Gap generation in the BCS model with finite range time dependent interaction

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Abstract. *In the [BCS] paper the theory of superconductivity was developed for the BCS model, in which the (instantaneous) interaction is only between fermions of opposite momentum and spin. Such model was analyzed by variational methods, finding that a superconducting behavior is energetically favorable. Subsequently it was claimed that in the thermodynamic limit the BCS model is equivalent to the (exactly solvable) quadratic mean field BCS model; a rigorous proof of this claim is however still lacking. In this paper we consider the BCS model with a finite range time dependent interaction, and we prove rigorously its equivalence with the mean field BCS model in the thermodynamic limit if the range is long enough, by a (uniformly convergent) perturbation expansion about mean field theory.*

1. Introduction and main results

Bardeen, Cooper and Schreifer [BCS] developed their theory describing superconductors by the *BCS model*, in which the interaction has infinite range and only fermions of opposite momentum and spin (*Cooper pairs*) interact; the Hamiltonian of this model is

$$H_{BCS} = \sum_\sigma \int_V d\vec{x} a^+_{\vec{x},\sigma} \left(-\frac{\partial^2_{\vec{x}}}{2m}\right) a^-_{\vec{x},\sigma} - \frac{\lambda}{V}[\int_V d\vec{x} a^+_{\vec{x},+} a^+_{\vec{x},-}][\int_V d\vec{y} a^-_{\vec{y},-} a^-_{\vec{y},+}] \tag{1.1}$$

where $a^\pm_{\vec{x},\sigma}$ are creation or annihilation fermionic field operators with spin σ in a d-dimensional box with side L and $V = L^d$, m is the mass and $\lambda > 0$ is the (attractive) coupling. The model is not solvable but it was shown in [BCS] that a superconducting phase is energetically favorable with respect to a normal phase. Later on it was realized that the properties of such superconducting phase are identical to the ones of the *mean field BCS* model, an exactly solvable model in which the interaction is quadratic and the Hamiltonian has the form

$$H_{MF} = |\Delta|^2 + \sum_\sigma \int_V d\vec{x} a^+_{\vec{x},\sigma} \left(-\frac{\partial^2_{\vec{x}}}{2m}\right) a^-_{\vec{x},\sigma} - \sqrt{\lambda}\Delta[\int_V d\vec{y} a^-_{\vec{y},-} a^-_{\vec{y},+}] - \sqrt{\lambda}\bar{\Delta}[\int_V d\vec{y} a^+_{\vec{y},+} a^+_{\vec{y},-}] \tag{1.2}$$

where Δ is a complex number to be determined minimizing the ground state energy (that is Δ solves the BCS gap equation). It has been argued in several papers, starting from [BR],[B],[H],that in the limit $V \to \infty$ the BCS model (1.1) and the mean field model (1.2) *have the same finite temperature correlation functions*; this seems quite natural also by analogy with lattice classical statistical mechanics in which infinite range interaction gives mean field behavior in the thermodynamic limit. Indeed many arguments has been given to support this claim in the last fifty years but, as far as I known, a rigorous proof is still lacking; aim of this paper is show that a simple proof of this claim can be given at least if the instantaneous interaction in the BCS model (1.1) is replaced with a long (but finite) range time dependent interaction

We consider then a generalization of the BCS model in which fermions are on on a cubic lattice with step 1 and a *time dependent* interaction between Cooper pairs is considered; indeed, as stressed for instance in [CEKO], a realistic model for superconductivity should include a bosonic Hamiltonian describing phonons and a boson-fermion interaction, which can be written in a purely fermionic model only if a *time dependent* interaction between fermions is included. It is well known [NO] that the two point (finite temperature and immaginary time) Schwinger function of the BCS model on a lattice can be written as *Grassmann functional integral* in the following way

$$< \psi^{\varepsilon}_{\mathbf{k},\sigma}\psi^{-\varepsilon'}_{\mathbf{k}',\sigma'} >_{L,\beta,h}= \frac{\int P(d\psi)e^{-\mathcal{V}-h\sum_{\sigma=\pm}\int d\mathbf{x}\psi^{\sigma}_{\mathbf{x},\sigma}\psi^{\sigma}_{\mathbf{x},-\sigma}}\psi^{\varepsilon}_{\mathbf{k},\sigma}\psi^{-\varepsilon'}_{\mathbf{k}',\sigma'}}{\int P(d\psi)e^{-\mathcal{V}-h\sum_{\sigma=\pm}\int d\mathbf{x}\psi^{\sigma}_{\mathbf{x},\sigma}\psi^{\sigma}_{\mathbf{x},-\sigma}}} \tag{1.3}$$

where $\int d\mathbf{x} = \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dx_0 \sum_{x\in\Lambda}$ and Λ is a d-dimensional lattice with step 1 and

$$\mathcal{V} = \frac{\lambda}{L^d}\int d\mathbf{x}\int d\mathbf{y} v(x_0-y_0)\psi^{+}_{\mathbf{x},+}\psi^{+}_{\mathbf{x},-}\psi^{-}_{\mathbf{y},-}\psi^{-}_{\mathbf{y},+} \tag{1.4}$$

In the above expression $\{\psi^{\pm}_{\mathbf{k},\sigma}\}$ is a set of *Grassmannian variables*, $\mathbf{k}\in\mathcal{D}_{L,\beta}$ where $\mathcal{D}_{L,\beta}\equiv\mathcal{D}_L\times\mathcal{D}_\beta$, with $\mathcal{D}_L\equiv\{\vec{k}=2\pi\vec{n}/L, n\in\mathbb{Z}^d, -[L/2]\le n_i\le[(L-1)/2]\}$ and $\mathcal{D}_\beta\equiv\{k_0=2(n+1/2)\pi/\beta, \vec{n}\in Z, -M\le n\le M-1\}$, $\sigma=\pm$ and $P(d\psi)$ is a linear functional on the generated Grassmann algebra such that

$$\int P(d\psi)\hat\psi^{-}_{\mathbf{k}_1,\sigma_1}\hat\psi^{+}_{\mathbf{k}_2,\sigma_2} = L^d\beta\delta_{\mathbf{k}_1,\mathbf{k}_2}\delta_{\sigma_1,\sigma_2}\hat g(\mathbf{k}_1)\ , \quad \hat g(\mathbf{k}) = \frac{1}{-ik_0+\varepsilon(k)-\mu}\ . \tag{1.5}$$

where

$$\varepsilon(\vec{k}) = \sum_{i=1}^{d}(1-\cos k_i) \tag{1.6}$$

is the *dispersion relation* and μ is the chemical potential. The *Grassmannian fields* $\psi^{\pm}_{\mathbf{x}}$ are defined by, if $\mathbf{x}=(x,x_0)$ with $x_0\in(-\frac{\beta}{2},\frac{\beta}{2}]$ and $x=(x_1,..,x_d)$ with $x_i=1,2,..,L$,

$$\psi^{\pm}_{\mathbf{x},\sigma} = \frac{1}{L^d\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}}\hat\psi^{\pm}_{\mathbf{k},\sigma}e^{\pm i\mathbf{k}\cdot\mathbf{x}}\ . \tag{1.7}$$

The external field h is introduced to break the number symmetry (it will be removed after the thermodynamic limit $L\to\infty$ will be taken) and $v(x_0-y_0)$ is a *Kac potential* with a *long but finite* range potential κ^{-1}; for definiteness we choose

$$v(t) = \frac{1}{\beta}\sum_{\substack{k_0=\frac{2\pi n_0}{\beta}\\ n_0=0,\pm1,\pm2,..,\pm M}} e^{-ik_0t}\frac{\kappa^2}{k_0^2+\kappa^2} \tag{1.8}$$

Finally M is an ultraviolet cutoff in the time direction introduced to make the Grassmann integral well defined, and (1.3) is a Schwinger function of an Hamiltonian model only in the limit $M\to\infty$; this implies in particular that *the limit $M\to\infty$ must be taken before the thermodynamic limit $V\to\infty$*.

As we mentioned above, it was claimed in [BR] that the quartic BCS model (not solvable) should be equivalent to the mean field BCS model (solvable) in the limit $L\to\infty$. In [BZT] indeed it was claimed the identity of the correlation functions for H_{BCS} and H_{MF} in the thermodynamic limit by using a diagrammatic approach showing by a graph by graph analysis that the difference of the two correlation functions vanishes as V^{-1}. However the validity of such an argument presupposes that the perturbation expansion is convergent and there are many examples in which properties established at a perturbative level are indeed not valid for lacking of convergence (the phenomenon of anomalies in QFT for instance, or perturbation series for resonances in classical mechanics). A similar perturbative argument in a more modern (RG) language has been given in [SHML], in which it is pointed out the similarity of the perturbative expansion of the BCS model with the so called $\frac{1}{N}$ expansion. In [B] and [H] it was argued that H_{BCS} can be replaced by H_{MF} in the infinite volume limit on the basis of the fact that certain commutators are vanishing in the limit. The convergence of H_{BCS} to H_{MF} was indeed proved in [TW] but only in a rather small subspace of states with no single particle excitation and in which the "gap equation" holds. Unluckily this result does not imply the convergence of the finite temperature correlation functions (involving the trace over a complete set of states); the question was then reconsidered later on in [T] in which the equivalence of the correlation functions with H_{BCS} and H_{MF} in the thermodinamic limit was finally proved but only if *the fermionic dispersion relation is assumed to be a constant* $\varepsilon(\vec{k}) = \text{const}$ (degenerate BCS model), as in this case H_{BCS} can be explicitely diagonalized. Finally in [M] a new proof of the equivalence based on a functional integral approach was given, but the analysis involves an unjustified exchange of the $L \to \infty$ limit with the $M \to \infty$ limit.

It is apparently surprising the difficulty in proving that an infinite range interaction like the one in (1.1) leads to a mean field behavior in the thermodynamic limit; indeed in classical statistical mechanics for spin lattice systems the proof of a similar statement is a two line computation. The difficulty in the quantum case can be clearly understood in the functional integral formulation (1.3); in such a representation *the interaction $\mathcal{V}$ is not factorized* contrary to what happens in the Hamiltonian formulation, and this make the model *not* exactly solvable. Of course by replacing $v(x_0 - y_0)$ in (1.3) with a constant (that is considering an *infinite* long range time interaction $\kappa^{-1} = \infty$) the interaction in the functional integral is factorized and the model becomes exactly solvable; mean field behavior in the thermodynamic limit is then easily established, by performing a saddle point analysis essentially identical to the one for long range spin systems, see [L].

Aim of this paper is to prove that even if the range κ^{-1} in (1.8) is *finite*, so that the interaction is not factorized and the model *not* solvable, the BCS model (1.3) is equivalent to the mean field BCS model if κ^{-1} is large enough, in the limit $V \to \infty$; that is the BCS model with time dependent interaction has a phase transition into a superconducting state described by the BCS theory.

Our main result is the following.

Theorem *Assume $\mu < 2$ and $\lambda > 0$; there exist $\beta_c(\lambda) > 0$ and $\kappa_0(\beta) > 0$ such that for $\beta \geq \beta_c(\lambda)$ and $0 < \kappa < \kappa_0(\beta)$ the Schwinger functions (1.3) with $v(x_0 - y_0)$ given by (1.8) are such that*

$$\lim_{h\to 0^+} \lim_{L\to\infty} < \psi^-_{\mathbf{k},\sigma} \psi^+_{\mathbf{k},\sigma} >_{L,\beta,h} = \frac{ik_0 + (\varepsilon(\vec{k}) - \mu)}{k_0^2 + (\varepsilon(\vec{k}) - \mu)^2 + \lambda\Delta^2} \tag{1.9}$$

$$\lim_{h\to 0^+} \lim_{L\to\infty} < \psi^+_{\mathbf{k},+} \psi^+_{-\mathbf{k},-} >_{L,\beta,h} = \frac{\sqrt{\lambda}\Delta}{k_0^2 + (\varepsilon(\vec{k}) - \mu)^2 + \lambda\Delta^2} \tag{1.10}$$

where $\Delta \equiv \Delta(\beta)$ is the real negative solution of the BCS gap equation

$$1 = \lambda \int \frac{d\vec{k}}{(2\pi)^d} \frac{tangh(\frac{\beta}{2}\sqrt{(\varepsilon(\vec{k}) - \mu)^2 + \lambda\Delta^2})}{2\sqrt{(\varepsilon(\vec{k}) - \mu)^2 + \lambda\Delta^2}} \tag{1.11}$$

and $\beta_c(\lambda)$ is the minimal β such that (1.11) admits a solution.

The above Theorem ensures that, at a fixed temperature β and for range κ^{-1} large enough, the BCS model has the same behavior of the BCS mean field model; in particular for $\beta \geq \beta_c$ a gap is generated and the particle number symmetry is broken as $< \psi^+_{\mathbf{k},+} \psi^+_{-\mathbf{k},-} >$ is different from zero; this means that there is a phase transition into a superconducting phase for temperatures low enough. As there is no smallness requirement on λ, the above result implies a gap generation for a range $O(1)$ of the time dependent interaction (by considering $\lambda = O(1)$).

The proof of the above statement is somewhat based on the original ideas in [BZT], as we introduce a perturbative expansion for the difference of the Schwinger function of the BCS or mean field model and we show that each order is vanishing in the limit as the inverse of the volume; the key point is that we avoid (contrary to [BZT]) an expansion in terms of Feynmann diagrams (which has bad convergence properties) but we consider a different expansion in terms of product of determinants (which one cannot expand otherwise the good convergence properties are lost) and we prove that at each order they are vanishing in the thermodynamic limit at least as V^{-1}; the convergence of the expansion is established via determinant bounds for fermionic expectations. Our perturbation theory about the mean field theory uses as a small parameter the inverse range κ of the Kac potential (1.8); this is a classical approach in classical statistical mechanics to prove phase transition beyond mean field theory, see for instance [LMP].

We can prove convergence only for small κ, as it turns out that $\kappa \leq C^{-1}\lambda^{-\frac{1}{2}}\beta^{-\frac{d+5}{2}}$ for a suitable constant C; of course it would be very interesting to prove convergence for larger κ up to $\kappa=\infty$, so otaining a real solution of the BCS model with instantaneous interaction; this would be possible for instance if one could improve the bound for the n-th order of the expansion $n!C^nV^{-n}$, obtained by the diagrammatic analysis, up to C^nV^{-n} (in the present paper we got only the bound C^nV^{-1}).

2. Partial Hubbard-Stratonovich transformation

In momentum space we can write the interaction $\mathcal{V}$ in the following way

$$\mathcal{V}=-\frac{\lambda}{(\beta V)^3}\sum_{\mathbf{k},\mathbf{k}'}\sum_{\substack{p_0=\frac{2\pi}{\beta}(n_0+1)\\ n_0\in Z}} v(p_0)\psi^+_{+,\mathbf{k}}\psi^+_{-,-\mathbf{k}+p_0}\psi^-_{-,\mathbf{k}'}\psi^-_{+,-\mathbf{k}'+p_0} \tag{2.1}$$

where we have used that $p_0=k_{0,1}+k_{0,2}=\frac{2\pi}{\beta}((n_{0,1}+n_{0,2})+1)$. We split the interaction $\mathcal{V}$ as sum over two terms

$$\mathcal{V}=\bar{\mathcal{V}}+\hat{\mathcal{V}} \tag{2.2}$$

$$\bar{\mathcal{V}}=-\frac{\lambda}{(\beta V)^3}\sum_{\mathbf{k},\mathbf{k}'}\psi^+_{+,\mathbf{k}}\psi^+_{-,-\mathbf{k}}\psi^-_{-,\mathbf{k}'}\psi^-_{+,-\mathbf{k}'} \tag{2.3}$$

$$\hat{\mathcal{V}}=-\frac{\lambda}{(\beta V)^3}\sum_{\mathbf{k},\mathbf{k}',|p_0|\geq\frac{2\pi}{\beta}} v(p_0)\psi^+_{+,\mathbf{k}}\psi^+_{-,-\mathbf{k}+p_0}\psi^-_{-,\mathbf{k}'}\psi^-_{+,-\mathbf{k}'+p_0} \tag{2.4}$$

Note that $\bar{\mathcal{V}}$ can be written as, $\varepsilon=\pm$

$$\bar{\mathcal{V}}=-2\Delta^+\Delta^- \qquad \Delta^\sigma\equiv\frac{\sqrt{\lambda}}{(2\beta V)^{1/2}}\mathcal{D}^\sigma=\frac{\sqrt{\lambda}}{(2\beta V)^{1/2}(\beta V)}\sum_{\mathbf{k}}\psi^\sigma_{\mathbf{k},\sigma}\psi^\sigma_{-\mathbf{k},-\sigma} \tag{2.5}$$

that is can be written as the product of the total number of Cooper pairs. Let us consider the *generating function* of the Schwinger functions

$$e^{\mathcal{S}_{L,\beta,h}(J)}=\int P(d\psi)e^{2\Delta^+\Delta^- -\hat{\mathcal{V}}}e^{-h\frac{\sqrt{2\beta V}}{\sqrt{\lambda}}\Delta^+ -h\frac{\sqrt{2\beta V}}{\sqrt{\lambda}}\Delta^-}e^{\int d\mathbf{x}\sum_\sigma[J^+_{\mathbf{x},\sigma}\psi^-_{\mathbf{x},\sigma}+\psi^+_{\mathbf{x},\sigma}J^-_{\mathbf{x},\sigma}]} \tag{2.6}$$

where $J^\pm$ are external Grasssmann field, so that

$$<\psi^\varepsilon_{\mathbf{x},\sigma}\psi^{\varepsilon'}_{\mathbf{y},\sigma}>=\frac{\partial^2}{\partial J^\varepsilon_{\mathbf{x},\sigma}\partial J^{\varepsilon'}_{\mathbf{y},\sigma'}}S(J)|_{J=0} \tag{2.7}$$

By using the identity (*Hubbard-Stratanovich transformation*) ($\phi=u+iv$, $\bar\phi=u-iv$, $u,v\in R$)

$$e^{2ab}=\frac{1}{2\pi}\int_{R^2}dudve^{-\frac{1}{2}|\phi|^2}e^{a\phi+b\bar\phi} \tag{2.8}$$

we can rewrite the above expression as

$$e^{\mathcal{S}_{L,\beta,h}(J)}=\frac{1}{2\pi}\int_{R^2}dudve^{-\frac{1}{2}|\phi|^2}\int P(d\psi)e^{-\hat{\mathcal{V}}}e^{(\phi-h\frac{\sqrt{2\beta V}}{\sqrt{\lambda}})\Delta^+ +(\bar\phi-h\frac{\sqrt{2\beta V}}{\sqrt{\lambda}})\Delta^-}e^{\int d\mathbf{x}\sum_\sigma[J^+_{\mathbf{x},\sigma}\psi^-_{\mathbf{x},\sigma}+\psi^+_{\mathbf{x},\sigma}J^-_{\mathbf{x},\sigma}]} \tag{2.9}$$

Performing the change of variables $(u,v)\to\sqrt{2\beta V}(u,v)$ we obtain

$$e^{\mathcal{S}_{L,\beta,h}(J)}=\frac{\beta V}{2\pi}\int_{R^2}dudve^{-\beta V(v^2+(u+\frac{h}{\sqrt{\lambda}})^2)}e^{-\beta V\mathcal{F}_{L,\beta,h}(u,v)+\mathcal{B}_{L,\beta,h}(u,v,\phi)} \tag{2.10}$$

where

$$e^{-\beta V\mathcal{F}_{L,\beta,h}(u,v)+\mathcal{B}_{L,\beta,h}(u,v,J)}=\int P(d\psi)e^{-\hat{\mathcal{V}}}e^{\sqrt{\lambda}\phi\mathcal{D}^+ +\sqrt{\lambda}\bar\phi\mathcal{D}^-}e^{\int d\mathbf{x}\sum_\sigma[J^+_{\mathbf{x},\sigma}\psi^-_{\mathbf{x},\sigma}+\psi^+_{\mathbf{x},\sigma}J^-_{\mathbf{x},\sigma}]} \tag{2.11}$$

and by definition $\mathcal{B}_{L,\beta,h}(u,v,J)$ is vanishing for $J=0$ so that $\mathcal{F}_{L,\beta,h}(u,v)$ is given by

$$e^{-\beta V\mathcal{F}_{L,\beta,h}(u,v)}=\int P(d\psi)e^{-\hat{\mathcal{V}}}e^{\sqrt{\lambda}\phi\mathcal{D}^+ +\sqrt{\lambda}\bar\phi\mathcal{D}^-} \tag{2.12}$$

We are interested in computing the two point Scwhinger function, given by (2.7)

$$<\psi^\varepsilon_{\mathbf{k},\sigma}\psi^{-\varepsilon'}_{\varepsilon'\mathbf{k},\varepsilon'}>=\frac{1}{Z_{L,\beta,h}}\int_{R^2}dudve^{-\beta V(v^2+(u+\frac{h}{\sqrt{\lambda}})^2)}e^{-\beta V\mathcal{F}_{L,\beta,h}(u,v)}S^{\varepsilon,\varepsilon'}_{L,\beta,h}(\mathbf{k},u,v) \tag{2.13}$$

where $S^{\varepsilon,\varepsilon'}_{L,\beta,h}(u,v)=\partial_{J^\varepsilon_{\mathbf{x}}}\partial_{J^{-\varepsilon'}_{\mathbf{y}}}\mathcal{B}(J,u,v)|_{J=0}$ and

$$Z_{L,\beta,h}=\int_{R^2}dudve^{-\beta V(v^2+(u+\frac{h}{\sqrt{\lambda}})^2)}e^{-\beta V\mathcal{F}_{L,\beta,h}(u,v)} \tag{2.14}$$

We will show in the following section that

$$\mathcal{F}_{L,\beta,h}(u,v)=t_{BCS}+\bar{\mathcal{F}}_{L,\beta,h}(u,v) \tag{2.15}$$

where, if $E(\vec{k})=\varepsilon(\vec{k})-\mu$

$$t_{BCS}=-\frac{1}{V}\sum_{\vec{k}}\frac{2}{\beta}\log\frac{cosh(\frac{\beta}{2}\sqrt{E^2(\vec{k})+\lambda|\phi|^2})}{cosh\frac{\beta}{2}E(\vec{k})} \tag{2.16}$$

is the free energy in the mean field BCS model [BCS] and $\bar{\mathcal{F}}_{L,\beta,h}$ is the perturbation to the mean field; we will show in the following section that, for $0<\kappa<\kappa_0(\beta)$, $\kappa_0(\beta)=C^{-1}\lambda^{-\frac{1}{2}}\beta^{\frac{-d-5}{2}}$, for a suitable constant C

$$|\bar{\mathcal{F}}_{L,\beta,h}(u,v)|\le C\frac{\lambda}{V}(\kappa^2\beta^2)\beta^{d+2} \tag{2.17}$$

hence $V\bar{\mathcal{F}}_{L,\beta,h}(u,v)$ it is uniformly bounded as $V\to\infty$; it is more convenient to call $V\bar{\mathcal{F}}_{L,\beta,h}(u,v)\equiv\hat{\mathcal{F}}_{L,\beta,h}(u,v)$ and we can write the two point Schwinger functions as

$$\frac{1}{Z_{L,\beta,h}}\int_{R^2}dudve^{-\beta V[v^2+(u+\frac{h}{\sqrt{\lambda}})^2+t_{BCS}(u,v)]}e^{-\beta\hat{\mathcal{F}}_{L,\beta,h}(u,v)}S^{\varepsilon,\varepsilon'}_{L,\beta,h}(\mathbf{k},u,v) \tag{2.18}$$

By the saddle point Theorem, for β large enough

$$\lim_{L\to\infty}\frac{e^{-\beta V(v^2+(u+\frac{h}{\sqrt{\lambda}})^2+t_{BCS}(u,v))}}{\int dudve^{-\beta V(v^2+(u+\frac{h}{\sqrt{\lambda}})^2+t_{BCS}(u,v))}}=\delta(u)\delta(v-v_0) \tag{2.19}$$

where u_0 is given by the negative (for $h > 0$) solution of

$$u_0[\lambda \int \frac{d\vec{k}}{(2\pi)^d} \frac{tangh(\frac{\beta}{2}\sqrt{E^2(k) + \lambda u_0^2})}{2\sqrt{(E^2(\vec{k}) + \lambda u_0^2)}} - 1] = \frac{2h}{\sqrt{\lambda}} \tag{2.20}$$

In the limit $h \to 0$ it reduces to the BCS equation (1.11). Moreover it hols also that $S^{\varepsilon,\varepsilon'}_{L,\beta,h} - S^{\varepsilon,\varepsilon',BCS}_{L,\beta,h} = O(\lambda\kappa^2\beta^{d+5}V^{-1})$ (see (3.49) below) so that the Theorem follows. Note also that even for finite L (but large enough) a gap is generated, as the above analysis immediately implies; in fact $< \psi^+\psi^+ >_{L,\beta} = < \psi^+\psi^+ >_\beta + O(V^{-1})$.

Remark. Note that one could perform the Hubbard-Stratonovich transformation for the full interaction $\mathcal{V}$ (not only for $\bar{\mathcal{V}}$) so writing the partition function as, if $\mathcal{N}_M$ is a normalization factor

$$Z_{L,\beta} = \mathcal{N}_M \int [\prod_{p_0} d\phi_{p_0}] e^{-\mathcal{F}_{L,\beta}(\{\phi\})} \tag{2.21}$$

with

$$\mathcal{F}_{L,\beta}(\{\phi\}) = \frac{1}{2}\sum_{p_0}(p_0^2\kappa^{-1}+1)|\phi_{p_0}|^2 - \log\int P(d\psi) e^{-\frac{\sqrt{\lambda}}{2(\beta V)^{\frac{3}{2}}}\sum_{\mathbf{k},p_0}[\phi_{p_0}\psi^+_{\mathbf{k},+}\psi^+_{-\mathbf{k}+p_0,-} + \bar{\phi}_{p_0}\psi^-_{\mathbf{k},-}\psi^-_{-\mathbf{k}+p_0,+}]} \tag{2.22}$$

A similar representation holds also for the Schwinger function. If one performs the limits in the *wrong way*, that is $L \to \infty$ with the ultraviolet cutoff M fixed, the evaluation of the Schwinger function becomes then immediate; in fact $\mathcal{F}_{L,\beta}$ has a global minimum corresponding to $\phi^*_{p_0} = \delta_{p_0,0}\phi^*$, with ϕ^* the solution of the BCS mean field solution; hence, keeping M finite and $L \to \infty$, the saddle point theorem can be applied and the mean field behaviour is immediately recovered for any κ. However the limit $M \to \infty$ must be taken *before* the thermodynamic limit, and the naive saddle point analysis cannot be applied (the corrections are $O(C^M V^{-1})$). The problem becomes equivalent to a statistical mechanical model of an infinite chain of continuous Gaussian spins coupled with a non standard weight function with large number of colors (small N^{-1}), and the analysis is quite harder. In certain somewhat similar problems it has been possible, by cluster expansion techniques and Peierls estimates, to prove that fluctuations around mean field are negligible for large N, see for instance [K],[KRM],[MVH]; if such methods could be applied to recover our result, or, more interestingly, if they could be applied to larger κ possibly up to the case of local interaction ($\kappa = \infty$), is at the moment an open problem.

3. Convergence of series expansion

3.1 *The partition function*

We can "absorb" the quadratic fermion term in the free interaction

$$\int P(d\psi) e^{\sqrt{\lambda}\phi\mathcal{D}^+ + \sqrt{\lambda}\bar{\phi}\mathcal{D}^-} e^{-\hat{\mathcal{V}}(\psi)} = e^{-\beta V t_{BCS}} \int P_\sigma(d\psi) e^{-\hat{\mathcal{V}}(\psi)} \tag{3.1}$$

where

$$\hat{\mathcal{V}}(\psi) = -\frac{\lambda}{V}\int d\mathbf{x}d\mathbf{y}\tilde{v}(x_0 - y_0)\psi^+_{\mathbf{x},+}\psi^+_{\mathbf{x},-}\psi^-_{\mathbf{y},-}\psi^-_{\mathbf{y},+} \tag{3.2}$$

and

$$\tilde{v}(x_0 - y_0) = \frac{1}{\beta}\sum_{k_0 \neq 0} e^{ik_0 t}\frac{\kappa^2}{\kappa^2 + k_0^2} \tag{3.3}$$

and, if $\sigma=\sqrt{\lambda}\phi$

$$P_\sigma(d\psi)=\prod_{\mathbf{k}}\frac{d\hat\psi^+_{\mathbf{k}}d\hat\psi^-_{\mathbf{k}}}{\mathcal{N}(\mathbf{k})}\left\{-\frac{1}{V\beta}\sum_{\mathbf{k}}\sum_{\varepsilon,\varepsilon'=\pm}\hat\psi^{\varepsilon}_{\varepsilon\mathbf{k},\varepsilon}T_{\varepsilon,\varepsilon'}\hat\psi^{-\varepsilon'}_{\varepsilon'\mathbf{k},\varepsilon'}\right\}\tag{3.4}$$

where $\mathcal{N}(\mathbf{k})$ is the normalization of $P_\sigma(d\psi)$ and

$$t_{BCS}=-\frac{1}{V\beta}2\sum_{\mathbf{k}}\log\frac{k_0^2+E^2(k)+|\sigma|^2}{k_0^2+E^2(k)}\tag{3.5}$$

and the 2×2 matrix $T(\mathbf{k})$ is given by

$$T(\mathbf{k})=\begin{pmatrix}-ik_0+E(\vec k) & \sigma\\ \bar\sigma & -ik_0-E(\vec k)\end{pmatrix}\ .\tag{3.6}$$

We can write t_{BCS} as (2.16) and of course $t_1\le\sqrt{|\lambda|}C[1+|\phi|]$. The propagator of $P_\sigma(d\psi)$ is given by, if $\varepsilon,\varepsilon'=\pm$

$$\int P_\sigma(d\psi)\psi^{\varepsilon}_{\mathbf{x},\varepsilon}\psi^{-\varepsilon'}_{\mathbf{y},\varepsilon'}\equiv g_{\varepsilon,\varepsilon'}(\mathbf{x}-\mathbf{y})=\frac{1}{V\beta}\sum_{\mathbf{k}}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}[T^{-1}(\mathbf{k})]_{\varepsilon,\varepsilon'}\ .\tag{3.7}$$

We decompose the propagator $g_{\varepsilon,\varepsilon'}(\mathbf{x}-\mathbf{y})$ into a sum of two propagators supported in the regions of k_0 "large" and "small", respectively. The regions of k_0 large and small are defined in terms of a smooth compact support function $H_0(t)$, $t\in\mathbb{R}$, such that

$$H_0(t)=\begin{cases}1 & \text{if } t<1/\gamma\ ,\\ 0 & \text{if } t>1\ ,\end{cases}\tag{3.8}$$

with $\gamma>1$. We define $h(k_0)=H_0(|k_0|)$ so that we can rewrite $\hat g_{\varepsilon,\varepsilon'}(\mathbf{x}-\mathbf{y})$ as:

$$g_{\varepsilon,\varepsilon'}(\mathbf{x}-\mathbf{y})=\hat g^{(u.v.)}_{\varepsilon,\varepsilon'}(\mathbf{x}-\mathbf{y})+g^{(i.r.)}_{\varepsilon,\varepsilon'}(\mathbf{x}-\mathbf{y})\tag{3.9}$$

where

$$g^{i.r.}_{\varepsilon,\varepsilon'}(\mathbf{x}-\mathbf{y})=\frac{1}{V\beta}\sum_{\mathbf{k}}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}h(k_0)[T^{-1}(\mathbf{k})]_{\varepsilon,\varepsilon'}\tag{3.10}$$

$$g^{u.v.}_{\varepsilon,\varepsilon'}(\mathbf{x}-\mathbf{y})=\frac{1}{V\beta}\sum_{\mathbf{k}}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}(1-h(k_0))[T^{-1}(\mathbf{k})]_{\varepsilon,\varepsilon'}\tag{3.11}$$

are called the infrared and the ultraviolet propagator. In the Appendix we show that

$$\int P_\sigma(d\psi^{u.v.})e^{-\hat{\mathcal{V}}(\psi^{i.r.}+\psi^{u.v})}=e^{-\mathcal{V}^0(\psi^{i.r.})}\tag{3.12}$$

with

$$\mathcal{V}^0=-\frac{\lambda}{V}\int d\mathbf{x}d\mathbf{y}\tilde v(x_0-y_0)\psi^+_{\mathbf{x},+}\psi^+_{\mathbf{x},-}\psi^-_{\mathbf{y},-}\psi^-_{\mathbf{y},+}+\sum_{n=1}^{\infty}\int d\mathbf{x}_1...\int d\mathbf{x}_{2n}W^0_{2n}(\mathbf{x}_1,..,\mathbf{x}_{2n})\prod_{i=1}^{2n}\psi^{\varepsilon_i}_{\mathbf{x}_i,\sigma_i}\tag{3.13}$$

with

$$\frac{1}{V\beta}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2n}|W^{(0)}_n(\mathbf{x}_1,..,\mathbf{x}_{2n})|\le C^n|\lambda|^{max(1,n-1)}(\kappa^2\beta^2)^{max(1,n-1)}\tag{3.14}$$

3.2 *Convergence of the infrared integration*

We define a distance $\mathbf{d}(\mathbf{x},\mathbf{y})_{L,\beta}=(d_\beta(x_0,y_0),d_L(x_1,y_1),...,d_L(x_n,y_n)$ as

$$d_\beta(x_0,y_0)=\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(x_0-y_0)\quad d_L(x_i,y_i)=\frac{L}{\pi}\sin\frac{\pi}{L}(x_i-y_i) \tag{3.15}$$

In order to perform the infrared integration we need the large distances behaviour of the infrared propagator.

Lemma *For any integer N the following bounds hold*

$$|g^{i.r.}_{\varepsilon,\varepsilon}(\mathbf{x}-\mathbf{y})|\leq\beta\frac{C_N}{1+[\beta^{-1}\mathbf{d}(\mathbf{x}-\mathbf{y})]^N} \tag{3.16}$$

$$|g^{i.r.}_{\varepsilon,-\varepsilon}(\mathbf{x}-\mathbf{y})|\leq\beta\frac{\sqrt{\lambda}|\phi|}{\sqrt{\lambda}|\phi|+\beta^{-1}}\frac{C_N}{1+[\beta^{-1}\mathbf{d}(\mathbf{x}-\mathbf{y})]^N} \tag{3.17}$$

Proof. The above bounds follows by integrating by parts. Consider integers $N_0,N_1,..,N_d$ and note that, if $i=1,..,d$

$$d_L(x_i,y_i)^{N_i}d_\beta(x_0,y_0)^{N_0}g^{i.r}_{\varepsilon,\varepsilon'}(\mathbf{x}-\mathbf{y})= \\ e^{-i\pi(xL^{-1}N_i+x_0\beta^{-1}N_0)}(-i)^{N_0+N_i}\frac{1}{V\beta}\sum_{\mathbf{k}}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\partial_k^{N_i}\partial_{k_0}^{N_0}\left[h(k_0)[T_0^{-1}(\mathbf{k}')]_{\varepsilon,\varepsilon'}\right]\ , \tag{3.18}$$

where ∂_k and ∂_{k_0} denote the discrete derivatives. The bound then easily follows noting that $[T_0^{-1}(\mathbf{k}')]_{\varepsilon,\varepsilon'}$ is bounded by $C\beta$ uniformly in σ and each derivatives over it is bounded by an extra β.■

We can write

$$\int P(d\psi^{(i.r.)})e^{-\mathcal{V}^0(\psi^{i.r.})}=e^{\sum_{n=0}^\infty(-1)^n\frac{1}{n!}\mathcal{E}^T(\mathcal{V}^0;n)} \tag{3.19}$$

where $\mathcal{E}^T$ are the fermionic truncated expectations

$$\mathcal{E}^T(X;n)=\frac{\partial^n}{\partial\lambda^n}\log\int P(d\psi)e^{\lambda X(\psi)}\Big|_{\lambda=0}\ . \tag{3.20}$$

We write (3.13) as

$$\sum_P\int d\mathbf{x}_P W(\mathbf{x}_P)\tilde\psi(P) \tag{3.21}$$

where P is the set of field labels appearing in (3.13), $W(\mathbf{x}_P)$ are the kernels in (3.13), that is $\lambda V^{-1}\tilde v(x_0-y_0)$ or $W(\mathbf{x}_1,..,\mathbf{x}_{2n})$ and

$$\tilde\psi(P)=\prod_{f\in P}\psi^{\varepsilon(f)}_{\mathbf{x}(f),\sigma(f)} \tag{3.22}$$

Then we get

$$\mathcal{E}^T(\mathcal{V}^0;n)=\sum_{P_1,...,P_n}\int d\mathbf{x}_{P_1}...\int d\mathbf{x}_{P_n}W(\mathbf{x}_{P_1})...W(\mathbf{x}_{P_n})\mathcal{E}^T(\tilde\psi(P_1),...,\tilde\psi(P_n)) \tag{3.23}$$

The fermionic truncated expectations can be expressed by the formula (see [GM] for example), if $s>1$,

$$\tilde{\mathcal{E}}^T(\tilde\psi(P_1),...,\tilde\psi(P_s))=\sum_T\prod_{l\in T}g_{\varepsilon_l,\varepsilon'_l}(\mathbf{x}_l-\mathbf{y}_l)\int dP_T(\mathbf{t})\det G^T(\mathbf{t})\ , \tag{3.24}$$

where

$$\tilde{\psi}(P) = \prod_{f \in P} \psi^{\varepsilon(f)}_{\mathbf{x}(f),\sigma(f)} \tag{3.25}$$

and

a) T is a set of lines forming an *anchored tree* between the cluster of points $P_1, .., P_s$ *i.e.* T is a set of lines which becomes a tree if one identifies all the points in the same clusters.

c) $\mathbf{t} = \{t_{i,i'} \in [0,1], 1 \le i, i' \le s\}$, $dP_T(\mathbf{t})$ is a probability measure with support on a set of $\mathbf{t}$ such that $t_{i,i'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i \in \mathbb{R}^s$ of unit norm.

d) $G^T(\mathbf{t})$ is a $(N-s+1) \times (N-s+1)$ matrix, $2N = |P_1| + .. + |P_s|$ whose elements are given by $G^T_{ij,i'j'} = t_{i,i'} g_{\varepsilon,\varepsilon'}(\mathbf{x}_{ij} - \mathbf{y}_{i'j'})$ with $(f^-_{ij}, f^+_{i'j'})$ not belonging to T.

If $s = 1$ the sum over T is empty, but we can still use the above equation by interpreting the r.h.s. as 1 if P_1 is empty, and $\det G(P_1)$ otherwise.

We bound the determinant using the well known *Gram-Hadamard inequality*, stating that, if M is a square matrix with elements M_{ij} of the form $M_{ij} =< A_i, B_j >$, where A_i, B_j are vectors in a Hilbert space with scalar product $< \cdot, \cdot >$, then

$$|\det M| \le \prod_i ||A_i|| \cdot ||B_i|| \,. \tag{3.26}$$

where $|| \cdot ||$ is the norm induced by the scalar product.

Let $\mathcal{H} = \mathbb{R}^s \otimes \mathcal{H}_0$, where $\mathcal{H}_0$ is the Hilbert space of complex four dimensional vectors $F(\mathbf{k}) = (F_1(\mathbf{k}), \ldots, F_4(\mathbf{k}))$, $F_i(\mathbf{k})$ being a function on the set $\mathcal{D}$, with scalar product

$$< F, G >= \sum_{i=1}^{4} \frac{1}{L\beta} \sum_{\mathbf{k}} F^*_i(\mathbf{k}) G_i(\mathbf{k}) \,. \tag{3.27}$$

and one checks that

$$G^T_{ij,i'j'} = t_{i,i'} g^{i.r.}_{\varepsilon_l,\varepsilon'_l}(\mathbf{x}_{ij} - \mathbf{y}_{i'j'}) =< \mathbf{u}_i \otimes A_{\mathbf{x}(f_{ij}),\varepsilon(f_{ij})}, \mathbf{u}_{i'} \otimes B_{\mathbf{x}(f_{i'j'}),\varepsilon(f_{i'j'})} > , \tag{3.28}$$

where $\mathbf{u}_i \in \mathbb{R}^s$, $i = 1, \ldots, s$, are the vectors such that $t_{i,i'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$, and

$$\begin{aligned} A_{\mathbf{x},\varepsilon}(\mathbf{k}) &= e^{i\mathbf{k}\mathbf{x}} \frac{\sqrt{h(k_0)}}{\sqrt{k_0^2 + E^2 + |\sigma|^2}} \cdot \begin{cases} (-ik_0 + E(k), 0, \sigma, 0), & \text{if } \varepsilon = +, \\ (0, \bar{\sigma}, 0, 1), & \text{if } \varepsilon = -, \end{cases} \\ B_{\mathbf{x},\varepsilon} &= e^{-i\mathbf{k}\mathbf{y}} \frac{\sqrt{h(k_0)}}{\sqrt{k_0^2 + E^2 + |\sigma|^2}} \cdot \begin{cases} (1, 1, 0, 0), & \text{if } \varepsilon = +, \\ (0, 0, 1, -ik_0 - E(k))), & \text{if } \varepsilon = -. \end{cases} \end{aligned} \tag{3.29}$$

Hence from (3.26), as $||A|| \le C$ and $||B|| \le \beta$ we find

$$|G^T_{ij,i'j'}| \le C_1^{N-s+1} \beta^{N-s+1} \tag{3.30}$$

where C_1 is an $O(1)$ constant.

By using the above formula in (3.23) we get

$$|\mathcal{E}^T(\mathcal{V}^0; n)| \le \sum_{P_1,\ldots,P_n} \beta^{\frac{1}{2}[|P_1|+..+|P_n|]} \int d\mathbf{x}_{P_1} \ldots \int d\mathbf{x}_{P_n} |W(\mathbf{x}_{P_1})| \ldots |W(\mathbf{x}_{P_n})| \sum_T [\prod_{l \in T} |\beta^{-1} g^{i.r.}_{\varepsilon,\varepsilon'}(\mathbf{x}_l - \mathbf{y}_l)|] \tag{3.31}$$

where we have used that $\int dP_T(\mathbf{t}) = 1$. The number of addends in $\sum_T$ is bounded by $n! C_2^n$.

In order to bound the integration over propagators we use antiperiodicity

$$\int_{-\beta}^{\beta} dr_0 g^{i.r.}_{\varepsilon,\varepsilon'}(\vec{r}, r_0) = \int_{-\beta/2}^{\beta/2} dr_0 g^{i.r.}_{\varepsilon,\varepsilon'}(\vec{r}, r_0) + \int_{|r_0|\geq\beta/2} dr_0 g^{i.r.}_{\varepsilon,\varepsilon'}(\vec{r}, r_0) =$$

$$\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dr_0 (g_{\varepsilon,\varepsilon'}(\vec{r}, r_0) + g^{i.r.}_{\varepsilon,\varepsilon'}(\vec{r}, r_0 - \beta)) \tag{3.32}$$

The tree T realizes a connection between all the $\mathcal{V}^0$, and we get the bound

$$\int \prod_{i=1}^{n} d\mathbf{x}_i \frac{1}{n!} \sum_T [\prod_{l\in T} |\beta^{-1} g^{i.r.}_{\varepsilon,\varepsilon'}(\mathbf{x}_l - \mathbf{y}_l)|] \leq (\beta V)\beta^{(n-1)(d+1)} \tag{3.33}$$

In order to perform the integration over the remaining coordinates we note that if $W(\mathbf{x}_P) = \lambda V^{-1} \tilde{v}(x_0)$ then

$$|\tilde{v}(x_0)| = |\frac{1}{\beta} \sum_{\substack{k_0\neq 0 \\ k_0=\frac{2\pi}{\beta}(n_0+1)}} e^{ik_0 t} \frac{\kappa^2}{\kappa^2 + k_0^2}| \leq \frac{1}{(2\pi)^2\beta} \sum_{n_0\neq 0} \frac{\kappa^2\beta^2}{n_0^2} \leq \beta^{-1} C(\kappa\beta)^2 \tag{3.34}$$

so that

$$\int d\mathbf{x} V^{-1} |\tilde{v}(x_0)| \leq C(\kappa\beta)^2 \tag{3.35}$$

if $C$is a suitable constant. On the other hand if $W(\mathbf{x}_P) = W(\mathbf{x}_1, .., \mathbf{x}_{2n})$ we use the bound (3.14); then we get, assuming $\kappa\beta \leq C^{-1}$ in order to sum over P_i

$$|\mathcal{E}^T(\mathcal{V}^0; n)| \leq n! \prod_{i=1}^{n} [\sum_{P_i} C^{P_i} |\lambda|^{max(1,|P_i|/2-1)} (\kappa^2\beta^2)^{max(1,|P_i|/2-1)}] (\beta V) \beta^{(n-1)(d+1)} \beta^{2n} \leq$$

$$(\beta V) n! C^n \lambda^n (\kappa^2\beta^2)^n \beta^{(d+3)n} \beta^{-(d+1)} \leq (\beta V) C^n \lambda^n (\kappa^2 \beta^{d+5})^n \beta^{-(d+1)} n!$$

Finally the following bound can be found, calling $\mathcal{F}_{L,\beta,h} = t_{BCS} + \bar{\mathcal{F}}_{L,\beta,h}$

$$|\bar{\mathcal{F}}_{L,\beta,h}| \leq C\lambda(\kappa^2\beta^{d+5})\beta^{-(d+1)}$$

assuming that $\kappa \leq C^{-1}\lambda^{-\frac{1}{2}}\beta^{\frac{-d-5}{2}} = \kappa_0(\beta)$ to assuring the convergence of the sum over n.

Remark. The above analysis immediately implyes a bound for the *effective potential* $\int P(d\psi^{(i.r.)}) e^{-\mathcal{V}^0(\psi^{i.r.}+\phi)}$ where ϕ is an external fermionic field. The kernels of the effective potential $W^{(n)}$ of order n with n^e external fields obey to the bound

$$\frac{1}{V\beta} \int d\mathbf{x}_1 .. d\mathbf{x}_n |W_n(\mathbf{x}_1, .., \mathbf{x}_n)| \leq C^n \lambda^n (\kappa^2\beta^{d+5})^n \beta^{-\frac{n^e}{2}} \beta^{-(d+1)} \tag{3.36}$$

as now the propagators are $2n - \frac{n_e}{2}$.

3.3 *Extracting a volume factor*

The above analysis says that $\bar{\mathcal{F}}_{L,\beta,h}$, which is the correction to the mean field, is given by a convergent expansion for sufficiently long range interaction $0 < \kappa < \kappa_0(\beta)$. We prove now that we can improve the above bounds by a factor V^{-1} so that the correction vanishes in the thermodynamic limit.

Consider first the case in which in (3.23) there is at least one kernel $W(P_i)$ associated to $\lambda V^{-1}\tilde v$. We can write, by using that for the fields in $\mathcal{E}^T$ holds the rule $\psi_{\mathbf{x}} = \int d\mathbf{x}' g(\mathbf{x},\mathbf{x}')\frac{\partial}{\partial\psi_{\mathbf{x}}}$

$$\frac{1}{n!}\frac{1}{\beta V}\mathcal{E}^T(\mathcal{V}^0;n) = \frac{1}{n!}\frac{1}{\beta V}\{\frac{\lambda}{V}\int d\mathbf{x}_1 d\mathbf{y}_1 d\mathbf{x}_a d\mathbf{x}_b d\mathbf{x}_c d\mathbf{x}_d \tilde v(x_{0,1}-y_{0,1}) \tag{3.37}$$

$$g^{i.r.}_{+,\varepsilon_a}(\mathbf{x}_1-\mathbf{x}_a)g^{i.r.}_{+,\varepsilon_b}(\mathbf{x}_1-\mathbf{x}_b)g^{i.r.}_{-,\varepsilon_c}(\mathbf{y}_1-\mathbf{x}_c)g^{i.r.}_{-,\varepsilon_d}(\mathbf{y}_1-\mathbf{x}_d)H^{(4,n)}_{\varepsilon_a,\varepsilon_b,\varepsilon_c,\varepsilon_d}(\mathbf{x}_a,\mathbf{x}_b,\mathbf{x}_c,\mathbf{x}_d)$$

$$+2\frac{\lambda}{V}\int d\mathbf{x}_1 d\mathbf{y}_1 d\mathbf{x}_a d\mathbf{x}_b g^{i.r.}_{+-}(\mathbf{x}_1-\mathbf{y}_1)\tilde v(x_{0,1}-y_{0,1})g^{i.r.}_{+,\varepsilon_a}(\mathbf{x}_1-\mathbf{x}_a)g^{i.r.}_{+,\varepsilon_b}(\mathbf{y}_1-\mathbf{x}_b)H^{(2,n)}_{\varepsilon_a,\varepsilon_b}(\mathbf{x}_a,\mathbf{x}_b)+$$

$$\sum_{\varepsilon=\pm}\frac{\lambda}{V}\int d\mathbf{x}_1 d\mathbf{y}_1 d\mathbf{x}_a d\mathbf{x}_b g^{i.r.}_{-\varepsilon,-\varepsilon}(\mathbf{0})\tilde v(x_{0,1}-y_{0,1})g^{i.r.}_{\varepsilon',\varepsilon_a}(\mathbf{x}_1-\mathbf{x}_a)g^{i.r.}_{\varepsilon',\varepsilon_b}(\mathbf{y}_1-\mathbf{x}_b)H^{(2,n)}_{\varepsilon_a,\varepsilon_b}(\mathbf{x}_a,\mathbf{x}_b)\}$$

where for $n>1$

$$H^{(4,n)}_{\varepsilon_a,\varepsilon_b,\varepsilon_c,\varepsilon_d}(\mathbf{x}_a,\mathbf{x}_b,\mathbf{x}_c,\mathbf{x}_d) = \frac{\partial}{\partial\psi^{\varepsilon_a}_{\mathbf{x}_a}}\frac{\partial}{\partial\psi^{\varepsilon_b}_{\mathbf{x}_b}}\frac{\partial}{\partial\psi^{\varepsilon_c}_{\mathbf{x}_c}}\frac{\partial}{\partial\psi^{\varepsilon_d}_{\mathbf{x}_d}}\mathcal{E}^T(\mathcal{V}^0;n-1) \tag{3.38}$$

$$H^{(2,n)}_{\varepsilon_a,\varepsilon_b}(\mathbf{x}_a,\mathbf{x}_b) = \frac{\partial}{\partial\psi^{\varepsilon_a}_{\mathbf{x}_a}}\frac{\partial}{\partial\psi^{\varepsilon_b}_{\mathbf{x}_b}}\mathcal{E}^T(\mathcal{V}^0;n-1) \tag{3.39}$$

Note that the last addend in (3.37) (corresponding to a *tadpole* contribution) is vanishing; in fact it can be written in momentum space as

$$\frac{1}{n!}\frac{\lambda}{V}g^{i.r.}(\mathbf{0})\frac{1}{\beta}\sum_{p_0\neq 0}\delta_{p_0,0}\hat v(p_0)\frac{1}{\beta V}\sum_{\mathbf{k}'}\hat g^{i.r.}_{\varepsilon',\varepsilon_a}(\mathbf{k}')\hat g^{i.r.}_{\varepsilon',\varepsilon_b}(\mathbf{k}'+p_0)H^{2,n}_{\varepsilon_a,\varepsilon_b}(\mathbf{k}',p_0)=0$$

The first addend in (3.37) can be bounded in the following way, remembering that $|g^{i.r.}_{\varepsilon,\varepsilon'}(\mathbf{x},\mathbf{y})|\leq\beta$

$$\leq C\frac{\lambda}{V}\beta^{-1}(\kappa\beta)^2\beta^2\sup_{\mathbf{x}_a}[\int d\mathbf{x}_1|g^{i.r}_{+,\varepsilon_a}(\mathbf{x}_1-\mathbf{x}_a)|]\sup_{\mathbf{x}_c}[\int d\mathbf{y}_1 g^{i.r}_{-,\varepsilon_c}(\mathbf{y}_1-\mathbf{x}_c)]$$

$$\frac{1}{n!}\frac{1}{\beta V}\int d\mathbf{x}_a d\mathbf{x}_b d\mathbf{x}_c d\mathbf{x}_d|H^{(4,n)}_{\varepsilon_a,\varepsilon_b,\varepsilon_c,\varepsilon_d}(\mathbf{x}_a,\mathbf{x}_b,\mathbf{x}_c,\mathbf{x}_d)|\,. \tag{3.40}$$

The bound for the last integral is given by (3.36) with $n^e=4$; hence the first addend in (3.37) obeys to the following bound

$$C^n\frac{\lambda^n}{V}\beta^{-1}(\kappa\beta)^2\beta^4\beta^{2(d+1)}(\kappa^2\beta^{d+5})^{n-1}\beta^{-2}\beta^{-(d+1)} \tag{3.41}$$

so that summing over n we have the bound $\frac{\lambda}{V}(\kappa\beta)^2\beta^{d+2}$.
Finally the second addend in (3.37) can be bounded by

$$\frac{\lambda}{V}(\kappa\beta)^2\beta\sup_{\mathbf{x}_a}[\int d\mathbf{x}_1|g^{i.r}_{+,\varepsilon_a}(\mathbf{x}_1-\mathbf{x}_a)|]\sup_{\mathbf{x}_b}[\int d\mathbf{y}_1|g^{i.r}_{+,\varepsilon_b}(\mathbf{y}_1-\mathbf{x}_b)|]\int d\mathbf{x}_a d\mathbf{x}_b|H^{2,n}_{\varepsilon_a,\varepsilon_b}(\mathbf{x}_a,\mathbf{x}_b)| \tag{3.42}$$

and again using (3.36) with $n^e=2$ we get that the second addend in (3.37) obeys to the following bound

$$C^n\frac{\lambda^n}{V}\beta^{-1}(\kappa\beta)^2\beta^3\beta^{2(d+1)}(\kappa^2\beta^{d+5})^{n-1}\beta^{-1}\beta^{-(d+1)}\,. \tag{3.43}$$

In the case there is no $W(P_i)$ associated to $\lambda V^{-1}\tilde v$ we can apply the same reasoning to one of the kernel $W(\mathbf{x}_1,..,\mathbf{x}_n)$. Summing over n (and computing explicitely the case $n=1$) we have the bound $\frac{\lambda}{V}(\kappa^2\beta^2)\beta^{d+2}$; then, for $\kappa\leq\kappa_0$ we get the bound

$$|\bar{\mathcal{F}}_{L,\beta,h}|\leq C\frac{\lambda}{V}(\kappa^2\beta^2)\beta^{d+2} \tag{3.44}$$

3.4 *The integration of S*

By performing the change of variables, if $\psi = (\psi^+, \psi^-)$ and g is the matrix propagator of $P_\sigma(d\psi)$, $\psi_{\mathbf{k}} \to \psi_{\mathbf{k}} + g\psi_{\mathbf{k}}$, we get for two point Schwinger function the formula

$$S_{\varepsilon,\varepsilon'}(\mathbf{k},u,v) = g_{\varepsilon,\varepsilon'}(\mathbf{k}) + \sum_{\varepsilon'',\varepsilon'''} g_{\varepsilon,\varepsilon''}(\mathbf{k}) V_{2;\varepsilon'',\varepsilon'''}(\mathbf{k}) g_{\varepsilon''',\varepsilon'}(\mathbf{k}) \tag{3.45}$$

where $V_{2;\varepsilon',\varepsilon'''}(\mathbf{k})$ is the kernel of the effective potential with two external fields; it can be bounded by

$$|V_{2;\varepsilon,\varepsilon'}(\mathbf{k})| \leq \frac{1}{\beta V}\int d\mathbf{x}\int d\mathbf{y}|V_{2;\varepsilon,\varepsilon'}(\mathbf{x},\mathbf{y})| \tag{3.46}$$

By using (3.36) we get that

$$|V_{2;\varepsilon,\varepsilon'}(\mathbf{k})| \leq C^n\lambda^n(\kappa^2\beta^{d+5})^n\beta^{-1}\beta^{-(d+1)} \tag{3.47}$$

We can improve the above bound as described in §3.3. If there is at least a $W = \lambda V^{-1}\tilde{v}$, we get

$$V_{2;\varepsilon,\varepsilon'}(\mathbf{x},\mathbf{y}) = \frac{\lambda}{V}\int d\mathbf{z}d\mathbf{z}'d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 d\mathbf{x}_4 g^{i.r}(\mathbf{z}-\mathbf{x}_1)g^{i.r}(\mathbf{z}-\mathbf{x}_2)g^{i.r}(\mathbf{z}'-\mathbf{x}_3)g^{i.r}(\mathbf{z}'-\mathbf{x}_4)V_6(\mathbf{x},\mathbf{y},\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)+$$

$$\frac{\lambda}{V}\int d\mathbf{x}d\mathbf{y}d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{z}d\mathbf{z}' g^{i.r}(\mathbf{z}-\mathbf{z}')g^{i.r}(\mathbf{z}-\mathbf{x}_1)g(\mathbf{z}'-\mathbf{x}_2)V_4(\mathbf{x},\mathbf{y},\mathbf{x}_1,\mathbf{x}_2)$$

where we have used that the tadpoles contributions is vanishing. The integral over $\mathbf{x},\mathbf{y}$ times $(\beta V)^{-1}$ of the first addend can be bounded, using also that $sup|g| \leq \beta$

$$\frac{\lambda}{\beta V}(\kappa\beta)^2\frac{1}{\beta V}\int d\mathbf{x}\int d\mathbf{y}d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 d\mathbf{x}_4\int d\mathbf{z}d\mathbf{z}'$$

$$|g^{i.r}(\mathbf{z}-\mathbf{x}_1)g^{i.r}(\mathbf{z}-\mathbf{x}_2)g^{i.r}(\mathbf{z}'-\mathbf{x}_3)g(\mathbf{z}-\mathbf{x}_4)||S(\mathbf{x},\mathbf{y},\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)| \leq \frac{\lambda}{\beta V}(\kappa\beta)^2\beta^2$$

$$(sup_{\mathbf{x}_1}|\int d\mathbf{z}g^{i.r}(\mathbf{z}-\mathbf{x}_1)|)(sup_{\mathbf{x}_3}|\int d\mathbf{z}'g^{i.r}(\mathbf{z}'-\mathbf{x}_3)|)\frac{1}{\beta V}\int d\mathbf{x}\int d\mathbf{y}d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 d\mathbf{x}_4|S(\mathbf{x},\mathbf{y},\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)| \tag{3.48}$$

By using (3.36) we get for (3.45) the bound $\frac{\lambda}{V}(\kappa\beta)^2\beta^{d+1}$. On the other hand the second addend is bounded by

$$\frac{1}{\beta V^2}\beta(sup_{\mathbf{x}_1}|\int d\mathbf{z}g^{i.r}(\mathbf{z}-\mathbf{x}_1)|)(sup_{\mathbf{x}_3}|\int d\mathbf{z}'g^{i.r}(\mathbf{z}'-\mathbf{x}_3)|)\int d\mathbf{x}\int d\mathbf{y}d\mathbf{x}_1 d\mathbf{x}_2|V_2(\mathbf{x},\mathbf{y},\mathbf{x}_1,\mathbf{x}_2)|$$

and by using (3.36) we get the same bound $\frac{\lambda}{V}(\kappa\beta)^2\beta^{d+1}$. If there are no vertex $\tilde{v}$, we can repaet the above argument on the kernels of (3.14). Hence from (3.45)

$$|S_{\varepsilon,\varepsilon'}(\mathbf{k},u,v) - g_{\varepsilon,\varepsilon'}(\mathbf{k})| \leq C\frac{\lambda}{V}(\kappa^2\beta^2)\beta^{d+3} \tag{3.49}$$

Appendix A1. The ultraviolet integration

The integration of the ultraviolet part (3.12) can be done by a multiscale analysis; it is quite standard and we refer to §3 of [BM] (or [GLM]) for details in a similar case. It is convenient to introduce an ultraviolet cut-off N by writing

$$g^{[1,N]}_{\varepsilon,\varepsilon'}(\mathbf{x}-\mathbf{y}) = \sum_{k=1}^{N} g^{(k)}_{\varepsilon,\varepsilon'}(\mathbf{x}-\mathbf{y}) \tag{3.1}$$

where

$$g^{(k)}_{\varepsilon,\varepsilon'}(\mathbf{x}-\mathbf{y}) = \frac{1}{V\beta}\sum_{\mathbf{k}\in\mathcal{D}_{\beta,L}} h_k(k_0)e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}g_{\varepsilon,\varepsilon'}(\mathbf{k}) \tag{3.2}$$

with $h_k(k_0) = H_0(\gamma^{-k}|k_0|) - H_0(\gamma^{-k+1}|k_0|)$. Note that $\lim_{N\to\infty} g^{[1,N]}(\mathbf{x}-\mathbf{y}) = g^{(u.v.)}(\mathbf{x}-\mathbf{y})$ and that, for any integer $K\geq 0$, $g^{(k)}(\mathbf{x}-\mathbf{y})$ satisfies the bound, for any integer K

$$|g^{(k)}_{\varepsilon,\varepsilon'}(\mathbf{x}-\mathbf{y})| \leq \frac{C_K}{1+(\gamma^k|x_0-y_0|+|\vec{x}-\vec{y}|)^K} \tag{3.3}$$

We associate to any propagator $g^{(k)}_{\varepsilon,\varepsilon'}(\mathbf{x},\mathbf{y})$ a Grassmann field $\psi^{(k)}$ and a Gaussian integration $P(d\psi^{(k)})$ with propagator $g^{(k)}(\mathbf{x},\mathbf{y})$. We can rewrite $\mathcal{V}^{(0)}$ as:

$$\mathcal{V}^{(0)}(\phi) + V\beta E_1 = -\lim_{N\to\infty}\log\int P(d\psi^{(1)})\cdots P(d\psi^{(N)})e^{-V(\psi^{[1,N]}+\phi)} \tag{3.4}$$

We can integrate iteratively the fields on scale $N, N-1,\ldots,h+1$ and after each integration we can rewrite the r.h.s. of (3.4) in terms of a new effective potential $\mathcal{V}^{(h)}$:

$$(3.4) = \lim_{N\to\infty}\left\{V\beta\sum_{j=h+1}^{N}E_j - \log\int P(d\psi^{(1)})\cdots P(d\psi^{(h)})e^{-\mathcal{V}^{(h)}(\psi^{[1,h]}+\phi)}\right\} \tag{3.5}$$

with $\mathcal{V}^{(h)}(\psi^{[1,h]})$ admitting a representation in terms of *trees* defined in the following way:

1) Let us consider the family of all trees which can be constructed by joining a point r, the *root*, with an ordered set of $n\geq 1$ points, the *endpoints* of the *unlabeled tree*, so that r is not a branching point. n will be called the *order* of the unlabeled tree and the branching points will be called the *non trivial vertices*. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order.

Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. It is then easy to see that the number of unlabeled trees with n end-points is bounded by 4^n.

We shall consider also the *labeled trees* (to be called simply trees in the following); they are defined by associating some labels with the unlabeled trees, as explained in the following items.

2) We associate a label $h\geq 0$ with the root and we denote $\mathcal{T}_{(h,N),n}$ the corresponding set of labeled trees with n endpoints. Moreover, we introduce a family of vertical lines, labeled by an an integer taking values in $[h,N]$, and we represent any tree $\tau\in\mathcal{T}_{(h,N),n}$ so that, if v is an endpoint or a non trivial vertex, it is contained in a vertical line with index $h_v > h$, to be called the *scale* of v, while the root is on the line with index h. The tree will intersect the vertical lines in set of points different from the root and the end-points; these points will be called *trivial vertices*. The set of the *vertices* of τ will be the union of the endpoints, the trivial vertices and the non trivial vertices. Note that, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$.

Moreover, there is only one vertex immediately following the root, which will be denoted v_0 and can not be an endpoint; its scale is $h+1$.

3) With each endpoint v of scale h_v we associate $\hat{\mathcal{V}}(\psi^{(\leq h_v+1)})$ (2.4). Given a vertex v which is not an end-point, $\mathbf{x}_v$ will denote the family of all space-time points associated with one of the endpoints following v.

4) We introduce a *field label* f to distinguish the field variables appearing in the terms $\hat{\mathcal{V}}$ associated with the endpoints. The set of field labels associated with the endpoint v will be called I_v. Analogously, if v is not an endpoint, we shall call I_v the set of field labels associated with the endpoints following the vertex v; $\mathbf{x}(f)$, and $\sigma(f)$ will denote the space-time point, the σ index and the ω index, respectively, of the field variable with label f.

We call *trivial tree* a tree containing only the root and an endpoint and we will define $V^{(h+1)}(\psi^{(\leq h+1)},\tau) = \hat{\mathcal{V}}(\psi^{(\leq h+1)})$.

The effective potential can be written then in the following way:

$$\mathcal{V}^{(h)}(\psi^{(\leq h)}) + L\beta\tilde{E}_{h+1} = \sum_{n=1}^{\infty} \sum_{\tau\in\mathcal{T}_{(h,N)n}} V^{(h)}(\tau, \psi^{(\leq h)}) \qquad (3.6)\ ,$$

where, if v_0 is the first vertex of the non trivial tree τ and $\tau_1, .., \tau_s$ ($s = s_{v_0}$) are the subtrees of τ with root v_0,
$V^{(h)}(\tau, \psi^{(\leq h)})$ is defined inductively by the relation, if $s > 1$

$$\begin{aligned} &V^{(h)}(\tau, \psi^{(\leq h)}) = \\ &\frac{(-1)^{s+1}}{s!}\mathcal{E}^T_{h+1}[V^{(h+1)}(\tau_1, \psi^{(\leq h+1)}); ..; V^{(h+1)}(\tau_s, \psi^{(\leq h+1)})]\ , \end{aligned} \qquad (3.7)$$

If $s = 1$ then $V^{(h)}(\tau, \psi^{(\leq h)}) = \mathcal{E}_{h+1}[V^{(h+1)}(\tau_1, \psi^{(\leq h+1)})]$ if τ_1 is not a trivial tree; on the contrary if τ_1 is trivial then $V^{(h)}(\tau, \psi^{(\leq k)}) = \mathcal{E}_{k+1}[\hat{\mathcal{V}}^{(h+1)}(\psi^{(\leq h+1)})] - \hat{\mathcal{V}}^{(h+1)}(\psi^{\leq(h)})$.

By iterating (3.7) we can write $V^{(h)}(\tau, \psi^{(\leq h)})$ in the following way. We associate with any vertex v of the tree a subset P_v of I_v, the *external fields* of v. These subsets must satisfy various constraints. First of all, if v is not an endpoint and $v_1, \ldots, v_{s_v}$ are the vertices immediately following it, then $P_v \subset \cup_i P_{v_i}$; if v is an endpoint, $P_v = I_v$. We shall denote Q_{v_i} the intersection of P_v and P_{v_i}; this definition implies that $P_v = \cup_i Q_{v_i}$. The subsets $P_{v_i} \backslash Q_{v_i}$, whose union will be made, by definition, of the *internal fields* of v, have to be non empty, if $s_v > 1$ or if $s_v = 1$ and v_1 is an endpoint.

Given $\tau \in \mathcal{T}_{(h,N)n}$, there are many possible choices of the subsets P_v, $v \in \tau$, compatible with all the constraints; we shall denote $\mathcal{P}_\tau$ the family of all these choices and $\mathbf{P}$ the elements of $\mathcal{P}_\tau$. Then we can write

$$V^{(h)}(\tau, \psi^{(\leq h)}) = \sum_{\mathbf{P}\in\mathcal{P}_\tau} V^{(h)}(\tau, \mathbf{P})\ ; \qquad (3.8)$$

$V^{(h)}(\tau, \mathbf{P})$ can be represented as

$$V^{(h)}(\tau, \mathbf{P}) = \int d\mathbf{x}_{v_0} \tilde{\psi}^{(\leq h)}(P_{v_0}) K^{(h+1)}_{\tau,\mathbf{P}}(\mathbf{x}_{v_0})\ , \qquad (3.9)$$

with $K^{(h+1)}_{\tau,\mathbf{P}}(\mathbf{x}_{v_0})$ defined inductively (recall that $h_{v_0} = h+1$) by the equation, valid for any $v \in \tau$ which is not an endpoint,

$$\begin{aligned} K^{(h_v)}_{\tau,\mathbf{P}}(\mathbf{x}_v) = \frac{1}{s_v!}\prod_{i=1}^{s_v}[K^{(h_v+1)}_{v_i}(\mathbf{x}_{v_i})]\ \cdot \\ \cdot\ \mathcal{E}^T_{h_v}[\tilde{\psi}^{(h_v)}(P_{v_1}\backslash Q_{v_1}), \ldots, \tilde{\psi}^{(h_v)}(P_{v_{s_v}}\backslash Q_{v_{s_v}})]\ , \end{aligned} \qquad (3.10)$$

where if v is an endpoint $K^{(h_v)}_v(\mathbf{x}_v)$ is the kernel $\lambda V^{-1}\tilde{v}(\mathbf{x})$.

By using the representation of the truncated expectation analogous to (3.20) and the Gram inequality we get that the contribution from a tree $\tau \in \mathcal{T}_{(1,h),n}$ associated to a kernel with $2l$ external legs can be bounded as (see §3.14 [BM] for details in a similar case):

$$\begin{aligned} &\frac{1}{V\beta}\int d\mathbf{x}_1 \cdots d\mathbf{x}_{2l} |W^{(h)}_{2l}(\tau; \mathbf{x}_1; \ldots; \mathbf{x}_{2l})| \leq \\ &\leq C^n |\lambda(\kappa\beta)^2|^n \gamma^{-h(n-1)} \prod_{v\ \text{not e.p.}} \gamma^{-(h_v - h_{v'})(n_v - 1 + z_v)}\ , \end{aligned} \qquad (3.11)$$

where v' is the vertex immediately preceding v on τ, n_v is the number of endpoints following v on τ and $z_v = 1$ if $n_v = 1$ and 0 otherwise. In deriving (3.11) we have used that, if v is a vertex with $n_v = 1$ (tadpole contribution), to such vertex is associated

$$\frac{\lambda}{V}\sum_\sigma \int d\mathbf{x} \int d\mathbf{y} \tilde{v}(x_0 - y_0) \psi^+_{\mathbf{x},-\sigma} \psi^-_{\mathbf{y},-\sigma} g^{(h_v)}_{\sigma,\sigma}(\mathbf{x} - \mathbf{y}) \qquad (3.12)$$

where we have used that the contraction of $\psi^+\psi^+$ or $\psi^-\psi^-$ in $\hat{\mathcal{V}}$ is vanishing by momentum conservation; then we can bound the kernel of (3.12) using the propagator $g^{(h_v)}_{\sigma,\sigma}(\mathbf{x}-\mathbf{y})$ to integrate over the coordinates (instead of the interaction) , so obtaining the bound $C\lambda(\kappa\beta)^2V^{-1}\gamma^{-h_v}$.

Then, proceeding as in §3.14 of [BM], one can sum over τ and the bound (3.14) is proved; moreover by proceeding as in §3.3 it is easy to see that we can extract a factor $O(V^{-1})$ from each kernel $W^{(h)}_{2l}$.

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