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Note

Limit Theorems for Multidimensional Markov Processes

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Abstract. An informal exposition of some recent results and conjectures.

A multidimensional Markov process (mdmp) is a dynamical system (K, m, T) where:

K space of the sequences of symbols from a finite alphabet $I = (a, b, \dots z)$ indexed by the elements $\eta \in Z^d \equiv$ lattice formed by the d-ples of integers. K is regarded as $K = \Pi_{\eta \in Z^d} I$ i.e. as a product space of copies of I; furthermore I is topologized by the discrete topology and K by the product topology.

T is the translation group acting, in the natural way, on K: if $\underline{\sigma} \in K$, $\underline{\sigma} = \{\sigma_\xi\}_{\xi \in Z^d}$ then $T_\eta \underline{\sigma} = \underline{\sigma}' = \{\sigma_{\xi+\eta}\}_{\xi \in Z^d}$, if $\eta \in Z^d$.

m is a regular complete probability measure on K whose σ-field contains all the open sets of K. Furthermore m has the "Markov property".

The Markov property can be easily expressed as a requirement on the conditional distributions associated with finite sets $\Lambda \subset Z^d$. Let $\underline{\sigma}_\Lambda = \{\sigma_\xi\}_{\xi \in \Lambda}$ $\underline{\sigma}' = \{\sigma_\xi\}_{\xi \in Z^d \setminus \Lambda}$; then, with obvious notations, $\underline{\sigma}_\Lambda \cup \underline{\sigma}' \in K$ and we can define $m_\Lambda(\underline{\sigma}_\Lambda / \underline{\sigma}')$ as the conditional probability that a configuration $\underline{\sigma} \in K$ coincides with $\underline{\sigma}_\Lambda$ inside Λ once it is known that, outside Λ, $\underline{\sigma}$, and $\underline{\sigma}'$ coincide. The Markov property is then the following [5, 17]:

mp for m-almost all $\underline{\sigma}_\Lambda \cup \underline{\sigma}'$ in K the functions $m_\Lambda(\underline{\sigma}_\Lambda/\underline{\sigma}')$ depend on $\underline{\sigma}'$ only through the values σ'_ξ with $\xi \in \partial\Lambda \equiv$ {set of lattice points not in Λ but located at unit distance from Λ}. Here Λ is an arbitrary finite subset of Z^d. Furthermore, $m_\Lambda(\underline{\sigma}_\Lambda/\underline{\sigma}') > 0$ m-a.e. $\forall \Lambda \subset Z^d$.

In the following we shall assume, for simplicity, that I is a two symbol alphabet $I = \{-1, +1\}$.

The following very interesting structure (and existence) theorem for mdmp holds: [5, 10, 15, 17].

Theorem. *All ergodic mdmp in d-dimensions can be obtained as follows:*

i) *choose* $d+1$ *real numbers* $\beta_1, \dots, \beta_d, h$;

ii) *choose* $\underline{\sigma}^0 \in K$;

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iii) *introduce for each square* $\Lambda \subset Z^d$, *centered at the origin, the measures* $P_{\Lambda,\underline{\sigma}^0}$ *on* K:

$$P_{\Lambda,\underline{\sigma}^0}(\underline{\sigma}) = P_{\Lambda,\underline{\sigma}^0}(\underline{\sigma}_\Lambda \cup \underline{\sigma}') = \delta^{(\Lambda)}_{\underline{\sigma}',\underline{\sigma}^0} \cdot \frac{\exp\left[\sum_{\xi,\eta \in \Lambda \cup \partial\Lambda} \beta_{\xi\eta}\sigma_\xi\sigma_\eta + h \sum_{\xi\in\Lambda} \sigma_\xi\right]}{\text{normalization}}$$

if $\underline{\sigma} = \underline{\sigma}_\Lambda \cup \underline{\sigma}'$, $\underline{\sigma}_\Lambda = \{\sigma_\xi\}_{\xi\in\Lambda}$, $\underline{\sigma}' = \{\sigma_\xi\}_{\xi\in Z^d\setminus\Lambda}$; *here* $\beta_{\xi\eta} = 0$ *unless* ξ, η *is a couple of nearest neighbours lying on the i-th coordinate axis of* Z^d, *in which case* $\beta_{\xi\eta} = \beta_i$; $\delta^{(\Lambda)}_{\underline{\sigma}',\underline{\sigma}^0}$ *is the Dirac measure* $\delta^{(\Lambda)}_{\underline{\sigma}',\underline{\sigma}_0} = 0$ *if* $\sigma_\xi \neq \sigma^0_\xi$ *for some* $\xi \in Z^d\setminus\Lambda$ *and* $\delta^{(\Lambda)}_{\underline{\sigma}',\underline{\sigma}_0} = 1$ *if* $\sigma_\xi \equiv \sigma^0_\xi$ *for* $\xi \in Z^d\setminus\Lambda$.

iv) *Consider the translation invariant, ergodic, weak limits of the measures* $P_{\Lambda,\underline{\sigma}^0}$ *when* $\Lambda \to \infty$. *For all* $\beta_1 \dots \beta_d h$ *there are suitable choices of* σ_0 *such that the measures* $P_{\Lambda,\underline{\sigma}^0}$ *have a limit as* $\Lambda \to \infty$.

The measures $P_{\Lambda,\underline{\sigma}^0}$ are called "finite Gibbs distributions with boundary condition σ^0".

A well known result is [5, 14, 15, 6].

Theorem. *If* $d = 1$ *there is one and only one Markov process with parameters* (β, h) *(see preceding theorem). Furthermore such a process is isomorphic to a Bernoulli scheme.*

A natural question is whether the above results concerning $d = 1$ extend to $d > 1$.

We shall restrict ourselves to the case $\beta_1 = \beta_2 = \dots = \beta_d = \beta > 0$ and write (β, h) instead of $(\beta_1, \beta_2, \dots, \beta_d, h)$. The equality condition is a "simplicity" condition; however the positivity condition is a real restriction and the qualitative aspects of the discussion which follows would radically change if one of the β's was negative.

The following theorem holds (see for instance, [9]).

Theorem. *If* $d \geqq 2$, $\exists \beta_c > 0$ *such that for all* $\beta > \beta_c$ *the process* $(\beta, 0)$ *is not unique. The process* (β, h) *is always unique if* $h \neq 0$ *and if* $h = 0$ *but* $\beta < \beta_c$.

The case $\beta = \beta_c$, $h = 0$ is an open problem if $d \geqq 3$. For $d = 2$ it is known that $(\beta_c, 0)$ is unique [19].

So we see that the situation in d-dimension is much more complicate.

Non uniqueness of the process (β, h) has a physical interpretation in statistical mechanics where the processes (β, h) describe mathematical models for the equilibrium properties of ferromagnets at temperature β^{-1} and in a magnetic field $h\beta^{-1}$. Non uniqueness corresponds to a phase transition: the ergodic mdmp (β, h) describe the pure phases of the magnet.

Of great importance, in Physics, is the theory of the fluctuations in a pure phase. Physicists think that in a "normal" situation the dispersion of a random variable which can be expressed as a sum of many "elementary" random variables should have a "normal gaussian distribution".

This statement can be made more precise in the context of the mdmp: consider a finite square $\Lambda \subset Z^d$ and consider

$$M_\Lambda(\underline{\sigma}) = \sum_{\xi\in\Lambda} \sigma_\xi \qquad \underline{\sigma} \in K .$$

Then the variables

$$v(\underline{\sigma}) = \frac{M_\Lambda(\underline{\sigma}) - \langle M_\Lambda(\cdot)\rangle}{|\Lambda|^{\frac{1}{2}}}$$

where $\langle\ \rangle$ means expectation with respect to an ergodic mdmp (β, h) should have an "essentially" gaussian distribution when Λ is large enough with the possible exception of few (β, h). If $d=1$ one has, in fact, the following theorem (see, for instance, [14]):

Theorem. *The (unique) measure (β, h) has the property that for a suitable choice of $\Delta = \Delta(\beta, h)$ the variable v introduced above has a probability distribution $P_\Lambda(v)$ such that*

$$\alpha)\qquad \lim_{\Lambda\to\infty} \int_a^b P_\Lambda(v)\,dv = \int_a^b \frac{e^{-\frac{y^2}{2\Delta}}}{\sqrt{2\pi\Delta}}\,dy \qquad \forall a, b \in (-\infty, +\infty),$$

$$\beta)\qquad \int_{-\infty}^{+\infty} e^{i\omega v} P_\Lambda(v)\,dv = e^{-\frac{\Delta}{2}\omega^2 + R_\Lambda} \qquad \forall \omega \in (-\infty, +\infty),$$

and $R_\Lambda \underset{\Lambda\to\infty}{\to} 0$ uniformly for ω in a compact subset of $(-\infty, +\infty)$ and proportionally to $|\Lambda|^{-\frac{1}{2}}$

$$\gamma)\qquad P_\Lambda(M_\Lambda(\underline{\sigma}) = 2k) = \frac{e^{-\frac{(2k - \langle M_\Lambda(\underline{\sigma})\rangle)^2}{2\Delta|\Lambda|}}}{\sqrt{2\pi\Delta|\Lambda|}}(1 + R'_\Lambda)$$

$\forall k$ integers and, furthermore, $R'_\Lambda \underset{\Lambda\to\infty}{\to} 0$ uniformly in k for

$$|2k - \langle M_\Lambda(\underline{\sigma})\rangle| < C|\Lambda|^{\frac{1}{2}} \qquad C > 0.$$

It is easy to see that γ) or β)$\to\alpha$).

So we see that, if $d=1$, the physicist's expectations are satisfied in the rather strong sense γ).

If $d>1$ the situation is not so simple and the following result is available [2, 4, 8].

(Central limit theorem for mdmp:)

Theorem. *If $h \neq 0$ and $\beta > 0$ then the obvious generalization to $d>1$ of the statement β) of the last theorem holds $\forall d$. If $h=0$ $\exists \beta'_0 < \beta''_0$ such that if $\beta < \beta'_0$ or $\beta > \beta''_0$ the ergodic processes $(\beta, 0)$ have the property α) of the theorem above. If $d=2$ then β'_0 can be taken equal to β''_0 and $\beta'_0 = \beta''_0 = \beta_c$.*

The method of proof of this theorem used in [8] is "non standard" in the sense that it uses methods rather different, in spirit, from the methods used in [14] for the proof of the one-dimensional limit theorem. In Refs. [1, 4] are obtained results sometimes much stronger than the ones mentioned in the last theorem but more restrictive on (β, h): the methods are, in some sense, close to the ones of Ref. [14].

It is now intersting to look for the limit theorems of more general nature.

First we decide to regard a probability measure m on K as a probability measure defined on (R = real line):

$$K_c = \prod_{\eta \in Z^d} R$$

(regarded as a topological product of copies of the real line with the usual topology) and carried by K regarded in the natural way as a subset of K_c.

Divide the lattice Z^d into boxes with side l, $l = 1, 2, \ldots$, and label each of the boxes by an element $x \in Z^d$, in a natural way. Then define the following random variables ("block spins"):

$$v_x = \frac{\sum_{\xi \in x} \sigma_\xi - \left\langle \sum_{\xi \in x} \sigma_\xi \right\rangle}{l^{\frac{1}{2} d \varrho}} \qquad x \in Z^d,$$

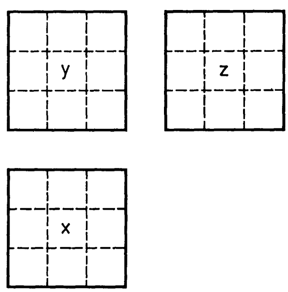

Fig. 1. Here $l = 3$, $d = 2$

where $\xi \in x$ means that ξ is in the box with label x and $\langle \cdot \rangle$ is the average with respect to a probability measure m on K_c.

In the following we shall call the regular probability measures m on K_c "random fields" if they are translation invariant and:

$$\int |\sigma_\xi| \, dm < \infty .$$

They will be called centered if $\int \sigma_\xi \, dm = 0$.

If m is a random field on K_c then it is easy to see that the random variables $\underline{v} = \{v_x\}_{x \in Z^d}$ are also distributed as a random field which will be called m_l.

The limit theorem problem discussed in the preceding sections can be now formulated as follows.

Problem. *Does the limit* $\lim_{l \to \infty} m_l$ *exist (in some sense) if* m *is a mdmp?*

In Refs. [7, 8] the following theorems are proved:

Theorem. *The ergodic mdmp* $(\beta, h) = m$ *considered in the Central limit theorem for mdmp are such that*

$$m_\infty = \lim_{l \to \infty} m_l$$

exists in the "natural weak sense" (see below) if $\varrho = 1$.

The weak sense of convergence is the following: let μ_∞, μ_n be a sequence of random fields and let $f: K_c \to R$ be a function such that

$$f(\underline{v}) = \phi(v_{x_1}, v_{x_2}, \ldots, v_{x_k}) \quad 0 < k < \infty, \; x_i \in Z^d$$

with ϕ being a continuous function with compact support. We say that $\lim_{n\to\infty} \mu_n = \mu_\infty$ if

$$\lim_{n\to\infty} \int f(\underline{v})\, d\mu_n = \int f(\underline{v})\, d\mu_\infty$$

for all the functions f of the type above (cylindrical compact functions).

Theorem. *In the same assumptions of the above theorem*

$$m_\infty(dv) = \prod_{x \in Z^d} \left(\frac{e^{-\frac{v_x^2}{2\Delta}}}{\sqrt{2\pi\Delta}}\, dv_x \right),$$

where Δ is the same as the Δ appearing in the Central limit theorem for mdmp.

The transformation $H_l^{(\varrho)}$ which maps a random field m on K_c into the centered random field m_l is called a "renormalization transformation" for the random field m.

The above two theorems can be formulated in terms of $H_l^{(\varrho)}$ as:

Theorem. *Under the same assumptions of the above two theorems*

$$\lim_{l\to\infty} H_l^{(1)} m = m_\infty \quad \textit{exists}, \tag{1}$$

$$H_l^{(1)} m_\infty \equiv m_\infty \qquad \forall l, \tag{2}$$

$$m_\infty \textit{ is a product of gaussian measures}. \tag{3}$$

In this formulation the above theorem should sound rather familiar to the probabilists: actually the above theorem justifies the following definition suggested in [12].

Definition. A "centered stable" random field is a centered random field which is invariant under the renormalization transformations $H_l^{(\varrho)}$ $l = 1, 2, \ldots$ for some $1 \leqq \varrho < 2$.

We remember here the usual definition of stable distribution [11] (adapted):

Definition. A centered probability measure P on R is "(centered) stable" if there is a number $2 > \varrho \geqq 1$ such that the probability distribution of the random variable $\dfrac{x_1 + x_2 + \cdots + x_n}{n^{\varrho/2}}$ is equal to P when x_i $i = 1, 2, \ldots, n$ are independently distributed with distribution P.

Clearly there is a one-to-one correspondence between the stable probability distributions P in the last sense and the stable 1-dim. random fields m whose probability measure is the product of identical factors equal to P.

The definition of a stable random field is a very interesting extension of the notion of stable distribution. The theorems of the last section say that the block spins of the pure phases are (at least if $d = 2$) distributed as a trivial stable random field, with $\varrho = 1$. Obviously we call "trivially stable" the stable random fields which are represented by a product of identical independent (stable) distributions.

As suggested in [12] the real interest of the notion of stable random field arises in the study of the mdmp $(\beta_c, 0)$.

It can be rigorously seen, if $d = 2$, that the above theorem concerning the convergence to a stable random field (with $\varrho = 1$) of the block spin distributions is no longer valid.

There are strong indications (but no rigorous proofs) that there exists $\varrho = \varrho(d)$ such that the mdmp $(\beta_c, 0)$ will have block spins which are asymptotically distributed as a stable random field with parameter ϱ. However it seems that such a stable random field is "non trivial" in the sense that it is not described by a product of independent stable distributions.

It appears, therefore, of great interest to develop a theory of the stable random fields, i.e. to provide:

i) Examples of such fields. It is, actually, not difficult to exhibit a stable measure in the form of a suitable gaussian measure on K_c which is not a product. However the indications offered by the non rigorous investigations of the block spin distributions for $(\beta_c, 0)$ lead to stable measures that are neither products nor gaussians at least for some values of d.

A nice example of one-dimensional non Gaussian stable random field can be obtained from the stationary sequence not satisfying the central limit theorem reported in Ref. [20].

ii) A classification of the possible random fields in analogy with the classification of the stable distributions [11].

iii) A description of the attraction domains of the stable random fields in analogy with the beautiful theory of the domains of attraction of the stable distributions [11].

Some indications on a possible direction of attack to these problems are contained in the papers [7, 8]. Some particular results concerning a similar problem can be found in [2]. The problems of classification and domain of attraction seem of particular interest because recently the physicists have developed a rich theory of the qualitative dependence of (β, h) on β and h in the neighborhood of $(\beta_c, 0)$ [13, 18] (for a more mathematical formulation of the theory see [2, 3]).

The hope is that the class of stable random fields is not too large and that this fact could be exploited to give a theoretical basis to what is known in physics as the "universal nature" of the critical phenomena [13].

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