

G. Benfatto, P. Falco, and V. Mastropietro
Università di Roma Tor Vergata

We present the first rigorous construction of the QFT Thirring model, for any value of the mass, in a functional integral approach, by proving that a set of Grassmann integrals converges, as the cutoffs are removed, to a set of Schwinger functions verifying the Osterwalder-Schrader axioms. The massless limit is investigated and it is shown that the Schwinger functions have different properties with respect to the ones of the well known exact solution: the Ward Identities have anomalies violating the anomaly non-renormalization property and additional anomalies, apparently unnoticed before, are present in the closed equation for the interacting propagator, obtained by combining a Schwinger-Dyson equation with Ward Identities.

PACS numbers: 03.70.+k, 11.10.Cd, 11.15.Tk, 11.30.Rd

I. INTRODUCTION

The Thirring model has been the subject of a very intense research in the last fifty years: it is one of the very few QFT models for which non-perturbative informations can be obtained and it shares with more realistic $d = 4$ models, like QED_4 , many features, as it is apparent from a classical perturbative Feynman graph analysis [20]. In the *massless case*, a complete set of correlations has been obtained via an exact solution [16, 18]; they verify the Wightman reconstruction axioms [6, 8] so that a QFT corresponding to the massless Thirring model can be constructed from them. In the *massive case*, the equivalence (at a perturbative level) with the Sine-Gordon model is known [7] and some eigenstates by Bethe-ansatz analysis have been found [5]; but a complete rigorous construction has never been performed.

In this paper we fill this gap by considering a set of Grassmann integrals regularized via suitable cutoffs and with a contact current-current interaction and proving that, removing cutoffs and for any value of the mass, they converge to a set of Schwinger functions verifying the Osterwalder-Schrader reconstruction axioms for Euclidean QFT [21]; this provides the first rigorous construction of the massive Thirring model. Moreover, even if in the massless case other constructions were known, we find in any case interesting to reach a complete non-perturbative construction of the Thirring model relying only on a functional integral approach, which could be the only possible one at higher dimensions or for more realistic models. We stress that our results are *non-perturbative*, in the sense that the Grassmann integrals are expressed in terms of series expansion whose *convergence is proved* (that is we resum the naive perturbation theory and we prove the convergence of the resummed expansion). Our results are smooth in the mass and the massless limit can be investigated; the natural question is whether or not the correlations of the massive Thirring model, which we can compute from a functional integral approach, pass over smoothly into the known correlations of the exact solution in the massless limit. Contrary to what was found in the well known paper [13], based on

Zimmermann's version of BPH subtraction scheme (the analysis was purely perturbative), we get that the correlations in the massless limit have *different properties* with respect to the exact solution. The Ward Identities (WI) have anomalies violating the *anomaly non-renormalization* property; moreover, *additional anomalies*, apparently unnoticed before, are present even in the closed equation for the interacting propagator, obtained by combining a Schwinger-Dyson equation with the Ward Identities.

II. THE EXACT SOLUTION

It is worth to recall the Johnson solution [16] of the massless Thirring model, describing $d = 2$ massless Dirac fermions with coupling $(\lambda/2) \int j_{\mathbf{x}}^{\mu} j_{\mathbf{x}}^{\mu}$, in order to compare its properties with the results from the functional integral approach. The solution is essentially based on a *self-consistency* argument: a number of reasonable requirements on the correlations is assumed, from which their explicit expression can be determined. Calling $G^{\mu}(\mathbf{z}; \mathbf{x}, \mathbf{y})$, $G^{\mu,5}(\mathbf{z}; \mathbf{x}, \mathbf{y})$ and $G(\mathbf{x}, \mathbf{y})$ the truncated vacuum expectations in the Minkowski space of the T -product of $Z^{-1} j_{\mathbf{z}}^{\mu} \psi_{\mathbf{x}} \bar{\psi}_{\mathbf{y}}$, $Z^{-1} j_{\mathbf{z}}^{\mu,5} \psi_{\mathbf{x}} \bar{\psi}_{\mathbf{y}}$ and $Z^{-1} \psi_{\mathbf{x}} \bar{\psi}_{\mathbf{y}}$ respectively, where Z is the *wave function renormalization*, the first assumption is the validity of WI of the form

$$\partial_{\mathbf{z}}^{\mu} G^{\mu}(\mathbf{z}; \mathbf{x}, \mathbf{y}) = -ia[\delta(\mathbf{z} - \mathbf{x}) - \delta(\mathbf{z} - \mathbf{y})]G(\mathbf{x}, \mathbf{y}) \quad (1)$$

$$\partial_{\mathbf{z}}^{\mu} G^{\mu,5}(\mathbf{z}; \mathbf{x}, \mathbf{y}) = -i\bar{a}[\delta(\mathbf{z} - \mathbf{x}) - \delta(\mathbf{z} - \mathbf{y})]\gamma^5 G(\mathbf{x}, \mathbf{y}) \quad (2)$$

where a, \bar{a} are parameters to be determined and $j_{\mathbf{z}}^{\mu}, j_{\mathbf{z}}^{\mu,5}$ are operators formally defined, via a point splitting procedure, from $\bar{\psi} \bar{\gamma}^{\mu} \psi$ and $\bar{\psi} \bar{\gamma}^{\mu} \bar{\gamma}^5 \psi$; $\bar{\gamma}^{\mu}, \bar{\gamma}^5$ are the Minkowski gamma matrices. Assuming also the validity of a *Schwinger-Dyson* equation and using the WI for G^{μ} , a *closed equation* for the two point Schwinger function is obtained, which reads, if $\hat{G}(\mathbf{k})$ denotes the Fourier transform of $G(\mathbf{x}, \mathbf{y})$:

$$\hat{G}(\mathbf{k}) = \hat{g}(\mathbf{k}) \left[\frac{1}{Z} - \lambda(a - \bar{a}) \int \frac{d^2 \mathbf{p}}{(2\pi)^2} \frac{\hat{G}(\mathbf{k} - \mathbf{p})}{\not{\mathbf{p}}} \right]. \quad (3)$$

This equation can be solved (at a very formal level, as one has to take a vanishing wave function renormalization) and $G(\mathbf{x}, 0)$ is found to be equal to $i(\bar{\gamma}^\mu \partial_\mu)^{-1}(|\mathbf{x}|/x_0)^{-\eta}$, where x_0 is an arbitrary constant with the dimension of a length and η is a critical index related to the coefficients a, \bar{a} by

$$\eta = \frac{\lambda}{2\pi}(a - \bar{a}). \quad (4)$$

Finally, by a self-consistency argument involving also the four point Schwinger function, the explicit values for the anomalies were found:

$$a^{-1} = 1 - \frac{\lambda}{2\pi}, \quad \bar{a}^{-1} = 1 + \frac{\lambda}{2\pi}. \quad (5)$$

The above equation is particularly significant, as it says the anomalies do not receive contributions from higher orders; this property is called *anomaly non-renormalization* and it holds, as a statement valid at all order in perturbation theory, in realistic models like QED_4 (it is the content of the well known *Adler-Bardeen theorem* [1]). The order by order analysis of [1] can be adapted to the Thirring model [14] and (5) is indeed obtained; hence the validity of (5) has been considered *non-perturbative* verification of the *Adler-Bardeen theorem* applied to the Thirring model. However, it should be noticed that the regularization and the assumptions in the exact solution or in the perturbative (or functional integral) approach are different, hence there is no guarantee that the same results should be found in the two approaches, see for instance [2].

III. NON-PERTURBATIVE RENORMALIZATION

Our starting point is the generating functional of the truncated Schwinger functions, $W_{\kappa,K}(A, \eta)$, defined so that $e^{W_{\kappa,K}(A, \eta)}$ is the following *Grassmann integral*:

$$\frac{1}{\mathcal{N}} \int P_{\kappa,K}(d\psi) e^{\int d\mathbf{x} \left[-\frac{1}{2} j_{\mathbf{x}}^\mu j_{\mathbf{x}}^\mu + \frac{1}{\sqrt{Z_K}} (\bar{\eta}_{\mathbf{x}} \psi_{\mathbf{x}} + \bar{\psi}_{\mathbf{x}} \eta_{\mathbf{x}}) + j_{\mathbf{x}}^\mu A_{\mathbf{x}}^\mu \right]} \quad (6)$$

where \mathcal{N} is a renormalization constant, η, A are external fields, $\psi_{\mathbf{x}}, \bar{\psi}_{\mathbf{x}}$ are Grassmann variables, $j_{\mathbf{x}}^\mu = \bar{\psi}_{\mathbf{x}} \gamma^\mu \psi_{\mathbf{x}}$ and $P_K(d\psi)$ is the Grassmann integration with propagator

$$g_{\kappa,K}(\mathbf{x}, \mathbf{y}) = \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \chi_{\kappa,K}(\mathbf{k}) \frac{e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{\mathbf{k} + \mu_K}, \quad (7)$$

where the *smooth cutoff function* $\chi_{\kappa,K}(\mathbf{k})$ selects the momenta $\kappa \leq |\mathbf{k}| \leq K$, with $\kappa < 1, K > 1$. Finally Z_K and μ_K are the bare wave function renormalization and mass and γ^5, γ^μ are the Euclidean gamma matrices. The Schwinger functions are defined as

$$S_{n,m}^{K,k} = \frac{\partial^{n+m} W_{\kappa,K}(A, \eta)}{\partial A_{z_1}^{\mu_1} \dots \partial A_{z_m}^{\mu_m} \partial \bar{\eta}_{y_1} \partial \eta_{y_1} \dots \partial \bar{\eta}_{y_n} \partial \eta_{y_n}} \Big|_0. \quad (8)$$

In particular

$$G_{\kappa,K}^\mu(\mathbf{z}; \mathbf{x}, \mathbf{y}) = \frac{\partial^3 W_{\kappa,K}}{\partial A_{\mathbf{z}}^\mu \partial \bar{\eta}_{\mathbf{x}} \partial \eta_{\mathbf{y}}} \Big|_0, \quad G_{\kappa,K}(\mathbf{x}, \mathbf{y}) = \frac{\partial^2 W_{\kappa,K}}{\partial \bar{\eta}_{\mathbf{x}} \partial \eta_{\mathbf{y}}} \Big|_0 \quad (9)$$

and $G_{\kappa,K}^\mu(\mathbf{z}; \mathbf{x}, \mathbf{y}) = -i\varepsilon_{\mu,\nu} G_{\kappa,K}^{\mu,5}(\mathbf{z}; \mathbf{x}, \mathbf{y})$.

The presence of the ultraviolet cutoff K and the infrared cutoff κ makes the functional integral (6) well defined; to carry out the renormalization program at non-perturbative level we have to prove that there exist K -depending bare parameters such that, in the limit $\kappa \rightarrow 0, K \rightarrow \infty$, the Schwinger functions verify the Osterwalder-Schrader axioms, (OS), [21]. Our basic result [4] is the following theorem.

Theorem 1 *Given λ small enough and $\mu \geq 0$, there exist bare parameters*

$$Z_K = K^{-\eta}(1 + O(\lambda^2)) \quad , \quad \mu_K = \mu K^{-\bar{\eta}}(1 + O(\lambda)), \quad (10)$$

with $\eta = a\lambda^2 + O(\lambda^4)$ and $\bar{\eta} = -b\lambda + o(\lambda^2)$, $a, b > 0$, such that $\lim_{\kappa^{-1}, K \rightarrow \infty} S_{n,m}^{K,k}$ exist at coinciding points and verify the Osterwalder-Schrader axioms.

In particular, in the massless case $\mu = 0$,

$$\lim_{\kappa^{-1}, K \rightarrow \infty} G(\mathbf{x}, \mathbf{y})_{\kappa,K} = (1 + f_\lambda) \frac{\not{x} - \not{y}}{|\mathbf{x} - \mathbf{y}|^{2+\eta}}. \quad (11)$$

with $f_\lambda = O(\lambda)$ independent from \mathbf{x}, \mathbf{y} .

The above theorem says that, by choosing properly the bare wave function and mass renormalization, with a singular behavior as the cutoffs are removed, one gets a set of finite Schwinger functions, for which a QFT for the massive Thirring model is obtained via the reconstruction theorem in [21]. The proof is based on the new methods introduced in [3], which allow us to overcome the well known technical problem posed by the combination of a non-perturbative setting based on multiscale analysis [12, 22] with the necessity of exploiting cancellations due to the local symmetries. Such cancellations are established by suitable WI valid at each scale and, contrary to the WI formally valid when all the cutoffs are removed, they have corrections due to the cutoffs. The crucial role of WI in the construction of the theory is a feature that the Thirring model shares with realistic models like QED or the Electroweak theory in $d = 4$, requiring WI even to prove the perturbative renormalizability. Note that this feature is absent in the models previously rigorously constructed by functional integral methods, like the massive *Yukawa* model [19] or the massive *Gross-Neveu* model [11]. A delicate point in the proof in [4] of the above theorem is the verification of the positivity axiom, as the cutoff we have chosen destroy the positivity definiteness and it would be quite difficult to prove it directly after the cutoffs are removed. We overcome this problem by considering a functional integral with a *lattice regularization* containing a Wilson term (to avoid a spurious singularity); in this case the positivity property is automatically satisfied and one has to choose the parameters $\lambda_a(\lambda)$,

$Z_a(\lambda)$, $\mu_a(\lambda)$, if a is the lattice step, in order to control the limit $a \rightarrow 0$. We have then showed that it is possible to choose them so that $\lim_{k^{-1}, K \rightarrow \infty} S_{n,m}^{K,k} = \lim_{a \rightarrow 0} S_{n,m}^a$, hence positivity follows. Note finally that it has been claimed that the functional integral (6) can be exactly computed for massless fermions and no cutoffs, see for instance [9], by introducing a boson field and using that that fermionic determinants are *quadratic* if the Bose field is regular. In this way one recovers the same results of the exact solution, but mathematically this procedure is not justified (the functional integral is over all the possible fields) and, as we will see below, it leads to wrong

conclusions.

IV. ANOMALY RENORMALIZATION

We shall only consider the WI in the massless case, $\mu_K = 0$; the presence of the cutoff function $\chi_{\kappa,K}$ breaks the continuous symmetries, $\psi_{\mathbf{x}} \rightarrow e^{i\alpha_{\mathbf{x}} + i\alpha_{\mathbf{x}}^5 \gamma^5} \psi_{\mathbf{x}}$ and $\bar{\psi}_{\mathbf{x}} \rightarrow \bar{\psi}_{\mathbf{x}} e^{-i\alpha_{\mathbf{x}} + i\alpha_{\mathbf{x}}^5 \gamma^5}$, so that there are corrective terms with respect to the formal WI (obtained formally neglecting all cutoffs):

$$\mathbf{p}^\mu \widehat{G}_{\kappa,K}^\mu(\mathbf{p}; \mathbf{k}) = \widehat{G}_{\kappa,K}(\mathbf{k} - \mathbf{p}) - \widehat{G}_{\kappa,K}(\mathbf{k}) + \int \frac{d^2 \mathbf{k}'}{(2\pi)^2} C_{\kappa,K}^\mu(\mathbf{k}', \mathbf{k}' - \mathbf{p}) \langle \bar{\psi}_{\mathbf{k}'} \gamma^\mu \psi_{\mathbf{k}' - \mathbf{p}} | \psi_{\mathbf{k}} | \bar{\psi}_{\mathbf{k} - \mathbf{p}} \rangle_{\kappa,K} \quad (12)$$

$$\mathbf{p}^\mu \widehat{G}_{\kappa,K}^{\mu,5}(\mathbf{p}; \mathbf{k}) = \gamma^5 \widehat{G}_{\kappa,K}(\mathbf{k} - \mathbf{p}) - \gamma^5 \widehat{G}_{\kappa,K}(\mathbf{k}) + \int \frac{d^2 \mathbf{k}'}{(2\pi)^2} C_{\kappa,K}^\mu(\mathbf{k}', \mathbf{k}' - \mathbf{p}) \langle \bar{\psi}_{\mathbf{k}'} \gamma^\mu \gamma^5 \psi_{\mathbf{k}' - \mathbf{p}} | \psi_{\mathbf{k}} | \bar{\psi}_{\mathbf{k} - \mathbf{p}} \rangle_{\kappa,K}; \quad (13)$$

the function $C_{\kappa,K}^\mu(\mathbf{k}_+, \mathbf{k}_-)$ is given by $(\chi_{\kappa,K}^{-1}(\mathbf{k}_-) - 1) \mathbf{k}_-^\mu - (\chi_{\kappa,K}^{-1}(\mathbf{k}_+) - 1) \mathbf{k}_+^\mu$; $|$'s single out the cluster of fields w.r.t. which the truncation of the expectation is taken. The above expression can be perturbative checked at lowest orders using the (trivial) identities

$$\mathbf{p}^\mu \widehat{g}_{\kappa,K}(\mathbf{k} - \mathbf{p}) \gamma^\mu \Gamma \widehat{g}_{\kappa,K}(\mathbf{k}) = \Gamma [\widehat{g}_{\kappa,K}(\mathbf{k} - \mathbf{p}) - \widehat{g}_{\kappa,K}(\mathbf{k})] + C_{\kappa,K}^\mu(\mathbf{k}, \mathbf{k} - \mathbf{p}) \widehat{g}_{\kappa,K}(\mathbf{k} - \mathbf{p}) \gamma^\mu \Gamma \widehat{g}_{\kappa,K}(\mathbf{k}), \quad (14)$$

for $\Gamma = 1, \gamma^5$.

The last addenda in the above WI involves the average of an highly non-local and complex operator, but removing cutoffs, as proven in [4], they can be written in a remarkable simple form.

Theorem 2 *In the same hypothesis of Theorem 1,*

$$\int \frac{d^2 \mathbf{k}'}{(2\pi)^2} C_{\kappa,K}^\mu(\mathbf{k}', \mathbf{k}' - \mathbf{p}) \langle \bar{\psi}_{\mathbf{k}'} \gamma^\mu \psi_{\mathbf{k}' - \mathbf{p}} | \psi_{\mathbf{k}} | \bar{\psi}_{\mathbf{k} - \mathbf{p}} \rangle_{\kappa,K} = \alpha_+ \mathbf{p}^\mu \widehat{G}_{\kappa,K}^\mu(\mathbf{p}; \mathbf{k}) + H_{\kappa,K}(\mathbf{p}; \mathbf{k}) \quad (15)$$

$$\int \frac{d^2 \mathbf{k}'}{(2\pi)^2} C_{\kappa,K}^\mu(\mathbf{k}', \mathbf{k}' - \mathbf{p}) \langle \bar{\psi}_{\mathbf{k}'} \gamma^\mu \gamma^5 \psi_{\mathbf{k}' - \mathbf{p}} | \psi_{\mathbf{k}} | \bar{\psi}_{\mathbf{k} - \mathbf{p}} \rangle_{\kappa,K} = -\alpha_- \mathbf{p}^\mu \widehat{G}_{\kappa,K}^{\mu,5}(\mathbf{p}; \mathbf{k}) + H_{\kappa,K}^5(\mathbf{p}; \mathbf{k}) \quad (16)$$

where α_+ and α_- are suitable functions of λ such that

$$\alpha_\pm = \frac{\lambda}{2\pi} \pm c_2 \lambda^2 + O(\lambda^3) \quad (17)$$

and c_2 strictly negative; moreover, for fixed non-zero \mathbf{k}, \mathbf{p} , $\lim_{\kappa^{-1}, K \rightarrow \infty} H_{\kappa,K}(\mathbf{p}; \mathbf{k}) = \lim_{\kappa^{-1}, K \rightarrow \infty} H_{\kappa,K}^5(\mathbf{p}; \mathbf{k}) = 0$

Hence the WI we get in the functional integral approach have the same form as those found in the exact approach, see (1) and (2), but the anomaly coefficients, instead of by (5), are given by $a^{-1} = 1 - \frac{\lambda}{2\pi} + c_+ \lambda^2 + O(\lambda^3)$, $\bar{a}^{-1} = 1 + \frac{\lambda}{2\pi} + c_+ \lambda^2 + O(\lambda^3)$. The anomaly coefficients are not linear in the bare coupling (*the anomaly*

non-renormalization is violated), contrary to (5); hence the theory found in the massless limit starting from the functional integral (6) has different properties with respect to the one constructed by the exact solution. The presence of such anomaly renormalization can be checked in standard perturbation theory, by calculating the two graphs of Fig. 1, but the proof of the non-perturbative bounds of Theorem 2 requires a careful mathematical analysis, see [4].



FIG. 1: First and second order contribution to the anomaly; The small circle represents C_ω

V. ADDITIONAL ANOMALIES

The two point function $\widehat{G}_{k,K}(\mathbf{k})$ verifies the SD equation

$$\frac{\widehat{G}_{k,K}(\mathbf{k})}{\widehat{g}_{\kappa,K}(\mathbf{k})} = \frac{1}{Z_K} - \lambda \int \frac{d^2 \mathbf{p}}{(2\pi)^2} \gamma^\mu \widehat{G}_{k,K}^\mu(\mathbf{p}; \mathbf{k} - \mathbf{p}). \quad (18)$$

Inserting the explicit expression of $\widehat{G}_{\kappa,K}^\mu$ obtained from the WI, we obtain

$$\frac{\widehat{G}_{k,K}(\mathbf{k})}{\widehat{g}_{\kappa,K}(\mathbf{k})} = \frac{1}{Z_K} - \lambda(a - \bar{a}) \int \frac{d^2\mathbf{p}}{(2\pi)^2} \frac{\widehat{G}_{k,K}(\mathbf{k} - \mathbf{p})}{\mathbf{p}} \quad (19)$$

$$- \int \frac{d^2\mathbf{p}}{(2\pi)^2} \left[\frac{aH_{\kappa,K}(\mathbf{p}; \mathbf{k} - \mathbf{p})}{\mathbf{p}} + \frac{\bar{a}\gamma^5 H_{\kappa,K}^5(\mathbf{p}; \mathbf{k} - \mathbf{p})}{\mathbf{p}} \right].$$

If the last term in (19) were vanishing in the limit $\kappa^{-1}, K \rightarrow \infty$, one would get a closed equation for $\widehat{G}(\mathbf{k})$, which is identical to the closed equation obtained in [16], that is (3). *This is not what happens*; as already noticed, $H_{\kappa,K}(\mathbf{p}; \mathbf{k})$ and $H_{\kappa,K}^5(\mathbf{p}; \mathbf{k})$ are vanishing in the limit $\kappa^{-1}, K \rightarrow \infty$ at \mathbf{k}, \mathbf{p} fixed, but *not* if \mathbf{p} is integrated. Intuitively this can be understood by noting that the integral involves momenta close to the u.v. cutoff scale K , where $H_{\kappa,K,\omega,\omega'}^{2,1}$ is not small at all. In other words: *even if the WI and the SD equation are true, in the limit $\kappa^{-1}, K \rightarrow \infty$, the closed equation obtained by combining the two identities is not verified; this is a new anomaly* which is hard to see in a purely perturbative approach and in fact it was never noticed before. One could guess that the fact that the last term in (19) is not vanishing in the limit of removed cutoffs should imply that there is no closed equation for $\widehat{G}(\mathbf{k})$. Instead, we proved in [4] another crucial identity.

Theorem 3 *In the same hypothesis of previous theorems, the integral in (19) is equal to*

$$\sigma \frac{1}{Z_K} + \rho \frac{\widehat{G}_{k,K}(\mathbf{k})}{\widehat{g}_{\kappa,K}(\mathbf{k})} + R_{\kappa,K}(\mathbf{k}) \quad (20)$$

with $O(\lambda_K^2)$, non-zero σ and ρ ; furthermore, for fixed non-zero \mathbf{k} , $\lim_{\kappa^{-1}, K \rightarrow \infty} R_{\kappa,K}(\mathbf{k}) = 0$.

By inserting (20) in (19) we get a closed equation which is *different* from the one assumed in [16]; in particular, the relation (4) is replaced by

$$\eta = \frac{\lambda}{2\pi} \frac{a - \bar{a}}{1 + \rho}. \quad (21)$$

VI. CONCLUSIONS

We stress that one could construct a QFT corresponding to the Thirring model also starting from a *non-local* interaction $\int dx dy v(\mathbf{x} - \mathbf{y}) j_{\mathbf{x}}^\mu j_{\mathbf{y}}^\mu$ with $\hat{v}(\mathbf{k}) = e^{-\mathbf{k}^2/\Lambda^2}$. If the cut-offs are removed in the order $\Lambda \rightarrow \infty, K \rightarrow \infty$, the results are the same as depicted in the previous sections. On the contrary, if the cut-offs are removed in the opposite order, $K \rightarrow \infty, \Lambda \rightarrow \infty$, the results of the exact solution are recovered. This means that the Adler-Bardeen theorem is not in contrast with our result: if one assumes, as in [1], that the interaction is mediated by a boson field and removes the fermionic u.v. cutoff before the bosonic one, then the anomaly non-renormalization holds; in the opposite case new features appear.

Finally, the combination of WI and SD equations is a rather general technique; it is used, for instance, in QED in [10, 17] and in condensed matter physics in [15]. We have seen that such a method can really be implemented in a full non-perturbative approach, but taking care of unexpected anomalous features. It would be very interesting to see if such features occur in 4-dimensional models.

Acknowledgments We are indebted with K. Gawedzki and G. Gallavotti for enlightening discussions. P.F. gratefully acknowledges the support of the Erwin Schrödinger Institute for Mathematical Physics (Vienna).

-
- [1] Adler S.L., Bardeen W.A.: *Phys. Rev.* **182** 1517 (1969)
 - [2] Akiyama A., Futami F.: *Phys. Rev. D* **46** 798 (1992).
 - [3] Benfatto G., Mastropietro V.: *Rev. Math. Phys.* **13** 1323 (2001) *Comm. Math. Phys.* **231** 97 (2002) *Comm. Math. Phys.* **258** 609 (2005).
 - [4] Benfatto G., Falco P., Mastropietro V.: hep-th/0606177
 - [5] Bergknoff H., Thacker H. B.: *Phys. Rev. Lett.* **42** 135 (1979).
 - [6] Carey A.L., Ruijsenaars S.N.M., Wright J.D.: *Comm. Math. Phys.* **99** 347 (1985).
 - [7] Coleman S.: *Phys. Rev. D* **11** 2088 (1975).
 - [8] Dell'Antonio G.F., Frishman Y., Zwanziger, D.: *Phys. Rev. D* **6** 988 (1972).
 - [9] Furuya K., Gamboa Saravi S., Schaposnik F. A.: *Nucl. Phys. B* **208** 159 1982.
 - [10] Green H.S., Cartier J.F., Broyles A.A.: *Phys. Rev. D* **18**, 1102-1109 (1978)
 - [11] Gawedzki K., Kupianinen A.: *Comm. Math. Phys.* **102** 1 (1985)
 - [12] Gallavotti G.: *Rev. Mod. Phys.* **55** 471-562 (1985)
 - [13] Gomes M., Lowenstein J.H.: *Nucl. Phys. B* **45** 252-266 1972.
 - [14] Georgi H., Rawls J.M.: *Phys. Rev. D* **3** 874 (1971)
 - [15] Ioffe L.B., Lidsky D., Altschuler B.L. *Phys. Rev. Lett.* **73** 472 (1994)
 - [16] Johnson K.: *Nuovo Cimento* **20** 773 (1961).
 - [17] Johnson K., Baker M., Willey R.: *Phys. Rev.* **136** 1111 (1964)
 - [18] Klaiber B.: In: Quantum theory and statistical physics. Vol X A. Barut A.O. and Brittin. W.F. (1968).
 - [19] Lesniewski A.: *Comm. Math. Phys.* **108** 437 (1987).
 - [20] Mueller A.H., Trueman T.L. *Phys. Rev. D* **4** 1635 (1971)
 - [21] Osterwalder K., Schrader R.: *Comm. Math. Phys.* **31** 83 (1973).
 - [22] Polchinski J. *Nucl. Phys. B* **231** 269 (1984)