varnothing$ is defined by $\delta_\varnothing(f) = f(\varnothing)$.

From the definitions (7) of $\mathfrak{L}_a$, one verifies also that

$$\mathfrak{L}_{na}(\tau_{na} f) = f(\mathfrak{L}_{na} 1), \quad f \in C(K_+), \quad n \ge 0, \tag{41}$$

and so, using (40), (41), (25), and (26), we have

$$\lim_{\substack{y_1\to\infty\\ y_2\to\infty}} \bar{\gamma}_{(0,y_1+y_2)}(\tau_{y_1}(\tau_b A)_+) = \lim_{\substack{y_1\to\infty\\ y_2\to\infty}} \frac{(e^{-(y_1+y_2)P}\mathfrak{L}^*_{y_1+y_2}\delta_\varnothing)(\tau_{y_1}(\tau_b A)_+)}{(e^{-(y_1+y_2)P}\mathfrak{L}^*_{y_1+y_2}\delta_\varnothing)(1)}$$
$$= h(\varnothing)^{-1} \lim_{\substack{y_1\to\infty\\ y_2\to\infty}} (e^{-y_2 P}\mathfrak{L}^*_{y_2}\delta_\varnothing)((\tau_b A)_+(e^{-y_1 P}\mathfrak{L}_{y_1} 1)) = \nu(h(\tau_b A)_+).$$

Hence we have found

$$\bar{\gamma}(A) = \nu(h(\tau_b A)_+) \quad \text{if} \quad A \in C_{(b,b')}; \tag{42}$$

this formula proves also the translational invariance of $\bar{\gamma}$.

Let now $A_1 \in C_{(b_1,b_1')}$, $A_2 \in C_{(b_2,b_2')}$. Then, using the following formula easily deduced from definition (9),

$$(\tau_x f)(X)\nu(dX) = (e^{-xP}\mathfrak{L}^*_x f\nu))(dX), \quad f \in C(K_+), \tag{43}$$

and the property that $\mathfrak{L}^*_x \nu = e^{xP}\nu$, one can deduce [taking also into account (25), (26), and (42)] that

$$\lim_{x\to\infty} \bar{\gamma}(A_1 \tau_x A_2) = \lim_{x\to\infty} \bar{\gamma}((\tau_{b_1}A_1)_+ \tau_x(\tau_{b_2}A_2)_+)$$
$$= \lim_{x\to\infty} \int h(X)(\tau_{b_1}A_1)_+(X)(\tau_x(\tau_{b_2}A_2)_+)(X)\nu(dX)$$
$$= \lim_{x\to\infty} \int h(X)(\tau_{b_1}A_1)_+(X)(e^{-xP}\mathfrak{L}^*_x(\tau_{b_0}A_2)_+\nu)(dX)$$
$$= \int h(X)(\tau_{b_1}A_1)_+(X)\nu(dX) \int h(X)(\tau_{b_2}A_2)_+\nu(dX) = \bar{\gamma}(A_1)\bar{\gamma}(A_2).$$

Now we prove the more general cluster property (6) by induction. Suppose it is true for the product of $A_2 \in C_{(b_2,b_2')} \cdots A_n \in V_{(b_n,n')}$, and let $A_1 \in C_{(b_1,b_1')}$; using (43) and the uniformity of the convergence of $(e^{-x_1 P}\mathfrak{L}_{x_1} h(\tau_{b_1}A)_+)$ to $h\bar{\gamma}(A_1)$, and the fact that

$$\|(\tau_{b_2}A_2)_+ \cdots \tau_{x_2+\cdots+x_{n-1}}(\tau_{b_n}A_n)_+\| \le \|A_2\| \cdots \|A_n\|,$$

we have:

$$\lim_{x_i\to\infty} \bar{\gamma}(A_1\tau_{x_1}A_2 \cdots \tau_{x_1+\cdots+x_{n-1}}A_n) = \lim_{x_i\to\infty} \bar{\gamma}((\tau_{b_1}A_1)\tau_{x_1}(\tau_{b_2}A_2) \cdots \tau_{x_2+\cdots+x_{n-1}}(\tau_{b_n}A_n))$$
$$= \lim_{x_i\to\infty} \int h(X)(\tau_{b_1}A_1)_+(X)\{\tau_{x_1}[(\tau_{b_2}A_2)_+ \cdots \tau_{x_2+\cdots+x_{n-1}}(\tau_{b_n}A_n)_+]\}(X)\nu(dX)$$
$$= \lim_{x_i\to\infty} \int h(X)(\tau_{b_1}A_1)_+(X)\{e^{-x_1 P}\mathfrak{L}^*_{x_1}[(\tau_{b_2}A_2)_+ \cdots \tau_{x_2+\cdots+x_{n-1}}(\tau_{b_n}A_n)_+]\nu\}(dX)$$
$$= \lim_{x_i\to\infty} \int [e^{-x_1 P}\mathfrak{L}_{x_1}(h(\tau_{b_1}A_1)_+)](X)[(\tau_{b_2}A_2)_+ \cdots \tau_{x_2+\cdots+x_{n-1}}(\tau_{b_n}A_n)_+](X)\nu(dX)$$
$$= \lim_{x_i\to\infty\, i>1} \int \bar{\gamma}(A_1)(\tau_{b_2}A_2)_+(X) \cdots (\tau_{x_2+\cdots+x_{n-1}}(\tau_{b_n}A_n)_+)(X)\nu(dX)$$
$$= \bar{\gamma}(A_1) \cdots \bar{\gamma}(A_n).$$

Finally the restrictive hypothesis that $A_i \in C_{(b_i, b_i')}$ can be released by density arguments.

5. DESCRIPTION OF THE EQUILIBRIUM STATE $\bar{\gamma}$

We have seen in the preceding section that h and ν determine completely the equilibrium state of the system considered. In this section we describe $\bar{\gamma}$ by means of a family of density distributions[10] in the case that Φ has finite range.

We remark that if the range of the interaction is between na and $(n+1)a$, then the function h has the property that $h(Y) = h(Y \cap [0, (n+1)a))$; this is because $h = \lim_n \lambda^{-n} \mathfrak{L}_{na} 1$ and $\lambda^{-n}\mathfrak{L}_{na}1$ depends only on $Y \cap [0, (n+1)a)$.

Now the state $\bar{\gamma}$ can be described by the family of density distributions $f_L(x)$ on $[0, La)$ which have the meaning of probability densities (with respect to the measure dX, $X \subset [0, La)$) for finding the configuration X inside $[0, La)$ irrespective of what happens outside.

If $L \geq n + 1$, these probabilities can be defined as

$$f_L(X) = \lim_{\substack{m_1 \to \infty \\ m_2 \to \infty}} Z^{-1}_{(-m_1 a, (L+m)a)} \int_{\substack{X_1 \subset [-m_1 a, 0) \\ X_2 \subset [La, (L+m_2)a)}} dX_1\, dX_2 \times e^{-U(X_1 | X)} e^{-U(X)} e^{-U(X_2 | X)}. \quad (44)$$

One can define a family of operators $\tilde{\mathfrak{L}}_x$ on $C(K_-)$ which are the analogies of the $\mathfrak{L}_x$ for left-semi-infinite systems as

$$(\tilde{\mathfrak{L}}_x f)(Y) = \int_{X \subset [-x, 0)} e^{-U(X | \tau_{-x} Y)} f(X \cup \tau_{-x} Y)\, dX, \qquad Y \in K_-. \quad (45)$$

The theory of these operators is exactly the same as that for $\mathfrak{L}_x$, so there exists $\tilde{h} \in C(K_-)$ such that $\tilde{\mathfrak{L}}_x \tilde{h} = e^{xP}\tilde{h}$, and $\tilde{h}(Y) = \lim_{x\to\infty} e^{-xP}\tilde{\mathfrak{L}}_x 1(Y)$ (uniformly in $Y \in K_-$). An explicit expression for $\tilde{h}$ in terms of eigenfunctions of operators of the type of $\mathfrak{L}$ can be given by considering the potential $\mathfrak{J}\Phi$ defined as the mirror image of Φ, and let $h^{\mathfrak{J}}$ be the eigenfunction of the operator $\mathfrak{L}$ corresponding to $\mathfrak{J}\Phi$; then one can prove that

$$\tilde{h}(\tau_{-La} Y) = \lim_{m\to\infty} (e^{-amP}\tilde{\mathfrak{L}}_{ma}1)(\tau_{-La}Y) = h^{\mathfrak{J}}(\mathfrak{J}Y), \quad (46)$$

where $\mathfrak{J}Y$ is the configuration obtained from $Y \subset [0, La)$ by reflecting $\tau_{-La}Y$ around the origin.

[10] D. Ruelle, J. Math. Phys. 8, 1657, 1967.

Now (44) can be written in terms of $\mathfrak{L}$ and $\tilde{\mathfrak{L}}$ as

$$f_L(X) = \lim_{\substack{m_1 \to \infty \\ m_2 \to \infty}} (\mathfrak{L}^*_{(m_1+m_2+L)a}\delta_\varnothing)(1)^{-1}(\mathfrak{L}_{m_1 a}1)(Y) \times e^{-U(Y)}(\tilde{\mathfrak{L}}_{m_2 a}1)(\tau_{-La}Y); \quad (47)$$

hence,

$$f_L(X) = e^{-PLa}h(\varnothing)^{-1}h(Y)h^{\mathfrak{J}}(\mathfrak{J}Y)e^{-U(Y)}. \quad (48)$$

One easily verifies, using the properties of h and $h^{\mathfrak{J}}$ as eigenvectors of $\mathfrak{L}$ and $\tilde{\mathfrak{L}}$, the normalization and compatibility conditions implicit in the meaning of f_L.

If we consider now the functional on the set $E \cap \mathfrak{L}^\perp$ of the translationally invariant measures on K defined as the difference between the mean entropy[11] and the mean energy, then, as a consequence of the differentiability properties of the pressure, one finds that this functional attains its maximum at one unique point of $E \cap \mathfrak{L}^\perp$ which coincides with $\bar{\gamma}$.[12,13] The value of this maximum is P.

This last property can easily be verified by writing

$$s(\bar{\gamma}) - U(\bar{\gamma}) = \lim_{L\to\infty} La^{-1} \int_{[0, La)} f_L(x)[-lg f_L(X) - U(X)]\, dX,$$

and using (48).

We mention without producing the explicit calculations that one can find a sufficiently large class of elements $\epsilon \subset E \cap \mathfrak{L}^\perp$ and suitably parametrize its elements so that the variational equations corresponding to the extremum problem $\max_{\rho\in\epsilon} s(\rho) - U(\rho)$ give rise to the integral equation $\lambda^{-1}\mathfrak{L}_a h = h$ and to the expression (48) for the state maximizing $s(\rho) - U(\rho)$. In this context one could use the results of this paper to guarantee that the state in $\mathcal{E}$ maximizing $s(\rho) - U(\rho)$ is the true equilibrium state.[14]

ACKNOWLEDGMENTS

We are deeply indebted to D. Ruelle for suggesting the problem, for enlightening discussions, and for reading the manuscript. We also thank M. L. Motchane for his kind hospitality at IHES. One of us (S. M.-S.) is indebted to CNRS for financial support.

[11] D. Robinson and D. Ruelle, Commun. Math. Phys. 5, 324 (1967).

[12] D. Ruelle, Commun. Math. Phys. 5, 324 (1967).

[13] G. Gallavotti and S. Miracle-Sole, Ann. Inst. Henri Poincaré 8, 287 (1968).

[14] S. Miracle-Sole, Thesis, University Aix-Marseille, France, 1968.