QUASI PARTICLES AND SCALING PROPERTIES AT THE FERMI SURFACE

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The results presented in this talk were developed in collaboration with G. Benfatto.

The Fermi gas hamiltonian is:

$$H = \Big\{\sum_{i=1}^{N}\Big(\frac{-\Delta_{\mathbf{x_i}}}{2m}-\frac{p_F^2}{2m}\Big)\Big\}+\Big[\sum_{i\neq j}^{1,N}\lambda(\mathbf{x}_i-\mathbf{x}_j)+\sum_{i=1}^{N}\Big(\alpha\Big(\frac{-\Delta_{\mathbf{x_i}}}{2m}-\frac{p_F^2}{2m}\Big)+\nu\Big)\Big]$$
$$\equiv \{H_0\}+[V] \tag{1}$$

where λ is a pair potential, stable and with range p_0^{-1}; m is the particles mass; α,ν are two physical parameters equal to prefixed values, (e.g. we use α,ν to fix the Fermi sphere radius to be p_F and the particles mass to be m even in presence of interaction).

The basic quantity is the pair Schwinger function:

$$S(\mathbf{x},t)=\lim_{\beta\to\infty,L\to\infty}\frac{Tr\, e^{-(\beta-t)H}\,\Phi^-_{\mathbf{x}}\,e^{-tH}\,\Phi^+_{\mathbf{x}}}{Tr\, e^{-\beta H}} \tag{2}$$

where L= size of the container, $\Phi^\pm_{\mathbf{x}}$ are creation and annihilation operators.

The free case, i.e. $\lambda=0$, $\alpha=0$, $\nu=0$, is explicitly computable and the two basic parameters m, p_F describe the nature of the singularity at large distance a, in momentum space, on the set $k_0=0$, $|\mathbf{k}|=p_F$ ("Fermi surface"):

$$S(\mathbf{x},t)=\frac{1}{(2\pi)^4}\int d^3\mathbf{k}dk_o\frac{e^{-i(\mathbf{k}\cdot\mathbf{x}+k_0(t+0^-))}}{-ik_o+\frac{\mathbf{k}^2-p_F^2}{2m}} \tag{3}$$

We see that if we study $\hat S_0(k)^{-1}$, "propagator on scale $(2^{-h}p_0)^{-1}$", i.e. for k such that:

$$2^{2h-2}p_0^2 < K_0^2+\Big(\frac{\mathbf{k}^2-p_F^2}{2m}\Big)^2\le 2^{2h}p_0^2 \tag{4}$$

we see that the contribution of such momenta to S_0 is proportional to:

$$2^h\, p_0^{-1}\, e^{i\theta} e^{-p_F\omega\cdot\mathbf{x}} e^{-i2^h p_0(cos\theta t + sin\theta\omega\cdot\mathbf{x}} \tag{5}$$

where:

$$\theta = \text{arctg}\frac{\varepsilon(\mathbf{k})}{k_0} \quad , \varepsilon(\mathbf{k}) \equiv \frac{\mathbf{k}^2 - p_F^2}{2m} \quad , \omega = \frac{\mathbf{k}}{|\mathbf{k}|} \tag{6}$$

hence it is very regular for $(\mathbf{x}, t) \sim 2^{-h}\, p_0^{-1}$, "on scale h", except for the factor $\exp -i\, p_F\omega\cdot\mathbf{x}$ oscillating on scale p_F^{-1}, h-independent.

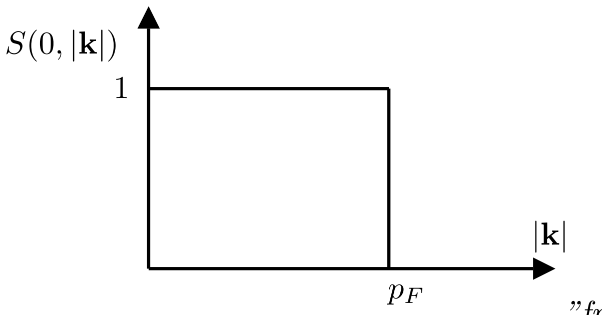
We can expect that on the fixed "scale h" (i.e. in the region (4)) the perturbation V changes "very little" the $\hat{S}^{-1}(k)$, i.e. that it becomes:

$$Z_h(-ik_0 + \varepsilon(\mathbf{k}) + O_h(k_0^2 + \varepsilon(\mathbf{k})^2)) \tag{7}$$

if $\alpha\,, \nu$ are tuned so that the zero is at $|\mathbf{k}| = p_F, k_0 = 0$, with:

$$\hat{S}^{-1}(0, p_F) = 0\,, \quad \frac{\partial_{|\mathbf{k}|}\, \hat{S}^{-1}}{\partial_{-ik_0}\, \hat{S}^{-1}}\,(0, p_F) = \frac{p_F}{m} \tag{8}$$

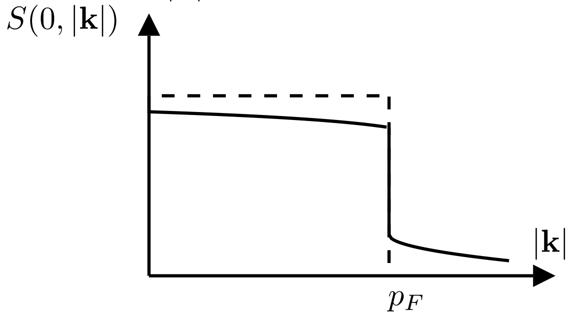
But it is clear that small corrections on each scale can produce completely different behaviour, compared to (3), (5) above, as $h \to -\infty$ (i.e. a very different long distance behaviour of $S(\mathbf{x}, t)$. A picture of the free case is:



(9)

"free Fermi gas"

This can become, without violating "physical continuity" and if α, ν are tuned so that the singularity is at $k_0 = 0, |\mathbf{k}| = p_F$ with curvature at p_F equal to p_F/m:



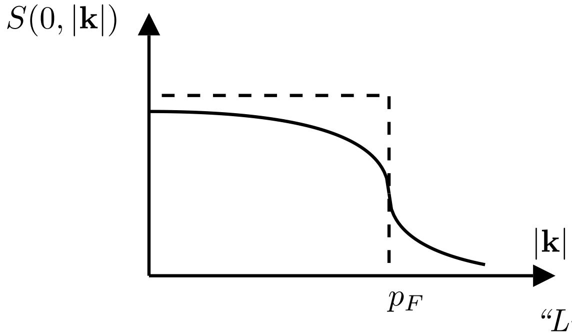
(10)

"normal Fermi gas"

if $Z_h \to Z_\infty$ with Z_∞ depending on the interaction λ. Or, if $Z_h/Z_{h-1} \to 2^{2\eta}$, $\eta > 0$, so that $Z_h \sim \mu 2^{-2\eta h} \to +\infty$, it can become:

$$\mu\,\frac{(k_0^2 + \varepsilon(\mathbf{k})^2)^\eta\, p_0^{-2\eta}}{(-ik_0 + \varepsilon(\mathbf{k}) + \ldots)} \tag{11}$$

with a graph:



(12)

"Luttinger Fermi gas"

with η function of λ (as well as μ). The latter name comes from the fact that, as suggested by Tomonaga, Luttinger and as proved by Mattis-Lieb, the one dimensional Luttinger model provides an example of such a gas, see [ML].

There are other possibilities, like the BCS-Fermi gas, for a discussion of it see [FT], [FKT].

To investigate which is the case that is actually realized we try to establish

the relationship between Z_h and Z_{h-1}. The hope being that such a relatio n is non singular and essentially scale independent and therefore computable by perturbation theory methods. Then the problem would be converted into that of studying the sequences Z_h that such a relation can generate as the "initial data", λ, α, ν vary.

It is easy to convince oneself that Z_h will not "evolve autonomously", i.e. it will not be possible to find a "scale independent", or approximately such, function $\bar{B}$ with the property that $Z_{h-1} = \bar{B}(Z_h)$.

The question than becomes to find how many other "relevant" quantities one has to attach to Z_h to form a set x_h of parameters describing the Schwinger functions on scale h and "evolving" with h in an autonomous way, i.e. via $x_{h-1} = B_h(x_h)$, or via similar relation with possibly memory effects, i.e. via $x_{h-1} = B_h(x_h, x_{h+i}, \ldots, x_o)$. In the latter case the memory should be of "short range", of course.

A well known method to generate the "relevant variables" and to compute B_h, the "beta functional", to all orders is to convert the original problem into a functional integral problem.

For Fermi gases there is a well known representation of the generic Schwinger function:

$$S(x_1\,\sigma_1, \ldots, x_n\,\sigma_n) = \lim_{\beta\to\infty, L\to\infty} \frac{Tr\, e^{-(\beta-t_1)H}\Phi_{\mathbf{x}_1} e^{-(t_1-t_2)H}\Phi_{\mathbf{x}_2}\ldots}{Tr\, e^{-\beta H}} \tag{13}$$

Let $\psi^{\pm}\,, \varphi^{\pm}_x$ be a set of gassmanian variables, and set:

$$\begin{aligned} V_0(\psi) = \int \lambda(\mathbf{x}-\mathbf{x}')\,\delta(t-t')\psi^+_x\,\psi^+_{x'}\,\psi^-_{x'}\,\psi^-_x\,dxdx + \\ + \int \psi^+_x \Big(\alpha(-\frac{\Delta_{\mathbf{x}}}{2m} - p_F^2) + \nu\Big)\;\psi^-_x\,dx \end{aligned} \tag{14}$$

Define the gassmanian integration:

$$\int P(d\psi)\,\psi^+_{x_1}\,\psi^-_{x_{2n}} = \text{Wick rule with propagator}\, S_0(x)\,, \text{see (3)} \tag{15}$$

then setting:

$$e^{-V_{eH}(\varphi)} = \int e^{-V_0(\psi+\varphi)}\, P(d\psi) \tag{16}$$

one defines on "effective potential" which is trivially related to the "truncated" Schwinger functions. In fact if

$$V_{eff}(\varphi) = \sum \int W_{eff}(x_1 \ldots x_{2n})\, \varphi^1_{x_1} \ldots \varphi^-_{x_{2n}} \tag{17}$$

the kernels W_{eff} are related by a convolution with S_0 to the truncated Schwinger functions. Precisely the relation is generated by the identity:

$$S^T(\varphi) \equiv (S_0\varphi^+ \cdot \varphi^-) - V_{eff}(S_0 * \varphi) \tag{18}$$

In particular (18) yields, for the "two fields" case, one in Fourier transform:

$$\hat{S}(k) = \frac{1}{-ik_0 + \varepsilon(\mathbf{k})} \Big(1 + \frac{\hat{W}_{eff}(k)}{-ik_0 + \varepsilon(\mathbf{k})}\Big) \tag{19}$$

We see that if:

$$\hat{W}_{eff}(k_0, \mathbf{k}) = a - ik_0 b + c\varepsilon(\mathbf{k}) + O(k_0^2 + \varepsilon(\mathbf{k})^2) \tag{20}$$

then we must tune α, ν, given λ, so that:

$$a = 0\,, \qquad b = c = \zeta_\infty \tag{21}$$

leading, for $k_0 \sim 0$, $|\mathbf{k}| \sim p_F$ (i.e. $\varepsilon(\mathbf{k} \sim 0)$:

$$\hat{S}(k) = \frac{1 + \zeta_\infty}{-ik_0 - \varepsilon(\mathbf{k})} \big(1 + O(|k_0|, |\varepsilon(\mathbf{k})|)\big) \tag{22}$$

A natural approach to study V_{eff} is to slice momentum space into layers:

$$\begin{aligned} &I_1,\, I_0,\, I_{-1}, \ldots \\ &I_i = \{k_0, \mathbf{k} | k_0^2 + \varepsilon(\mathbf{k})^2 \geq p_0^2\} \\ &I_j = \{k_0, \mathbf{k} | p_0^2\, 2^{2j-2} \leq k_j^2 + \varepsilon(\mathbf{k})^2 \leq p_0^2\, 2^{25}\} \quad j = 0, -1, -2, \end{aligned} \tag{23}$$

and decompose S_0 as

$$S_0(x) = \sum_{h=1}^{-\infty} \frac{1}{(2\pi)^4} \int_{I_h} \frac{e^{-ikx}\, dk}{-ik_0 + \varepsilon(\mathbf{k})^2} \equiv \sum_{h=1}^{-\infty} g^{(h)}(x) \tag{24}$$

and the fields as

$$\psi_x = \psi_x^{[1]} + \psi_x^{[0]} + \psi_x^{[-1]} + \ldots \tag{25}$$

where $\psi_x^{[j]}$ are new grassmanian variables with propagators

$$\langle \psi_x^{[j]}\, \psi_y^{[l]} \rangle = \delta_{jl}\, g^{(j)}(x - y) \tag{26}$$

In this way

$$e^{-V_{eff}(\varphi)} = \int e^{-V_0(\varphi+\psi^{[1]}+\psi^{[0]}+\psi^{[-1]}+\ldots)} \prod_{j=-n}^{1} P(d\psi^{[1]}) \tag{27}$$

Hence it is natural to define the effective potential on scale $h = 0, -1,$ by "partial integration" on the high frequency (or small scale) "components" $\psi^{(1)}, \psi^{(0)}, \psi^{(-1)}, \ldots, \psi^{(h+1)}$ of the field:

$$e^{-V_{eff}^{(h)}(\varphi)} = \int e^{-V_0(\varphi+\psi^1+\psi^0+\ldots+\psi^{h+1})} \prod_{j=1}^{h+1} P(d\psi^{(j)}) \tag{28}$$

(remember that $h \le 0$).

Before proceeding we remark that the field $\psi^{(1)}$, i.e. the high frequency field, plays a rather special role as, physically, the integration over $\psi^{(1)}$ has to do with "ultraviolet problems", which in our case translate into stability problems of the interaction. Such problems are not interesting for our purposes. In dimension $d = 1$ they are trivial, as they are also in $d > 1$ if one is willing to use a dispersion relation $\varepsilon(\mathbf{k})$ growing fast enough in $\mathbf{k}$ as $\mathbf{k} \to \infty$: the latter models are obvious continuous versions of the Hubbard model.

Therefore to avoid getting involved in an analysis that has little to do with our "low frequency" or "infrared problem" we just set $\psi^{(1)} \equiv 0$ in (28) so that the propagator is, in fact:

$$g(\mathbf{x}, t) = \sum_{h=-\infty}^{0} g^{(h)}(\mathbf{x}, t) \tag{29}$$

and the product in (28) starts at $j = 0$.

Another convenient modification is to replace the slicing via the sharp cuts of the integration domains in (24) with a more gentle cut: this is not an approximation but it is well known to be very convenient (as it avoids oscillations in position space of the propagators). For instance we interpret $\int_{I_n}$ as:

$$g^{(h)}(x) = \frac{1}{2h)^4} \int \frac{(e^{-p_0^{-2}2^{2h}(k_0^2+\varepsilon(\mathbf{k})^2)} - e^{-p_0^{-2}2^{2h+2}(k_0^2+\varepsilon(\mathbf{k})^2)})}{-ik_0 + \varepsilon(\mathbf{k})} e^{ikx}\, dk \tag{28}$$

We are now in a very familiar situation (see [GN], [G], [GF], [P]). However we cannot apply <u>any</u> of the methods in the quoted papers as they all suppose $g^{(h)}(x)$ to be "scaling as $h \to \infty$", i.e. they all require that, for some a, b:

$$g^{(h)}(x) \sim 2^{ah}\, \bar{g}(2^{bh}\, p_0 x) \tag{29}$$

It is easy to see, essentially from (5), that this is false: no scaling is possible as the presence of a second length scale p_F^{-1}, besides p_0^{-1}, forbids it.

However [BG] prove that it is possible, if $d =$ odd, to represent $g, g^{(h)}$ i n (29) as:

$$g(x) = \int d\vec{\omega}\, e^{-i\vec{\omega}\cdot\mathbf{x}p_F}\; g(x, \vec{\omega}) \tag{30}$$

with:

$$g(x,\vec\omega)=\sum_{h=-\infty}^{0}(2^h g_0(2^h x p_0,\vec\omega)+0_h(2^{2h})) \tag{31}$$

and $g_0(\xi,\vec\omega)$ decays very fast as $\xi\to\infty$ and it is smoot h and vanishes at $\xi=0$. The corrections $0(2^{2h})$ can also be explicitly studied, however their influence on the bounds that we shall study is simply som

e redefinition of the constants in the bounds.

For d even such a decomposition does not seem possible: in any d it is howev er possible to find a decomposition of $g(x,\vec\omega)$ of the form

$$\sum_{h=-\infty}^{0} 2^h \bar g_0(2^h_{p_0} t, 2^h_{p_0}\,\vec\omega\cdot\mathbf{x})+O_h(2^{2h}) \tag{32}$$

with $\bar g_0$ decaying very fast as $\xi\to\infty$, smooth and vanishing at $\mathbf{x}=0$. The remainder terms contain contributions which vanish only as powers of $(\vec\omega\cdot\mathbf{x})^{-1}$ but they are $O(e^{-(2^{-h})const})$. We conjecture that using a decomposition like (32), possible in all dimensions, the same bounds discussed below are still valid (but the matter requires considerable work to be completed, if possible).

Coming back to (31), it allows us to represent the Fermi fields as:

$$\psi^{\pm}_x=\int_{|\vec\omega|=1} d\vec\omega\, e^{\pm i\,p_F\vec\omega\cdot\mathbf{x}}\,\psi^{\pm}_{x,\vec\omega} \tag{33}$$

where the $\psi^{\pm}_{x,\vec\omega}$ are new grassmanian variables which we call "quasi particle" fields. They admit a "scaling decomposition":

$$\psi^{\pm}_{x\vec\omega}=\sum_{h=-\infty}^{0}\psi^{\pm(h)}_{x\vec\omega} \tag{34}$$

and the propagators are:

$$\langle\psi^{-(h)}_{x\vec\omega},\,\psi^{+(h')}_{x',\vec\omega'}\rangle=\delta_{hh'}\,\delta(\vec\omega-\vec\omega')2^h\,g_0(2^h p_0 x,\vec\omega) \tag{34}$$

since the quasi particles fields scale (almost) exactly (i.e. if we neglect the correction terms in (31) or (32)), we chose to think our problem as a theory of quasi particles interacting via a very special potential, namely:

$$\begin{aligned}&\int d\vec\omega_1\,d\vec\omega_2\,d\vec\omega_3\,d\vec\omega_4\,\psi^{+}_{x\vec\omega_1}\,\psi^{+}_{y\vec\omega_2}\,\psi^{+}_{y\vec\omega_3}\,\psi^{+}_{x\vec\omega_4}\,dxdy\\ &\qquad\lambda(\mathbf{x}-\mathbf{y})\delta(t-t')exp-i\,p_F(\vec\omega_1-\vec\omega_4)\cdot\mathbf{x}-i\,p_F(\vec\omega_2-\vec\omega_3)\mathbf{y}\\ &+\int d\vec\omega_1\,d\vec\omega_2\,e^{i\,p_F(\vec\omega_2-\vec\omega_2)\cdot\mathbf{x}}\psi^{+}_{x\vec\omega}[\alpha(\partial+i\vec\omega\,\frac{\Delta_{\mathbf{x}}}{p_F})+v]\psi^{-}_{x\vec\omega_2}\end{aligned} \tag{35}$$

Of course the Schwinger functions of such a theory will depend also on the quasi momenta $\vec\omega_1,\ldots,\vec\omega_n$ and they have a physical meaning only if suitable integrals over such quasi momenta are performed.

Furthermore the theory regarded as a theory of quasi particles has a built in symmetry; namely the effective potential on scale h

$$
\begin{aligned}
e^{-V^{(h)}_{eff}(\psi)} &= \int P(d\psi^{(h+1)}_{x\vec\omega})\dots P(d\psi^{(0)}) \\
e^{-V(\psi+\psi^{(h+1)}+\ldots+\psi^{(0)})}
\end{aligned}
\tag{36}
$$

has to be such that, after integrating over all the $\vec\omega$'s, one should be left with expressions involving only "particle fields"; for instance the collection of <u>all</u> local terms which are quadratic in the ψ's must add up to:

$$
\int v(\vec\omega_1,\vec\omega_2)e^{-i\,p_F(\vec\omega_1-\vec\omega_2)\mathbf{x}}\,\psi^+_{x\vec\omega_1}\,\psi^-_{x\vec\omega_2}\,dx\,d\vec\omega_1\,d\vec\omega_2 \tag{37}
$$

must be such that $v(\vec\omega_1,\,\vec\omega_2)\equiv\nu'$ so that (37) becomes

$$
\begin{aligned}
&\nu'(\int e^{-i\,p_F\vec\omega_1\cdot\mathbf{x}}\,d\vec\omega_1\,\psi^+_{x\vec\omega_1})\,(\int e^{+i\,p_F\vec\omega_2\cdot\mathbf{x}}\,d\vec\omega_2\,\psi^-_{x\vec\omega_2}) = \\
&= \nu' \int \psi^+_x\,\psi^-_x\,dx
\end{aligned}
\tag{38}
$$

This implies infinitely many identities to which we refer as "quasi particles symmetry".

Another symmetry is the "gauge symmetry" which means, in this case, that only an equal number of ψ^+ and ψ^- fields can appear in the expansion of the effective potentials.

At this point one can guess, on the basis of the experience on Renormalization Group methods in quantum field theory or statistical mechanics that the "relevant part" of the effective potential should be defined by a "localization operator" L defined to be

$$
\begin{aligned}
&L\quad \psi^+_{x_1\,\vec\omega_1}\dots\psi^-_{x_{2n}\,\vec\omega_{2n}} = 0 \qquad n\ge 3\\
&L\quad \psi^+_{x_1\,\vec\omega_1}\,\psi^+_{x_2\,\vec\omega_2}\,\psi^-_{x_3\,\vec\omega_3}\,\psi^-_{x_4\,\vec\omega_4} = \frac{1}{2}\sum_{j=1}^{2}\psi^+_{x_j\,\vec\omega_1}\,\psi^+_{x_j\,\vec\omega_2}\,\psi^-_{x_j\,\vec\omega_3}\,\psi^-_{x_j\,\vec\omega_4}\\
&L\quad \psi^+_{x_1\,\vec\omega_1}\,\psi^-_{x_2\,\vec\omega_2} = (\psi^+_{x_2\vec\omega_1} + (x_2-x_1)\cdot D_{\vec\omega_1}\,\psi^+_{x_2\vec\omega_1})\,\psi^-_{x_2\vec\omega_2}
\end{aligned}
\tag{39}
$$

where $D_{\vec\omega} = (\partial_t, \partial - i\vec\omega\,\frac{\Delta_{\mathbf{x}}}{p_F})$ is so defined, rather than $D=(\partial_t,\partial)$, to keep track of the quasi particle symmetry in a more efficient way.

Therefore we can write, using (39) the effective potential

$$
V^{(h)}_{eff} = L\,V^{(h)}_{eff} + (1-L)V^{(h)}_{eff} = L\,V^{(h)}_{eff} + V^{(h)}_{irrelevant} \tag{40}
$$

with $L\,V^{(h)}_{eff}\equiv V^{(h)}_{relevant}$ having, necessarily the form:

$$
\begin{aligned}
&V^{(h)}_{relevant} = L\,V^{(h)}_{eff} = \int \lambda^{(h)}(\vec\omega_1\,\vec\omega_2\,\vec\omega_3\,\vec\omega_4)\,e^{-i\,p_F(\vec\omega_1+\vec\omega_2-\vec\omega_3-\vec\omega_4)\cdot\mathbf{x}}\\
&\psi^+_{x\vec\omega_1}\,\psi^+_{x\vec\omega_2}\,\psi^-_{x\vec\omega_3}\,\psi^-_{x\vec\omega_4}\,dx\,d\vec\omega_1\,d\vec\omega_2\,d\vec\omega_3\,d\vec\omega_4 + \\
&+\int 2^h\,\nu_h\,e^{-i\,p_F(\vec\omega_1-\vec\omega_2)\cdot\mathbf{x}}\;\;psi^+_{x\vec\omega_1}\,\psi^+_{x\vec\omega_2} + \\
&+\int e^{-i\,p_F(\vec\omega_1-\vec\omega_2)\cdot\mathbf{x}}\,\psi^+_{x\underline{\mathrm{f}}\vec\omega_1}(\alpha_h\,\vec\omega_2\cdot D_{\vec\omega_2} + \zeta_h\,\partial_t)\psi^-_{x\vec\omega_2}
\end{aligned}
\tag{41}
$$

The above definitions may look somewhat arbitrary. But they will be interesting if one will be able to prove that the variables $\kappa_h = (\lambda_h, v_h, \alpha_h, \zeta_h)$ verify a scale independent autonomous recursion relation and that they determine entirely the effective potential.

More precisely it should be time that:
1) $V_{irr}^{(h)}$ should be expressible in terms of the relevant terms on scales $h+1, \ldots, 0$ and of the initial parameters (i.e. pair potential λ as well as α, v).
2) The relevant parameters $\kappa_h = (\lambda_h, \alpha_h, \zeta_h, \nu_h)$ should verify a relation:

$$\begin{aligned} \lambda_{h-1} &= \lambda_h + B_h^1(\kappa_h, \kappa_{h+1}, \ldots, \kappa_0) \\ \nu_{h-1} &= 2\nu_h + B_h^2(\kappa_h, \kappa_{h+1}, \ldots, \kappa_0) \\ \alpha_{h-1} &= \alpha_h + B_h^3(\kappa_h, \kappa_{h+1}, \ldots, \kappa_0) \\ \zeta_{h-1} &= \zeta_h + B_h^4(\kappa_h, \kappa_{h+1}, \ldots, \kappa_0) \end{aligned} \tag{42}$$

where B_h^j are formal power series with coefficients of order n uniformly bounded in h in the sense that of $|\kappa_h| \le R$, for all h, t hen

$$\sum_{c \in \; all \; n-th \; order \; terms \; in \; B_h^i} |c| \le R^n \, b_n^i, \qquad \forall h \le 0 \tag{43}$$

The main result in [BG] is that:

$$|b_n^i| \le (n-1)! \, B^{n-1} \, D_i \qquad \forall h \le 0 \tag{44}$$

for suitably chosen $B, D > 0$ (depending only on the range p_0 of the potential) . The h independence of the bounds (44) will be said to mean that the "beta function is perturbatively well defined".

Furthermore if $d = 3$ (and 2?), the $D_i, i = 2, 3, 4$, can be bounded by:

$$D_i \le \bar{D}_i \, 2^{-\varepsilon |h|} \qquad n \le 0, \; i = 2, 3, 4 \tag{45}$$

for some $\varepsilon > 0$.

Finally all the kernels, $W_{eff}^{(h)}(x_1 \vec{\omega}_1, \ldots, x_n \vec{\omega}_n)$ can be expanded in a power series in $\kappa_h, \kappa_{h+1}, \ldots, \kappa_0$ wit h coefficients, measured with a suitable norm, bounded as in (44) (with different D's).

This concludes our general rigorous results. Clearly they are just the beginning of a theory.

We now describe some heuristic consequences, a few particular rigorous results a nd several conjectures or possibilities.
a) One can do "consistent" perturbation theory by truncating the B's in (42) to order $p \ge 2$ and studying the resulting dynamical system. One has to look for initial data $(\lambda_0, \nu_0, \alpha_0, \zeta_0 = 0)$ such that κ_h, defined by the truncated (42) (meaningful because of (44)), such that $|\kappa_h| \le R$, for some R and alla $h \le 0$, and:

$$\begin{aligned} &\nu_h \to 0 \qquad\qquad h \to -\infty \\ &\alpha_h - \zeta_n \to 0 \qquad \zeta_h \to \zeta_\infty, \qquad h \to -\infty \end{aligned} \tag{46}$$

which would mean, heuristically, that the gas is a normal Fermi gas (with $Z_\infty^{-1} = 1 + \zeta_\infty$).

For instance truncation to second order plus the inequality (45) implies that the flow of (κ_h) is "driven" by that of λ_h-alone. And the recursion for λ_h (setting $\alpha_n, \zeta_h, \nu_h = 0$ for simplicity) is, in a second order truncation:

$$\begin{aligned}
&\lambda_{h-1}(\vec\omega_1\,\vec\omega_2\,\vec\omega_3\,\vec\omega_4) = \lambda_h(\vec\omega_1\,\vec\omega_2\,\vec\omega_3\,\vec\omega_4) + \\
&-\beta\int d\vec\omega\,d\vec\omega'\,\delta_h^{(2)}(\vec\omega_1+\vec\omega_2-\vec\omega-\vec\omega')\cdot \\
&\cdot\lambda_h(\vec\omega_1\,\vec\omega_2\,\vec\omega\,\vec\omega')\,\lambda_h(\vec\omega'\,\vec\omega\,\vec\omega_3\,\vec\omega_4) + \\
&-\beta\int d\vec\omega\,d\vec\omega'\,\{\delta_h^{(2)}(\vec\omega_1-\vec\omega_3+\vec\omega-\vec\omega') \\
&\lambda_h(\vec\omega_1\,\vec\omega\,\vec\omega_3\,\vec\omega')\,\lambda_h(\vec\omega_2\,\vec\omega'\,\vec\omega_4\,\vec\omega) + \\
&+ \textit{three more "antisymmetrization terms''}\}
\end{aligned} \tag{47}$$

where $\beta > 0$ is a suitable constant and:

$$\begin{aligned}
&\delta_h^{(2)}(\mathbf{u}) \equiv 2^{-(d-1)h}\,G(2^{-h}\mathbf{u};\vec\omega,\vec\omega') \\
&G(\xi;\vec\omega,\vec\omega')\,\propto\,[\frac{(\vec\omega-\vec\omega')^2}{4} + \vec\omega\cdot\xi\,\vec\omega'\cdot\xi]\,e^{-\xi^2}
\end{aligned} \tag{48}$$

the first integral operator in (47) is the "direct" term while the others form the "exchange" term.

The (47) is a difficult equation to study. If the initial interaction λ_0 has Fourier transform $\hat\lambda_0$ such that:

$$\begin{aligned}
&\hat\lambda_0(p_F(\vec\omega-\vec\omega')) - \hat\lambda_0(p_F(\vec\omega+\vec\omega')) = \\
&\quad = \sum_{\substack{l=1\\ \ell=odd}}^{\infty}(2\ell+1)\,P_\ell(\vec\omega\cdot\vec\omega')\,\hat\lambda_\ell^0
\end{aligned} \tag{49}$$

(if $d = 3$) is an expansion with some $\hat\lambda_\ell^0 < 0$ and if the exchang e term is neglected then one can see, easily, that λ_h bounded for all $h \le 0$ is not possible. On the other hand of the interaction is repulsive ($\lambda_0 \ge 0$) then one can prove $\hat\lambda_\ell^0 \ge 0$ and, if the exchange term is neglected, there is a bounded solution to (47) because it becomes:

$$(\hat\lambda_\ell)_{h-1} = (\lambda_\ell)_h - \beta(\hat\lambda_\ell)_h^2 \tag{50}$$

But it is unclear to what extent one can really neglect the exchange term. One can in fact show that if one fixes $\vec\omega_1, \vec\omega_2, \vec\omega_3, \vec\omega_4$ and if $|\lambda_h| \le R$ then the exchange term tends to zero exponentially. On the other hand one can always find (h-dependent) configurations $\vec\omega_1^h, \vec\omega_2^h, \vec\omega_3^h, \vec\omega_4^h$ on which the exchange term could stay of $O(1)$, (for suitable λ_h).

A rigorous analysis of this simple second order truncation would be very interesting, we believe.

b) One can ask whether there are "fixed points" of (42). This should be define d by a sequence κ_h tending to some limit as $h \to -\infty$ and such that the limi t k_∞ verifies $\nu_\infty = 0$ and:

$$0 = B^1(\kappa_\infty, \kappa_\infty, \ldots) \tag{51}$$

with $\alpha_\infty, \zeta_\infty$ suitably chosen. This is somewhat difficult as the above mentioned non uniformity properties of the exchange integral generate interpretation problems.

c) Maybe the fixed point equation can be solved by singular (in the $\vec\omega$'s) λ's. A question of great interest is whether there are fixed

points such that the effective potential has the form

$$V_L = \int \lambda \psi^+_{x,\vec\omega}\psi^-_{x,\vec\omega}\psi^+_{x,-\vec\omega}\psi^-_{x,-\vec\omega} \tag{52}$$

or:

$$V_{BCC} = \int \lambda \vec\omega\cdot\vec\omega^1 \psi^+_{x\vec\omega}\psi^+_{x-\vec\omega}\psi^-_{x-\vec\omega}\psi^-_{x\vec\omega} \tag{52}$$

At second order a direct search for such (lines of) fixed points seems feasible and this could mean that there are three possibilities:

α) Normal Fermi gas ($\lambda = 0$)

β) Luttinger Fermi gas ($\lambda \neq 0$, $V^{(-\infty)} \equiv V_L$

γ) BCC Fermi gas ($\lambda \neq 0$, $V^{(-\infty)} \equiv V_{BS}$

Case β) is particularly interesting as it could provide an example of a "Luttinger fluid", see also [A].

d) Finally we conclude by mentioning some work in progress:

1) $d = 1$: "no factorials" in (44), "convergence of the beta function" (G. Benfatto, G. Procacci, B.Scoppola)

2) $d > 1$: "no factorials" in the part of B^1 which cannot be bounded as in (45), i.e. which does not go exponentially to zero,

and some open problems:

3) convergence of the full beta function: no factorial in (44), if $d > 1$

4) extension to the electron gas in a periodic potential: what happens in the case of "filled bands"?

5) Once convergence is established one can ask about the existence of non trivial fixed points of type β) or γ) above. Note that such fixed points will break the system symmetries (as the effective potential cannot be expressed in terms of the particle fields).

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