Pendulum, Elliptic Functions and Relative Cohomology Classes

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September 17, 2009

Abstract

Revisiting canonical integration of the classical pendulum around its unstable equilibrium, normal hyperbolic canonical coordinates are constructed and an identity between elliptic functions is found whose proof can be based on symplectic geometry and global relative cohomology. Alternatively it can be reduced to a well known identity between elliptic functions. Normal canonical action-angle variables are also constructed around the stable equilibrium and a corresponding identity is exhibited.

Key words: *Elliptic Functions, Pendulum, Canonical Integrability, Relative Cohomology*[1]

1 Pendulum near the separatrix

The theory of Jacobian elliptic functions, for reference see [1], yields a complete calculation for the motion of a pendulum as a function of time. This is

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revisited here, to exhibit a few interesting properties of the elliptic integrals.

Write the pendulum energy, with inertia moment I and gravity constant g^2 (rather than the usual g), in the canonical coordinates (B, β) or as

$$\frac{B^2}{2I} - Ig^2(1 - \cos\beta) \stackrel{def}{=} H(B, \beta) \tag{1.1}$$

where the origin in β is set at the unstable equilibrium: the definition implies that g has dimension of inverse time and the Lyapunov exponents of the unstable equilibrium are $\pm g$. $B \stackrel{def}{=} I\dot\beta$ and β are canonical coordinates for the motions.

It is well known that near the unstable equilibrium of the pendulum $B = 0, \beta = 0$ it is possible to define a canonical transformation, mapping the origin into itself, introducing new local cordinates (p, q) such that

$$B = R_c(p, q), \quad \beta = S_c(p, q) \tag{1.2}$$

with R, S holomorphic in a polidisk $|p|, |q| < \kappa$ with $\kappa > 0$, and in terms of which the motion near $B = \beta = 0$ is described by a Hamiltonian G depending on the product pq only, of the form $\mathcal{U}(p \cdot q) = H(B, \beta)$ with $\frac{d\mathcal{U}}{d(pq)}(0) = g$.

The purpose of this paper, which includes an unpublished note [2] where a proof of the latter statement via the theory of elliptic functions was derived, is to provide an alternative proof of the main formula, Eq.5.1. The interest of the new derivation is that it is deductive in nature and it yields an interesting application of the theory of global relative cohomology classes: the latter allows to perform calculations discarding systematically a large number of quantities that can be shown immediately that will give, if kept and evaluated, no eventual contribution.

The approach of both proofs presented in this paper is not the simplest if one is just interested to know the existence of normal hyperbolic coordinates: existence of R, S could be easily established without deriving their "explicit" expressions for p and q in terms of elliptic functions. Here we also correct a few errors in the earlier attempt made in [3, Appendix 9] (where the main formula was not derived).

The natural correspondence between the hyperbolic fixed point of the pendulum and its elliptic fixed point is briefly reported in Appendix C and leads to the construction of the normal canonical coordinates for the small oscillations, hence to the action-angle variables.

2 Solution in terms of elliptic integrals

Motions near the unstable equilibrium have a quite different nature depending on the sign of the total energy $H(B,\beta) = U$: the ones with $U < 0$ are "oscillations" (their motions do not encompass the full perimeter of the circle) while the ones with $U > 0$ are "librations". Therefore it will not be possible to find global action-angle coordinates: motions near the separatrix (which with our conventions has $U = 0$) require other coordinates to be expressed in a simple way.

Introduce the variables that appear in the theory of Jacobi's elliptic functions

$$
\begin{aligned}
k' &= \sqrt{1-k^2}, \qquad & h' &= \frac{k}{\sqrt{1+k^2}}, \qquad & h &= \sqrt{1-h'^2}\\
U &= 2g^2 I \frac{1}{k^2}, \qquad & u &= t\sqrt{\frac{U}{2I}} \qquad & g_0 &= g\frac{\pi}{2\,h'\,\mathbf{K}(h)}
\end{aligned}
\tag{2.1}
$$

where, $\mathbf{K}(k) \stackrel{def}{=} \int_0^{\frac{\pi}{2}} (1-k^2\sin^2\alpha)^{-\frac{1}{2}} d\alpha$. Hence the separatrix has $k = +\infty$ and $U = 0$; and the data *above the separatrix* correspond to $U > 0$ (or $k > 0$). Note that $g_0(0) = g$ because $\mathbf{K}(0) = \frac{\pi}{2}$ (as $U = 0$ corresponds to $k = \infty$ and $h' = 1, h = 0$); the following formulae become singular as $U \to 0$, but the singularity is only apparent and it will disappear from all relevant formulae derived or used in the following.

Other important quantities in the elliptic functions theory are, see the references [1, (8.198.1)],[1, (8.198.2),(8.146)],

$$
\begin{aligned}
x' \stackrel{def}{=} \xi(h) &= e^{-\pi\mathbf{K}(h')/\mathbf{K}(h)} = \lambda + 2\lambda^5 + 15\lambda^9 + ..\\
\lambda \stackrel{def}{=} \frac{1}{2}\frac{1-\sqrt{h'}}{1+\sqrt{h'}} &= \frac{\sum_{n=0}^{\infty}\xi(h)^{(2n+1)^2}}{1+2\sum_{n=1}^{\infty}\xi(h)^{4n^2}}
\end{aligned}
\tag{2.2}
$$

where $\xi(k')$ denotes here what in [1] would be $q(k')$ ([1, (8.146.1),(8.194.2)]).

In terms of the above conventions we have, directly from the definitions of $\mathrm{am}\,, \mathrm{cn}\,, \mathrm{sn}\,, \mathrm{dn}$ (Jacobi's elliptic functions, [1, (8.14)]), and from the equations of motion:

$$
\beta(t) = 2\,\mathrm{am}\,(u, ik),\ u = \frac{tg}{k}, \qquad B(t) = I\dot\beta = \frac{2Ig}{k}\,\mathrm{dn}\,(u, ik) \tag{2.3}
$$

([1, (8.143),(8.141)]). So that the action B is given as a function of time

$$
B(t) = \frac{2Ig}{k\ \mathrm{dn}\,(\frac{u}{h}, h')} = 2Ig\,\frac{\mathrm{cn}(-i\frac{u}{h}, h)}{k\ \mathrm{dn}\,(-i\frac{u}{h}, h)}, \tag{2.4}
$$

([1, (8.153.9),(8.153.3)]) assuming that initial data are assigned with $\beta = 0$.

The t dependence of $B(t)$ is naturally expressed via the argument $\frac{u}{h} = \frac{gt}{kh\mathbf{K}(h)}$, if the second of Eq.(2.4) is used, since $kh \equiv h'$, see Eq.(2.1). This explains the important role that the quantity

$$g_0(x') \stackrel{def}{=} g_0 \equiv \frac{\pi}{2}\frac{g}{kh\mathbf{K}(h)} = \frac{\pi}{2}\frac{g}{h'\mathbf{K}(h)} \tag{2.5}$$

will play in the following analysis. The $g_0(x')$ admits a rather simple product expansion, [1, (8.197.1),(8.197.4)],

$$g_0(x') = g\prod_{n=1}^{\infty}\Big(\frac{1+x'^n}{1-x'^n}\Big)^2 \tag{2.6}$$

and its logarithmic derivative is $4\sum_{n=1}^{\infty}\frac{nx'^{n-1}}{1-x'^{2n}}$ so that $x'\frac{d}{dx'}\log g_0(x')$ is $\frac{1}{2}\frac{d^2}{dz^2}\log\theta_4(z,x')\Big|_{z=0}$, where $\theta_4(z,x')$, [4, p.463,489], is a Jacobi's theta function.

It is also convenient to remark that in a motion with energy U it will be

$$H(B(t),\beta(t)) \equiv U = 2g^2 I\frac{1}{k^2} \tag{2.7}$$

3 Power series representation

From the theory of elliptic functions the evolution $B(t), \beta(t)$ with any initial data *above the separatrix* (*i.e.* with $\beta(0) = 0$ and $B(0) = I\dot\beta(0)$ corresponding to a given value of h, with $U > 0$), can be expressed as $B(t) = \overline{R}(\gamma,\delta)$ and $\beta(t) = \overline{S}(\gamma,\delta)$ with $\gamma = e^{g_0 t}, \delta = e^{-g_0 t}$ and, taking into account [1, (8.146.11)],

$$\overline{R}(\gamma,\delta) = -4g_0 I\sum_{n=1}^{\infty}\Big[\frac{(-1)^n\,\xi^{n-\frac{1}{2}}\,(\gamma^{2n-1}+\delta^{2n-1})}{1-\xi^{2n-1}}\Big] \tag{3.1}$$

with $\xi \equiv \xi(h)$. Definitions in Eq. (2.1),(2.2) yield

$$g_0(\xi) = g\frac{\pi}{2}\frac{1}{\sqrt{1-h^2}\,\mathbf{K}(h)} = g\,(1+\frac{1}{4}h^2+\ldots) \tag{3.2}$$

which is analytic in h^2 by [1, (8.113.1)] near $h = 0$.

Eq.(2.2) implies that $\xi = \lambda + O(\lambda^5)$ is analytic in λ near $\lambda = 0$ so that $h^2 = 16\lambda + \ldots = 16\xi + \ldots$. Therefore

$$g_0 = (1 + 4\xi + 12\xi^2 + \ldots)\,g \tag{3.3}$$

is analytic in ξ near $\xi = 0$.

The evolution of φ is then a consequence of Eq.(3.1) which leads to an expression for $\overline{S}$ by the remark that $\overline{R} = g_0 I(\gamma\partial_\gamma - \delta\partial_\delta)\overline{S}$ (just expressing that B is I times the derivative of β): namely

$$\overline{S}(\gamma,\delta) = -4\sum_{n=1}^{\infty} \frac{(-1)^n \xi^{n-\frac{1}{2}}}{1-\xi^{2n-1}} \frac{\gamma^{2n-1}-\delta^{2n-1}}{2n-1} \tag{3.4}$$

and, after developing in powers of ξ the denominators and resumming,

$$\begin{aligned}\overline{S} =& 4\sum_{m=0}^{\infty} (\,\arctan(\xi^m\,\delta\sqrt{\xi}) - \arctan(\xi^m\,\gamma\sqrt{\xi}))\\ \overline{R} =& 4I\,g_0\sum_{m=0}^{\infty}(\tfrac{\xi^m\gamma\sqrt{\xi}}{1+(\xi^m\,\gamma\,\sqrt{\xi})^2} + \tfrac{\xi^m\delta\sqrt{\xi}}{1+(\xi^m\,\delta\,\sqrt{\xi})^2})\end{aligned} \tag{3.5}$$

The first formula reminds of one found by Jacobi which he commented by saying that " *inter formulas elegantissimas censeri debet* ", [4, p.509] (*i.e.* " *it should be counted among the most elegant formulae* ".

Note that g_0 depends only on ξ, see Eq.(3.2), which would be surprising if the mechanical interpretation was not taken into account. The Eq.(3.5) exhibits the convergence of the map $(B,\beta)\longleftrightarrow(\xi,\gamma)$, since $\xi < 1$ in the region above the separatrix: in the latter region Eq.(3.5) provides a convergent expansion of the solution.

4 Hyperbolic Coordinates

Motions with initial coordinate $\beta(0) \neq 0$ also admit a rather simple representation. Remark that all pendulum motions with $\dot\beta > 0$ (hence different from the two equilibria) pass at some time through a phase space point with $\beta = 0$. If $\dot\beta$ is their velocity at that moment we can find a quantity ξ such that $\dot\beta, \beta$ are given by Eq.(3.5) with $\gamma = \delta = 1$ Therefore they can be represented, at least as long as $U > 0, \dot\beta > 0$ by introducing the *dimensionless* variables $q' = \gamma\sqrt{\xi}, p' = \delta\sqrt{\xi}$ and allowing δ, γ to be arbitrary. Then the motions will be $t \to (p' e^{g_0 t}, q' e^{-g_0 t})$ showing that the motion can be represented by the following two functions,

$$\begin{aligned}S' =& 4\sum_{m=0}^{\infty} (\,\arctan((p'q')^m\,q') - \arctan((p'q')^m\,p')),\\ R' =& 4I\,g_0\sum_{m=0}^{\infty}(\tfrac{(p'q')^m p'}{1+((p'q')^m\,p')^2} + \tfrac{(p'q')^m q'}{1+((p'q')^m\,q')^2})\end{aligned} \tag{4.1}$$

The motions $t \to (p' e^{g_0 t}, q' e^{-g_0 t})$ solve the equations of motion if p', q' (*i.e.* γ, δ) are positive. But the equations of motion are analytic, hence the formulae Eq.(4.1) together with $t \to (p' e^{g_0 t}, q' e^{-g_0 t})$, with $g_0 = g_0(p'q')$, give solutions of the pendulum equations independently of the sign of p', q', provided the series converge. The convergence requires $|p'q'| < 1$: which represents many data, in particular those in the vicinity of the separatrix.

Hence in the domain where $|p'q'| < 1$ the motion is linearized in the sense developed in [5] but it is not yet in symplectic coordinates.

The coordinates can be called "hyperbolic" being suitable to describe motions near the separatrix (where $p'q' = 0$). We also see that time evolution preserves both volume elements $dBd\beta$ and $dp'dq'$; which means that the Jacobian determinant $\frac{\partial(B,\beta)}{\partial(p',q')}$ must be a function constant over the trajectories, hence a function $D(x')$ of $x' \stackrel{def}{=} p'q'$. Note that $D(x')$ has dimension of an action.

It is then possible to change coordinates setting $p = a(x')p', q = a(x')q'$ and choose $a(x)$ so that the Jacobian determinant for $(B, \beta) \longleftrightarrow (p, q)$ is $\equiv 1$. A brief calculation shows that this is achieved by fixing

$$a^2(x') = \frac{1}{x'} \int_0^{x'} D(y) dy, \tag{4.2}$$

which is possible for x small because, from Eq.(3.5) and (2.2), it is $D(0) = 32Ig > 0$. Therefore the variables, which will have the dimension of a, hence of a square root of an action,

$$p = p' \, a(x'), \qquad q = q' \, a(x'), \tag{4.3}$$

have Jacobian determinant 1 with respect to (B, β) and the map $(B, \beta) \longleftrightarrow (p, q)$ is area preserving, hence *canonical*. The Hamiltonian Eq.(1.2) becomes a function $\mathcal{U}(x)$ of $x = pq$ and the derivative of the energy with respect to x has to be $g_0(x')$ (because the p, q are canonically conjugated to B, β). Note that x has the dimension of an action, while p, q are, dimensionally, square roots of action.

This allows us to find $D(x')$: by imposing that the equations of motion for the (p, q) canonical variables have to be the Hamilton's equations with Hamiltonian $\mathcal{U}(x) \stackrel{def}{=} U(x') \equiv H(B, \beta)$ it follows that $\frac{d\mathcal{U}(x)}{dx} = g_0(x')$, *i.e.* $\frac{dU(x')}{dx'} \frac{dx'}{dx} = g_0(x')$ or $\frac{dU(x')}{dx'} = g_0(x')(\frac{d}{dx'}(x' \, a(x')^2)) = g_0(x')D(x')$ by the above expression for $a(x')$. The just obtained relation, together with Eq.(2.2), gives

$$D(x') = g_0(x')^{-1} \frac{d}{dx'} U(x') \tag{4.4}$$

which is an explicit expression for the Jacobian $\frac{\partial(B,\beta)}{\partial(p',q')} \equiv \frac{\partial(R,S)}{\partial(p',q')} = \frac{\partial(p,q)}{\partial(p',q')}$ (note that the Jacobian between (B, β) and (p, q) is identically 1 by construction). Eq.(4.4) is dimensionally correct because x' is dimensionless so that $U(x')$ has the correct dimension (*i.e.* energy).

The function $U(x')$ is in Eq.(2.7) where $k^2 = \frac{h'^2}{h^2}$, by Eq.(2.1), is related to $x' = \xi(h)$ by Eq.(2.2), so that [1, (8.197.3),(8.197.4)],

$$U(x') = 2g^2 I \frac{1}{k^2} = 2g^2 I \frac{h^2}{h'^2} = 32 I g^2 x' \prod_{n=1}^{\infty} \Big(\frac{1 + x'^{2n}}{1 - x'^{(2n-1)}}\Big)^8 \tag{4.5}$$

To complete the determination of the canonical hyperbolic coordinates it remains to find an expression for $D(x'), \mathcal{U}(x)$ in terms of the elliptic functions to obtain the canonical variable and the Hamiltonian in closed form (rather than as power series as done so far).

5 Determination of the Jacobian. Remarks

It is remarkable that the function a^2 defined above, hence such that $D(x') = \frac{d}{dx'}(x' \, a^2(x'))$, seems to be simply

$$a^2(z) = 8I \frac{d}{dz} g_0(z), \tag{5.1}$$

in a common holomorphy domain, for both sides, around $z = 0$. This is suggested by the agreement of the first 200 coefficients of the expansion of the two sides in powers of z: however this is not a proof and the relation Eq.(5.1) holds because it can be seen to be equivalent to an identity on elliptic functions, as discussed in Appendix B below, or it can be independently derived from symplectic geometry, as discussed in Appendix A below.

Remarks: (1) The expansion of $D(x')$ in powers of x' can be derived from Eq.(4.5),(2.6), while that of $a^2(x')$ is obtained from Eq.(5.1) and, again, Eq.(2.6).
(2) It is perhaps natural to guess that the function $a(x')^2$ should be closely related to $g_0(x')$; this is a guide to its determination as it becomes, then, natural to look for it among the derivatives of g_0 with respect to x'. By dimensional analysis all x'-derivatives of $I g_0$ have the same dimension as a^2.

Looking *also* at the derivatives of g_0 as candidates for a^2 is an idea due to one of us (PG). This follows a similar line of thought which led to a conjecture on the canonical integrability of the "Calogero lattice", [6], whose proof was discovered in two subsequent works [7] and [8].

Other peculiarities are, setting $32Ig = 1, g = 1$,

(1) The function $g_0(x'), U(x')$, hence $\frac{d}{dx'}g_0(x'), D(x')$, have Taylor coefficients in powers of x' which *are* all positive integers as it follows from the relations Eq.(4.5) and Eq.(2.6), while $\mathcal{U}(x) - x$ seems to have alternating sign Taylor coefficients:

$$\mathcal{U}(x) - x = 2x^2 - 4x^3 + 20x^4 - 132x^5 + 1008x^6 + \dots \tag{5.2}$$

where $\mathcal{U}(x)$ is obtained by power series inversion of $x = x'a(x')^2$ and from $\mathcal{U}(x) = U(x')$ together with Eq.(4.5).
(2) The function $U(x')$, energy of the pendulum expressed as a function of x', has also the form

$$U(x') = 32Ig_0^2[p'U_{x'}(p') + q'V_{x'}(q')][p'V_{x'}(p') + q'U_{x'}(q')] \stackrel{def}{=} x'\, f(x') \tag{5.3}$$

which, remarkably, has by Eq.(4.5) to depend only upon x', and have the form $x'f(x')$ for some f. This is not *a priori* evident, unless the mechanical interpretation is kept in mind, from the expressions found for U, V, namely

$$U_{x'}(z) = \sum_{\ell=0}^{\infty} \frac{{x'}^{2\ell}}{1 + (x'^{2\ell}z)^2}, \qquad V_{x'}(z) = \sum_{\ell=0}^{\infty} \frac{{x'}^{2\ell+1}}{1 + (x'^{2\ell+1}z)^2}, \tag{5.4}$$

(3) Existence of an analytic canonical map integrating, near the hyperbolic point, the system with energy Eq.(1.1) into one with Hamiltonian $\mathcal{U}(pq) = g\,pq + O((pq)^2)$ is well known: it can be established without an explicit calculation by perturbation analysis, see [3, Appendix A3], for instance .

Appendices

A A deductive proof of Eq.(5.1)

A first proof for this formula can be based directly on the equalities on symplectic forms:

$$dB \wedge d\beta = D(\xi)dp' \wedge dq' = \frac{d}{d\xi}[\xi a(\xi)^2]dp' \wedge dq'. \tag{A1}$$

In principle, the computation of $dS'(p', q') \wedge dR'(p', q')$ should provide the value of the function D. But the computation gets too involved. The idea is to compute only the class of the volume forms in the relative cohomology of the function $\xi = p'q'$. This type of computation is well-known in the theory of limit cycles of plane vector fields [9], [10] but is perhaps more novel in the context of Hamiltonian dynamics.

Recall that any holomorphic 2-form $\phi(p', q')dp' \wedge dq'$ has a unique decomposition

$$\phi(p', q')dp' \wedge dq' = \psi(\xi)dp' \wedge dq' + d\xi \wedge d\eta. \tag{A2}$$

Remark that the function $\psi(\xi) = \sum_{n=0}^{\infty} \phi_{nn}(p'q')^n$ is obtained from $\phi(p', q') = \sum_{n,m=0}^{\infty} \phi_{nm} p'^m q'^n$ by collecting all terms in the series of equal exponents for both variable p' and q' and $\eta = \sum_{n \neq m=0}^{\infty} \frac{\phi_{nm} p'^n q'^m}{m-n}$. This is a consequence of the identity

$$p'^n q'^m dp' \wedge dq' = d(p'q') \wedge d(\frac{p'^n q'^m}{m-n}). \tag{A3}$$

by linearity.

The class of cohomology of the 2-form $\phi(p', q')dp' \wedge dq'$ relative to the function ξ is defined as the quotient of the holomorphic 2-forms modulo the 2-forms of type $d\xi \wedge d\eta$ for η holomorphic. It is conveniently represented by $\psi(\xi)dp' \wedge dq'$ (or equivalently by the function $\psi(\xi)$). In the local version near an isolated singularity, this is a very special case of a general theory due to Brieskorn and Sebastiani, [11].

There is another equivalent representation of the cohomology class of a 2-form $\phi(p', q')dp' \wedge dq'$. Write

$$\phi(p', q')dp' \wedge dq' = d[f(p', q')p'q'\omega], \tag{A4}$$

with

$$\omega = \frac{1}{2} d\log(\frac{q'}{p'}) = \frac{1}{2}[\frac{dq'}{q'} - \frac{dp'}{p'}]. \tag{A5}$$

Then this yields:

$$\phi(p', q') = f(p', q') + \frac{1}{2}(p' \frac{\partial f}{\partial p'} + q' \frac{\partial f}{\partial q'}), \tag{A6}$$

which defines a $1-1$ linear correspondence between $\sum_{m,n} \phi_{mn} p'^m q'^n$ and $\sum_{m,n} f_{mn} p'^m q'^n$ by

$$\phi_{mn} = f_{mn}[1 + \frac{1}{2}(m+n)]. \tag{A7}$$

Remark that this 1-1 transformation defines by restriction a 1-1 correspondence between $\psi(\xi) = \sum \phi_{nn}(p'q')^n$ and $F(\xi) = \sum f_{nn}(p'q')^n$. In other words, the cohomology class of the 2-form $\phi(p',q')dp' \wedge dq'$ is uniquely defined by the function $F(\xi)$ or equivalently by $\psi(\xi)$.

The application to our problem of finding an expression for $a(\xi)$ can be implemented by remarking that

$$D(\xi)dp' \wedge dq' = \frac{d}{d\xi}[\xi a(\xi)^2]dp' \wedge dq' = \frac{d}{d\xi}[\xi a(\xi)^2]d\xi \wedge \omega = d[\xi a^2(\xi)\omega]. \tag{A8}$$

So that, by Eq.(A1), the cohomology class of the symplectic form $dB \wedge d\beta$ with respect to ξ ($i.e.$ $D(\xi)$) corresponds to the function $a(\xi)^2$ via Eq.(A7), and we can use the formulae Eq.(3.1) to compute it.

The symplectic form can be written

$$dB \wedge d\beta = dR'(p',q') \wedge dS'(p',q') = d[R'(p',q')dS'(p',q')], \tag{A9}$$

and this yields

$$\begin{aligned} R'(p',q')dS'(p',q') &= 16Ig_0(\xi) \\ &\cdot \{\sum_{m=0}^{+\infty}[\frac{(p'q')^m p'}{1+((p'q')^m p')^2} + \frac{(p'q')^m q'}{1+((p'q')^m q')^2}]\} \\ &\cdot \{\sum_{l=0}^{+\infty}[\frac{(p'q')^l dq'}{1+((p'q')^l q')^2} - \frac{(p'q')^l dp'}{1+((p'q')^l p')^2}]\} \end{aligned} \tag{A10}$$

modulo $d(p'q')$.

In this double sum, it is convenient to first isolate the terms corresponding to both $l = m = 0$ finding;

$$\begin{aligned} &16Ig_0(\xi)[\frac{p'}{1+p'^2} + \frac{q'}{1+q'^2}][\frac{dq'}{1+q'^2} - \frac{dp'}{1+p'^2}] \\ &= 16Ig_0(\xi)[\frac{p'dp'}{(1+p'^2)^2} - \frac{q'dq'}{(1+q'^2)^2} + \frac{p'dq' - q'dp'}{(1+q'^2)(1+p'^2)}]. \end{aligned} \tag{A11}$$

The term

$$16Ig_0(\xi)[\frac{p'dp'}{(1+p'^2)^2}-\frac{q'dq'}{(1+q'^2)^2}] \tag{A12}$$

gives 0 in the relative cohomology. The term

$$16Ig_0(\xi)\frac{p'dq'-q'dp'}{(1+q'^2)(1+p'^2)}=32g_0(\xi)\frac{p'q'}{(1+q'^2)(1+p'^2)}\omega \tag{A13}$$

contributes with

$$32Ig_0(\xi)p'q'\sum_{i=0}^{+\infty}(p'q')^{2i}\omega, \tag{A14}$$

and hence

$$32Ig_0(\xi)\frac{1}{1-(p'q')^2}\,\omega \tag{A15}$$

in calculating the relative cohomology by Eq.(A4),(A7).

The other terms of the double sum can be written

$$\begin{aligned}
&\sum_{l,m;l+m\geq 1}(\frac{(p'q')^{l+m}}{\{1+[(p'q')^m p']^2\}\{1+[(p'q')^l q']^2\}}\\
&\qquad+\frac{(p'q')^{l+m-1}q'^2}{\{1+[(p'q')^m q']^2\}\{1+[(p'q')^l q']^2\}})p'dq'\\
-&\sum_{l,m;l+m\geq 1}(\frac{(p'q')^{l+m}}{\{1+[(p'q')^m p']^2\}\{1+[(p'q')^l q']^2\}}\\
&\qquad+\frac{(p'q')^{l+m-1}p'^2}{\{1+[(p'q')^m p']^2\}\{1+[(p'q')^l p']^2\}})q'dp',
\end{aligned} \tag{A16}$$

hence they have the form:

$$\begin{aligned}P(q',p')p'dq'+Q(q',p')q'dp'=&\alpha(q',p')(p'dq'+q'dp')\\
&+\beta(q',p')(p'dq'-q'dp'),\end{aligned} \tag{A17}$$

with $\alpha=\frac{P+Q}{2}$, $\beta=\frac{P-Q}{2}$. Notice then that the term $\alpha(q',p')(p'dq'+q'dp')$ gives 0 in relative cohomology. The term $\beta(q',p')(p'dq'-q'dp')$ yields:

$$
\begin{aligned}
8Ig_0(\xi) &\sum_{l,m;l+m\geq 1}(p'dq'-q'dp')\\
&\cdot\Big(\frac{2(p'q')^{l+m}}{\{1+[(p'q')^m p']^2\}\{1+[(p'q')^l q']^2\}}\\
&+\frac{(p'q')^{l+m-1}q'^2}{\{1+[(p'q')^m q']^2\}\{1+[(p'q')^l q']^2\}}\\
&+\frac{(p'q')^{l+m-1}p'^2}{\{1+[(p'q')^m p']^2\}\{1+[(p'q')^l p']^2\}}\Big).
\end{aligned}
\tag{A18}
$$

The series formed by the addends

$$
\begin{aligned}
&\frac{(p'q')^{l+m-1}q'^2}{\{1+[(p'q')^m q']^2\}\{1+[(p'q')^l q']^2\}}\\
&+\frac{(p'q')^{l+m-1}p'^2}{\{1+[(p'q')^m p']^2\}\{1+[(p'q')^l p']^2\}}
\end{aligned}
\tag{A19}
$$

does not contain any monomials with equal exponents for both p' and q', hence it does not contribute to the relative cohomology. The only term left is

$$
8Ig_0(\xi)\sum_{l,m;l+m\geq 1}\Big(\frac{2(p'q')^{l+m}}{\{1+[(p'q')^m p']^2\}\{1+[(p'q')^l q']^2\}}\Big)(p'dq'-q'dp') \tag{A20}
$$

which contributes

$$
\begin{aligned}
&16Ig_0(\xi)\sum_{l,m;l+m\geq 1}\xi^{l+m}\sum_{i=0}^{+\infty}\xi^{[2(l+m)+2]i}(p'dq'-q'dp')\\
&=16Ig_0(\xi)\sum_{l,m;l+m\geq 1}\frac{\xi^{l+m}}{1-\xi^{2(l+m)+2}}(p'dq'-q'dp').
\end{aligned}
\tag{A21}
$$

This last double sum can be computed using new indices l and $k=l+m$ and this yields

$$
16Ig_0(\xi)\sum_{k=1}^{+\infty}\sum_{l=0}^{k}\frac{\xi^k}{1-\xi^{2(k+1)}}(p'dq'-q'dp'). \tag{A22}
$$

This contributes to the relative cohomology, in the correspondence Eq.(A7),

$$
32Ig_0(\xi)\sum_{k=1}^{+\infty}\frac{(k+1)\xi^k}{1-\xi^{2(k+1)}}\xi\omega. \tag{A23}
$$

After adding the contribution from the $l = m = 0$ term and changing $k+1$ into k, this yields

$$32 I g_0(\xi) \sum_{k=1}^{+\infty} \frac{k\xi^{k-1}}{1-\xi^{2k}}\,\omega. \tag{A24}$$

Finally one should notice that the logarithmic derivative of g_0 can be expressed, as commented after Eq.(2.6), by

$$\frac{dg_0(\xi)}{d\xi} = 4 g_0(\xi) \sum_{k=1}^{+\infty} \frac{k\xi^{k-1}}{1-\xi^{2k}}, \tag{A25}$$

leading to the equality, recalling Eq.(A8):

$$a^2(\xi) = 8I\frac{dg_0(\xi)}{d\xi}. \tag{A26}$$

B Alternative proof of Eq.(5.1) via an identity between elliptic functions

Alternatively the formula can be reduced to a well known identity between elliptic functions.

Calling $\mathbf{E}(k) = \int_0^{\frac{\pi}{2}} (1-k^2\sin^2\alpha)^{\frac{1}{2}} d\alpha$ it is $\mathbf{E}(h) = hh'^2\frac{d\mathbf{K}(h)}{dh} + h'^2\mathbf{K}(h)$, see [1, (8.123.2], and $\mathbf{E}(h)\mathbf{K}(h') + \mathbf{E}(h')\mathbf{K}(h) - \mathbf{K}(h)\mathbf{K}(h') = \frac{\pi}{2}$, see [1, (8.122]; the latter "Legendre's relation", [4, p.520], combined with $\frac{dh'}{dh} = -\frac{h}{h'}$ yields the identity

$$h'h^2\Big(\mathbf{K}(h)\frac{d\mathbf{K}(h')}{dh'} - \mathbf{K}(h')\frac{d\mathbf{K}(h)}{dh'}\Big) = \frac{\pi}{2}\,. \tag{B1}$$

This can be used to obtain an expression for $\frac{d\log x'}{dh}$: keeping in mind $x' = e^{-\pi\mathbf{K}(h')/\mathbf{K}(h)}$ it is $\frac{d\log x'}{dh'} = -\pi(\frac{1}{\mathbf{K}(h)}\frac{d\mathbf{K}(h')}{dh'} - \frac{\mathbf{K}(h')}{\mathbf{K}(h)^2}\frac{d\mathbf{K}(h)}{dh'})$ which is transformed into $\frac{d\log x'}{dh'} = \log x'\,(\frac{1}{\mathbf{K}(h')}\frac{d\mathbf{K}(h')}{dh'} - \frac{1}{\mathbf{K}(h)}\frac{d\mathbf{K}(h)}{dh'})$.

Form Eq.(B1) it follows, therefore,

$$\frac{d}{dh'}\log x' = \frac{\pi}{2}\frac{\log x'}{h'h^2\mathbf{K}(h)\mathbf{K}(h')} \qquad \frac{d}{dh}\log x' = -\frac{\pi}{2}\frac{\log x'}{hh'^2\mathbf{K}(h)\mathbf{K}(h')}\,. \tag{B2}$$

and the corresponding derivatives with respect to h are obtained by multiplying both sides by $-\frac{h}{h'}$.

To establish Eq.(5.1) consider the relation,

$$\frac{d}{dh}\Big\{hh'^2\,\frac{d\mathbf{K}(h)}{dh}\Big\} - h\mathbf{K}(h) = 0 \tag{B3}$$

see [1, (8.124.1]. This implies by simple algebra, and keeping in mind that $\frac{h}{h'} = -\frac{dh'}{dh}$, the following identity

$$h\,\mathbf{K}(h) = h'^3\,\frac{d}{dh}\Big(hh'^2(-\frac{1}{h'}\frac{d}{dh}\mathbf{K}(h) + \frac{h}{h'^3}\mathbf{K}(h))\Big) \tag{B4}$$

which is a known linear equation, solved by $\mathbf{K}(h)$. This can be rewritten, since $\frac{h}{h'} = -\frac{dh'}{dh}$, as

$$\begin{aligned}\frac{h}{h'^3}\,\mathbf{K}(h) &= \frac{d}{dh}\Big(hh'^2(-\frac{1}{h'}\frac{d}{dh}\mathbf{K}(h) - \frac{1}{h'^2}\frac{dh'}{dh}\mathbf{K}(h))\Big)\\ &= \frac{d}{dh}\Big(hh'^2\mathbf{K}(h)^2\frac{d}{dh}\frac{1}{h'\mathbf{K}(h)}\Big)\end{aligned} \tag{B5}$$

Remarking that $\frac{2h}{h'^4} \equiv \frac{d}{dh}\frac{h^2}{h'^2}$, Eq.(B5) implies, multiplying both sides by $\frac{2}{h'\mathbf{K}(h)}$,

$$\begin{aligned}\frac{d}{dh}(\frac{h}{h'})^2 &= \frac{2}{h'\mathbf{K}(h)}\frac{d}{dh}\Big(hh'^2\mathbf{K}(h)^2(\frac{d}{dh}\frac{1}{h'\mathbf{K}(h)})\Big)\\ &= \frac{2\pi}{h'\mathbf{K}(h)}\frac{d}{dh}\Big(\frac{hh'^2\mathbf{K}(h)\mathbf{K}(h')}{\pi\mathbf{K}(h')/\mathbf{K}(h)}(\frac{d}{dh}\frac{1}{h'\mathbf{K}(h)})\Big)\end{aligned} \tag{B6}$$

and by the first of Eq.(B2) multiplied by $\frac{dh'}{dh} = -\frac{h}{h'}$ this is, using $k^2 = \frac{h^2}{h'^2}$

$$\begin{aligned}\frac{d}{dh}\frac{1}{k^2} &= \frac{\pi^2}{h'\mathbf{K}(h)}\frac{d}{dh}\Big(\frac{dh}{d\log x'}(\frac{d}{dh}\frac{1}{h'\mathbf{K}(h)})\Big)\\ &= \frac{\pi^2}{h'\mathbf{K}(h)}\frac{d}{dh}\Big(x'\frac{d}{dx'}\frac{1}{h'\mathbf{K}(h)}\Big)\end{aligned} \tag{B7}$$

and multiplying by $2Ig^2\frac{dh}{dx'}$ it follows

$$2Ig^2\frac{d}{dx'}\frac{1}{k^2} = 8I\frac{\pi g}{2}\frac{1}{h'\mathbf{K}(h)}\frac{d}{dx'}(x'\frac{d}{dx'}\frac{\pi g}{2}\frac{1}{h'\mathbf{K}(h)}) \tag{B8}$$

and setting $a(x')^2 \stackrel{def}{=} 8I\frac{d}{dx'}\frac{\pi g}{2}\frac{1}{h'\mathbf{K}(h)} \equiv 8I\frac{d}{dx'}g_0(x')$ the last relation is $\frac{d}{dx'}U(x') = g_0(x')\frac{d}{dx'}(x'a(x')^2)$ so that Eq.(4.4) and Eq.(4.2) imply Eq.(5.1).

C Pendulum at the stable equilibrium: action-angle coordinates

From the above results it is straightforward to find the canonical transformation that converts the pendulum Hamiltonian in its normal form around the stable equilibrium point. The Hamiltonian is now given by Eq.(1.1) with the substitution: $g = ig_s$. It is natural to define $k_s = ik$ in order to use the same set of equations from the unstable case. The system energy is then $U_s = 2g_s^2/k_s^2$ and large values of k_s correspond now to small oscillations around the equilibrium point.

Finally, it is convenient to define

$$\begin{aligned} k_s' &= \sqrt{1-k_s^2}, \qquad h_s'(k_s) = \frac{k_s}{\sqrt{k_s^2-1}}, \qquad h_s = \sqrt{1-{h_s'}^2} \\ h'(k) &= \frac{1}{h_s'(k_s)}, \qquad h(k) = \frac{ih_s(k_s)}{h_s'(k_s)} \end{aligned} \tag{C1}$$

and one finds:

$$\begin{aligned} g_0^{(s)}(h_s) &= -ig_0(h) = \frac{\pi}{2}\frac{g_s}{\mathbf{K}(h_s)} \\ x_s'(h_s) &= e^{-\pi\mathbf{K}(h_s')/\mathbf{K}(h_s)} = -x'(h) \end{aligned} \tag{C2}$$

where we have used [1, (8.128)].

With these conventions, and going through computations similar to the ones performed to study the unstable point, the relations found for the latter can be converted into the corresponding ones for the equilibrium point. In particular, by choosing $p' = \sqrt{x_s'}\cos(g_0^{(s)}t)$ and $q' = \sqrt{x_s'}\sin(g_0^{(s)}t)$ the transformation given by Eq.(4.1) is now:

$$\begin{aligned} S_s' =& 4i\sum_{m=0}^{\infty}(-1)^m\big(\arctan(({p'}^2+{q'}^2)^m\,(p'+iq')) \\ & -\arctan(({p'}^2+{q'}^2)^m\,(p'-iq'))\big), \\ R_s' =& -4I\,g_0^{(s)}\sum_{m=0}^{\infty}(-1)^m\Big(\frac{({p'}^2+{q'}^2)^m(p'+iq')}{1-(({p'}^2+{q'}^2)^m\,(p'+iq'))^2} \\ & +\frac{({p'}^2+{q'}^2)^m(p'-iq')}{1-(({p'}^2+{q'}^2)^m\,(p'-iq'))^2}\Big) \end{aligned} \tag{C3}$$

where the relation $R_s' = g_0^{(s)}I(p'\partial_{q'} - q'\partial_{p'})S_s'$ holds. And the energy can be written (see Eq.(4.5)):

$$U_s(x_s') = 2g_s^2 I\frac{1}{k^2} = 32Ig_s^2x_s'\prod_{n=1}^{\infty}\Big(\frac{1+{x_s'}^{2n}}{1+{x_s'}^{(2n-1)}}\Big)^8 \tag{C4}$$

The transformation $(B, \beta) \rightarrow (p', q')$ is not canonical. The canonical variables, (p, q), can be found by looking for a function $a_s(x'_s)$ (which depends on the constant of motion x'_s) such that $(p, q) = (a_s(x'_s)p', a_s(x'_s)q')$ and the Jacobian of the transformation is one. It is, as in the hyperbolic case,

$$a_s^2(z) = -16I \frac{d}{dz} g_0^{(s)}(z), \tag{C5}$$

Finally, the normal form of the Hamiltonian now reads:

$$\mathcal{U}_s(x) = 32Ig_s^2\, W(\frac{x}{64Ig_s}) \tag{C6}$$

where

$$W(z) = z\,(1 - 2z - 4z^2 - 20z^3 - 132z^4 - 1008z^5 \ldots) \tag{C7}$$

which can be compared with the hyperbolic case expression Eq.(5.2):

$$\mathcal{U}(x) = -32Ig^2\, W(-\frac{x}{32Ig}) \tag{C8}$$

Action-angle coordinates $(A, \alpha) \in \mathbb{R}_+ \times \mathbb{T}$ around equilibrium are related to (p, q) by $p = \sqrt{2A}\cos\alpha, q = \sqrt{2A}\sin\alpha$.

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