ON THE φ_4^4 - PROBLEM

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Abstract

In this paper we illustrate the recent techniques developed to treat renormalizable interactions by developing an example which, although implicit in the literature, has not been pointed out. We consider the formal perturbation expansion of $\lambda\varphi_4^4$-hierarchical with $\lambda > 0$ (i.e. the "unstable" case) and show that one can construct a one parameter family P_λ, $\lambda \in [0,\lambda_o]$, of real stochastic processes whose effective potential (and Schwinger functions) admit the formal renormalized series as asymptotic series: this shows that P_λ can be a family of probability measure and does not have to be necessarily complex valued as it is in the constructions based on analytic continuation from $\lambda < 0$ to $\mathrm{Re}\lambda > 0$.

The construction seems to be generalizable to more interesting hierarchical models (i.e. having non-trivial S-matrix): however as it stands, it is bound to produce in the Euclidean case a theory violating the Osterwalder-Schrader positivity (which would be kept only in the hierarchical cases).

§ 1 The Model: "Non-Perturbative Formulation"

In this paper, the hierarchical φ_4^4 model is considered.

The hierarchical model was introduced by Dyson [1] in a Statistical Mechanics context and by Wilson [2] in Field Theory.

It played a major role in the development of the renormalization group methods both from the Theoretical Physics' viewpoint [3] and from the Mathematical Physics' [4,5,6,7,8,9].

In [10], appendix G, a concise introduction to the model is given, together with its renormalizability properties.

The free field with cut-off γ^N, $\gamma = 2$ (say) is, in dimension $d > 2$:

$$\varphi_x^{(\leq N)} = \sum_{j=0}^{N} \varphi_x^{(j)} \qquad x \in R^d , \quad d > 2 . \tag{1.1}$$

The fields $\varphi^{(j)}$ have to be thought of as gaussian random fields independently distributed over j . Their x-dependence is specified by

$$\varphi_x^{(j)} = \sqrt{\gamma^{(d-2)j}}\; z_\Delta \qquad \text{if} \quad x \in \Delta \tag{1.2}$$

where Δ is a tessera of a pavement Q_j of the unit cube U built with cubes of side size γ^{-j} . The z_Δ is a gaussian random variable with covariance $\frac{1}{2}$ if $j > 0$: if $j = 0$, i.e. $\Delta \equiv U$, we take the covariance to be some $c \neq \frac{1}{2}$, and not $c = \frac{1}{2}$, in order to simplify some formulas.

The φ_d^4-problem is related to the family of stochastic processes (i.e. probability measures on the space of the fields $\varphi_x^{(\le N)}$) defined by

$$\exp\left(\sum_{j=0}^{2} \int_U r^{(j)}(N) \;\; \varphi_x^{(\le N)2j} \, dx \right) P(d\varphi^{(\le N)}) \tag{1.3}$$

where $P(d\varphi^{(\le N)})$ is the probability distribution of $\varphi^{(\le N)}$, i.e. by (1.1), (1,2):

$$P(d\varphi^{(\le N)}) = \left(\prod_{j>0}^{N} \prod_{\Delta \in Q_j} e^{-z_\Delta^2} \frac{dz_\Delta}{\sqrt{\pi}} \right) \left(e^{-\frac{z_U^2}{2c}} \frac{dz_U}{\sqrt{2\pi c}} \right) \tag{1.4}$$

(as said above, the case $j = 0$, $U \equiv \Delta$, is treated differently to simplify later formulae).

The $r^{(j)}(N)$ are called "bare couplings".

More generally, one could consider expressions like

$$e^{V^{(N)}} \; dP = (\exp \int_U w_D(\varphi_x^{(\le N)}) d^d x) \;\; P(d\varphi^{(\le N)}) \tag{1.5}$$

for w_D arbitrary (bounded above).

The φ_d^4-problem, "ultraviolet problem", is to find, and analyze in some detail, the set of probability measures on the fields which can be reached as limit points of sequences of measures like (1.3) when $N \to \infty$: the general "ultraviolet" problem of scalar field theories is to find the limit points of sequences of measures like (1.5) when $N \to \infty$.

Although it is known that if $d = 1,2,3$ such set of limit points is rather large and consists of non-trivial objects (i.e. of non-gaussian stochastic processes), this is not the case if $d = 4$ (or $d > 4$), where sometimes it has even been conjectured that only gaussian measures build up the set of limit points ("strong triviality conjecture", see [11]).

To study the limit as $N \to \infty$ of (1.3), (1.5), one defines $V^{(K)}(\varphi^{(\le K)})$ as

$$\exp V^{(K)}(\varphi^{(\le K)}) = \int \exp V^{(N)}(\varphi^{(\le N)}) \; P(d\varphi^{(K+1)}) \ldots P(d\varphi^{(N)}) \tag{1.6}$$

and $V^{(K)}$ will be called the "effective potential" on scale K, or "renormalized potential on scale K corresponding to the bare potential $V^{(N)}$ on the cut-off scale N.

Using the fact that $\varphi^{(\le N)}$ is constant over the tesserae Δ of size γ^{-N}, $\Delta \in Q_N$, the integral defining $V^{(N)}$ in (1.5) can obviously be written as a sum:

$$\int_U w_D(\varphi_x^{(\le N)})d^dx \equiv \sum_\Delta \gamma^{-dN} w_D(\gamma^{\frac{d-2}{2}N} x_\Delta) \equiv \sum_\Delta w^{(N)}(x_\Delta) \qquad (1.7)$$

where the "normalized field" corresponding to Δ has been introduced as

$$X_\Delta = \frac{\varphi_x^{(\le N)}}{\gamma^{\frac{d-2}{2}N}} \qquad \text{if} \quad x \in \Delta \in Q_N \, . \qquad (1.8)$$

The field X_Δ is called normalized because $E(\varphi_x^{(\le N)z}) = \text{const.}\ \gamma^{(d-2)N}$ (the constant being $1/2(\gamma^{d-2}-1)$ if c is chosen conveniently: $c = 1/2(\gamma^{d-2}-1)$).

The $w^{(N)}$ defined in (1.7) will be called the "dimensionless potential" on scale N (the subscript D in w_D stands, in fact, for "dimensional").

The identity $\varphi_x^{(\le N)} = \varphi_x^{(\le N-1)} + \varphi_x^{(N)}$ can be written, for dimensionless fields, as:

$$X_\Delta = \alpha z_\Delta + \beta X_\Delta \, , \qquad (1.9)$$

if $x \in \Delta \subset \Delta'$, $\Delta \in Q_N$, $\Delta' \in Q_{N-1}$, $N \ge 1$, and

$$\alpha^2 + \beta^2 = 1 \, , \quad \beta = \gamma^{-\frac{d-2}{2}} \qquad (1.10)$$

provided c in (1.4) is conveniently chosen as $c = 1/2(\gamma^{d-2}-1)$. If $c = 1/2$, then $\beta = \gamma^{-\frac{d-2}{2}} + O(\gamma^{-N})$ a change which would only result in messier expressions.

The main property of the hierarchical model is that, if $V^{(N)}$ is given by (1.7), then $V^{(N-1)}$ is also given by the r.h.s. of (1.7) with $w^{(N)}$ replaced by a new function $w^{(N-1)}$ defined by

$$e^{w^{(N-1)}(x)} = \left(\int e^{-z^2} \frac{dz}{\sqrt{\pi}} e^{w^{(N)}(x\beta+2\alpha)}\right)^{\gamma^d} \qquad (1.11)$$

expressing the "strict locality" of the hierarchical model.

The proof of (1.11) is straightforward from (1.4), (1.8), (1.9), (1.10) (see [6], [7] and [11] appendix G).

We call $\mathcal{N}$ the operator (1.11) on $f = e^w$:

$$\mathcal{N}\, f(x) = (\Gamma(\beta) f(x))^{\gamma^d}$$
$$\Gamma(\beta) f(x) = \int e^{-z^2}\, f(\alpha z + \beta x)\, \frac{dz}{\sqrt{\pi}} \tag{1.12}$$

and $\Gamma(\beta)$ is a well-known linear operator ("hypercontractive operator") which acts on f by multiplying its n-th Hermite coefficient by β^n :

$$\Gamma(\beta) H_n(x) = \beta^n H_n(x) \qquad \forall n = 0,1,\ldots \tag{1.13}$$

where H_n are the Hermite polynomials defined by:

$$e^{-\frac{\alpha^2}{4}+\alpha x} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} H_n(x) \quad \forall \alpha, x \in R\ . \tag{1.14}$$

This remark implies that the linearization of $\mathcal{N}$ near $f = 1$, which is a trivial fixed point of $\mathcal{N}$, is

$$D_1\mathcal{N}\, F(x) = \gamma^d\, \Gamma(\beta)\, F(x) \qquad \text{so that}$$
$$D_1\mathcal{N}\, H_n(x) = \gamma^d \beta^n H_n(x) \equiv \gamma^{\sigma(n)} H_n(x) \quad \sigma(n) = d - \frac{d-2}{2} n \tag{1.15}$$

suggesting representing w as a Hermite series, as we shall often do (this is called a "Wick ordered representation" of w(x)).

The general ultraviolet problem can be rephrased as follows: describe the set of points $w^{(0)}$ of the form

$$\exp w^{(0)}(x) = \lim_{N\to\infty} \mathcal{N}^N w^{(N)}(x) \tag{1.16}$$

and the problem of φ_d^4 can be put in the same way with the extra restriction that $w^{(N)}$ is a fourth order polynomial:

$$w^{(N)}(x) = \sum_{j=0}^{2} \lambda^{(j)}(N)\, H_{2j}(x)\ . \tag{1.17}$$

Here $\lambda^{(j)}(N)$ will be called the "dimensionless bare couplings".

The above formulation of the φ_d^4 ultraviolet problem will be called "non-perturbative" following a trend become recently widespread.

In [11] such a formulation of the φ_d^4 problem has been criticized (in the corresponding Euclidean version) as being too restrictive. In fact, the "perturbative" formulation defined below will be richer.

§ 2 Form Factors and the Perturbative Formulation of the φ_d^4-Problem

The effective potential on scale k is described by the "dimensionless" function $w^{(k)}$ related to $w^{(N)}$ by

$$w^{(k)} = \log \mathcal{N}^{N-k} (\exp w^{(N)}) \, . \tag{2.1}$$

We define the form factors of $w^{(k)}$ by considering the Hermite expansion for $w^{(k)}$

$$w^{(k)}(k) = \sum_{j=0}^{2} \lambda^{(j)}(k)\, H_{2j}(x) + \sum_{j=3}^{\infty} \lambda^{(j)}(k)\, H_{2j}(x) \tag{2.2}$$

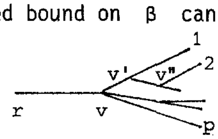
and the three coefficients $\underline{\lambda}(k) = (\lambda^{(0)}(k), \lambda^{(1)}(k), \lambda^{(2)}(k))$ will be called the "form factors" on scale k .

Then in [12], [10], [13] it has been proved that $\forall d \geq 4$:

(1) if $j \geq 3$ then $\lambda^{(j)}(k)$ is a formal power series in terms of $\underline{\lambda}(k+1), \ldots, \underline{\lambda}(N)$ and $\underline{\lambda}(N+1) \equiv \underline{0}$, $\underline{\lambda}(N+1) \equiv \underline{0}$, ... with N-independent coefficients. Such coefficients are non-negative and will be denoted $\beta^{(j)}(h_1, \ldots, h_p;\, \alpha_1, \ldots, \alpha_p)$ so that

$$\lambda^{(j)}(k) = \sum_{p=2}^{\infty} \sum_{\substack{\underline{h}=(h_1,\ldots,h_p \\ h_i \geq k+1}} \sum_{\alpha_1,\ldots,\alpha_p}^{0,2} \beta^{(j)}(\underline{h};\underline{\alpha}) \lambda^{(\alpha_1)}(h_1) \ldots \lambda^{(\alpha_p)}(h_p). \tag{2.3}$$

In fact, a very detailed bound on β can be provided, as described below.

(2) If θ denotes a tree  and if to each of its vertices v , non-extremal, one associates an integer h_v ("frequency or scale of the vertex v") so that $h_r \equiv k$, $h_{v'} > h_v$ if $v' > v$ (the tree being oriented from r upwards) then, if p is the number of extremal vertices of θ :

$$\beta^{(j)}(h_1, \ldots, h_p;\, \alpha_1, \ldots, \alpha_p) \leq C^p (p-1)! \sum_{\theta} \sum_{\underline{h}}^{*} \prod_{v>r} \gamma^{-\bar{\rho}(h_v - h_w)} \tag{2.4}$$

where the sum runs over all the ways to associate "frequency labels" h_v to the vertices $v \in \theta$ which are not extremal; furthermore the p extremal vertices are, respectively, directly attached to vertices of θ carrying labels $h_1, \ldots, h_p$; $\bar{\rho}$ is a suitably chosen positive number and w is the vertex immediately preceding v in θ .

(3) The $\underline{\lambda}(k)$ themselves obey a formal power series relation; $\alpha = 0,1,2$:

$$\lambda^{(\alpha)}(k) = \gamma^{\sigma(\alpha)} \lambda^{(\alpha)}(k+1) + \sum_{p=2}^{\infty} \sum_{\substack{\underline{h}=(h_1,\ldots,h_p) \\ N \geq h_i \geq k+1}} \sum_{\underline{\alpha}=(\alpha_1,\ldots,\alpha_p)}^{0,1,2} \beta^{(\alpha)}(\underline{h};\underline{\alpha}) \prod_{j=1}^{p} \lambda^{(\alpha_j)}(h_j) \tag{2.5}$$

and $\beta^{(\alpha)}(\underline{h};\underline{\alpha}) \geq 0$ obey a bound like (2.4); the $\sigma(\alpha)$'s $\sigma(\alpha) = d - 2\alpha(d-2)/2$. The $\beta^{(\alpha)}$ depend only on $h_v - k$ $v \in \theta$.

(4) The Eq. (2.5) can be thought of as an equation for $\underline{\lambda}(k)$. As such it is "inhomogeneous" if $\underline{\lambda}(0) \equiv \underline{\lambda}$ is considered known and, if $d=4$, it admits a recursive solution in the form of a formal power series in $\underline{\lambda} \equiv \underline{\lambda}(0)$, with coefficients $\underline{\ell}^{(n)}(k) = (\ell^{(\underline{n})0}(k), \ell^{(\underline{n})1}(k), \ell^{(\underline{n})2}(k))$:

$$\lambda^{(\alpha)}(k) = \gamma^{-\sigma(\alpha)k}\,\lambda^{(\alpha)} + \sum_{\substack{n_0,n_1,n_2\\ |\underline{n}|\geq 2}} \ell^{(n_0,n_1,n_2)\alpha}(k) \prod_{j=0}^{3} (\lambda^{(j)})^{n_j} \tag{2.6}$$

or, more compactly

$$\underline{\lambda}(k) = \gamma^{-\underline{\sigma}k}\,\underline{\lambda} + \sum_{|\underline{n}|\geq 2} \ell^{(\underline{n})}(k)\ \underline{\lambda}^{\underline{n}} \tag{2.7}$$

and for all N

$$|\underline{\ell}^{(n)}(k)| \leq (|\underline{n}|-1)!\ C^{|\underline{n}|-1} \sum_{j=0}^{|\underline{n}|-1} \frac{(\beta k)^j}{j!} \tag{2.8}$$

which follows from (2.4), (2.5) and $\sigma(\alpha) = d-\alpha\,\frac{d-2}{2} \leq 0$ (because $d=4$), alone [12], [10].

(5) The explicit form of (2.5), "to second order", is

$$\lambda^{(2)}(k) = \lambda^{(2)}(k+1) + \beta\,\lambda^{(2)}(k+1)^2 + \beta_1\,\lambda^{(2)}(k+1)\,\lambda^{(1)}(k+1) + \ldots$$

$$\lambda^{(2)}(k) = \gamma^2\,\lambda^{(1)}(k+1) + \beta_2\,\lambda^{(2)}(k+1)^2 + \beta_3\lambda^{(1)}(k+1)^2 + \beta_4\lambda^{(1)}(k+1)\lambda^{(2)}(k+1) + \ldots$$

$$\lambda^{(0)}(k) = \gamma^4\,\lambda^{(0)}(k+1) + \beta_5\,\lambda^{(2)}(k+1)^2 + \beta_6\,\lambda^{(1)}(k+1)^2 + \ldots \tag{2.9}$$

where $\beta, \beta_1, \beta_2, \ldots$ are (computable) positive constants and d has been taken 4 (if $d>4$ the first equation is modified by multiplying the linear term by γ^{4-d} and the third by replacing γ^4 by γ^d).

(6) The same results hold if $d \geq 4$ and $w^{(N)}$ has the form

$$w^{(N)}(k) = \sum_{j=0}^{n} \lambda^{(j)}(N)\ H_{2j}(x) \tag{2.10}$$

provided one now introduces the dimensionless form factors on scale k by a formula like (2.2) with 2 replaced by n and 3 by $n+1$; furthermore, in the analogue of (2.5), (2.3) one chooses $\alpha, \alpha_1, \ldots, \alpha_p$ to vary in $0,1,\ldots,n$ and j in $n+1, n+2, \ldots$. However, there is no obvious analogue of (4) above because (2.5) no longer admits a formal solution as a power series in $\underline{\lambda}(0)$ if $n>2$ or if $d>4$.

Nevertheless, an analogue of (4) can be formulated as follows [13]: if $d=4$, $n>2$, Eq. (2.5) admits a formal power series solution in powers of $\lambda^{(0)}(0)$, $\lambda^{(1)}(0)$, $\lambda^{(2)}(0)$, $\lambda^{(3)}(N),\dots,\lambda^{(n)}(N)$. But the resulting formal series has the remarkable property that its coefficients involving non-trivially $\lambda^{(j)}(N)$, $j\geq 3$, vanish as $N\to\infty$ [13]. If $d>4$ also $\lambda^{(2)}(0)$ has to be replaced by $\lambda^{(2)}(N)$ and $j\geq 3$ by $j\geq 2$.

The latter property says that it is not different, from a formal viewpoint, to consider (2.10) with $n>2$ rather than (1.17), if $d=4$: "formally all polynomial theories in $d=4$ coincide with the φ_4^4-theories".

This suggests that the appropriate way to define φ_4^4 seems to be the old-fashioned way ("perturbative definition") of defining φ_4^4 via the result (4) above. In other words, one defines φ_4^4-theory a family $P_{\lambda,\mu,\nu}$ of stochastic processes parametrized by three parameters $(\lambda,\mu,\nu)\equiv(\lambda^{(2)}(0),\ \lambda^{(1)}(0),\ \lambda^{(0)}(0))$ varying in a set Ω such that $\underline{0}\in\partial\Omega$ and such that the effective potential $w^{(k)}(x)$ on scale k has an asymptotic expansion in (λ,μ,ν) near $\underline{0}$ agreeing to all orders with the one constructed formally when $N\to\infty$ via (4), see [10], [11], for each fixed x.

We say that $P_{\lambda,\mu,\nu}$, $(\lambda,\mu,\nu)\in\Omega$, is "trivial" if $(\lambda,\mu,\nu)\in\Omega$ implies $\lambda\equiv 0$, [11].

Observe that this definition neither prescribes the sign of λ nor the form of $w^{(N)}$: this is in contrast with the definition ("non-perturbative") of § 1 which prescribes $w^{(N)}$, see (1.11), and also fixes the sign of $\lambda^{(2)}(N)$ to be ≤ 0.

In spite of what the wording seems to suggest to many, the perturbative approach is harder (in fact much harder, probably) than the non-perturbative one.

In this paper, it is proved that a family $P_{\lambda,\mu,\nu}$ exists, and is non-trivial, in a region Ω containing a vicinity of the origin restricted by the condition $\lambda>0$.

For instance, one can show the existence of a family $P_{\lambda,0,0}$ for small $\lambda>0$ with $\lambda^{(j)}(k)$ given, for $j\geq 3$, by a C^∞-function of λ admitting the usual perturbation series as asymptotic expansion at $\lambda=0$.

A similar result was proved in [14] to low order of perturbation theory: however, in [14] $P_{\lambda,0,0}$ is constructed by analytic continuation from the $\lambda<0$ cut-off case with $w^{(N)}$ given exactly by (1.17): this ultimately implies, at least in [14], see, however, [15], that $P_{\lambda,0,0}$ is probably not a probability measure (being probably complex as it can be defined only if $\mathrm{Re}\,\lambda>0$, $\mathrm{Im}\,\lambda\neq 0$ and taking the limit $\mathrm{Im}\,\lambda\to 0$: it remains unclear whether the result is real, leaving aside the harder positivity problem).

On the other hand, the techniques of [14], [9] are extensively used in this work as they proved a good scheme to treat renormalizable theories: they are combined with the general theory of the "beta-function" developed in [12], [11], [13] to show the validity of perturbation theory to all orders. This work can be considered as a simple example of the renormalization group approach to constructive field theory in the renormalizable cases which follows closely the schemes already successfully developed in the constructions of φ_d^4, $d = 2,3$ in [6], [7], [8] based on the detailed analysis of only few orders of perturbation theory. As shown in [14], the new idea of the "control of remainders" by the use of analyticity properties, proposed in [9] allows to carry the techniques of superrenormalizable theories to the renormalizable cases.

That scalar fields could exist in $d=4$ and be non-trivial for $\lambda > 0$ has been hinted (even very recently) by many authors [16], [11], [18], [15]: the proposal based on adding higher order powers of φ to the "bare action" $W^{(N)}$ with dimensionally appropriate bare constants seem to have failed to realize that the resulting theory might be φ_4^4 (a remark in [13], see iii) of theorem 3). If the theory discussed in [17] is non-gaussian, it might just be the full Euclidean φ_4^4.

§3 Construction of a Scalar Field with $\lambda > 0$

Let $\bar\lambda_k > 0$ be a sequence obeying

$$\bar\lambda_{k-1} = \bar\lambda_k + \beta\, \bar\lambda_k^2 + \kappa_1\, \bar\lambda_k^{3-\varepsilon_1} \qquad k = 1,2,\ldots \tag{3.1}$$

where β is the positive coefficient appearing in (2.9) and $\kappa_1 > 0$, $\varepsilon_1 > 0$ will be fixed later.

Assume also that for some $\bar\lambda > 0$:

$$\bar\lambda_k = \frac{\bar\lambda}{1+\beta\,\bar\lambda k}\,\left(1 + 0\left(\left(\frac{\bar\lambda}{k+1}\right)^{1-\varepsilon_1}\right)\right) \qquad k = 0,1,\ldots\ . \tag{3.2}$$

It is easy to show that (3.1) implies that if $\bar\lambda_0$ is small enough, then (3.2) holds for some $\bar\lambda = 0(\bar\lambda_0)$; vice versa if $\bar\lambda_0$ is given a priori, positive and small enough, then one can define $\bar\lambda_2, \bar\lambda_2, \ldots$ so that (3.1), (3.2) hold for $k = 1,2,\ldots$.

Let, for some $\varepsilon_0 > 0$ to be fixed later:

$$B_k = \bar\lambda_k^{-\varepsilon_0}\ . \tag{3.3}$$

Then, fixed $k \in [0,N]$, consider a function f_k defined for $|x| \le B_k$, $x \in \mathbb{C}$

$$f_k(x) = \exp\left(\sum_{j=0}^{3} \tilde\lambda^{(j)}(k)\, H_{2j}(x) + \nu_k(x)\right) \tag{3.4}$$

with

$$\begin{aligned} &(a) \quad |\tilde{\lambda}^{(2)}(k)| \le \lambda_k \\ &(b) \quad |\tilde{\lambda}^{(j)}(k)| \le \kappa_o \, \bar{\lambda}_k^2 \qquad\qquad j = 0,1,3 \end{aligned} \tag{3.5}$$

(c) $\nu_k(x)$ is holomorphic in $|x| \le B_k$, divisible by x^8 and

$$|\nu_k(x)| \le \kappa_o \, \bar{\lambda}_k^{3-\varepsilon_1}$$

where κ_o will be fixed later.

We now consider the stochastic process (here $\chi(|x|<B)=1$ if $|x|<B$ and $= 0$ otherwise):

$$P(d\varphi) = \text{"}\lim_{N\to\infty}\text{"} \left(\prod_{\Delta\in Q_N} f_N(x_\Delta) \right) \left(\prod_{k=0}^{N} \prod_{\Delta\in Q_N} \chi(|x_\Delta| \le B_k) \, e^{-z_\Delta^2} \frac{dz_\Delta}{\sqrt{\pi}} \right) \tag{3.6}$$

where f_N is a function verifying (a), (b), (c) with $\nu_N \equiv 0$. We want to show that if $\tilde{\lambda}^{(0)}(N)$, $\tilde{\lambda}^{(1)}(N)$ are conveniently chosen, then the limit in (3.6) exists in the sense that the distribution of the zero scale field in (3.6) converges to some $f_0(x)\ \chi(|x| < B_0)$ with f_0 of the form (3.4) obeying (3.5). The " " mean that possibly one has to consider a subsequence of $N \to \infty$ in (3.6).

The presence of the field cut-offs in (3.6) changes the above ultraviolet problem formulated in terms of the map N. If $\chi_k(x) \equiv \chi(|x| < B_k)$ let

$$N_k\, f(x) \equiv \chi_k\, N(\chi_{k+1} \cdot f)(x), \tag{3.7}$$

then the effective potentials on scale k for (3.6) are given by

$$f_k(x) = N_k\, N_{k+1} \cdots N_{N-1}\, f_N(x) \tag{3.8}$$

(rather than by $f_k = N^{N-k} f_N$).

The limit in (3.6) will exist, thus defining a stochastic process, if f_N can be so chosen that the limit in (3.8) exists for all k's and x's.

We show first that $N_k f_k$ has the form (3.4), (3.5) with k replaced by $k-1$ and (b) replaced:

$$|\tilde{\lambda}^{(j)}(k-1)| \le 2\gamma^{\sigma(j)}\kappa_o\, \bar{\lambda}_{k-1}^{-2} \qquad j = 0,1,3 \tag{3.9}$$

where $\sigma(j) = d-j(d-2)$ and $d = 4$.

Furthermore, let $\{\le 2\}$ denote a truncation to second order in $\bar{\lambda}_k$ of a polynomial in $\tilde{\lambda}^{(j)}(k)$, $j = 0,1,2,3$, when $\tilde{\lambda}^{(2)}(k)$, $j = 0,1,3$, are regarded of second order (see (3.5), (a), (b) for motivation on this point). Then

$$\sum_{j=0}^{3} \tilde{\lambda}^{(j)}(k-1)\, H_{2j}(x) = \left[\gamma^d \sum_{p=1}^{2} \frac{E^T}{p!}\left(\sum_{j=0}^{3} \tilde{\lambda}^{(j)}(k)\, H_{2j}(d\cdot+\beta x);p\right)\right]^{\{\leq 2\}} + \qquad (3.10)$$

$$+ \text{"higher orders in } \bar{\lambda}_k\text{"}$$

where E, E^T denote, respectively, the expectation (i.e. integration) and the truncated expectation with respect to the measure $e^{-z^2}\frac{dz}{\sqrt{\pi}}$.

The (3.10) can be written explicitly:

$$\tilde{\lambda}^{(2)}(k-1) = \tilde{\lambda}^{(2)}(k) + \beta\, \tilde{\lambda}^{(2)}(k)^2 + O(\bar{\lambda}_k^{3-\varepsilon_1})$$

$$\tilde{\lambda}^{(1)}(k-1) = \gamma^2\, \tilde{\lambda}^{(2)}(k) + \beta_2\, \tilde{\lambda}^{(2)}(k)^2 + O(\bar{\lambda}_k^{3-\varepsilon_1})$$

$$\tilde{\lambda}^{(0)}(k-1) = \gamma^4\, \tilde{\lambda}^{(0)}(k) + \beta_5\, \tilde{\lambda}^{(2)}(k)^2 + O(\bar{\lambda}_k^{3-\varepsilon_1})$$

$$\tilde{\lambda}^{(3)}(k-1) = \frac{1}{\gamma^2}\, \tilde{\lambda}^{(3)}(k) + \beta_7\, \tilde{\lambda}^{(2)}(k)^2 + O(\bar{\lambda}_k^{3-\varepsilon_1}) \qquad (3.11)$$

where $\beta_7 > 0$ and the other coefficients $\beta, \beta_2, \ldots$ have been introduced in (2.9) and, furthermore:

$$|O(\bar{\lambda}_k^{3-\varepsilon_1})| \leq \kappa_1\, \frac{\bar{\lambda}_k^{3-\varepsilon_1}}{2} \qquad (3.12)$$

provided $\varepsilon_0, \varepsilon_1$ are chosen conveniently and $\bar{\lambda}_0$ is small enough.

Assume for the time being that $N_k f_k$ has the properties (3.9) through (3.12). We first show how this allows us to complete the proof.

First we fix the value of the free parameter κ_0 so that

$$\gamma^2\kappa_0 - \beta_2 > \kappa_0\,, \quad \gamma^4\kappa_0 - \beta_5 > \kappa_0\,,\ \frac{1}{\gamma^2}\kappa_0 + \beta_7 < \kappa_0 \qquad (3.13)$$

and then we suppose that $\bar{\lambda}_0$ (hence $\bar{\lambda}$ and $\bar{\lambda}_k$, see (3.2)) is so small that (3.13) remains true even if the r.h.s. are increased or decreased by $\frac{\kappa_1}{2}\, \bar{\lambda}_k^{3-\varepsilon_1}$, $\forall k \geq 0$. Then it follows from the "expansivity" of the $\tilde{\lambda}^{(1)}, \tilde{\lambda}^{(0)}$ linear terms in (3.11) that the square

$$D_N = [-\kappa_0\, \bar{\lambda}_N^2\,,\ \kappa_0\, \bar{\lambda}_N^2]^2 \qquad (3.14)$$

in the plane $(\lambda_N^{(0)}, \lambda_N^{(1)})$ will have an image under N_N covering D_{N-1} in the sense that the $(\tilde\lambda^{(0)}_{N-1}, \tilde\lambda^{(1)}_{N-1})$ will cover D_{N-1} as $\tilde\lambda^{(0)}_N$, $\tilde\lambda^{(1)}_N$ cover D_N at fixed $\tilde\lambda^{(2)}_N$, $\tilde\lambda^{(3)}_N$, ν_N obeying (a), (b), (c) in (3.5).

Iterating this remark and assuming the above properties of $\mathcal{N}_k f_k$, $f \sim f_k$ verifying (3.4), (3.5), it follows the existence of a sequence of subsets of D_N:

$$D_N > \mathcal{D}_{N-1} > \mathcal{D}_{N-2} > \ldots > \mathcal{D}_0 \tag{3.15}$$

such that

$$\mathcal{N}_k \mathcal{N}_{k+1} \cdots \mathcal{N}_{N-1} \mathcal{D}_{N-k} = D_{N-k} . \tag{3.16}$$

This proves that for each choice of $\tilde\lambda^{(2)}(N) \in [0,\bar\lambda_N]$ it is possible to choose $\tilde\lambda^{(0)}(N)$, $\tilde\lambda^{(1)}(N)$ and $\tilde\lambda^{(3)}(N) = 0$ (say), $\nu_N(x) \equiv 0$ (say) so that (3.4), (3.5) hold for all $k \le N$ and $\tilde\lambda^{(0)}(0) = \nu$, $\tilde\lambda^{(1)}(0) = \mu$ are prescribed in D_0 (e.g. $\nu = \mu = 0$).

It is then easy to infer that this actually implies the existence of the limit (3.6), possibly considering a subsequence.

Therefore, it remains to show that κ_1, ε_o, ε_1 can be fixed so that $\mathcal{N}_k f_k$ has the properties listed between (3.9) and (3.12) if f_k has the properties (3.4), (3.5).

Fix $\eta > 0$ so that (recall $d = 4$, $\gamma = 2$, $\beta = \gamma^{-\frac{d-z}{2}}$)

$$\tau \equiv 2\,\gamma^d \left(\frac{\beta}{1-\eta}\right)^8 \left(-1 + \frac{1-\eta}{\beta}\right) < 1 . \tag{3.17}$$

Then for $|x| < (1-\eta) B_k \beta^{-1}$:

$$\mathcal{N}\chi_k f_k(x) = \Big(\int_{|\alpha z + \beta x| < B_k} f_k(dz + \beta x)\, e^{-z^2} \frac{dz}{\sqrt{\pi}} \Big)^{\gamma^d} =$$

$$= \Big(\int_{|z|<\eta B_k/\alpha} f_k(\alpha z + \beta x)\, e^{-z^2} \frac{dz}{\sqrt\pi} + \int_{\substack{|z|>\eta B_k/\alpha \\ |\alpha z+\beta x|<B_k}} f_k(\alpha z+\beta x)\, e^{-z^2} \frac{dz}{\sqrt\pi} \Big)^{\gamma^d} =$$

$$= (1-\rho_k)^{\gamma^d}\, \hat{\mathcal{E}}(f_k(\alpha\cdot+\beta x))^{\gamma^d} \left(1 + \frac{\int_{\substack{|z|>\eta B_k/\alpha \\ |\alpha z+\beta x|<B_k}} f_k(\alpha z+\beta x) e^{-z^2} \frac{dz}{\sqrt\pi}}{(1-\rho_k)\, \hat{\mathcal{E}}(f_k(\alpha\cdot+\beta x))} \right)^{\gamma^d}$$

where $\hat{\mathcal{E}}(\cdot)$ denotes the expectation with respect to the measure

$\chi(|z| < \eta B_k/\alpha) e^{-z^2} \frac{dz}{\sqrt{\pi}}$ /norm and $\mathcal{E}^T(\cdot)$ will later denote the truncated expectation with respect to the same measure; finally $(1-\rho_k)$ is the normalization constant

$$\rho_k = \int_{|z|>\eta B_k/\alpha} e^{-z^2} \frac{dz}{\sqrt{\pi}} = O(e^{-(\eta B_k \alpha)^2/z}) . \tag{3.19}$$

Using the bounds (3.4), (3.5), (3.19), (3.3) that for $|x| < (1-\eta)B_k\beta^{-1}$

$$N\chi_k f(x) = \hat{\mathcal{E}}(f_k(\alpha\cdot + \beta x))^{\gamma^d} \left(1-\rho_k+\theta\rho_k\, e^{2(\bar{\lambda}_k B_k^4 + 4\kappa_o \bar{\lambda}_k^2 B_k^6)T}\right) \tag{3.20}$$

where T is such that $|H_{2j}(x)| \le T\, B^{2j}$ for all $|x| \le B$, $(B > 1)$, $j = 0,1,2,3$ and θ is an unspecified number between 0 and 1 (introduced to keep (3.20) an equality rather than an inequality).

By using the definition of the truncated expectations $\hat{\mathcal{E}}^T$ we see that

$$\hat{\mathcal{E}}^T(f_k(\alpha\cdot + \beta x)) = \exp\left[\sum_{p=1}^{2} \frac{1}{p!} \hat{\mathcal{E}}^T \left(\sum_{j=0}^{2} \tilde{\lambda}^{(j)}(k)\, H_{2j}(\alpha\cdot + \beta x) + \nu_k(\alpha\cdot + \beta x); p\right) + R\right] \tag{3.21}$$

where for some $\bar{\kappa} > 0$

$$\begin{aligned} |R| &\le \bar{\kappa}(\bar{\lambda}_k B_k^4 + 3\kappa_o\, \bar{\lambda}_k^2\, B_k^6 + \kappa_o\, \bar{\lambda}_h^{3-\varepsilon_1})^3\, T^3 \le \\ &\le \delta\, \kappa_o\, \bar{\lambda}_k^{3-\varepsilon_1} \end{aligned} \tag{3.22}$$

where the estimate of the remainder of the cumulant expansion has been made by simply estimating $\sum_{j=0}^{2} \tilde{\lambda}^{(j)}(k)\, H_{2j}(\alpha z + \beta x) + {}_k(\alpha z + \beta x)$ by its maximum in $|x| < (1-\eta)B_k\beta^{-1}$, $|z| < \eta B_k/\alpha$, using (3.5); the (3.22) holds provided

$$3 - 12\varepsilon_o > 3-\varepsilon_1 , \quad \varepsilon_1 < 1 \tag{3.23}$$

e.g. if $\varepsilon_o = \frac{\varepsilon_1}{13}$, $\varepsilon_1 = \frac{1}{2}$.

The number δ in (3.22) can be fixed a priori once (3.23) holds, if $\bar{\lambda}_o$ is taken small enough (depending on δ): we take $\delta = \frac{1}{2}$, say.

The sum of the truncated expectations in (3.21) can be rewritten as, $(|\theta_\alpha| < 1)$:

$$\sum_{p=1}^{2} \frac{1}{p!}\, \varepsilon^T \left(\sum_{j=0}^{3} \tilde{\lambda}^{(j)}(k)\, H_{2j}(\alpha\cdot+\beta x); p\right) + \hat{\mathcal{E}}(\nu_k(\alpha\cdot+\beta x) +$$

$$+ \theta_1\, T_1\, \bar{\lambda}_k^{4-\varepsilon_1}\, B_k^4 \equiv \theta_2(T_1\, \bar{\lambda}_k^{4-\varepsilon}\, B_k^4 + T_2\rho_k\, \bar{\lambda}_k\, B_k^4) +$$

$$+ \hat{E}(\nu_k(\alpha\cdot + \beta x) + \sum_{p=1}^{2} \frac{1}{p!} E^T \Big(\sum_{j=0}^{3} \tilde{\lambda}^{(j)}(k) \; H_{2j}(\alpha\cdot + \beta x);p\Big) \tag{3.24}$$

because replacing E^T by E^T changes the expectation value of a polynomial of degree ≤ 6 by a quantity involving ρ_k times the maximum of the polynomial in the integration domain specified in E^T which, in our case, is of the order of $\bar{\lambda}_k B_k^4 << 1$ in the region $|z| < \eta B_k/\alpha$, $|x| < (1-\eta)B_k/\beta$.

Therefore, (3.24) can be written for $|x| < (1-\eta)B_k/\beta$ as:

$$\Big[\sum_{p=1}^{2} \frac{1}{p!} E^T \Big(\sum_{j=0}^{3} \tilde{\lambda}^{(j)}(k) \; H_{2j}(\alpha\cdot + \beta x);p\Big)\Big]^{\{\leq 2\}} + \gamma^{-d}\, \tilde{\nu}_{k-1}(x)$$

and (3.25)

$$|\nu_{k-1}(x)| \leq \kappa_o \, \bar{\lambda}_k^{3-\varepsilon_1} \, (1+2\delta)\gamma^d < 2\gamma^d \, \kappa_o \, \bar{\lambda}_k^{3-\varepsilon_1}$$

(recalling the choice of $\delta < \frac{1}{2}$) and here one might again have to restrict further $\bar{\lambda}_o$ to be small so that

$$\kappa_o \, \delta \, \bar{\lambda}^{3-\varepsilon_1} + \bar{\kappa}_o \, \bar{\lambda}^{3-\varepsilon_1} + T_1 \, \bar{\lambda}_k^{4-\varepsilon_1} \, B_k^4 + T_2 \, \bar{\lambda}_k \, B_k^4 \rho_k \leq$$

$$\leq \kappa_o \, \bar{\lambda}^{3-\varepsilon_1} \, (1+2\delta) \, . \tag{3.26}$$

The factor γ^{-d} in (3.25) has been inserted because f_{k-1} is the exponential of the first of (3.25) raised to the power γ^d.

We now observe that $(N\chi_k \, f_k)(x)$ is an entire function of x because:

$$N\chi_k f_k(x) = \int_{|y|<B_k} f_k(y) \; e^{-(y-\beta x)^2/\alpha^2} \frac{dy}{\alpha\sqrt{\pi}} \, . \tag{3.27}$$

Hence it easily follows from our bounds that $\tilde{\nu}_{k-1}$ is holomorphic in $|x| < B_k(1-\eta)/\beta$ and therefore we can consider its first 6 Taylor coefficients

$$t_j = \frac{1}{2\pi i} \oint \tilde{\nu}_{k-1}(\zeta) \, \frac{d\zeta}{\zeta} \, \frac{1}{\zeta^{2j}} \qquad j = 0,1,2,3 \tag{3.28}$$

where the contour of integration is any circle C_r around the origin and radius $r \leq B_k(1-\eta)/\beta$; we choose $r = 1$: hence (3.25) implies

$$|t_j| \leq (\kappa_o \, \bar{\lambda}_k^{3-\varepsilon_1} \, \gamma^d \, 2) \, . \tag{3.29}$$

Therefore the coefficients of the Hermite polynomial of order < 8 in the representation of type (3.4) of $f_{k-1}(x)$ are given by (3.11) with the $O(\cdot)$ bounded by

$$2\gamma^d \kappa_o \bar{\lambda}_k^{3-\varepsilon_1} T_3 \equiv \frac{\kappa_1}{2} \bar{\lambda}_k^{3-\varepsilon_1} \tag{3.30}$$

for some T_3, a numerical constant: this determines κ_1.

The "remainder"

$$\nu_{k-1}(x) = \frac{1}{2\pi i} \oint \frac{d\zeta}{\zeta - x} \left(\frac{x}{\zeta}\right)^8 \tilde{\nu}_{k-1}(\zeta) \tag{3.31}$$

can be estimated by choosing $|\zeta| = B_k(1-\eta)/\beta$ as integration contour and $|x| < B_{k-1} \le B_k$: it is

$$|\nu_{k-1}(x)| \le 2\kappa_o\gamma^d \bar{\lambda}_k^{3-\varepsilon_1} \left(\frac{\beta}{1-\eta}\right)^8 \left(\frac{1-\eta}{\beta} - 1\right) = \tau \kappa_o \bar{\lambda}_k^{3-\varepsilon_1} < \kappa_o \bar{\lambda}_{k-1}^{3-\varepsilon_1} . \tag{3.32}$$

This proves (3.12), (3.11) and (c) in (3.5). Property (a) of (3.5) follows from (3.12) and the first of (3.11) by (3.1).

Property (b) in (3.5) with k replaced by $(k-1)$ follows from the last of (3.11), for $j = 3$, and (3.12) if $\bar{\lambda}_o$ is taken small enough.

Hence our construction of the measures in (3.6) is complete and such measures will have, by construction, effective potentials analytic on their support (bounded, because $|x| \le B_k$ is the support of f_k): in fact, they are, by our construction, uniformly bounded (as $N \to \infty$) on each fixed scale k. This allows us to use Vitali's theorem to find a subsequence $N_j \to \infty$ such that the limit in (3.6) exists.

Actually, it would not be difficult to prove that there is no need to consider subsequences, but we leave aside this (minor) question.

We can pick $\tilde{\lambda}^{(0)}(N)$, $\tilde{\lambda}^{(1)}(N)$ so that $\tilde{\lambda}^{(0)}(0)$, $\tilde{\lambda}^{(1)}(0) = 0$: with this choice, the measures in (3.6) can be thought of as parametrized by $\lambda \equiv \tilde{\lambda}^{(2)}(0)$: the resulting family of stochastic processes will be denoted $P_{\lambda,0,0}(d\varphi)$. Note that we still have to check that $\lambda \neq 0$ if $\tilde{\lambda}^{(2)}(N)$ is taken conveniently.

§ 4 Proof that the Scalar Field Constructed in § 3 is φ_4^4 with $\lambda > 0$

To prove that the field in § 3 is non-trivial and "admits an asymptotic expansion in $\lambda^{(2)}(0) \equiv \lambda$, $\lambda^{(1)}(0) \equiv \mu \equiv 0$, $\lambda^{(0)}(0) \equiv \nu \equiv 0$ identical to that of φ_4^4" we proceed as follows.

Fix a sequence of bare constants $\lambda^{(2)}(N)$, $\lambda^{(1)}(N)$, $\lambda^{(0)}(N)$ such that $(\lambda^{(0)}(0), \lambda^{(1)}(0), \lambda^{(0)}(0))$ have a limit, as $N \to \infty$, $(\lambda,0,0)$.

If we choose

$$\lambda^{(2)}(N) \equiv \frac{\bar{\lambda}/2}{1+\beta\frac{\bar{\lambda}}{2} N} < \bar{\lambda}_N = \frac{\bar{\lambda}}{1+\beta\bar{\lambda}N} \left(1 + O((N+1)^{-(1+\varepsilon_1)}\right) \tag{4.1}$$

where the inequality holds if N is large enough and $\bar{\lambda}$ is small enough (independently on N), then (3.11) implies

$$\tilde{\lambda}^{(2)}(k-1) \geq \tilde{\lambda}^{(2)}(k) + \beta\tilde{\lambda}^{(2)}(k)^2 - \frac{\kappa_1}{2} \quad \bar{\lambda}_k^{3-\varepsilon} \tag{4.2}$$

which easily implies, if $\bar{\lambda}_o$, i.e. $\bar{\lambda}$, is small enough

$$\tilde{\lambda}^{(2)}(0) \geq \frac{1}{4} \ \bar{\lambda} \tag{4.3}$$

proving that $\lambda > 0$.

We show now the asymptoticity of perturbation theory and the identity of the family $P_{\lambda,0,0}(d\varphi)$ of stochastic processes with the formal perturbation expansion of φ_4^4 .

Consider (2.5) and its formal power series solution (2.6).

Since the coefficients $\beta^{(\alpha)}(\underline{h},\underline{\alpha})$ are "translation invariant" in their frequency dependence (i.e. they are functions of h_v-k) it follows that, if $N \to \infty$,

$$\lambda(k+p) = \gamma^{-\underline{\sigma}(p-1)} \ \underline{\lambda}(k+1) + \sum_{|\underline{n}|\geq 2} \underline{\ell}^{(\underline{n})}(p) \ \underline{\lambda}(k+1)^{\underline{n}} \tag{4.4}$$

where K+1 plays now the role of 0 in (2.6).

This expression can be reinserted into (2.5) to obtain a formal relation between $\underline{\lambda}(k)$ and $\underline{\lambda}(k+1)$ of the type:

$$\underline{\lambda}(k) = \gamma^{\underline{\sigma}} \ \underline{\lambda}(k+1) + \sum_{|\underline{n}|\geq 2} \underline{L}^{\underline{n}} \ (\underline{\lambda}(k+1))^{\underline{n}} \ . \tag{4.5}$$

The reader can check that by substituting (4.4) into (2.5) and using the bound (2.8) on $\underline{\ell}^{\underline{n}}(p)$ one falls exactly on the same expressions met in the proof that (2.8) holds, see § 19 of [10]; this immediately implies a bound on $\underline{L}^{\underline{n}}$ of the form:

$$0 \leq \ |\underline{L}^{\underline{n}}| \ \leq (|\underline{n}|-1)! \ \bar{C}^{|\underline{n}|-1} \tag{4.6}$$

the details are in [10].

The formula (4.5) gives us a formal function

$$\gamma^{\underline{\sigma}}\underline{\lambda} + \underline{B}(\underline{\lambda}) \equiv \gamma^{\underline{\sigma}} \ \underline{\lambda} + \sum_{|\underline{n}|\geq 2} \underline{L}^{\underline{n}} \ \underline{\lambda}^{\underline{n}} \tag{4.7}$$

of three variables: it is the "beta function" of φ_4^4 .

We now assume that for any $p \geq 2$, $|x| \leq B_k$:

$$f_k(x) = \exp \sum_{j=0}^{p+1} \tilde{\lambda}_p^{(j)}(k)\, H_{2j}(x) + \nu_{k,p}(x) \tag{4.8}$$

where for some $\kappa_p > 0$

$$|\tilde{\lambda}_p^{(j)}(k)| \leq \kappa_p \quad \bar{\lambda}_k^{j-1} \qquad 2j > 4$$

$$|\tilde{\lambda}_p^{(2)}(k)| \leq \kappa_p \quad \bar{\lambda}_k$$

$$|\tilde{\lambda}_p^{(j)}(k)| \leq \kappa_p \quad \bar{\lambda}_k^{2} \qquad j = 0,1$$

$$|\ _{k,p}(x)| \leq \kappa_p \quad \bar{\lambda}_k^{p+1-\varepsilon_p} \quad 0 < \varepsilon_p = 1 \ . \tag{4.9}$$

Then, proceeding as in § 3, it follows that

$$\sum_{j=0}^{p+1} \tilde{\lambda}_p^{(j)} \ (K-1)\ H_{2j}(x) = \left[\sum_{q=1}^{p} \frac{\Upsilon^d}{q!}\, \mathcal{E}^T\left(\sum_{j=0}^{p+1} \tilde{\lambda}_p^{(j)}(k)\ H_{2j}(\beta x + \alpha\cdot);q\right)\right]^1 + \tag{4.10}$$

$$+ \text{ "higher order terms"}$$

and "higher order terms" means $O(\bar{\lambda}_k^{p+1-\varepsilon_p})$, $\quad 0 < \varepsilon_p < 1$.

As a matter of fact we can replace $\tilde{\lambda}_p^{(j)}(k)$ by "p-independent" quantities: in fact, it follows from (4.8) that if $f_k(x)$ is defined $\equiv 1$ for $|x| > B_k$ and

$$\log f_k(x) = \sum_{j=0}^{\infty} \lambda^{(j)}(k)\ H_{2j}(x) \tag{4.11}$$

is the Hermite expansion of $\log f_k$, then the (4.9) imply:

$$\tilde{\lambda}_p^{(j)}(k) = \lambda^{(j)}(k) + O(\bar{\lambda}_k^{p+1-\varepsilon_p}) \ .$$

The (4.9) also imply that

$$f_k(v) = \exp \sum_{j=0}^{p+1} \lambda^{(j)}(k)\ H_{2j}(x) + \bar{\ }_{\kappa,p}(x) \tag{4.12}$$

and

$$|\bar{\nu}_{k,p}(x)| \leq \bar{\kappa}_p \quad \bar{\lambda}_k^{p+1-\varepsilon_p} \ . \tag{4.13}$$

Finally it is clear that, by the uniqueness of the formal perturbation series the relation (4.10) <u>must</u> coincide with

$$\underline{\lambda}(k-1) = \gamma^{\underline{\sigma}} \underline{\lambda}(k) + \{B\, \underline{\lambda}(k)\}^{\{\le p\}} + O(\bar{\lambda}_k^{p+1-\varepsilon_p}) \tag{4.14}$$

where $\{\cdot\}^{\{\le p\}}$ is the truncation of the formal power series defining the beta-function (4.7) to the order p .

The (4.14) implies that the dependence of $\underline{\lambda}(k)$ on $\underline{\lambda}(0)$ has to start as prescribed by perturbation theory for φ_4^4 : in fact, one just has to iterate (4.14) at fixed k a finite number of times, X_0 express $\underline{\lambda}(k)$ in terms of $\underline{\lambda}(0)$ and of functions of $\underline{\lambda}(p)$ of formal degree higher than p .

This shows that all the form factors admit an expansion in $\underline{\lambda}(0)$ which is asymptotic and coincides with that of φ_4^4 .

It remains to analyze the $f_k(x)$ fixed k,x and the Schwinger functions: we do not enter into the details of the proof that, as a consequence of the above asymptotic properties of the form factors, also the effective potentials and the Schwinger functions are expressed by an asymptotic series in $\underline{\lambda}(0)$ coinciding with the one of φ_4^4 .

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