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LINDSTEDT SERIES FOR PERIODIC SOLUTIONS OF BEAM EQUATIONS WITH QUADRATIC AND VELOCITY DEPENDENT NONLINEARITIES

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Abstract. We prove the existence of small amplitude periodic solutions, for a large Lebesgue measure set of frequencies, in the nonlinear beam equation with a weak quadratic and velocity dependent nonlinearity and with Dirichelet boundary conditions. Such nonlinear PDE can be regarded as a simple model describing oscillations of flexible structures like suspension bridges in presence of an uniform wind flow. The periodic solutions are explicitly constructed by a convergent perturbative expansion which can be considered the analogue of the Lindstedt series expansion for the invariant tori in classical mechanics. The periodic solutions are defined only in a Cantor set, and resummation techniques of divergent powers series are used in order to control the small divisors problem.

1. Introduction and Main Results

1.1. The search of periodic solutions in nonlinear wave equations has attracted a wide interest in recent times. In the finite dimensional case the problem has its analogous in the study of periodic orbits close to elliptic equilibrium points: results of existence have been obtained in such a case starting from Lyapunov [20]. Systems with infinitely many degrees of freedom (as the nonlinear wave equation, the beam equation, the nonlinear Schrödinger equation and other PDE systems) have been studied much more recently; the problem is much more difficult because of the presence of a *small divisors problem*, which is absent in the finite dimensional case, and one has to prove an infinite dimensional KAM theorem to overcome such difficulty. Periodic or quasi periodic solutions in PDEs have been obtained for instance by Wayne [21], Kuksin [18], Kuksin and Poeschel [19] by KAM methods and by Craig and Wayne [9] and Bourgain [5],[7],[6] by a Lyapunov-Schmidt decomposition and Newton iteration scheme; this last method is flexible and it has been applied in more than one dimension and for non hamiltonian PDEs. More recently periodic solutions in PDEs have been constructed [13], [14] by convergent power series expansion similar to the Lindstedt series for KAM tori [10],[11] using resummation techniques of divergent power series to control the small divisor problem;

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this method has the advantage of being constructive and it can be also extended to non hamiltonian PDEs.

Generally in the literature the nonlinear terms in the PDEs are assumed *odd* and *velocity-independent*, as such features considerably simplify the analysis. A velocity dependent non linearity has been considered only in [5], in which the string equation with a nonlinear term u_t^2 and *periodic* boundary conditions is considered. The recent papers [3] and [4] consider the massless string equation, under Dirichlet boundary conditions, with velocity independent (but otherwise quite general) nonlinearities.

Aim of this paper is to construct periodic solutions in a *beam equation* with an even and velocity dependent nonlinearity and *Dirichelet* boundary conditions;

$$\begin{cases} v_{tt} + \partial_x^4 v + \mu v = av^2 + bv_t^2, \\ v(0,t) = v(\pi,t) = 0, \end{cases} \tag{1.1}$$

where a, b, μ are suitable parameters. As it will appear clear in the following, our results could be easily extended to include more general nonlinearities. With respect to [5], we have considered the beam instead of the wave equation, leading to a simpler small divisor problem; on the other hand Dirichelet boundary conditions and even nonlinearities introduce various regularity problems which are not present in the case of periodic boundary conditions considered in [5].

The interest of (1.1) lies moreover in the fact that it can be regarded as a simple model describing oscillations of flexible structures; for instance, see [16],[8], a suspension bridge subjected to elastic forces due to suspensions and to forces caused by a uniform wind-flow has been described by a beam equation with a nonlinear terms quadratic in v (describing the anharmonic elastic forces) and depending also from v_t (to take into account the forces due to the wind flow). Another applications of PDE with this kind of nonlinear terms is in [17] to describe the oscillations of the atmosphere on the flat earth. In the literature there is no proof of existence of periodic solutions in a large set for such a problem. We will construct such solutions generalizing to the present case the approach based on Lindstedt series expansion already adopted first in [12] to prove the existence of periodic solutions in a zero measure set, and later on generalized to construct periodic solutions in a large measure set in [12], [13],[14].

Equation (1.1) has an elliptic fixed point at $v = 0$ with frequencies $\omega_m = \sqrt{m^4 + \mu}$ so that for $a = b = 0$ every solution of (1.1) can be written as

$$v(x,t) = \sum_{m=0}^{\infty} A_m \cos(\omega_m t + \theta_m) \sin mx, \tag{1.2}$$

where θ_m is an arbitrary phase. In particular if $\mu \notin \mathbb{Q}$ the fixed point is non-degenerate so that the only ω_1 periodic solutions are:

$$A_1 \cos \omega_1 t \sin x\,, \quad A_1 \in \mathbb{R} \tag{1.3}$$

We will prove that there are periodic solutions of the nonlinear PDE (1.1) with $a \neq 0, b \neq 0$ with frequency $\Omega = \omega_1 + \varepsilon$, for any small ε in a Cantor set, which are ε-close to (1.3); this is possible provided that we choose a proper A_1 which is $O(\sqrt{\varepsilon})$.

1.2. To face the small divisor problem, some Diophantine conditions must be imposed on the mass μ.

Definition 1. *We call $M(\gamma)$, $\gamma \leq 2^{-6}$, the set $\mu \in [0, \mu_0]$, $\mu_0 = \frac{1}{8}$ verifying the following Diophantine condition*

$$|\omega_1 n \pm \omega_m| \geq \gamma |n|^{-\tau_0} \qquad \forall n \in \mathbb{Z} \setminus \{0\} \text{ and } \forall m \in \mathbb{N} \setminus \{1\} \tag{1.4}$$
$$|\omega_1 n \pm \omega_m \pm \omega_{m'}| \geq \gamma |n|^{-\tau_0} \qquad \forall n \in \mathbb{Z} \setminus \{0\} \text{ and } \forall m, m' \in \mathbb{N} \setminus \{1\}$$

It will be shown in Appendix A1 that the set of μ verifying (1.4), for some positive γ, is of measure $O(\mu_0)$ provided that $\tau_0 \geq 4$ and γ is small enough.

Our main result is the following Theorem.

Theorem 1. *Generically in a, b, for any $\mu \in M(\gamma)$ there exists an $\varepsilon_0 > 0$ and a Cantor set $\mathcal{C}(\gamma) \subset (0, \varepsilon_0)$ verifying $\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}$ meas $(\mathcal{C}(\gamma) \cap (0, \varepsilon)) = 1$ such that for all $\varepsilon \in \mathcal{C}(\gamma)$ there exists a periodic solution $v(x, \Omega t) : \mathbb{T}^2 \to \mathbb{R}$ of (1.1) , with $\Omega = \omega_1 + \varepsilon$, of the form*

$$v(x, \Omega t, \varepsilon) = \sqrt{\varepsilon} u(x, \Omega t; \varepsilon) = \sqrt{\varepsilon} \sum_{n \in \mathbb{Z}} \sum_{m=1}^{\infty} e^{in\Omega t} \sin(mx) u_{n,m} \tag{1.5}$$

with $u_{n,m} = u_{-n,m}$ and

$$|u_{n,m}| \leq \sqrt{\varepsilon} \frac{C_0 e^{-\sigma |n|}}{m^7}, \quad (n, m) \neq (\pm 1, 1), \quad u_{\pm 1,1} = O_\varepsilon(1) \tag{1.6}$$

with suitable constants C_0, σ.

Note that in presence of odd nonlinearities, like v^3, one can continue the periodic solution in an analytic solution both is space and time, see [13] ; on the contrary, in presence of even or velocity depending nonlinearities, like in the present case, the periodic solutions are *not analytic* in space and this lack of regularity is reflected in some complications in their constructions.

1.3. By inserting (1.6) in (1.1) we get a closed equation for the coefficients $u_{n,m}(\varepsilon) \equiv u_{n,m}$

$$u_{n,m} \left[-\Omega^2 n^2 + \omega_m^2 \right] = \sqrt{\varepsilon} \hat{f}_{n,m}(u). \tag{1.7}$$

where ω_m is defined in (1.2) and

$$\hat{f}_{n,m}(u) = \frac{\Omega}{2\pi^2} \int_0^\pi dx \int_0^{2\pi/\Omega} dt \sin(mx) e^{-in\Omega t} (a u^2 + b u_t^2) \tag{1.8}$$

More explicitly, see Appendix A2, (1.8) can be written as

$$\hat{f}_{n,m} = \sum^*_{\substack{n_1, m_1 \\ n_2, m_2}} v_{m,m_1,m_2} \delta_{n_1+n_2,n} (a + b(i\Omega n_1)(i\Omega n_2)) u_{n_1,m_1} u_{n_2,m_2} \tag{1.9}$$

where $\sum^*$ means that the sum is over m, m_1, m_2 such that $m \pm m_1 \pm m_2 = odd$ and

$$v_{m,m_1,m_2} = \frac{4 m m_1 m_2}{\pi (m^2 - (m_1 - m_2)^2)(m^2 - (m_1 + m_2)^2)} \tag{1.10}$$

One could try to write a power series expansion in ε for $u(x,t)$, using (1.7) to get recursive equations for the coefficients. However by proceeding in this way one finds that the coefficient of order k is given by a sum of terms some of which of order $O(k!^{\alpha})$, for some constant α. This is the same phenomenon occurring in the Lindstedt series for invariant KAM tori [10],[11] in the case of quasi-integrable Hamiltonian systems; in such a case however one can show that there are *cancellations* between the terms contributing to the coefficient of order k, which at the end admits a bound C^k, for a suitable constant C. On the contrary such cancellations are absent in the present case and we have to proceed in a different way, essentially equivalent to a *resummation*.

We write

$$\begin{cases} \eta u_{1,1} \equiv \eta q = \hat{f}_{1,1}(u) & \text{if } (|n|,m)=(1,1) \\ \left[-\Omega^2 n^2 + \omega_m^2 + n\nu_{n,m}\right] u_{n,m} \equiv g_{n,m}^{-1} u_{n,m} = \eta(\hat{f}_{n,m}(u) + n l_{n,m} u_{n,m}) & \text{otherwise} . \end{cases} \quad (1.11)$$

Naturally equation (1.11) coincides with (1.12) provided that:

$$\eta = \sqrt{\varepsilon} \quad \nu_{n,m} = \eta l_{n,m} \quad (1.12)$$

We introduce the following definition.

Definition 2. *We define Λ the set of (n,m) such that $|\omega_1|n| - m^2| \leq 1 + \varepsilon_0|n|$.*

We define $\mathcal{D}$, subset of $(\varepsilon,\nu) \in \mathbb{R}^+ \times l_\infty$, as

$$\mathcal{D} := \{(\varepsilon,\nu) : 0 < \varepsilon < \varepsilon_0 \,,\ \max_{n,m} |\nu_{n,m}| < c\varepsilon_0,\ \nu_{n,m} = 0 \text{ if } (n,m) \notin \Lambda\} \quad (1.13)$$

For any $\mu \in M(4\gamma)$, $\tau > \tau_0+5$, we define a subset $\mathcal{D}(\gamma) \subset \mathcal{D}$ of couples $(\varepsilon,\nu) \in \mathcal{D}$ verifying the following Diophantine conditions

$$\left|\Omega n \pm \sqrt{\omega_m^2 + n\nu_{n,m}}\right| \geq \gamma |n|^{-\tau} \qquad \forall n \in \mathbb{Z} \setminus \{0\} \text{ and } \forall m \in \mathbb{N} \setminus \{1\} \quad (1.14)$$

$$\left|\Omega(n_2 - n_1) \pm \sqrt{\omega_{m_1}^2 + n_1\nu_{n_1,m_1}} \pm \sqrt{\omega_{m_2}^2 + n_2\nu_{n_2,m_2}}\right| \geq \gamma |n_2 - n_1|^{-\tau}$$
$$\forall n_1, n_2 \in \mathbb{Z} \setminus \{0\} \text{ and } \forall m_1 \neq m_2 \in \mathbb{N} : \quad (n_i,m_i) \in \Lambda \,, \ \ i = 1,2 \quad (1.15)$$

We call (1.14) and (1.15), respectively the first and second Melnikov conditions. Our strategy (similar to the one followed in [12],[13],[14]) in order to prove Theorem 1 is the following:

1) First we consider (ε,ν) as independent parameters belonging to $D(\gamma)$ (so that the Melnikov conditions are verified) and we show that it is possible to *find* an appropriate $l_{n,m}(\eta,\varepsilon,\nu)$, well defined for $|\eta| \leq \eta_0$ and $(\varepsilon,\nu) \in D(\gamma)$, such that (1.11) admits a solution $u_{n,m}$ analytic in η; both $u_{n,m}$ and $l_{n,m}$ are expressed by convergent power series in η. Using a technique inspired by [4], we extend $l_{n,m}$ to a C^1 function,$l^E_{n,m}$,defined on the square $\mathcal{D}$; $l^E_{n,m}$ coincides with $l_{n,m}$ in the set $D(2\gamma)$.

2)The solution $u_{n,m}$ defined above is a solution of (1.7) only if (1.12) is verified; we show (Proposition 2) that we can find $\nu = \nu(\varepsilon)$ so that (1.12) is verified for all $(\varepsilon,\nu(\varepsilon)) \in D(2\gamma)$; more precisely $\nu(\varepsilon)$ solves the equation $\nu_{n,m} = \sqrt{\varepsilon} l^E_{n,m}(\sqrt{\varepsilon},\varepsilon,\nu)$: hence replacing $\nu_{n,m}$ with $\nu_{n,m}(\varepsilon)$ in the expansion for $u_{n,m}$ we get the solution of (1.7).

Proposition 1. *Assume that* $\mu \in M(4\gamma)$ *and* $(\nu, \varepsilon) \in D(\gamma)$. *Let* C_0, C_1, C_2, σ *be positive constants. It is possible to find a sequence*

$$\{l_{n,m}(\eta, \varepsilon, \nu)\}_{(n,m)\in\mathbb{Z}^2\setminus\{(\pm1,1)\}} \tag{1.16}$$

such that:

(i) There exists a unique solution $u(\eta, \nu, \varepsilon; x, t)$, *analytic in* t *and* C^5 *in* x, *of equation (1.11);* u *is analytic in* η *for* $|\eta| \le \eta_0$ *and is such that:*

$$|u(\eta, \nu, \varepsilon; x, t) - u_{1,1}(\nu, \varepsilon) \cos \Omega t \sin x| \le |\eta| C_0. \tag{1.17}$$

for a proper $u_{1,1}(\nu, \varepsilon,)$.

(ii) The sequence $l_{n,m}(\eta, \varepsilon, \nu)$ *is analytic in* η *and uniformly bounded for* $(\varepsilon, \nu) \in \mathcal{D}(\gamma)$:

$$|l(\eta, \varepsilon, \nu)|_\infty \equiv \max_{n,m} |l_{n,m}| \le C_1 |\eta|. \tag{1.18}$$

(iii) The functions $u_{n,m}(\eta, \varepsilon, \nu)$ *and* $l_{n,m}(\eta, \varepsilon, \nu)$ *can be extended to* C^1 *functions, denoted by* $u^E_{n,m}(\eta, \varepsilon, \nu)$, $l^E_{n,m}(\eta, \varepsilon, \nu)$, *on the set* $\mathcal{D}$, *such that*

$$l^E_{n,m}(\eta, \varepsilon, \nu) = l_{n,m}(\eta, \varepsilon, \nu) \quad \forall\, (\varepsilon, \nu) \in \mathcal{D}(2\gamma) \tag{1.19}$$

The same is true for $u^E_{n,m}$.

(iv) $l^E_{n,m}(\eta, \varepsilon, \nu)$ *respects the bounds:*

$$|l^E(\eta, \varepsilon, \nu)|_\infty \le |\eta| C_2\,, |\partial_\varepsilon l^E(\eta, \varepsilon, \nu)|_\infty \le |\eta| C_2\,, |\partial_{\nu_{n,m}} l^E(\eta, \varepsilon, \nu)|_\infty \le |\eta| C_2\,, \tag{1.20}$$

$$|\sum_{(n,m)\in\Lambda} \partial_{\nu_{n,m}} l^E(\eta, \varepsilon, \nu)|_\infty \le |\eta| C_2\,, \quad |u^E_{n,m}(\eta, \varepsilon, \nu)| \le |\eta| \frac{1}{m^7} C_2 e^{-\sigma|n|} \tag{1.21}$$

Once we have proved Proposition 1, we solve the compatibility equation for the extended counterterm function $l^E_{n,m}(\eta = \sqrt{\varepsilon}, \varepsilon, \nu)$ which is well defined provided that we choose ε_0 so that $\varepsilon_0 < \eta_0^2$.

Proposition 2. *For all* $(n, m) \ne (\pm1, 1)$, *exist* C^2 *functions* $\nu_{n,m}(\varepsilon) : (0, \varepsilon_0) \to (-c\varepsilon_0, c\varepsilon_0)$ *such that*

(i) $\nu_{n,m}(\varepsilon)$ *verifies*

$$\nu_{n,m}(\varepsilon) = \sqrt{\varepsilon}\, l^E_{n,m}(\sqrt{\varepsilon}, \varepsilon, \nu_{n,m}(\varepsilon)); \tag{1.22}$$

and is such that

$$|\nu_{n,m}(\varepsilon)| \le C\varepsilon\,, \quad |\partial_\varepsilon \nu_{n,m}(\varepsilon)| \le C \tag{1.23}$$

for a suitable constant C;

(ii) the set $\mathcal{C} \equiv \mathcal{C}(2\gamma)$ *defined by* $\varepsilon \in (0, \varepsilon_0)$ *and the conditions:*

$$\left|\Omega n - m^2\right| > 4\gamma |n|^{-\tau_0} \tag{1.24}$$

$$\left|\Omega n \pm \sqrt{\omega_m^2 + n\nu_{n,m}(\varepsilon)}\right| \ge 2\gamma |n|^{-\tau} \qquad \forall n \in \mathbb{Z} \setminus \{0\} \text{ and } \forall m \in \mathbb{N} \setminus \{1\} \tag{1.25}$$

$$\left|\Omega(n_2-n_1)\pm(\sqrt{\omega_{m_1}^2+n_1\nu_{n_1,m_1}(\varepsilon)}\pm\sqrt{\omega_{m_2}^2+n_2\nu_{n_2,m_2}(\varepsilon)})\right|\geq 2\gamma|n_2-n_1|^{-\tau} \tag{1.26}$$

$\forall n, n_2 \in \mathbb{Z}\setminus\{0\}$ *and* $\forall m_1, m_2 \in \mathbb{N}$ $\quad m_1-m_2\neq 0, |\omega_1|n_i|-m_i^2|\leq 1+\varepsilon_0|n_i|, i=1,2$ *has large relative Lebesgue measure, namely* $\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}$ *meas* $(\mathcal{C}(\gamma)\cap(0,\varepsilon))=1$.

1.4. Theorem 1 is an easy consequence of Proposition 1 and 2.

Proof of the Theorem 1. We start by choosing γ and $\mu\in M(4\gamma)$ and keep ε_0 as a parameter; by Proposition 1 (i) for all $(\varepsilon,\nu)\in\mathcal{D}(\gamma)$ we can find a sequence $l_{n,m}$ so that there exists a unique solution $u(\eta,\nu,\varepsilon;x,t)$ of (1.11) for all $|\eta|\leq\eta_0$ where η_0 depends only on γ for ε_0 small enough. By Proposition 1 (iii) the sequence $l_{n,m}$ and the solution $u(\eta,\nu,\varepsilon;x,t)$ can be extended to C^1 functions (denoted by l^E, u^E) for all $(\varepsilon,\nu)\in\mathcal{D}$. Moreover $l^E_{n,m}(\varepsilon,\nu)=l_{n,m}(\varepsilon,\nu)$, $u^E_{n,m}(\varepsilon,\nu)=u_{n,m}(\varepsilon,\nu)$ for all $(\varepsilon,\nu)\in\mathcal{D}(2\gamma)$.

Equation (1.11) coincides with our original eq.(1.7) provided that the compatibility equations (1.12) are satisfied. Now we fix $\varepsilon_0<\eta_0^2$ so that $l^E_{n,m}(\eta=\sqrt{\varepsilon},\varepsilon,\nu)$ and $u^E_{n,m}(\eta=\sqrt{\varepsilon},\varepsilon,\nu)$ are well defined. By Proposition 2 (i) there exists a sequence $\nu_{n,m}(\varepsilon)$ which satisfies the extended compatibility eq.(1.12). Finally by Proposition 2(ii) the Cantor set $\mathcal{C}(2\gamma)$ is well defined and of large relative measure.

For all $\varepsilon\in\mathcal{C}(2\gamma)$ we have that the couple $(\varepsilon,\nu(\varepsilon))$ is by definition in $\mathcal{D}(2\gamma)$ so that by Proposition 1(iii):

$$\begin{aligned} l_{n,m}(\sqrt{\varepsilon},\varepsilon,\nu(\varepsilon)) &= l^E_{n,m}(\sqrt{\varepsilon},\varepsilon,\nu(\varepsilon)) \\ u(\sqrt{\varepsilon},\varepsilon,\nu(\varepsilon);x,t) &= u^E(\sqrt{\varepsilon},\varepsilon,\nu(\varepsilon);x,t). \end{aligned} \tag{1.27}$$

so that $u(\sqrt{\varepsilon},\varepsilon,\nu(\varepsilon);x,t)$ solves eq.(1.11) for $\eta=\sqrt{\varepsilon}$. So by Proposition 2(i) $\nu(\varepsilon)$ solves the true compatibility eq.(1.12)

$$\nu_{n,m}(\varepsilon)=\sqrt{\varepsilon}l_{n,m}(\sqrt{\varepsilon},\varepsilon,\nu(\varepsilon)) \tag{1.28}$$

for all $\varepsilon\in\mathcal{C}(2\gamma)$. Then $\sqrt{\varepsilon}u(\sqrt{\varepsilon},\varepsilon,\nu(\varepsilon);x,t)$ is a true non trivial solution of our eq.(1.1) in $\mathcal{C}(2\gamma)$. ■

In the rest of the paper we prove Proposition 1 and 2.

2. Lindstedt Series and Tree Expansion.

2.1. In this section we find a formal solution $u_{n,m}$ of eq.(1.11) as power series on η; the solution $u_{n,m}$ is parameterized by the coefficients $l_{n,m}$ and it will be written in the form of a tree expansion.

We assume for $l_{n,m}(\eta,\varepsilon,\nu), u_{n,m}(\eta,\varepsilon,\nu)$ with $(n,m)\neq(\pm 1,1)$, a formal series expansion in η:

$$l_{n,m}(\eta,\varepsilon,\nu)=\sum_{k=2}^{\infty}\eta^{k-1}l^{(k)}_{n,m}\,,\qquad u_{n,m}(\eta,\varepsilon,\nu)=\sum_{k=1}^{\infty}\eta^k u^{(k)}_{n,m} \tag{2.1}$$

for all $(n,m)\neq(\pm1,1)$. By definition we set $q=u^{(0)}_{\pm1,1}$ and $u^{(k)}_{\pm1,1}=0$. Inserting the series expansion in the second eq.(1.11) we obtain the recursive equations:

$$u^{(k)}_{n,m}=g_{n,m}\Big(n\sum_{r=2}^{k-1}l^{(r)}_{n,m}u^{(k-r)}_{n,m}+\sum^{*}_{\substack{n_1,m_1\\ n_2,m_2}}\sum_{k_1+k_2=k-1}\delta_{n_1+n_2,n}(a-b\Omega^2n_1n_2)$$

$$v_{m,m_1,m_2} u^{(k_1)}_{n_1,m_1} u^{(k_2)}_{n_2,m_2}\Big) \equiv g_{n,m}\Big(n \sum_{r=2}^{k-1} l^{(r)}_{n,m} u^{(k-r)}_{n,m} + F^{(k)}_{n,m}\Big) \tag{2.2}$$

where $g_{n,m}$ (defined in eq.(1.11) is called the propagator. It holds the following Lemma.

Lemma 1. *For all $(n,m) \neq (\pm 1, 1)$ we have that $u^{(k)}_{n,m} = 0$ when $|n| > k+1$ or m is even.*

Proof. We proceed by induction. By definition $F^{(0)}_{n,m} = 0$ so that $u^{(0)}_{n,m} = 0$ if $(n,m) \neq (\pm 1, 1)$.

Now suppose that our claim holds for all $(n,m) \neq (\pm 1, 1)$ and $r < k$. Equations (2.2) are recursive so that $F^{(k)}_{n,m}$ is a quadratic polynomial sum of monomials of the form $v(m_1,m_2,m) u^{(h_1)}_{n_1,m_1} u^{(h_2)}_{n_2,m_2}$ such that the m_i are odd, $|n_i| < h_i$, $n = n_1 + n_2$ and $h_1 + h_2 = k-1$. This implies that $F^{(k)}_{n,m}$ can be nonzero only if $n = n_1 + n_2 \leq h_1 + h_2 + 2 = k+1$. In the same way the linear terms $l^{(r)}_{n,m} u^{k-r-1}_{n,m}$ can be non zero only if $|n| \leq k-1$.

Finally $\sum^*$ in eq. (2.2) is restricted to $m_1 + m_2 + m$ odd and by the inductive hypothesis m_1 and m_2 are odd so that m must be odd as well. ■

We introduce a smooth partition of the unity in the following way. Let $\chi(x)$ be a C^∞ non-increasing function such that $\chi(x) = 0$ if $|x| \leq \gamma$ and $\chi(x) = 1$ if $|x| \geq 2\gamma$; moreover $|\chi'(x)| \leq \gamma^{-1}$. Let $\chi_h(x) = \chi(2^h x) - \chi(2^{h+1} x)$ for $h \geq 0$, and $\chi_{-1}(x) = 1 - \chi(x)$; then

$$1 = \chi_{-1}(x) + \sum_{h=0}^{\infty} \chi_h(x) = \sum_{h=-1}^{\infty} \chi_h(x). \tag{2.3}$$

Calling

$$x_{n,m}(\varepsilon,\nu) = |\Omega n| - \sqrt{\omega_m^2 + n\nu_{n,m}} \tag{2.4}$$

we define

$$g_{n,m,h} = \chi_h(x_{n,m}(\varepsilon,\nu)) g_{n,m}(\varepsilon,\nu) \tag{2.5}$$

Note that if $\chi_h(x) \neq 0$ for $h \geq 0$ one has $2^{-h-1}\gamma \leq |x| \leq 2^{-h+1}\gamma$, while if $\chi_{-1}(x) \neq 0$ one has $|x| \geq \gamma$. Therefore $g_{n,m,h}(\varepsilon,\nu) = 0$ whenever $2^{-h-1}\gamma \leq |x_{n,m}(\varepsilon,\nu)| \leq 2^{-h+1}\gamma$ is not verified. Moreover if $g_{n,m,h}(\varepsilon,\nu) \neq 0$ and $g_{n,m,h'}(\varepsilon,\nu) \neq 0$ then necessarily $|h - h'| \leq 1$. Inserting (2.3) in (2.2) we get

$$u^{(k)}_{n,m} = \sum_h g_{n,m,h} F^{(k)}_{n,m} + n \sum_{h=-1}^{\infty} \sum_{r=2}^{k-1} l^{(r)}_{n,m} g_{n,m,h} u^{(k-r)}_{n,m} \equiv \sum_{h=-1}^{\infty} u^{(k)}_{n,m,h} \tag{2.6}$$

2.2. Eq.(2.6) can be applied recursively untill we obtain $u^{(k)}_{n,m}$ as a (formal) polynomial in the variables $g_{n,m,h}$, q and $l^{(r)}_{n,m}$ with $r < k$. It turns out that $u^{(k)}_{n,m}$ can be written as sum over *trees* (see Lemma 3 below) defined in the following way.

A (connected) graph $\mathcal{G}$ is a collection of points (vertices) and lines connecting all of them. The points of a graph are most commonly known as graph vertices, but may also be called nodes or points. Similarly, the lines connecting the vertices of a graph are most commonly known as graph edges, but may also be called branches or simply lines, as we shall do. We denote with $V(\mathcal{G})$ and $L(\mathcal{G})$ the set of vertices and the set of lines, respectively. A path between two vertices is a subset of $L(\mathcal{G})$ connecting the two vertices. A graph is planar if it can be drawn in a plane without graph lines crossing.

Definition 3. *A* tree *is a planar graph* $\mathcal{G}$ *containing no closed loops (cycles). One can consider a tree* $\mathcal{G}$ *with a single special vertex* v_0*: this introduces a natural partial ordering on the set of lines and vertices, and one can imagine that each line carries an arrow pointing toward the vertex* v_0*. We can add an extra (oriented) line* ℓ_0 *exiting the special vertex* v_0*; the added line will be called the* root line*. In this way we obtain a* rooted tree θ *defined by* $V(\theta) = V(\mathcal{G})$ *and* $L(\theta) = L(\mathcal{G}) \cup \ell_0$*. A* labeled tree *is a rooted tree* θ *together with a label function defined on the sets* $L(\theta)$ *and* $V(\theta)$*.*

We shall call *equivalent* two rooted trees which can be transformed into each other by continuously deforming the lines in the plane in such a way that the latter do not cross each other (i.e. without destroying the graph structure). We can extend the notion of equivalence also to labeled trees, simply by considering equivalent two labeled trees if they can be transformed into each other in such a way that also the labels match.

Given two *nodes* $v, w \in V(\theta)$, we say that $w \prec v$ if v is on the path connecting w to the root line. We can identify a line with the nodes it connects; given a line $\ell = (v, w)$ we say that ℓ enters v and comes out of w.

In the following we shall deal mostly with labeled trees: for simplicity, where no confusion can arise, we shall call them just trees.

We call *internal nodes* the vertices such that there is at least one line entering them. We call *end-points* the vertices which have no entering line. We denote with $L(\theta)$, $V_0(\theta)$ and $E(\theta)$ the set of lines, internal nodes and end-points, respectively. Of course $V(\theta) = V_0(\theta) \cup E(\theta)$.

We call $\Theta^{(k)}_{n,m}$ the set of all the possible trees *of order* k defined according to the following rules.

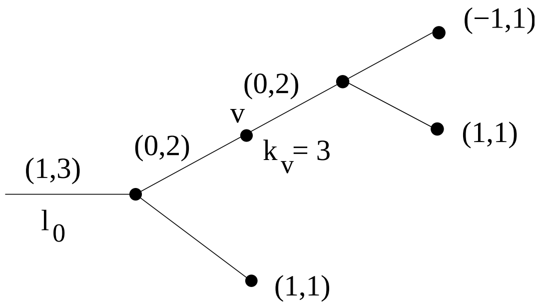

Fig.1. A tree $\theta \in \Theta^{(4)}_{3,4}$

(1) To each end-point $v \in E(\theta)$ one associates the mode label (n_v, m_v), with $m_v = 1$ and $n_v = \pm 1$, such that

$$\sum_{v \in E(\theta)} n_v = n. \tag{2.7}$$

we associate to each end-node a factor $\eta_v = q$ and an order $k_v = 0$.

(2) To each line $\ell \in L(\theta)$ one associates the mode label (n_ℓ, m_ℓ) where one has

$$n_\ell = \sum_{w \in E_\ell} n_w \tag{2.8}$$

where E_ℓ are the endpoints of the subtree with root given by ℓ.

(3) To each line $\ell \in L(\theta)$ one associates the scale label $h_\ell \in \mathbb{N} \cup \{-1, 0\}$. If two lines ℓ, ℓ' have the same mode label $(n_\ell, m_\ell) = (n'_\ell, m'_\ell)$ then $|h_\ell - h_{\ell'}| \le 1$. If ℓ exits an end-node then $h_\ell = -1$.

(4) To each node $v \in V_0(\theta)$ is associated a type label $t_v = a$ or b; For each node $v \in V_0(\theta)$ one has $s_v = 1, 2$ entering lines.

If $s_v = 1$ the momenta of the exiting and entering line are necessarily the same and the type label is by definition a. To v is associated an order $k_v \in [2, \infty)$ and a factor $\eta_v = n_\ell l^{(k_v)}_{n_\ell, m_\ell, h_\ell}$ where ℓ is the line exiting v.

If $s_v = 2$ then necessarily $k_v = 1$. Calling m, m_1, m_2 the momenta m_ℓ respectively of the lines exiting and entering v, to v is associated a factor $\eta_v = a v_{m,m_1,m_2}$ if v is of type a and $\eta_v = -b\Omega^2 v_{m,m_1,m_2}$if v is of type b.

(5) To each line entering an a node and to the root line of each tree, we associate the *propagator*

$$g_\ell \equiv g_{n_\ell, m_\ell, h_\ell}(\varepsilon, \nu) = \begin{cases} \dfrac{\chi^{(h_\ell)}(|\Omega n_\ell| - \sqrt{\omega^2_{m_\ell} + n_\ell \nu_{n_\ell, m_\ell}})}{-\Omega^2 n^2_\ell + \omega^2_{m_\ell} + n_\ell \nu_{n_\ell, m_\ell}}, & (n_\ell, m_\ell) \neq (\pm 1, 1), \\ 1 & (n_\ell, m_\ell) = (\pm 1, 1). \end{cases} \tag{2.9}$$

To each line entering a b node we associate

$$g_\ell \equiv n_\ell g_{n_\ell, m_\ell, h_\ell}(\varepsilon, \nu) = \begin{cases} \dfrac{n_\ell \chi^{(h_\ell)}(|\Omega n_\ell| - \sqrt{\omega^2_{m_\ell} + n_\ell \nu_{n_\ell, m_\ell}})}{-\Omega^2 n^2_\ell + \omega^2_{m_\ell} + n_\ell \nu_{n_\ell, m_\ell}} & (n_\ell, m_\ell) \neq (\pm 1, 1), \\ n_\ell & (n_\ell, m_\ell) = (\pm 1, 1). \end{cases} \tag{2.10}$$

Only the lines coming out from the end-points can have momentum $(n_\ell, m_\ell) = (\pm 1, 1)$.

(6) Finally we define the order of a tree as:

$$k(\theta) = \sum_{v \in V(T)} k_v. \tag{2.11}$$

Note that $|n_\ell| < k(\theta) - \sum_{v : s_v = 1} k_v$.

By the support properties of χ_h and bounding the denominator of g_ℓ with $C|n_\ell| 2^{-h_\ell}$, we get

$$|g^\ell| \le C 2^{h_\ell + 1} \tag{2.12}$$

The divisors can be small only if $n_\ell \simeq m^2_\ell$, as explained by the following Lemma.

Lemma 2. *If $g_\ell \neq 0$ and $h_\ell \ge 0$ then*

$$|\omega_1 |n_\ell| - m^2_\ell| \le 1 + \varepsilon_0 |n_\ell| \tag{2.13}$$

Proof. Equation (2.13) is equivalent to $(\omega_1 - \varepsilon_0)|n_\ell| - 1 \le m^2_\ell \le (\omega_1 + \varepsilon_0)|n_\ell| + 1$; we claim that if $m^2_\ell > (\omega_1 + \varepsilon_0)|n_\ell| + 1$ or $m^2_\ell < (\omega_1 - \varepsilon_0)|n_\ell| - 1$ then $n_\ell \neq [\Omega^{-1} m^2_\ell]$ ([...] denotes the closest integer); in fact if $n_\ell = \Omega^{-1} m^2_\ell + x$ with $|x| \le \frac{1}{2}$ then as $\omega_1 - \varepsilon_0 < \Omega < \omega_1 + \varepsilon_0$ we have that:

$$(\omega_1 - \varepsilon_0)(|n_\ell| - \frac{1}{2}) \le m^2_\ell \le (\omega_1 + \varepsilon_0)(|n_\ell| + \frac{1}{2}) \qquad (2.14.)$$

as $\omega_1 + \varepsilon_0 \leq 2$.

By contradiction assume that (2.13) is not true; then $n_\ell \neq [\Omega^{-1} m_\ell^2]$; then we can write $n = \Omega^{-1} m_\ell^2 + k + x$ with $|x| \leq \frac{1}{2}$, $|k| \geq 1$ so that

$$|\Omega n_\ell - \sqrt{\omega_{m_\ell}^2 + n_\ell \nu_{n_\ell, m_\ell}}| \geq |\Omega n_\ell - m_\ell^2| - |m_\ell^2 - \sqrt{\omega_{m_\ell}^2 + n_\ell \nu_{n_\ell, m_\ell}}|$$
$$\geq |\Omega n_\ell - m_\ell^2| - \frac{\mu_0 + |n_\ell \nu|_\infty}{m_\ell^2} \geq |k| - \frac{1}{2} - m_\ell^{-2}(\mu_0 + (\Omega^{-1} m_\ell^2 + k + \frac{1}{2})\varepsilon_0) \geq \frac{1}{8} > \gamma \tag{2.15}$$

in contradiction with $g^\ell \neq 0$ and $h_\ell \geq 0$. ■

The coefficients $u_{n,m}^{(k)}$ can be represented as sum over the trees defined above; this is in fact the content of the following Lemma.

Lemma 3. *$u_{n,m}^{(k)}$ solving (2.6) can be written as*

$$u_{n,m}^{(k)} = \sum_{\theta \in \Theta_{n,m}^{(k)}} Val(\theta), \tag{2.16}$$

where

$$Val(\theta) = \Big(\prod_{\ell \in L(\theta)} g_\ell \Big) \Big(\prod_{v \in V(\theta)} \eta_v \Big) . \tag{2.17}$$

Proof. The proof is done by induction on k. If $k = 1$ it holds by (2.6), recalling that $u_{n,m}^{(0)} = q \delta_{n,\pm 1} \delta_{m,1}$ (see Fig.2)

$$u_{n,m}^{(1)} = \sum_{h=-1}^{\infty} g_{n,m,h} \sum_{n_1 = \pm 1} v_{m,1,1}(a - b\Omega^2 n_1 (n - n_1)) u_{n_1,1}^{(0)} u_{n-n_1,1}^{(0)} = \sum_{\theta \in \Theta_{n,m}^{(1)}} \mathrm{V}al(\theta) \, . \tag{2.18}$$

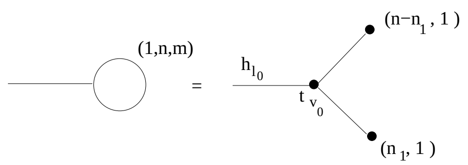

Fig.2. Graphical representation of (2.18) for $k = 1$; the sum over n_1, h_{ℓ_0}, t_{v_0} is understood.

From (2.6), (2.2) and the inductive hypothesis we have that $u_{n,m}^{(k)}$ is given by

$$u_{n,m}^{(k)} = \sum_{h=-1}^{\infty} g_{n,m,h} \{ \sum_{n_1, m_1, m_2, k_1} v_{m,m_1,m_2}(a - b\Omega^2 n_1 (n - n_1))$$
$$\sum_{\theta_1 \in \Theta_{n_1,m_1}^{(k_1)}} \mathrm{V}al(\theta_1) \sum_{\theta_2 \in \Theta_{n-n_1,m_2}^{(k-k_1)}} \mathrm{V}al(\theta_2) + n \sum_{r=2}^{k-1} l_{n,m}^{(r)} \sum_{\theta_3 \in \Theta_{n,m}^{(k-r)}} \mathrm{V}al(\theta_3) \} \tag{2.19}$$

which can be expressed graphically from Fig.3.

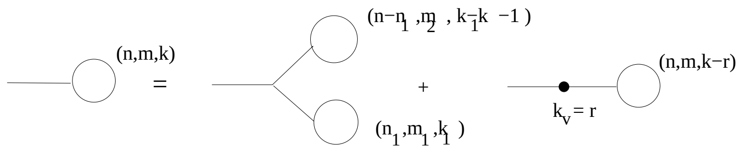

Fig. 3.Graphical representation of (2.19); the sum over n_1, m_1, k_1, m_2, r is understood.

Given a tree $\theta \in \Theta^{(k)}_{n,m}$ such that $s_{v_0} = 2$, $h_{l_0} = h$, let $\theta_1 \in \Theta^{(k_1)}_{n_1,m_1}$, $\theta_2 \in \Theta^{(k-k_1)}_{n-n_1,m_2}$ be the subtrees whoose root lines enter in v_0; if v_0 is of type a by (2.17) one has that:

$$\mathrm{V}al(\theta) = a g_{n,m,h} v(m, m_1, m_2) \mathrm{V}al(\theta_1) \mathrm{V}al(\theta_2); \tag{2.20}$$

if v_0 is of type b let n_1 be the momentum of θ_1. By our definitions we have that:

$$\mathrm{V}al(\theta) = -b\Omega^2 n_1(n - n_1) g_{n,m,h} v(m, m_1, m_2) \mathrm{V}al(\theta_1) \mathrm{V}al(\theta_2); \qquad (2.21.)$$

(recall that the root line of a tree is always an a-line.) Finally given a tree $\theta \in \Theta^{(k)}_{n,m}$ such that $s_{v_0} = 1$, $k_{v_0} = r$, $h_{l_0} = h$ let θ_3 be the subtree whoose root line enters v_0, by (2.17) one has that:

$$\mathrm{V}al(\theta) = n g_{n,m,h} l^{(r)}_{n,m} \mathrm{V}al(\theta_3)\ . \tag{2.22}$$

Hence inserting (2.20), (2.22) in (2.19) we get (2.16). ■

3. Choice of the Parameters $l_{n,m}$.

3.1. In the preceding section we have found a power series expansion for $u_{n,m}$ solving (1.11) and parametrized by $l_{n,m}$. However for generic values of $l_{n,m}$ such expansion is not convergent, as one can easily identify contributions at order k which are $O(k!^\alpha)$, for a suitable constant α. In this section we show that it is possible to choose the parameters $l_{n,m}$ in a proper way to cancel such “dangerous” contributions; in order to do this we have to identify the dangerous contributions and this will be done through the notion of *clusters* and *resonances*.

Definition 4. *Given a tree $\theta \in \Theta^{(k)}_{n,m}$ a cluster T is a connected set of nodes which are linked by a continuous path of lines with the same scale label h_T or a lower one and which are maximal; we shall say that the cluster has scale h_T. We shall denote by $V(T)$ and $E(T)$ the set of nodes and the set of end-points, respectively, which are contained inside the cluster T, and with $L(T)$ the set of lines connecting them.*

Therefore an inclusion relation is established between clusters, in such a way that the innermost clusters are the clusters with lowest scale, and so on. Each cluster T has an arbitrary number of lines entering it (incoming lines), but only one or zero line coming from it (outcoming or *root* line); we shall denote the latter (when it exists) with ℓ^1_T, and we shall denote by $h^{(e)}_T$ the scale of the outcoming external line of T.

Definition 5. *A cluster T with $|V(T)| > 1$, with only one incoming line ℓ_T such that one has*

$$n_{\ell^1_T} = n_{\ell_T} \ \textit{and}\ m_{\ell^1_T} = m_{\ell_T} \tag{3.1}$$

will be called resonance of scale h. In such a case we shall call a resonant line the root line ℓ_T^1.

The propagators on the path between the external lines of T have the form, $\alpha_\ell=(0,1)$

$$\frac{n_\ell^{\alpha_\ell}\chi^{(h_\ell)}(|\Omega n^0_\ell+x|-\sqrt{\omega^2_{m_\ell}+n_\ell\nu_{n_\ell,m_\ell}})}{-(\Omega n^0_\ell+x)^2+\omega^2_{m_\ell}+n_\ell\nu_{n_\ell,m_\ell}}\Big|_{x=\Omega n_{\ell_T}} \tag{3.2}$$

and we can consider the value of T as a function of $m,n,x=\Omega n_{\ell_T}$. The contribution of a resonance T of a tree θ is given by, calling $(n_{\ell_T},m_{\ell_T})=(n,m)$:

$$\mathcal{V}^h_T(\Omega n,m,n)=\Big(\prod_{\ell\in T}g_\ell\Big)\Big(\prod_{v\in V(T)}\eta_v\Big). \tag{3.3}$$

with $h=h_T^{(e)}$.

We define the localization operation acting on the resonances T in the following way; if $|\omega_1|n|-m^2|\le 1+\varepsilon_0|n_0|$ and $(n_\ell,m_\ell)\neq(n,m)$ for all $\ell\in T$ then

$$\mathcal{L}\mathcal{V}^h_T(\Omega n,m,n)=\mathcal{V}^h_T(sign\,(n)\sqrt{(\omega^2_m+n\nu_{n,m})},m,n) \tag{3.4}$$

and $\mathcal{L}=0$ otherwise. We split each resonance as

$$\mathcal{V}^h_T(\Omega n,m,n)=\mathcal{L}\mathcal{V}^h_T(\Omega n,m,n)+\mathcal{R}\mathcal{V}^h_T(\Omega n,m,n) \tag{3.5}$$

where $\mathcal{R}=1-\mathcal{L}$; we call $\mathcal{L}\mathcal{V}^h_T(\Omega n,m)$ *local resonances.* The action of $\mathcal{L}$ is then to replace, in the path connecting the external lines of T, the variable x with

$$\bar\omega_{m,n}=sign\,(n)\sqrt{(\omega^2_m+n\nu_{n,m})} \tag{3.6}$$

Definition 6. *The trees* $\theta_T\in\mathcal{R}^{(k)}_{h,n,m}$ *are defined as the trees* $\theta\in\Theta^{(k)}_{h,n,m}$ *with the following modifications: a) there is a single end node, called e, such that* $(n_e,m_e)=(n,m)\neq(\pm1,1)$*; to e is associated* $\eta_e=1/m_e^3$*. If* ℓ_e *be the line exiting from e,* ℓ_e *has associated* $g_{\ell_e}=1$ *if it enters an a node and* $g_{\ell_e}=n_e$ *if it enters a b node;*
b)the root line l_0 *has* $(n_{\ell_0},m_{\ell_0})=(n,m)$ *and* $g_{\ell_0}=1$*;*
c) for all lines $\ell\in\theta$*:* $\max_{\ell\in L(\theta)\setminus\{\ell_0,\ell_e\}}(h_\ell)=h$.

The definition of value of such tree is identical to the one given in (2.17).

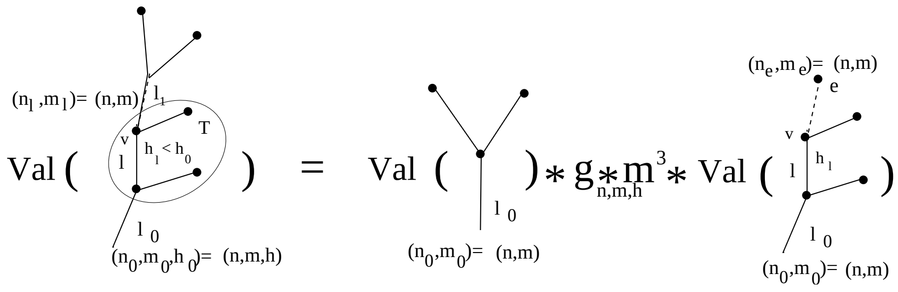

Fig.4. We associate to the resonance T (enclosed in an ellipse) the tree $\theta_T\in\mathcal{R}$, and vice-versa.

Given a resonance T, there exists a unique $\theta_T \in \mathcal{R}^{(k)}_{h,n,m}$ such that (see Fig. 4)

$$\mathcal{V}_T(\Omega n, m, n) = m^3 \mathrm{V}al(\theta_T) \tag{3.7}$$

where $\theta_T \in \mathcal{R}_{n,m,h}$ if the external line enters an a node and $n\mathcal{V}_T(\Omega n, m, n) = m^3 \mathrm{V}al(\theta_T)$ if the external line enters an b node.

3.2 With a suitable choice of the parameters $l_{n,m}$ the functions $u^{(k)}_{n,m}$ can be rewritten as sum over "renormalized" trees defined below.

Definition 7. *We define the set of renormalized trees $\Theta^{(k)}_{R,n,m}$ defined as the trees in $\Theta^{(k)}_{n,m}$ defined in §2 with the following differences: a)to each resonance T we apply the $\mathcal{R}$ operation; b)the nodes with $s_v = 1$ have associated $\eta_v = n_\ell l^{(k_v)}_{n_\ell, m_\ell, h_\ell}$ where ℓ is the line entering v. In the same way we define $\mathcal{R}^{(k)}_{R,h,n,m}$. We call resonant lines the lines coming out of a resonance or a node with $s_v = 1$.*

It holds the following result.

Lemma 4. *For all k, n, m it holds:*

$$u^{(k)}_{n,m} = \sum_{\theta \in \Theta^{(k)}_{R,n,m}} Val(\theta) \tag{3.8}$$

with

$$n l^{(k)}_{n,m,h} = -m^3 \sum_{h_1 \geq h} \sum_{\theta \in \mathcal{R}^{(k)}_{R,n,m,h_1}} \mathcal{L}\, Val(\theta) \tag{3.9}$$

provided that we choose $l^{(k)}_{n,m} = l^{(k)}_{n,m,-1}$ in (2.6).

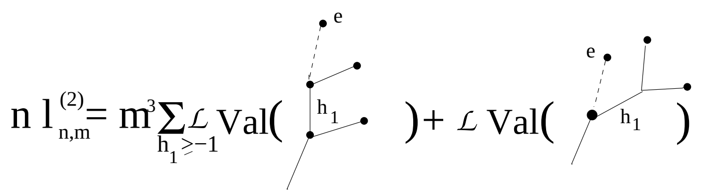

Fig.5. The counterterm $l^{(2)}_{n,m}$.

Proof. First note that by definition $l_{n,m,h} = 0$ if $|\omega_1 n - m^2| \geq 1 + \varepsilon_0 |n|$. We proceed by induction. For $k = 1, 2$ (3.8) surely holds as $\Theta^{(1,2)}_{R,n,m} \equiv \Theta^{(1,2)}_{n,m}$. Then we assume that (3.8) holds for all $r < k$; by (2.6)

$$u^{(k)}_{n,m,h} = g_{n,m,h}(F^{(k)}_{n,m} + n \sum_{r=2}^{k-1} l^{(r)}_{n,m} u^{(k-r)}_{n,m}) \tag{3.10}$$

$F^{(k)}_{n,m}$ is a function of $u^{(r')}_{n'.m',h'}$ with $r' < k$, where, by the inductive hypothesis, the $u^{(r')}_{n'.m',h'}$ are written as sum over trees in $\Theta_{R,n',m'}$. $g_{n,m,h} F^{(k)}_{n,m}$ is given by sum over $\theta \in \Theta^{(k)}_{n,m}$ with $s_{v_0} = 2$, and the root line can be resonant or not. If l_0 is non-resonant then $\theta \in \Theta^{(k)}_{R,n,m}$. If l_0 is resonant we split the biggest resonance in the form (3.5); if $\mathcal{L} = 0$ necessarily there is an inner resonance (whose resonant line is the root line) and again we apply (3.5) and surely $\mathcal{L} \neq 0$. We split $g_{n,m,h} F^{(k)}_{n,m}$ as sum of two terms; one, which we denote by $G^{(k)}_{n,m}$, which is the sum over all trees belonging to $\Theta_{R,n,m}$ with $s_{v_0}=2$ and the second which is sum of trees with value

$$\mathrm{V}al(\theta)=g_{n,m,h_{\ell_0}}[\mathcal{L}\mathrm{V}al(\theta_T)]\mathrm{V}al(\theta_1) \tag{3.11}$$

with $\theta_T\in\mathcal{R}^{(r)}_{R,h_1,n,m}$ and $\theta_1\in\Theta^{(k-r)}_{R,n,m}$. We get

$$F^{(k)}_{n,m}=m^3 g_{n,m,h}\sum_{r=2}^{k-1}u^{k-r}_{n,m}\sum_{h_1<h}(\mathcal{L}\sum_{\theta\in\mathcal{R}^{(r)}_{R,n,m,h_1}}\mathrm{V}al(\theta))+G^{(k)}_{n,m} \tag{3.12}$$

which inserted in (3.10) and using (3.9) gives

$$u^{(k)}_{n,m,h}=g_{n,m,h}m^3\sum_{r=2}^{k-1}u^{(k-r)}_{n,m}(\sum_{h_1<h}\sum_{\theta\in\mathcal{R}^{(r)}_{R,n,m,h_1}}\mathcal{L}\mathrm{V}al(\theta))+G^{(k)}_{n,m} \tag{3.13}$$

$$-m^3\sum_{r=2}^{k-1}u^{(k-r)}_{n,m}(\sum_{h_1\geq -1}\sum_{\theta\in\mathcal{R}^{(r)}_{R,n,m,h_1}}\mathcal{L}\mathrm{V}al(\theta))=g_{n,m,h}G^{(k)}_{n,m}+ng_{n,m,h}\sum_{r=2}^{k-1}u^{(k-r)}_{n,m}l_{n,m,h}\ .$$

By definition $G^{(k)}_{n,m}$ is a sum over all $\theta\in\Theta^{(k)}_{R,n,m}$ with $s_{v_0}=2$ while the last term in (3.13) is the sum over all $\theta\in\Theta^{(k)}_{R,n,m}$ with $s_{v_0}=1$ so that (3.8) is proved. ■

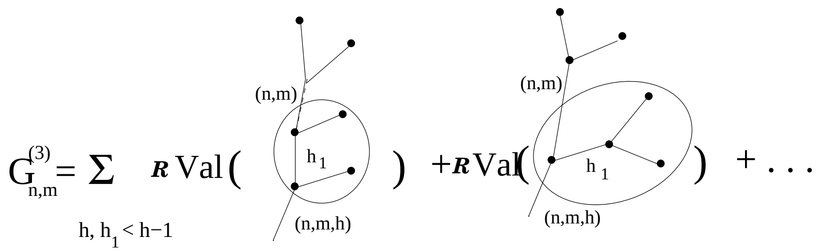

Fig.6. The term $G^{(3)}_{n,m}$, the dots represent sums over trees with $s_v=2$ and non resonant root line.

4. Bruno Lemmas and Bounds for the Expansion

4.1. In the previous section we have shown that, with a suitable choice of the parameters $l_{n,m}$, we can express $u_{n,m}$ as sum over trees belonging to $\Theta^{(k)}_{R,n,m}$; we show in this section that such expansion is indeed convergent if η is small enough and $\varepsilon,\nu\in D(\gamma)$ (see Definition 1).

Given a tree $\theta\in\Theta^{(k)}_{R,n,m}$, we call $S(\theta,\gamma)$ the set of $(\varepsilon,\nu)\in\mathcal{D}$ such that: for all $\ell\in L(\theta)$:

$$2^{-h_\ell-2}\gamma<|\Omega|n_\ell|-\sqrt{\omega^2_{m_\ell}+n_\ell\nu_{n_\ell,m_\ell}}|<2^{-h_\ell+2}\gamma. \tag{4.1}$$

In other words we can have $\mathrm{V}al(\theta)(\varepsilon,\nu)\neq 0$ only if $(\varepsilon,\nu)\in S(\theta,\gamma)$.

We call $D(\theta,\gamma)\subset\mathcal{D}$ the set of (ε,ν) such that, if $\alpha_1=\pm1$, $\alpha_2=\pm1$:

$$|x_{n_\ell,m_\ell}|=\left|\Omega|n_\ell|-\sqrt{\omega^2_{m_\ell}+n_\ell\nu_{n_\ell,m_\ell}}\right|\geq\gamma|n_\ell|^{-\tau} \tag{4.2}$$

$$|x^{\alpha_1,\alpha_2}_{n_{l_1},m_{l_1},n_{l_2},m_{l_2}}| = \Big|\Omega(n_{\ell_1}-n_{\ell_2}) + \alpha_1\sqrt{\omega^2_{\ell_1}+n_{\ell_1}\nu_{n_{\ell_1},m_{\ell_1}}} + \alpha_2\sqrt{\omega^2_{\ell_2}+n_{\ell_2}\nu_{n_{\ell_2},m_{\ell_2}}}\Big|$$
$$\geq \gamma|n_{\ell_1}-n_{\ell_2}|^{-\tau} \qquad \forall|\omega_1 n_{\ell_i} - m^2_{\ell_i}| < 1+\varepsilon_0|n_{\ell_i}| \tag{4.3}$$

for all lines $\ell_1, \ell_2 \in L(\theta)$ such that $n_{\ell_1} \neq n_{\ell_2}$ This means $D(\theta,\gamma)$ is the set of (ε,ν) verifying the Melnikov conditions in θ.

Calling $L_0(\theta), V_0(\theta)$ the set of lines, node and end-points not contained in any resonance, and $S_0(\theta)$ the *maximal resonance*, *i.e.* the resonances which are not contained in any other resonance, we can write $\mathrm{V}al(\theta)$ with $\theta \in \Theta_{R,n,m}$ as

$$\mathrm{V}al(\theta) = \Big(\prod_{\ell\in L_0(\theta)} g_\ell^{(h_\ell)}\Big)\Big(\prod_{v\in V_0(\theta)}\eta_v\Big)\Big(\prod_{T\in S_0(\theta)}\mathcal{R}\mathcal{V}_T^{h^e_T}(\Omega n_{\ell_T}, m_{\ell_T}, n_{\ell_T})\Big), \tag{4.4}$$

and by definition

$$\mathcal{R}\mathcal{V}_T^{h^e_T}(\Omega n_{\ell_T}, m_{\ell_T}, n_{\ell_T}) = \mathcal{V}_T^{h^e_T}(\Omega n_{\ell_T}, m_{\ell_T}, n_{\ell_T}) - \mathcal{V}_T^{h^e_T}(sign(n_{\ell_T})$$
$$\sqrt{(\omega^2_{m_{\ell_T}} + n_{\ell_T}\nu_{n_{\ell_T},m_{\ell_T}}}), m_{\ell_T}, n_{\ell_T}), \tag{4.5}$$

and $\mathcal{V}_T^{h^e_T}(\Omega n_{\ell_T}, m_{\ell_T}, n_{\ell_T})$ is given by

$$\mathcal{V}_T^{h^e_T}(\Omega n_{\ell_T}, m_{\ell_T}, n_{\ell_T}) = \Big(\prod_{\ell\in L_0(T)} g_\ell^{(h_\ell)}\Big)\Big(\prod_{v\in V_0(T)}\eta_v\Big)\Big(\prod_{T'\in S_0(T)}\mathcal{R}\mathcal{V}_{T'}^{h^e_{T'}}(\omega n_{\ell_{T'}}, m_{\ell_{T'}})\Big). \tag{4.6}$$

In order to bound $\mathrm{V}al(\theta)$ in (4.4) we will use the following result.

Lemma 5 (Bruno Lemma). *Given a tree* $\theta \in \Theta^{(k)}_{R,n,m}$*, we have that* $D(\theta,\gamma) \cap S(\theta,\gamma) \neq 0$ *if and only if the scales* h_ℓ *of* θ *respect*

$$N_h(\theta) \leq \max\{0, 2K(\theta)2^{(2-h)/\tau} - 1\} + S_h(\theta) + M_h(\theta), \tag{4.7}$$

where $N_h(\theta)$ *is the number of lines with scale greater or equal than* h*,* $K(\theta) \leq k(\theta)$ *is the number of non resonant lines,* $S_h(\theta)$ *is the number of resonances* T *in* θ *with* $h^{(e)}_T = h$ *and* $M_h(\theta)$ *is the number of vertices with* $s_v = 1$ *in* θ *such that the scale of the exiting line is* h*.*

The proof of the above Lemma is in Appendix A3. By the above lemma we can prove the following result.

Lemma 6. *Assume that there exist a constant* C *such that one has* $|l^{(k)}_{h,n,m}| \leq q^{2k}C^{k-1}2^{-h}$*, for any* n, m *and all* $h \geq 0$*. Then for all* $(\varepsilon,\nu) \in \mathcal{D}(\theta,\gamma)$ *it holds that, for a suitable constant* D

$$|\,Val(\theta)| \leq D^k q^{2k}\Big(\prod_{\substack{v\in V(T)\\ s_v=2}}|\eta_v|\Big) \tag{4.8}$$

Proof. Consider a tree with fixed scales h_ℓ and momenta n_ℓ, m_ℓ. In order to take into account the $\mathcal{R}$ operation we write (4.5) as, if $\bar\omega_{n,m} = sign\,(n)\sqrt{\omega^2_m + n\nu_{n,m}}$

$$\mathcal{R}\mathcal{V}_T^{h^e_T}(\Omega n_{\ell_T}, m_{\ell_T}, n_{\ell_T}) = \Big(\Omega n_{\ell_T} - \bar\omega_{n_{\ell_T},m_{\ell_T}}\Big)$$
$$\int_0^1 \mathrm{d}t\partial\mathcal{V}_T^{h^e_T}(\Omega n_{\ell_T} + t(\Omega n_{\ell_T} - \bar\omega_{n_{\ell_T},m_{\ell_T}}), m_{\ell_T}, n_{\ell_T}), \tag{4.9}$$

where ∂ denotes the derivative with respect to the argument $\omega n_{\ell_T} + t(\omega n_{\ell_T} - \tilde{\omega}_{m_{\ell_T}})$.

By (4.6) we see that the derivatives can be applied either on the propagators in $L_0(T)$, or on the $\mathcal{R}\mathcal{V}_{T'}^{h^e_{T'}}$. In the first case there is a factor $2^{-h_T^{(e)}+h_T}$: $2^{-h_T^{(e)}}$ is obtained from $\omega n_{\ell_T} - \bar{\omega}_{n_{\ell_T}, m_{\ell_T}}$ while $\partial g^{(h_T)}$ is bounded proportionally to 2^{2h_T}; in the second case note that $\partial_t \mathcal{R}\mathcal{V}_{T'}^{h^e_{T'}} = \partial_t \mathcal{V}_{T'}^{h^e_{T'}}$ as $\mathcal{L}\mathcal{V}_{T'}^{h^{(e)}_{T'}}$ is independent of t; if the derivative acts on the propagator of a line $\ell \in L(T)$, we get a gain factor

$$2^{-h_T^{(e)}+h_{T'}} \leq 2^{-h_T^{(e)}+h_T} 2^{-h_{T'}^{(e)}+h_{T'}}, \tag{4.10}$$

as $h_{T'}^{(e)} \leq h_T$. We can iterate this procedure until all the $\mathcal{R}$ operations are applied on propagators; at the end (i) the propagators are derived at most one time; (ii) the number of terms so generated is $\leq k$; (iii) to each resonance T a factor $2^{-h_T^{(e)}+h_T}$ is associated.

Assuming that $|l_{h,n,m}^{(k)}| \leq q^{2k} C^{k-1} 2^{-h}$ with $\gamma C > 1$ and recalling definition (2.11), for any θ one obtains:

$$\begin{aligned} |\mathrm{V}al(\theta)| \leq C^k q^{2k} \Big(\prod_{h=0}^{\infty} \exp \Big[h \log 2 \Big(4k 2^{-(h-2)/\tau} + S_h(\theta) + M_h(\theta) \Big) \Big] \Big) \\ \Big(\prod_{\substack{T \\ h_T^{(e)} \geq 3}} 2^{-h_T^{(e)}+h_T} \Big) \Big(\prod_{h=0}^{\infty} 2^{-h M_h(\theta)} \Big) \Big(\prod_{\substack{v \in V(\theta) \\ s_v = 2}} |\eta_v| \Big) \end{aligned} \tag{4.11}$$

where the first factor is a bound for $\prod_h 2^{h N_h(\theta)}$; moreover $\prod_{h=0}^{\infty} 2^{-h M_\nu^h(\theta)}$ takes into account the factors 2^{-h} arising from the running coupling constants $l_{h,n,m}^{(k)}$ and the action of $\mathcal{R}$ produces, as discussed above, the factor $\prod_T 2^{-h_T^{(e)}+h_T}$. Note that

$$\big(\prod_{h=0}^{\infty} 2^{h S_h(\theta)} \Big) \Big(\prod_T 2^{-h_T^{(e)}} \Big) = 1 \tag{4.12}$$

Moreover it holds that

$$\prod_{\substack{T \\ h_T^{(e)} \geq 3}} 2^{h_T} \leq \prod_{h=0}^{\infty} 2^{h 4k 2^{-(h-2)/\tau}} \tag{4.13}$$

as for any derivative produced by the $\mathcal{R}$ operation and acting on a propagator at scale h there is surely a non resonant propagator at the same scale (otherwise the maximal clusters contained in a resonance are all resonances and $\mathcal{R} = 1$). Then we can write (4.11) as

$$|\mathrm{V}al(\theta)| \leq (q^2 D_1)^k 2^{4k \sum_{h=0}^{\infty} h 2^{-(h-2)/\tau}} \Big(\prod_{\substack{v \in V(\theta) \\ s_v = 2}} |\eta_v| \Big) \tag{4.14}$$

from which (4.8) immediately follows.■

In order to bound the factors $|\eta_v|$ we will use the following result proven in the Appendix A4.

Lemma 7. *For all trees $\theta \in \Theta^{(k)}_{R,n,m}, \mathcal{R}^{(k)}_{R,h,n,m}$ with $s_v = 2$ for all v one has that*

$$\sum_{\{m_\ell\}} \prod_{v\in V(\theta)} |\eta_v| \le \frac{C_2^k}{m^3} \tag{4.15}$$

where m is the momentum associated to the root line, $\sum_{\{m_\ell\}}$ is the sum over the values of the momentum m_ℓ and C_2 depends only on a, b.

Finally we have to prove that $l^{(k)}_{n,m,h} \le C^{k-1}2^{-h}q^{2k}$.

Given a tree $\theta \in \mathcal{R}_{R,n,m}$, we call $\tilde{S}(\theta,\gamma)$ set of $(\varepsilon,\nu) \in \mathcal{D}$ such that 4.1 holds for all $l \in L(\theta)$ not on the path between e and v_0 and:

$$2^{-h_\ell-1}\gamma < ||\Omega n^0_\ell - \bar{\omega}_{n,m}| - \sqrt{\omega^2_{m_\ell} + n_\ell \nu_{n_\ell,m_\ell}}| < 2^{-h_\ell+1} \tag{4.16}$$

holds for $\ell \ne \ell_e$ on the path between e and v_0, namely $\mathcal{L}\mathrm{V}al(\theta) = 0$ outside $\tilde{S}(\theta,\gamma)$. Finally let $\tilde{D}(\theta,\gamma) \subset \mathcal{D}$ be the set of couples (ε,ν) such that 4.2 holds for all ℓ not in the path connecting e to ℓ_0, and 4.3 holds for all $\ell_1, \ell_2 \in L(\theta)$ such that $n_{\ell_1} \ne n_{\ell_2}$ and moreover either both ℓ_1, ℓ_2 are on the path connecting e to ℓ_0 or they both arent on such path.

First of all, the following generalization of Lemma 5 holds.

Lemma 8. *Given tree $\theta \in \mathcal{R}^{(k)}_{R,h,n,m}$, we have that $\tilde{D}(\theta,\gamma) \cap \tilde{S}(\theta,\gamma) \ne 0$ if and only if the number of lines in θ with scales h_ℓ verifies*

$$N_h(\theta) \le 2(K(\theta)-1)2^{(2-h)/\tau} + S_h(\theta) + M_h(\theta). \tag{4.17}$$

It is an immediate consequence of the previous Lemma the following result.

Lemma 9. *Given a tree $\theta \in \mathcal{R}^{(k)}_{R,h,n,m}$, and supposing that, for a suitable constant C, $l^{(r)}_{n,m,h'} \le C^{r-1}q^{2r}2^{-h'}$ for all $r < k$ then for $(\varepsilon,\nu) \in \tilde{D}(\theta,\gamma)$ it holds that*

$$|\mathcal{L}\,Val(\theta)| \le |n| C^{k-1} q^{2k} 2^{-h} \prod_{\substack{v\in V(\theta)\\ s_v=2}} |\eta_v| C_2^{-k_1} \tag{4.18}$$

where k_1 is the number of lines exiting a node with $s_v = 2$.

Proof. The proof is essentially identical to the one of Lemma 6; the factor n comes from the definition of $\mathrm{V}al(\theta)$ in the case when the external line enters a b node. To extract the factor 2^{-h} we recall that there is at least a non resonant line $\ell \ne \ell_0$ on scale $h_\ell = h, h-1$ which does not exit a node with $s_v = 1$. By Lemma 8 we have that $k_1 - 1 \ge K(\theta) - 1 > 2^{\frac{h-1}{\tau}}$, so that $2^{k_1-1}2^{-h} > 1$. Then

$$|\mathcal{L}\mathrm{V}al(\theta)| \le q^{2k}2^{-h}(2DC_2^2)^{k_1-1}C^{k-k_1} \prod_{\substack{v\in V(\theta)\\ s_v=2}} |\eta_v| C_2^{-k_1} \le 2^{-h}C^{k-1} \prod_{\substack{v\in V(\theta)\\ s_v=2}} |\eta_v| C_2^{-k_1} \tag{4.19}$$

provided that $2C\gamma > 1$ and we choose $C = 2DC_2^2$. Finally the factor D as in Lemma 6 is of the form $D = \bar{D}\gamma^{-1}$ with $\bar{D} > 1$ a pure (ε and γ independent) constant ■.

As aconsequence of (3.9) and Lemma 7 and 9 it follows that $|l^{(k)}_{h,n,m}| \le q^{2k}C^{k-1}2^{-h}$.

Lemma 10. *For η_0 small enough, the following bounds hold for all $(\varepsilon,\nu)\in D(\gamma)$:*

$$|l_{n,m,h}|<C_1\eta 2^{-h},\;\; |l_{n,m}|<C_1\eta, |u_{n,m}|<C_0|\eta|\frac{e^{-\sigma|n|}}{m^7}\;,\;(n,m)\neq(\pm1,1), \quad (4.20)$$

where (n',m') are such that $|\omega_1|n'|-(m')^2|\le 1+\varepsilon_0|n'|$ as otherwise $\nu_{n',m'}\equiv 0$ by definition.

Proof. By definition $D(\gamma)$ is contained in all $D(\theta,\gamma)$ and in all $\tilde D(\theta,\gamma)$ so that we can use Lemma 6 and Lemma 9 to bound the values of trees. First we fix an unlabeled tree θ and sum over the values of the labels. Fixed (ε,ν) and given (n_ℓ,m_ℓ) there are only two possible values for each h_ℓ such that $\mathrm{V}al(\theta)\neq 0$. So we can sum up on the possible scale values obtaining a factor 2^k. First we fix the tree $\theta\in\Theta_{R,n,m,h}$ and sum up the m_ℓ labels as in Lemma 7, we obtain a factor m^{-3}. Then we sum up on the possible values the momentum of lines exiting an end node, we obtain 4^k; finally we bound by $\bar C^k$ the number of unlabeled trees. The bound for $l_{n,m,h}$ is obtained by:

$$|l_{n,m,h}|\le\frac{m^3}{n}\sum_{k=2}^{\infty}\eta^{k-1}\sum_{h_1\ge h}\sum_{\theta\in\mathcal R^{(k)}_{R,h_1,n,m}}|\mathcal L\mathrm{V}al(\theta)|\le\eta C_1\sum_{h_1\ge h}2^{-h_1}. \quad (4.22)$$

By Lemma 1 $u^{(k)}_{n,m}=0$ if $|n|>k$, so that, using Lemma 7:

$$|u_{n,m}|\le\sum_{k=1}^{\infty}\eta^k|u^{(k)}_{n,m}|\le\sum_{k=|n|}^{\infty}\eta^kC^k\frac{1}{m^3}, \quad (4.21)$$

In order to get a better decay in m we simply note that if $|n|\le\frac{m^2}{4}$ then , if ℓ_0 is the root line, ℓ_0 is surely an a-line, $h_{\ell_0}=-1$ and $|g_{\ell_0}|\le Cm^{-4}$; if $|n|\ge\frac{m^2}{4}$ of course $\eta^{|n|}\le C\eta^{\frac{|n|}{2}}m^{-4}$. Then we get an extra m^{-4} in (4.21) so that the bound in (4.20) is found. ■

5. Whitney Extension and Implicit Function Theorems

5.1. In this section we extend the function $u_{n,m}$,$l_{n,m}$, defined in $D(\gamma)$ to the larger set $\mathcal D$.

Lemma 11. *Given $\theta\in\mathcal R^{(k)}_{R,h,n,m}$, we can extend $Val(\theta)$ to a function, called $Val^E(\theta)$, defined and C^∞ in $\mathcal D$ such that the bounds of Lemma 10 hold for any $(\varepsilon,\nu)\in\mathcal D$, $\mathcal L\,Val(\theta)=\mathcal L\,Val^E(\theta)$ for any $(\varepsilon,\nu)\in\mathcal D(\theta,2\gamma)\subset\mathcal D(\theta,\gamma)$ and finally $\mathcal L\,Val^E(\theta)=0$ for $(\varepsilon,\nu)\in\mathcal D\setminus\mathcal D(\theta,\gamma)$. We then define:*

$$l^{(k)E}_{n,m,h}=\sum_{h_1\ge h}\sum_{\theta\in\mathcal R^{(k)}_{R,h_1,n,m}}\mathcal L\,Val^E(\theta) \quad (5.1)$$

and l^E is differentiable in $(\varepsilon,\nu)\in\mathcal D$ and, if C_1 is a suitable constant

$$|\partial_\varepsilon l^E_{n,m}|<C_1\eta\;,\;|\partial_{\nu_{n',m'}}l^E_{n,m}|<C_1\eta,\quad|\sum_{(n',m')\in\Lambda}\partial_{\nu_{n',m'}}l^E_{n,m}|<C_1\eta \quad (5.2)$$

In the same way, given $\theta \in \Theta^{(k)}_{R,n,m}$, we define the extended value $Val^E(\theta)$ and therefore $u^E_{n,m}$.

Proof. We prove first the statement for the (more difficult) case $\theta \in \mathcal{R}^{(k)}_{R,h,n,m}$. We use the C^∞ compact support function $\chi(t) : \mathbb{R} \to \mathbb{R}^+$, defined in the previous section. Recall that $\chi(t)$ equal to 0 if $|t| < \gamma$ and 1 if $|t| \geq 2\gamma$, and $|\partial_t \chi| \leq \gamma^{-1}$. We proceed by induction let us suppose that we have proved Lemma 11 for $r < k$ and therefore defined $l^{(r)E}_{n,m,h}$ for $r < k$. Given a tree $\theta \in \mathcal{R}^{(k)}_{R,h,n,m}$,

$$\mathcal{L}\mathrm{V}al^E(\theta) = \prod^{*}_{\ell \in L(\theta)} \chi(|x_{n_\ell,m_\ell}||n_\ell|^\tau) \prod_{\alpha_1,\alpha_2} \prod^{**}_{\ell_1,\ell_2 \in L(\theta)} \chi(|x^{\alpha_1,\alpha_2}_{n_{\ell_1},m_{\ell_1},n_{\ell_2},m_{\ell_2}}||n_{\ell_1} - n_{\ell_2}|^\tau)) \mathcal{L}\mathrm{V}al(\theta) \tag{5.3}$$

where $\prod^*_{\ell \in L(\theta)}$ is the product on the lines not on the path between e and v_0 and $\prod^{**}_{\ell_1,\ell_2 \in L(\theta)}$ is the product on the couples $\ell_1, \ell_2 \in L(\theta)$ such that: $n_{\ell_1} \neq n_{\ell_2}$ and either both ℓ_1, ℓ_2 are on the path connecting e to v_0 or they both are not on such path. Finally in each node v with $s_v = 1$ we set $\eta_\ell = l^E_{n,m,h}$.

1. By definition $\mathrm{V}al^E(\theta) = \mathrm{V}al(\theta)$ for $(\varepsilon, \nu) \in \mathcal{D}(\theta, 2\gamma)$ as in this set the χ in the above formula are identically equal to 1;

2. By definition $\mathrm{supp}(\mathrm{V}al^E(\theta)) \subset \tilde{\mathcal{D}}(\theta, \gamma)$ as the χ in the above formula are identically equal to 0 in the complementary to $\tilde{\mathcal{D}}(\theta, \gamma)$;

Finally we define

$$l^{E(k)}_{n,m,h}(\varepsilon, \nu) = \sum_{h_1 = h}^{\infty} \sum_{\theta \in \mathcal{R}_{n,m,h_1}} \mathcal{L}\mathrm{V}al^E(\theta)(\varepsilon, \nu) \tag{5.4}$$

which respects the bounds in Lemma 10. In order to prove (5.2) we proceed by induction. Given a tree $\theta \in \mathcal{R}^{(k)}_{R,h_1,n,m}$, the derivatives act on the nodes with $s_v = 1$ which carry the factor $l^{(r)}_{n',m',h'}$ with $r < k$ so we can apply the inductive hypothesis. On the lines ℓ not on the path e, v_0 we get

$$|\partial_\varepsilon g_\ell| \leq C |n_\ell| 2^{2h_\ell}\,, \qquad |\partial_{\nu_{n',m'}} g_\ell| \leq C 2^{2h_\ell} \tag{5.5}$$

and we use that $|n_\ell| \leq k$. On the lines ℓ on the path e, v_0 the propagator is given by $\mathcal{L}g_\ell$, defined in (3.2) with x replaced by $\bar{\omega}_{n,m}$, so that

$$|\partial_\varepsilon \mathcal{L}g_\ell| \leq |n^{(0)}_\ell| 2^{2h_\ell}\,, \qquad |\partial_{\nu_{n',m'}} \mathcal{L}g_\ell| = C 2^{2h_\ell}; \tag{5.6}$$

where we have used that, by definition of $\mathcal{D}$, $|\omega_1 n' - (m')^2| \leq 1 + \varepsilon_0 |n'|$. Finally we consider the derivatives of the χ functions which produce in the bounds a factor $|n^0_\ell|^{\tau+1}$, all this factors are bounded by $k^{\tau+1} \leq C^k$, so that the derivatives of $\mathrm{V}al(\theta)$ respect the bounds (4.18). As this bounds are uniform (independent from (n,m)) so that $l^E_{n,m,h}$ is C^1 function of (ε, ν).

Moreover $\partial_{\nu_{n',m'}} l^{(k)E}_{n,m}(\varepsilon, \nu)$ is non vanishing only if $|\omega_1 n' - (m')^2| \leq 1 + \varepsilon_0 |n'|$ and if $|n| - k < |n'| < |n + k|$; hence

$$\sum_{n',m' \in \Lambda} |\partial_{\nu_{n',m'}} l^{(k)E}_{n,m}(\varepsilon, \nu)| \leq C_0 k^{3/2} \max_{n',m' \in \Lambda} |\partial_{\nu_{n',m'}} l^{(k)E}_{n,m}(\varepsilon, \nu)| \leq C_1^k$$

where Λ was defined in Definition 2.

In the same way for $\theta \in \Theta_{R,n,m}$

$$\mathrm{V}al^E(\theta) = \prod_{\ell\in L(\theta)} \chi(x_{n_\ell,m_\ell}|n_\ell|^\tau) \prod_{\alpha_1,\alpha_2} \prod_{\ell_1,\ell_2\in L(\theta)} \chi(x^{\alpha_1,\alpha_2}_{n_{\ell_1},m_{\ell_1},n_{\ell_2},m_{\ell_2}}|n_{\ell_1}-n_{\ell_2}|^\tau)\mathrm{V}al^E(\theta) \tag{5.7}$$

and finally

$$u^E_{n,m} = \sum_{k=1}^\infty \eta^k \sum_{\theta\in\Theta_{R,n,m}} \mathrm{V}al(\theta). \tag{5.8}$$

■

5.2. *Proof of Proposition 1.* Lemma 10 and Lemma 11 imply that $l^E_{n,m}$ and $u^E_{n,m}$ are C^1 in (ε,ν) for $(\varepsilon,\nu)\in\mathcal{D}$ and analytic in η, q for $|q|\le q_0$ and $\eta\le\varepsilon_0$ such that $Dq_0^2\varepsilon_0 \ll 1$.

Inserting in the first of (1.11) the expansion for u^E and l^E we get the following equation for q

$$q = \sum_{k=2}^\infty \eta^{k-2} \sum^{*}_{\substack{n_1,m_1\\ n_2,m_2}} \sum_{k_1+k_2=k} \delta_{n_1+n_2-1}(a-b\Omega n_1 n_2)v_{1,m_1,m_2} u^{(k_1)E}_{n_1,m_1} u^{(k_2)E}_{n_2,m_2}. \tag{5.9}$$

Indeed the leading order of (2.5r) is

$$q = -\frac{1}{2}q^3(a-b\Omega^2)\sum_m v^2_{1,1,m}(2ag_{0,m} + (a-2b\Omega^2)g_{2,m}) + O(\eta). \tag{5.10}$$

One can easily verify that for all $(\varepsilon,\nu)\in\mathcal{D}$ $|x_{0,m}|, |x_{2,m}| > \frac{1}{2}$ so that $u_{0,m} = u^E_{0,m}$. So (the equation for q is equivalent to:

$$q = (A + O(\varepsilon_0))q^3 + \eta F(q,\varepsilon,\nu,\eta) \tag{5.11}$$

We then exclude those values of a,b for which $A\le 0$. Equation (5.11) is clearly invertible near $\eta = 0$ if $A > 0$, so that we obtain $q = q(\eta,\varepsilon,\nu)$ analytic in η and C^1 in (e,ν). This completes the proof of Proposition 1.

Notice that if $A < 0$ then we would only need to consider $\Omega = \sqrt{1+\mu-\varepsilon}$ with as usual $\varepsilon > 0$. ■

6. Proof of Proposition 2

6.1. In order to prove the first part of Proposition 2, we consider the extended compatibility equation 1.12:

$$\nu_{n,m} = \eta l^E_{n,m}(\varepsilon,\nu,\eta) \equiv \sum_{k=2}^\infty \eta^k l^{(k)}_{n,m}, \tag{6.1}$$

where we have substituted q with $q(\varepsilon,\nu,\eta)$.and $l^E_{n,m}(\varepsilon,\nu,\eta)$ is a C^1 function with bounded Jacobian (see (5.2)) so that we can solve (6.1) by the implicit function theorem for $\eta < \eta_0$ small enough. We obtain a function $\nu(\varepsilon,\eta)$ defined for $\varepsilon\in(0,\varepsilon_0)$, $|\eta|\le\eta_0$ and of order η^2. Moreover $\nu_{n,m}(\varepsilon,\eta) = 0$ if $|\omega_1|n| - m^2| \ge 1 + \varepsilon_0|n|$.

From (6.1) we get

$$\partial_\varepsilon \nu_{n,m}(\varepsilon,\eta) = \eta(\partial_\varepsilon l^E_{n,m} + \sum_{\substack{n',m' \\ |\omega_1|n'|-(m')^2|\le 1+\varepsilon_0|n'|}} \partial_{\nu_{n',m'}} l^E_{n,m} \partial_\varepsilon \nu_{n',m'}(\varepsilon,\eta))$$

so that

$$|\partial_\varepsilon \nu_{n,m}(\varepsilon,\eta)|_\infty \le \eta^2 C\,, \qquad |\partial_\eta \nu_{n,m}(\varepsilon,\eta)|_\infty \le \eta C$$

Finally we set $\eta = \sqrt{\varepsilon}$ and obtain the desired bounds.■

6.2. We have now to bound the measure of $\mathcal{C}(\gamma)$.

We define $\mathcal{I}_1$ the set of $\varepsilon \in (0,\varepsilon_0)$ verifying for any (n,m),

$$|\Omega n - \sqrt{\omega_m^2 + n\nu_{n,m}(\varepsilon)}| \le \frac{C_0}{|n|^{\tau_0}} \tag{6.2}$$

with $C_0 = 2\gamma$. When (1.15) is satisfied by Lemma 2 there exists two constants such that

$$c_1\sqrt{n} \le m \le c_2\sqrt{n} \tag{6.3}$$

Moreover one must have (by using also (1.4))

$$\begin{aligned} 2C_0|n|^{-\tau_0} &\le |\omega_1 n - \omega_m| \\ &\le |\omega_1 n + \varepsilon n - \sqrt{\omega_m + n\nu_{n,m}(\varepsilon)}| + |\varepsilon n| + |\sqrt{\omega_m^2 + n\nu_m(\varepsilon)} - \omega_m| \\ &\le C_0|n|^{-\tau} + C\varepsilon_0|n|, \end{aligned} \tag{6.4}$$

which implies, for $|n| > 1$ and $\tau > \tau_0 + 1$,

$$|n| \ge \mathcal{N}_0 \equiv \left(\frac{C_0}{C\varepsilon_0}\right)^{1/(\tau_0+1)}, \tag{6.5}$$

We can define a map $t \to \varepsilon(t)$ such that

$$f_{n,m}(\varepsilon(t)) = \Omega n - \sqrt{\omega_m^2 + n\nu_{n,m}(\varepsilon)} = \frac{2\gamma t}{|n|^\tau}, \quad t \in [-1,1] \tag{6.6}$$

describes the interval defined by (6.2); then one has

$$\int_{\mathcal{I}_1} d\varepsilon = \sum_{|n|\ge\mathcal{N}_0, 0\le m\le c_2\sqrt{n}} \int_{-1}^{1} \mathrm{d}t \left|\frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t}\right|, \tag{6.7}$$

We have from the definition of $f_{n,m}$:

$$\frac{\mathrm{d}f_{n,m}}{\mathrm{d}t} = \frac{\mathrm{d}f_{n,m}}{\mathrm{d}\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{2\gamma}{|n|^\tau}, \tag{6.8}$$

We need a lower bound on

$$|\frac{\mathrm{d}f_{n,m}(\varepsilon)}{\mathrm{d}\varepsilon}| = |n + n\frac{\partial_\varepsilon \nu_{n,m}}{2\sqrt{\omega_m^2 + n\nu_m}}| \ge |n| - \frac{C|n|}{\omega_m} \tag{6.9}$$

By (6.5) and the fact that, by (6.3), $\frac{|n|}{\omega_m} \le \bar{C}$ we get for ε_0 small enough

$$|\frac{\mathrm{d}f_{n,m}(\varepsilon)}{\mathrm{d}\varepsilon}| > \frac{|n|}{2} \tag{6.10}$$

We substitute in (6.7):

$$\int_{\mathcal{I}_1} \mathrm{d}\varepsilon \le \sum_{|n|\ge \mathcal{N}_0, 0<m\le c_2|n|^{1/2}} \frac{C_1}{|n|^{\tau+1}} \le C_2 \varepsilon_0^{(\tau-\frac{1}{2})\frac{1}{\tau_0+1}}.$$

So the Cantor set of the ε verifying (1.25) has relative measure $\to 1$ as $\varepsilon_0 \to 0$ if $\tau > \tau_0 + \frac{5}{2}$.

6.3. We define $\mathcal{I}_2$ the set of $\varepsilon \in (0, \varepsilon_0)$ belonging to $\mathcal{I}_1$ and verifying, for $m_1 \ne m_2$, the condition $|\omega_1|n_i| - m_i^2| \le 1 + \varepsilon_0|n_i|$, $i = 1, 2$

$$|\Omega|n_2 - n_1| \pm \sqrt{\omega_{m_1}^2 + n_1\nu_{n_1,m_1}(\varepsilon)} \mp \sqrt{\omega_{m_2}^2 + n_2\nu_{n_2,m_2}(\varepsilon)}| \le \frac{2\gamma}{|n_2 - n_1|^{\tau_0}} \tag{6.11}$$

Of course $|n_i| \le C_1 m_i^2$, for $i = 1, 2$.

For simplicity we choose the signs in (6.11) as in $-,+$ in (the other case is done in the same way); then (6.11) can be verified for some ε only if $m_1 > m_2$. It holds that

$$m_1^2 - m_2^2 \le (\omega_1 + \varepsilon_0)|n| + 1 \tag{6.12}$$

where $n = n_1 - n_2$ The proof is by contradiction; if it is not true then $m_1^2 - m_2^2 > (\omega_1 + \varepsilon_0)|n| + 1$ which implies $|n| \ne [\Omega^{-1}|m_1^2 - m_2^2|]$, where [...] denotes the closest integer. Then

$$\begin{aligned} |\Omega|n| - \sqrt{\omega_{m_1}^2 + n_1\nu_{n_1,m_1}} + \sqrt{\omega_{m_2}^2 + n_2\nu_{n_2,m_2}}| &\ge |\Omega n - m_1^2 + m_2^2| - \frac{\mu_0}{m_1^2} - \frac{\mu_0}{m_2^2} \\ - (\frac{n_1}{m_1^2} + \frac{n_2}{m_2^2})c_1\varepsilon_0| &\ge |\Omega n - m_1^2 + m_2^2| - \frac{1}{4}(1 + C\varepsilon_0) \ge \frac{1}{8} \end{aligned} \tag{6.13}$$

as $|n_i| \le C_1 m_i^2$, for $i = 1, 2$, in contradiction with (6.11).

Then by (6.12) we get $m_1 + m_2 \le C_2 \frac{|n|}{m_1 - m_2} \le C_2|n|$ as $m_1 - m_2 \ge 1$; hence $m_1 \le C_3|n|$ and $m_2 \le C_3|n|$.

Finally when the conditions (6.11) are satisfied, one has, for $n_1 - n_2 = n$ and $C_0 = 2\gamma$

$$\begin{aligned} 2C_0|n|^{-\tau_0} &\le |\omega_1 n - (\omega_{m_2} - \omega_{m_1})| \\ &\le |\omega_1 n + \varepsilon n - \sqrt{\omega_{m_2} + n_2\nu_{n_2,m_2}} + \sqrt{\omega_{m_1} + n_1\nu_{n_1,m_1}}| \\ &\quad + |\sqrt{\omega_{m_1} + n_1\nu_{n_1,m_1}} - \omega_{m_1}| + |\sqrt{\omega_{m_2} + n_2\nu_{n_2,m_2}} - \omega_{m_2}| + \varepsilon_0|n| \\ &\le C_0|n|^{-\tau} + \varepsilon_0|n| + \frac{|n_1||\nu_{n_1,m_1}|}{m_1^2} + \frac{|n_2||\nu_{n_2,m_2}|}{m_2^2}, \end{aligned} \tag{6.14}$$

now as $|n_i| \le c_2 m_i^2$ we have that

$$|n| \ge \mathcal{N}_1 \equiv (C_8\varepsilon_0)^{1/(\tau_0+1)}. \tag{6.15}$$

We define the map $t \to \varepsilon(t)$ implicitly by:

$$f_{n,n_1,m_1,m_2} \equiv \Omega n - \sqrt{\omega_{m_1}^2 + n_1\nu_{m_1}} + \sqrt{\omega_{m_2}^2 + n_2\nu_{m_2}} = \frac{2\gamma t}{|n|^\tau} \tag{6.16}$$

We write

$$\int_{\mathcal{I}_2} \mathrm{d}\varepsilon = \sum_{\substack{|n|\le(C_8\varepsilon_0)^{1/(\tau_0+1)} \\ m_1,m_2\le C_3|n|;|n_1|\le Cm_1^2}} \int_{-1}^{1} \mathrm{d}t \left|\frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t}\right| \tag{6.17}$$

We need a lower bound on

$$\begin{aligned} |\frac{\mathrm{d}f_{n,n_1,m_1,m_2}(\varepsilon)}{\mathrm{d}\varepsilon}| &= |n - \frac{n_1\partial_\varepsilon \nu_{n_1,m_1}}{2\sqrt{\omega_{m_1}^2 + n_1\nu_{m_1}}} + \frac{n_2\partial_\varepsilon \nu_{m_2}}{2\sqrt{\omega_{m_2}^2 + n_2\nu_{nm_2}}}| \\ &\ge |n| - \frac{(C+\varepsilon_0)|n_1|}{\omega_{m_1}} - \frac{(C+\varepsilon_0)|n_2|}{\omega_{m_2}} \ge \frac{|n|}{2} \end{aligned} \tag{6.18}$$

where we have used that $\frac{|n_i|}{\omega_{m_i}}$ is bounded by a constant and we have chosen ε_0 small enough. Hence we get

$$\int_{\mathcal{I}_2} \mathrm{d}\varepsilon = \sum_{|n|\le(C_8\varepsilon_0)^{1/(\tau_0+1)}} C|n|^{-\tau-1+4} \le \varepsilon_0^{\frac{-\tau+4}{\tau_0+1}} \tag{6.19}$$

so the Cantor set of the ε verifying (1.26) has relative measure $\to 1$ as $\varepsilon_0 \to 0$ if $\tau > \tau_0 + \frac{5}{2}$. Finally we define $\mathcal{I}_3$ the set of $\varepsilon \in (0, \varepsilon_0)$ verifying

$$|\Omega|n| \pm \sqrt{\omega_{m_1}^2 + n_1\nu_{m_1}(\varepsilon)} \pm \sqrt{\omega_{m_2}^2 + n_2\nu_{m_2}(\varepsilon)}| \le \frac{2\gamma}{|n|^{\tau_0}} \tag{6.20}$$

and one proceeds as above with the only difference that (6.20) can be true only if $|m_1^2 + m_2^2| \le C_2|n|$ hence $m_i \le C_2\sqrt{|n|}$, $i = 1, 2$.

Appendix A1. Measure of the set $M(\gamma)$

The analysis is very similar to the one in §6. We call J_1 the set of μ which do not satisfy the first condition in (1.4). J_1 is given by:

$$(1+\mu)|n| - \sqrt{m^4 + \mu} = t\frac{\gamma}{|n|^{\tau_0}}, \quad t \in (-1, 1); \tag{A1.1}$$

the left hand side can be smaller than 1 only if $n = [\frac{\sqrt{m^4+\mu}}{(1+\mu)}]$, where [...] is the closest integer; this implies that $m < c\sqrt{|n|}$ for a suitable constant c. Then (A1.1) defines the values $\mu = \mu(t)$ in J_1 so that:

$$\mathrm{m}eas(J_1) = \sum_{\substack{n \\ m<c\sqrt{|n|}}} \int_{-1}^{1} dt|\frac{d\mu(t)}{dt}| \le \sum_{\substack{n \\ m<c\sqrt{|n|}}} \frac{2\gamma}{|n|^{\tau_0+1}} \le 2\gamma \tag{A1.2}$$

as $|\partial_\mu[(1+\mu)|n| - \sqrt{m^4+\mu}]| \ge \frac{|n|}{2}$, and $\tau_0 > \frac{1}{2}$.

Let us call J_2 the set of μ such that

$$(1+\mu)|n| - \sqrt{m_1^4+\mu} + \sqrt{m_2^4+\mu} = t\frac{\gamma}{|n|^{\tau_0}}\,, \quad t \in (-1,1); \tag{A1.3}$$

the left hand side can be smaller than one only if $n = [\frac{\sqrt{m_1^4+\mu}-\sqrt{m_2^4+\mu}}{(1+\mu)}]$ which implies $|m_1^2 - m_2^2| < c|n|$ and therefore $m_1 + m_2 < c|n|$.

$$meas(J_2) = \sum_{\substack{n \\ m_1,m_2<c|n|}} \int_{-1}^{1} dt|\frac{d\mu(t)}{dt}| \le \sum_n \frac{2\gamma}{|n|^{\tau_0-1}} \le 2\gamma, \tag{A1.4}$$

as $\tau_0 > 2$. Finally we proceed in the same way for J_3 the set of μ such that

$$(1+\mu)|n| - \sqrt{m_1^4+\mu} - \sqrt{m_2^4+\mu} = t\frac{\gamma}{|n|^{\tau_0}}\,, \quad t \in (-1,1); \tag{A1.5}$$

We have proved that the complementary set to $M(\gamma)$ is of order $6\gamma < \frac{1}{8}$ provided that γ is small enough, that is $\gamma \le 2^{-6}$.

Appendix A2. Proof of (1.7)

The equation for the coefficients (1.10) follows immediately from

$$\int_0^\pi dx \sin(mx)\sin(m_1 x) = \pi\delta_{m,m_1} \tag{A2.1}$$

and

$$\int_0^\pi dx \sin(mx)\sin(m_1 x)\sin(m_2 x) = \sum_{\varepsilon,\varepsilon_1,\varepsilon_2=\pm} (\varepsilon\varepsilon_1\varepsilon_2)\frac{e^{i(\varepsilon m+\varepsilon_1 m_1+\varepsilon_2 m_2)\pi}-1}{i(\varepsilon m+\varepsilon_1 m_1+\varepsilon_2 m_2)} \tag{A2.2}$$

which is vanishing if $\pm m_1 \pm m_1 \pm m_2$ is even, while if it is odd it is equal to

$$\begin{aligned} &4[\frac{1}{m+m_1+m_2} - \frac{1}{m+m_1-m_2} - \frac{1}{m-m_1+m_2} + \frac{1}{m-m_1-m_2}] \\ &= \frac{8m_1m_2m}{(m^2-(m_1-m_2)^2)(m^2-(m_1+m_2)^2)} \end{aligned} \tag{A2.3}$$

Appendix A3. Proof of the Lemmas 5 and 8

In order to prove Lemma 5 we prove inductively the bound, for $\theta \in \Theta_{R,n,m}$

$$N_h^*(\theta) \le \max\{0, 2K(\theta)2^{(2-h)/\tau} - 1\}, \tag{A3.1}$$

where $N_h^*(\theta)$ is the number of non resonant lines. As we are supposing $Val(\theta) \neq 0$ it holds for any ℓ that $\gamma 2^{-h_\ell-1} \le |x_{n_\ell,m_\ell}| \le \gamma 2^{-h_\ell+1}$. This implies, by the first Diophantine condition, that θ can have a line on scale h only if $K(\theta) > 2^{(h-1)/\tau}$. Then one can have $N_h(\theta) \ge 1$ only if $K(\theta)$ is such that $K(\theta) > k_0 \equiv 2^{(h-1)/\tau}$: therefore for values $K(\theta) \le k_0$ the bound (4.9) is satisfied.

If $K(\theta) > k_0$, we assume that the bound holds for all trees θ' with $K(\theta') < K(\theta)$. Define $E_h = 2^{-1}(2^{(4-h)/\tau})^{-1}$: so we have to prove that $N_h^*(\theta) \le \max\{0, K(\theta)E_h^{-1} - 1\}$.

Call ℓ the root line of θ and $\ell_1, \dots, \ell_m$ the $m \ge 0$ lines on scale $\ge h$ which are the closest to ℓ

If the root line ℓ of θ is on scale $< h$ then

$$N_h^*(\theta) = \sum_{i=1}^{m} N_h^*(\theta_i), \tag{A3.2}$$

where θ_i is the subtree with ℓ_i as root line, hence the bound follows by the inductive hypothesis.

If the root line ℓ has scale $\ge h$ then $\ell_1, \dots, \ell_m$ are the entering line of a cluster T.

By denoting again with θ_i the subtree having ℓ_i as root line, one has

$$N_h^*(\theta) = 1 + \sum_{i=1}^{m} N_h^*(\theta_i), \tag{A3.3}$$

and the bound becomes trivial if either $m = 0$ or $m \ge 2$.

If $m = 1$ then one has a cluster T with two external lines ℓ and ℓ_1, with $h_{\ell_1}, h_\ell \ge h$ so that by (1.14):

$$\big||\Omega n_\ell| - \sqrt{\omega_{m_\ell} + n_\ell \nu_{m_\ell,n_\ell}}\big| \le 2^{-h+1}\gamma, \ \Big||\Omega n_{\ell_1}| - \sqrt{\omega_{m_{\ell_1}} + n_{\ell_1}\nu_{m_{\ell_1},n_{\ell_1}}}\Big| \le 2^{-h+1}\gamma, \tag{A3.4}$$

As ℓ is non resonant, surely $n_\ell \ne n_{\ell_1}$ (otherwise if $n_\ell = n_{\ell_1}$ then $m_\ell \ne m_{\ell_1}$ hence the two lines cannot have both scale $\ge h$). Hence by (1.26) one has, for suitable $\eta_\ell, \eta_{\ell_1} \in \{+,-\}$,

$$2^{-h+2}\gamma \ge \big|\Omega(n_\ell - n_{\ell_1}) + \eta_\ell\sqrt{\omega_{m_\ell} + n_\ell\nu_{m_\ell,n_\ell}} + \eta_{\ell_1}\sqrt{\omega_{m_{\ell_1}} + n_{\ell_1}\nu_{m_{\ell_1},n_{\ell_1}}}\big| \ge \gamma|n_\ell - n_{\ell_1}|^{-\tau}, \tag{A3.5}$$

so that $K(\theta) - K(\theta_1) > E_h$. Hence by using the inductive hypothesis

$$N_h^*(\theta) = 1 + N_h^*(\theta_1) \le 1 + K(\theta_1)E_h^{-1} - 1 \le 1 + \Big(K(\theta) - E_h\Big)E_h^{-1} - 1 \le K(\theta)E_h^{-1} - 1, \tag{A3.6}$$

hence the bound is proved also if the root line is on scale $\ge h$.

We prove Lemma 8 for $\mathcal{L}\mathrm{V}al(\hat\theta)$, $\hat\theta \in \mathcal{R}_{R,n,m,h}$. We consider the two subtrees entering in v_0; one, called $\tilde\theta$, does not contain the endpoint e and the bounds of the preceding lemma can be applied, so we consider the subtree θ containing e. We proceed inductively on h for θ proving that $N_h^*(\theta) \le 2K(\theta)2^{\frac{2-h}{\tau}}$; such bound and the bound (A3.1) for $\tilde\theta$ immediately implies (4.7) as $K(\hat\theta) = K(\tilde\theta) + K(\theta) - 1$.

In order to prove $N_h^*(\theta) \le 2K(\theta)2^{\frac{2-h}{\tau}}$ we define $k_0 = 2^{(h-1)/\tau}$. One has $N_h^*(\theta) = 0$ for $K(\theta) < k_0$. In fact if the line ℓ with scale h is not in the path, this follows from the first Melnikov condition. If such line is on the path we have that, if $\mathcal{L}\mathrm{V}al(\theta)$ is non vanishing

$$||\Omega n_\ell^0 + \bar\omega_{n,m}| - \sqrt{\omega_{m_\ell} + n_\ell\nu_{m_\ell,n_\ell}})| \le \gamma 2^{-h+1} \tag{A3.7}$$

so that by the second Melnikov condition:

$$K(\theta) \ge |n_\ell^0| \ge 2^{\frac{h-1}{\tau}}. \tag{A3.8}$$

Then for $K(\theta) < k_0$ the bound is satisfied; for $K \geq k_0$, we assume that the bound holds for all $K(\theta) = K' < K$, and we show that it follows also for $K(\theta) = K$. If $K(\theta) > k_0$, we assume that the bound holds for all trees θ' with $K(\theta') < K(\theta)$. Define $E_h = 2^{-1}(2^{(4-h)/\tau})^{-1}$: so we have to prove that $N_h^*(\theta) \leq K(\theta)E_h^{-1}$. If the root line ℓ of θ is on scale $< h$ then

$$N_h^*(\theta) = \sum_{i=1}^{m} N_h^*(\theta_i), \tag{A3.9}$$

where θ_i is the subtree with ℓ_i as root line, hence the bound follows by the inductive hypothesis. If the root line ℓ has scale $\geq h$ then $\ell_1, \dots, \ell_m$ are the entering line of a cluster T. The same occurs if the root line is on scale $\geq h$ and non-resonant, and, by calling $\ell_1, \dots, \ell_m$ the lines on scale $\geq h$ which are the closest to ℓ, one has $m \geq 2$: in fact in such a case at least $m-1$ among the subtrees $\theta_1, \dots, \theta_m$ verifies the bound (A3.1) so that

$$N_h^*(\theta) = 1 + \sum_{i=1}^{m} N_h^*(\theta_i) \leq 1 + E_h^{-1} \sum_{i=1}^{m} K(\theta_i) - (m-1) \leq E_h K(\theta), \tag{A3.10}$$

If $m = 0$ then $N_h^*(\theta) = 1$ and $K(\theta)2^{(2-h)/\tau} \geq 1$ because one must have $K(\theta) \geq k_0$. So the only non-trivial case is when one has $m = 1$. In this case ℓ, ℓ_1 are on the path connecting the external lines of the resonance

$$||\Omega n_\ell^0 + \bar{\omega}_{n,m}| - \sqrt{\omega_{m_\ell} + n_\ell \nu_{m_\ell, n_\ell}})| \leq \gamma 2^{-h+1} \tag{A3.11}$$

$$||\Omega n_{\ell_1}^0 + \bar{\omega}_{n,m}| - \sqrt{\omega_{m_{\ell_1}} + n_{\ell_1} \nu_{m_{\ell_1}, n_{\ell_1}}})| \leq \gamma 2^{-h+1} \tag{A3.12}$$

so that for suitable $\eta_\ell, \eta_{\ell_1} \in \{+, -\}$

$$\begin{aligned} 2^{-h+2}\gamma &\geq \left|\Omega(n_\ell^0 - n_{\ell_1}^0) + \eta_\ell \sqrt{\omega_{m_\ell} + n_\ell \nu_{m_\ell, n_\ell}} + \eta_{\ell_1} \sqrt{\omega_{m_{\ell_1}} + n_{\ell_1} \nu_{m_{\ell_1}, n_{\ell_1}}}\right| \\ &\geq \gamma |n_\ell^0 - n_{\ell_1}^0|^{-\tau}, \end{aligned} \tag{A3.13}$$

from which $K(\theta) - K(\theta_1) \geq |n_\ell^0 - n_{\ell_1}^0| \geq 2^{(h-2)/\tau}$ and by the analogous of (A3.6) the bound is proved.

Appendix A4. Proof of Lemma 7

In order to prove (4.15) we proceed by induction; consider a tree θ with k internal nodes and $s_v = 2$ for any v; we call ℓ_1, ℓ_2 the two lines entering v_0; we call $m_{\ell_1} = m_1$ and $m_{\ell_2} = m_2$ the root lines of two subtrees θ_1 and θ_2 with $k_1 \geq 0$ and $k_2 \geq 0$ vertices, and $k_1 + k_2 = k - 1$. If $k_1 = 0$ (or $k_2 = 0$) then one of the two cases holds:

1. ℓ_1 connects to an end-node so that $|\eta_v| = 1$ and $m_1 = 1$.
2. ℓ_1 connects to the external node so that $|\eta_v| = |\eta_e| = 1/m_e^3$ (this case is possible only of $\theta \in \mathcal{R}_{R,h,n,m}$).

So we can proceed with our inductive hypothesis and suppose that our bound holds for all trees with $0 \leq k_1 < k$ end-nodes. Without loss of generality we can suppose that $m_1 \geq m_2$. We perform the bound $m+m_1+m_2 > m$,$m+m_1-m_2 > m$ so that:

$$\sum_{m_1 \geq m_2}^{*} \frac{1}{|(m^2 - (m_1+m_2)^2)||(m^2 - (m_1 - m_2)^2)|} \frac{1}{m_1^2 m_2^2}$$

$$\leq \frac{1}{m^2} \sum_{m_1 \geq m_2}^{*} \frac{1}{m_1^2 m_2^2} \frac{1}{|m-m_1-m_2||m-m_1+m_2|} \tag{A4.1}$$

If $m_1 \leq \frac{m}{4}$ then the bound is trivial as:

$$\frac{1}{m^2} \sum_{\frac{m}{4} \geq m_1 \geq m_2}^{*} \frac{1}{|m-m_1+m_2||m-m_1-m_2|} \frac{1}{m_1^2 m_2^2} \leq \frac{8}{3m^4} \sum_{m_1, m_2}^{*} \frac{1}{m_1^2 m_2^2} \leq \frac{C_1}{m^4}. \tag{A4.2}$$

In the remaining terms we treat separately the cases $m_1 \leq m-1$, $m_1 > m$ and $m_1 = m$.

We notice that in the first case $|m-m_1+m_2| \geq m-m_1$, while in the second case $|m-m_1-m_2| \geq m_1-m$. We then obtain the bound:

$$\frac{1}{m^2} \sum_{m_1 > \frac{m}{4}, m_1 \geq m_2}^{*} \frac{1}{m_1^2 m_2^2} \frac{1}{|m-m_1-m_2||m-m_1+m_2|}$$

$$\leq \frac{1}{m^2} \Big(\sum_{m-1 \geq m_1 > \frac{m}{4}}^{*} \frac{1}{m_1^2(m-m_1)} (\sum_{m_2=1}^{\infty} \frac{1}{m_2^2|m-m_1-m_2|}) + \tag{A4.3}$$

$$\sum_{m_1 > m}^{*} \frac{1}{m_1^2(m_1-m)} (\sum_{m_2=1}^{\infty} \frac{1}{m_2^2|m-m_1+m_2|}) + \frac{1}{m^2} \sum_{m_2=1}^{\infty} \frac{1}{m_2^2} \Big)$$

Now we estimate the sums with integrals:

$$\sum_{n \neq A} \frac{1}{|A-n|n^2} \leq C_0 [\int_{x=1}^{A-1} \frac{1}{(A-x)x^2} + \int_{x=A+1}^{\infty} \frac{1}{(x-A)x^2} + \frac{C_2}{A^2}] \leq \frac{C_3}{A} \tag{A4.4}$$

$$\sum_{n \neq A} \frac{1}{(A-n)^2 n^2} \leq C_0 [\int_{x=1}^{A-1} \frac{1}{(A-x)^2 x^2} + \int_{x=A+1}^{\infty} \frac{1}{(x-A)^2 x^2} + \frac{C_2}{A^2}] \leq \frac{C_3}{A^2} \tag{A4.5}$$

We use the first bound on the sum over m_2 then in the sum over m_1 we obtain a series as in the second bound immediately implying

$$\sum_{m_1 > \frac{m}{4}, m_1 \geq m_2}^{*} \frac{1}{m_1^2 m_2^2} \frac{1}{|(m^2-(m_1+m_2)^2)||(m^2-(m_1-m_2)^2)|} \frac{1}{m_1^2 m_2^2} \leq \frac{\bar{C}}{m^4} \tag{A4.6}$$

This implies the inductive hypothesis, as θ has k vertices and $k_1+k_2 = k-1$, by choosing $C_1 = \bar{C}$ in (4.15).

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