Universality relations in non-solvable quantum spin chains

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Abstract

We prove the exact relations between the critical exponents and the susceptibility, implied by the Haldane Luttinger liquid conjecture, for a generic lattice fermionic model or a quantum spin chain with short range weak interaction. The validity of such relations was only checked in some special solvable models, but there was up to now no proof of their validity in non-solvable models.

1 Introduction and Main results

One dimensional (1D) electron systems can be experimentally realized [1, 2] and their properties can be measured with increasing precision. Realistic models are very difficult to study and most of the theoretical predictions for such systems (for some recent experiments see [3]) are based on a number of *conjectures*, whose mathematical proof is quite hard.

Kadanoff [4] and Luther and Peschel [5] proposed that a large class of interacting 1D fermionic systems, quantum spin chains or 2D spin systems belongs to the same universality class. The critical indices appearing in the correlations are not the same (on the contrary, the indices depend on all details of the Hamiltonian), but they verify universal *extended scaling relations* between them, with the effect that all indices can be expressed in terms of any one of them. Usually, such hypothesis is formulated by saying

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that there exists a quantity K, whose value depends on the model, such that the critical indices can be expressed by simple universal relations in terms of K. The validity of such relations can be verified in the *Luttinger model*, which was solved by Mattis and Lieb [7].

Haldane [5] observed that in general even the knowledge of a single exponent is lacking, while the thermodynamic quantities are usually much more accessible, both experimentally and theoretically. He conjectured that certain relations between the parameter K and thermodynamical quantities, like the compressibility, are universal properties in a large class of models which he named *Luttinger liquids*. In the case of models which can be analyzed by *Bethe ansatz* and belonging to such class, the Haldane conjecture allows the exact computation of critical indices; indeed the Bethe ansatz by itself allows only the (partially rigorous) computation of spectral properties but not of the exponents.

The Haldane relations can be verified in the case of the *Luttinger model*, where the exact solution of [7] allows to calculate all the spectral quantities and the correlations. In the case of the XYZ spin chain model, whose ground state energy can be computed by the Bethe ansatz [8], the relations can be verified *assuming* the validity of the Kadanoff extended relations. The Haldane conjecture, stating that such relations should be valid in a general class of models (solvable or non-solvable) has been the subject of an impressive number of studies, see *e.g.* [9] for a review; we mention the RG analysis in [10] (valid only for the Luttinger model) and the (heuristic) probabilistic approach in [11]. While such analyses give deep insights, a proof of the conjecture for generic non-solvable models is still lacking.

In recent times, some of the Kadanoff relations have been proved in [15] for several (solvable and non-solvable) planar spin models, by rigorous Renormalization Group methods. In this paper we will extend such results to prove one of the Haldane relations for generic non-solvable lattice fermionic models or quantum spin chains with short range weak interaction. For definiteness (but our results, as it is evident from the proof, could be easily extended to 1-d fermionic continuum models) we consider a quantum spin chain with a non local interaction, whose Hamiltonian is

$$H = -\sum_{x=1}^{L-1}[J_1 S^1_x S^1_{x+1} + J_2 S^2_x S^2_{x+1}] - h\sum_{x=1}^{L} S^3_x + \lambda \sum_{1\le x,y\le L} v(x-y) S^3_x S^3_y + U^1_L \ , \quad (1)$$

where $S^\alpha_x = \sigma^\alpha_x/2$ for $i = 1, 2, \ldots, L$ and $\alpha = 1, 2, 3$, σ^α_x being the Pauli matrices, and U^1_L, to be fixed later, depends on the boundary conditions; finally $v(x-y) = v(y-x)$ and $|v(x-y)| \le Ce^{-\kappa|x-y|}$. If $v(x-y) = \delta_{|x-y|,1}/2$ and $h = 0$, (1) is the hamiltonian of the XXZ spin chain in a zero magnetic field, which can be diagonalized by the Bethe ansatz [8]; the same is true for the general XYZ model, always for $h = 0$ [18], but in the other cases no exact solution is known.

It is well known that the operators $a_x^\pm \equiv \prod_{y=1}^{x-1}(-\sigma_y^3)\sigma_x^\pm$ are a set of anticommuting operators and that, if $\sigma_x^\pm = (\sigma_x^1 \pm i\sigma_x^2)/2$, we can write

$$\sigma_x^- = e^{-i\pi\sum_{y=1}^{x-1} a_y^+ a_y^-} a_x^- \ , \quad \sigma_x^+ = a_x^+ e^{i\pi\sum_{y=1}^{x-1} a_y^+ a_y^-} \ , \quad \sigma_x^3 = 2a_x^+ a_x^- - 1 \ . \qquad (2)$$

Hence, if we fix the units so that $J_1 = J_2 = 1$ we get

$$H = -\sum_{x=1}^{L-1} \frac{1}{2}[a_x^+ a_{x+1}^- + a_{x+1}^+ a_x^-] - h\sum_{x=1}^{L}(a_x^+ a_x^- - \frac{1}{2}) +$$
$$+\lambda \sum_{1\le x,y\le L} v(x-y)(a_x^+ a_x^- - \frac{1}{2})(a_y^+ a_y^- - \frac{1}{2}) + U_L^2 \ , \qquad (3)$$

where U_L^2 is the boundary term in the new variables. We choose it so that the fermionic Hamiltonian coincides with the Hamiltonian of a fermion system on the lattice with periodic boundary conditions.

If O_x is a local monomial in the S_x^α or $a_x^\pm$ operators, we call $O_{\mathbf{x}} = e^{Hx_0}O_x e^{-Hx_0}$ where $\mathbf{x} = (x, x_0)$; moreover, if $A = O_{\mathbf{x}_1} \cdots O_{\mathbf{x}_n}$, $< A >_{L,\beta} = Tr[e^{-\beta H}\mathbf{T}(A)]/Tr[e^{-\beta H}]$, $\mathbf{T}$ being the time order product, denotes its expectation in the grand canonical ensemble, while $< A >_{T;L,\beta}$ denotes the corresponding truncated expectation. We will use also the notation $< A >_T = \lim_{L,\beta\to\infty} < A >_{T;L,\beta}$.

In recent times, constructive Renormalization Group techniques, combined with asymptotic Ward Identities, have been applied to the XYZ model [12, 13]. The extension to the general spin chain model (1) is immediate and one can prove that, for small λ, $J_1 = J_2 = 1$ and large $\mathbf{x}$,

$$\langle a_{\mathbf{x}}^- a_{\mathbf{0}}^+ \rangle_T \sim g_0(\mathbf{x}) \frac{1 + \lambda f(\lambda)}{(x_0^2 + v_s^2 x^2)^{(\eta/2)}} \ , \qquad (4)$$

where $f(\lambda)$ is a bounded function, $\eta = a_0\lambda^2 + O(\lambda^3)$, with $a_0 > 0$, and

$$g_0(\mathbf{x}) = \sum_{\omega=\pm} \frac{e^{i\omega p_F x}}{-ix_0 + \omega v_s x} \ , \qquad (5)$$

$$v_s = v_F + O(\lambda) \quad , \quad p_F = \cos^{-1}(h+\lambda) + O(\lambda) \quad , \quad v_F = \sin p_F \ . \qquad (6)$$

From (4) we see that the interaction has two main effects. The first one is to change the value of the Fermi momentum from $\cos^{-1}(h)$ to p_F and the sound velocity from v_F in the non interacting case to v_s. The second effect is that the power law decay is changed; the 2-point function is asymptotically given by the product of the non-interacting one (with a different sound velocity) times an extra power law decay factor with non-universal index η.

It was also proved in [12, 13] that the spin-spin correlation in the direction of the 3-axis (or, equivalently, the fermionic density-density correlation) is given, for large $\mathbf{x}$, by

$$\langle S^{(3)}_{\mathbf{x}} S^{(3)}_{\mathbf{0}} \rangle_T \sim \cos(2p_F x)\Omega^{3,a}(\mathbf{x}) + \Omega^{3,b}(\mathbf{x}) \; , \tag{7}$$

$$\Omega^{3,a}(\mathbf{x}) = \frac{1 + A_1(\mathbf{x})}{2\pi^2[x^2 + (v_s x_0)^2]^{X_+}} \; , \tag{8}$$

$$\Omega^{3,b}(\mathbf{x}) = \frac{1}{2\pi^2[x^2 + (v_s x_0)^2]}\Big\{\frac{x_0^2 - (x/v_s)^2}{x^2 + (v_s x_0)^2} + A_2(\mathbf{x})\Big\} \; , \tag{9}$$

with $|A_1(\mathbf{x})|, |A_2(\mathbf{x})| \le C|\lambda|$ and $X_+ = 1 - a_1\lambda + O(\lambda^2)$, $a_1 > 0$. Finally, by using the results of [17], one can prove that the Cooper pair density correlation, that is the correlation of the operator $\rho^c_{\mathbf{x}} = a^+_{\mathbf{x}} a^+_{\mathbf{x}'} + a^-_{\mathbf{x}} a^-_{\mathbf{x}'}$, $\mathbf{x}' = (x+1, x_0)$, behaves as

$$\langle \rho^c_{\mathbf{x}} \rho^c_{\mathbf{0}} \rangle_T \sim \frac{1 + A_3(\mathbf{x})}{2\pi^2(x^2 + v_s^2 x_0^2)^{X_-}} \; , \tag{10}$$

with $X_- = 1 + a_1\lambda + O(\lambda^2)$, a_1 being the same constant appearing in the first order of X_+.

In the case $J_1 \neq J_2$ the correlations decay faster than any power with rate ξ such that

$$\xi \sim C|J_1 - J_2|^{\bar\nu} \; , \tag{11}$$

with $\bar\nu = 1 + a_1\lambda + O(\lambda^2)$, a_1 being again the same constant appearing in the first order of X_+.

Several physical quantities are expressed in terms of the Fourier transform of the correlations; in particular, if we call

$$\widehat\Omega(\mathbf{p}) = \lim_{\beta, L\to\infty} \int_{-\beta/2}^{\beta/2} dx_0 \sum_{x\in\Lambda} e^{i\mathbf{p}\mathbf{x}} \langle S^{(3)}_{\mathbf{x}} S^{(3)}_{\mathbf{0}} \rangle_{T;L,\beta} \; , \tag{12}$$

the *susceptibility* is given by

$$\kappa = \lim_{p\to 0} \widehat\Omega(0, p) \; . \tag{13}$$

Note that, in the fermion system, $\kappa = \kappa_c \rho^2$, where κ_c is the *fermionic compressibility* and ρ is the fermionic density, see *e.g.* (2.83) of [16] or (3.16) of [10].

Our results can be summarized by the following theorem.

Theorem 1.1 *For small λ there exists an analytic function $K(\lambda)$ such that*

$$X_+ = K \quad , \quad X_- = K^{-1} \; , \tag{14}$$

$$\bar\nu = \frac{1}{2 - K^{-1}} \quad , \quad 2\eta = K + K^{-1} - 2 \; , \tag{15}$$

with

$$K = 1 - \lambda \frac{\widehat{v}(0) - \widehat{v}(2p_F)}{\pi \sin p_F} + O(\lambda^2) \; . \tag{16}$$

Moreover,

$$\widehat{\Omega}(\mathbf{p}) = \frac{K}{\pi v_s} \frac{v_s^2 p^2}{p_0^2 + v_s^2 p^2} + R(\mathbf{p}) \; , \tag{17}$$

with $R(\mathbf{p})$ continuous and such that $R(\mathbf{0}) = 0$, so that

$$\kappa = \frac{1}{\pi} \frac{K}{v_s} \; . \tag{18}$$

The relations (14) are the extended scaling laws conjectured by Kadanoff [4] and Luther and Peschel [5]. The critical indices, as functions of λ, are non-universal and depend on all details of the model; however, such non-universality is all contained in the function $K(\lambda)$ (which is expressed in our analysis as a convergent power series expansion), and the indices have a simple universal expressions in terms of the parameter K.

From (17) we see that, analogously to what happens for the critical exponents, the *amplitude* of the dominant part, for $\mathbf{p} \to 0$, of the density-density correlation Fourier transform verifies an *universal* relation in terms of K and v_s; on the contrary no universal relation is expected to be true for the amplitude of the Fourier transform close to $(\pm 2p_F, 0)$.

The equation (18) is an universal relation connecting the susceptibility defined in (13) with K and v_s; it is one of the two relations conjectured by Haldane in [6] (see (3) of [6], where $v_N \equiv (\pi\kappa)^{-1}$ and $K \equiv e^{2\phi}$). Note that in the case of the XYZ model ($J_1 \neq J_2$) with $h = 0$ the exponent $\bar{\nu}$ has been computed by Baxter and it has been found, see (10.12.24) of [18], if $\cos \bar{\mu} = -J_3/J_1 = \lambda$,

$$\bar{\nu} = \frac{\pi}{2\bar{\mu}} = 1 + \frac{2\lambda}{\pi} + O(\lambda^2) \; . \tag{19}$$

From (14) $K^{-1} = e^{-2\phi} = 2(1 - \frac{\bar{\mu}}{\pi})$. Moreover from the Bethe ansatz solution [8] exact expressions for v_s and κ can be obtained,

$$v_s = \frac{\pi}{\bar{\mu}} \sin \bar{\mu} \qquad \kappa = [2\pi(\pi/\bar{\mu} - 1) \sin \bar{\mu}]^{-1} \; , \tag{20}$$

so that (18) is verified. In general κ, K, v_s depend on the magnetic field h and the specific form of the interaction $\widehat{v}(k)$ (such dependence is simple at first order, see (16), but in general quite complex), but our theorem shows that the Kadanoff and Haldane relations (14) and (18) are still true. This is the first example in which such relations are proven in generic non-solvable models.

In [15] a statement similar to (14), (15) has been proved in the case of planar spin models; the extension to the present case is straightforward. The main novelty of the paper is the proof of the Haldane relation (18), so we will focus on its derivation. The main ideas of our proof should be understood also from people who did not read our previous papers, only referring to them for the proof of several technical results that we need.

2 Proof of Theorem 1.1

As the interaction modifies the value of the Fermi momentum and of the sound velocity, it is convenient to include some part of the free hamiltonian in the interaction part, by writing (3) in the following way

$$H = H_0 + \nu \sum_{x=1}^{L} a_x^+ a_x^- - \delta \sum_{x=1}^{L} [\cos p_F a_x^+ a_x^- - (a_{x+1}^+ a_x^- + a_x^+ a_{x+1}^-)/2]$$
$$+\lambda \sum_{1\le x,y\le L} v(x-y) a_x^+ a_x^- a_y^+ a_y^- , \qquad (21)$$

with

$$H_0 = -\frac{v_s}{v_F} \sum_{x=1}^{L} \frac{1}{2} [a_x^+ a_{x+1}^- + a_{x+1}^+ a_x^- - 2\cos p_F a_x^+ a_x^-] \qquad (22)$$

and

$$\cos p_F = -\lambda - h - \nu \quad , \quad v_s = v_F(1+\delta) \; . \qquad (23)$$

Note that, if $H = H_0$, the Fourier transform of the 2-point function is singular at $\mathbf{k} = (\pm p_F, 0)$ and the sound velocity is v_s. The parameter ν is chosen as a function of λ and p_F, so that the singularity of the Fourier transform of the two-point function corresponding to H is fixed at $\mathbf{k} = (\pm p_F, 0)$; the first equation in (23) gives the value of h corresponding, in the model (3), to the chosen value of p_F. On the contrary, the parameter δ is an unknown function of λ and p_F, whose value is determined by requiring that, in the renormalization group analysis, the corresponding marginal term flows to 0; this implies that v_s is the sound velocity even for the full Hamiltonian H.

It is well known that the correlations of the quantum spin chain can be derived by the following Grassmann integral, see [12]:

$$e^{\mathcal{W}_M(J,\widetilde{J},\phi)} = \int P(d\psi) e^{-\mathcal{V}(\psi) + \int d\mathbf{x} [J_\mathbf{x} \rho_\mathbf{x} + \widetilde{J}_\mathbf{x} j_\mathbf{x}] + \int d\mathbf{x} [\phi_\mathbf{x}^+ \psi_\mathbf{x}^- + \psi_\mathbf{x}^- \psi_\mathbf{x}^+]} \; , \qquad (24)$$

where $\psi_\mathbf{x}^\pm$ and $\phi_\mathbf{x}^\pm$ are Grassmann variables, $J_\mathbf{x}$ and $\widetilde{J}_\mathbf{x}$ are commuting variables, $\int d\mathbf{x}$ is a shortcut for $\sum_{x\in\Lambda} \int_{-\beta/2}^{\beta/2} dx_0$, $P(d\psi)$ is a Grassmann Gaussian measure in the field variables $\psi_\mathbf{x}^\pm$ with covariance (the free propagator) given by

$$g_M(\mathbf{x}-\mathbf{y}) = \frac{1}{\beta L}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} \frac{\chi(\gamma^{-M}k_0)e^{i\delta_M k_0}e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{-ik_0+(v_s/v_F)(\cos p_F-\cos k)}\;, \tag{25}$$

where $\chi(t)$ is a smooth compact support function equal to 0 if $|t|\geq\gamma>1$ and equal to 1 for $|t|<1$, $\mathbf{k}=(k,k_0)$, $\mathbf{k}\cdot\mathbf{x}=k_0x_0+kx$, $\mathcal{D}_{L,\beta}\equiv\mathcal{D}_L\times\mathcal{D}_\beta$, $\mathcal{D}_L\equiv\{k=2\pi n/L, n\in\mathbb{Z}, -[L/2]\leq n\leq[(L-1)/2]\}$, $\mathcal{D}_\beta\equiv\{k_0=2(n+1/2)\pi/\beta, n\in Z\}$ and

$$\begin{aligned}\mathcal{V}(\psi) &= \lambda\int d\mathbf{x}d\mathbf{y}\tilde{v}(\mathbf{x}-\mathbf{y})\psi^+_{\mathbf{x}}\psi^+_{\mathbf{y}}\psi^-_{\mathbf{y}}\psi^-_{\mathbf{x}}+\nu\int d\mathbf{x}\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}- \\ &-\delta\int d\mathbf{x}[\cos p_F\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}-(\psi^+_{\mathbf{x}+\varepsilon_1}\psi^-_{\mathbf{x}}+\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}+\varepsilon_1})/2]\;,\end{aligned}$$

with $\varepsilon_1=(1,0)$, $\tilde{v}(\mathbf{x}-\mathbf{y})=\delta(x_0-y_0)v(x-y)$. Moreover

$$\rho_{\mathbf{x}}=\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}\quad,\quad j_{\mathbf{x}}=(2iv_F)^{-1}[\psi^+_{\mathbf{x}+\varepsilon_1}\psi^-_{\mathbf{x}}-\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}+\varepsilon_1}]\;. \tag{26}$$

Note that, due to the presence of the ultraviolet cut-off γ^M, the Grassmann integral has a finite number of degree of freedom, hence it is well defined. The constant $\delta_M=\beta/\sqrt{M}$ is introduced in order to take correctly into account the discontinuity of the free propagator $g(\mathbf{x})$ at $\mathbf{x}=0$, where it has to be defined as $\lim_{x_0\to 0^-}g(0,x_0)$; in fact our definition guarantees that $\lim_{M\to\infty}g_M(\mathbf{x})=g(\mathbf{x})$ for $\mathbf{x}\neq 0$, while $\lim_{M\to\infty}g_M(0,0)=g(0,0^-)$.

We shall use the following definitions:

$$\begin{aligned}G^{2,1}_\rho(\mathbf{x},\mathbf{y},\mathbf{z}) &= \lim_{-l,N\to\infty}\frac{\partial}{\partial J_{\mathbf{x}}}\frac{\partial^2}{\partial\phi^+_{\mathbf{y}}\partial\phi^-_{\mathbf{z}}}\mathcal{W}_M(J,\tilde{J},\phi)|_{J=\tilde{J}=\phi=0}\;, \\ G^{2,1}_j(\mathbf{x},\mathbf{y},\mathbf{z}) &= \lim_{-l,N\to\infty}\frac{\partial}{\partial\tilde{J}_{\mathbf{x}}}\frac{\partial^2}{\partial\phi^+_{\mathbf{y}}\partial\phi^-_{\mathbf{z}}}\mathcal{W}_M(J,\tilde{J},\phi)|_{J=\tilde{J}=\phi=0}\;, \\ G^2(\mathbf{y},\mathbf{z}) &= \lim_{-l,N\to\infty}\frac{\partial^2}{\partial\phi^+_{\mathbf{y}}\partial\phi^-_{\mathbf{z}}}\mathcal{W}_M(J,\tilde{J},\phi)|_{J=\tilde{J}=\phi=0}\;, \\ G^{0,2}_{\rho,\rho}(\mathbf{x},\mathbf{y}) &= \lim_{-l,N\to\infty}\frac{\partial^2}{\partial J_{\mathbf{x}}\partial J_{\mathbf{y}}}\mathcal{W}_M(J,\tilde{J},\phi)|_{J=\tilde{J}=\phi=0}\;.\end{aligned} \tag{27}$$

The Fourier transforms $\widehat{G}^2(\mathbf{k})$ and $\widehat{G}^{0,2}_{\rho,\rho}(\mathbf{p})$ of $G^2(\mathbf{y},\mathbf{z})$ and $G^{2,0}_{\rho,\rho}(\mathbf{x},\mathbf{y})$ are defined in a way analogous to the definition of $\widehat{\Omega}(\mathbf{p})$ in (12). Moreover, we define the Fourier transforms of $G^{2,1}_\alpha$, $\alpha=\rho,j$, so that

$$G^{2,1}_\alpha(\mathbf{x},\mathbf{y},\mathbf{z}) = \frac{1}{L\beta}\sum_{\mathbf{k},\mathbf{p}} e^{i\mathbf{p}\mathbf{x}-i(\mathbf{k}+\mathbf{p})\mathbf{y}+i\mathbf{k}\mathbf{z}}\widehat{G}^{2,1}_\alpha(\mathbf{k},\mathbf{k}+\mathbf{p})\;. \tag{28}$$

The Grassmann integral (24) has been analyzed in [12, 13] by Renormalization Group methods; by choosing properly the counterterms ν and δ, one gets expression which are uniformly analytic in β, L, M. The correlations obtained from the Grassmann integral coincide with the correlations of the Hamiltonian model (21) as $M \to \infty$. By such analysis the asymptotic expressions (7) and (10) are proved, and the critical indices η, X_+, X_-, and $\bar\nu$ can be represented as power series in the variable $r = \lambda_{-\infty}/v_s$, where $\lambda_{-\infty} = \lambda + O(\lambda^2)$ is the *asymptotic effective coupling*. Such series are *convergent* for r small enough and their coefficients are *universal*, that is model independent. Moreover, v_s and $\lambda_{-\infty}$ can be represented as power series of λ, convergent near $\lambda = 0$ and depending on all details of the model, so that this property is true also for the critical indices. The fact that the critical indices can be represented as universal functions of a single parameter implies that they can be all expressed in terms of only one of them; however, to compute explicitly such relations, by only using the complicated expansions in terms of r, looks impossible.

The key observation is to take advantage from the gauge symmetries present in the theory in the formal scaling limit. We introduce a continuum fermion model, essentially coinciding with the formal scaling limit of the fermion model with hamiltonian (21) (which is a QFT model), regularized by a non local fixed interaction, together with an infrared γ^l and ultraviolet γ^N momentum cut-offs, $-l, N \gg 0$. The limit $N \to \infty$, followed from the limit $l \to -\infty$, will be called the *limit of removed cut-offs*. The model is expressed in terms of the following Grassmann integral:

$$\begin{aligned} e^{\mathcal{W}_{l,N}(J,\widetilde J,\phi)} &= \int P_Z(d\psi^{[l,N]}) e^{-\mathcal{V}^{(N)}(\sqrt{Z}\psi^{[l,N]})+\sum_{\omega=\pm}\int d\mathbf{x}[Z^{(3)}J_{\mathbf{x}}+\omega \widetilde Z^{(3)}\widetilde J_{\mathbf{x}}]\rho_{\mathbf{x},\omega}} \cdot \\ &\cdot \; e^{Z\sum_{\omega=\pm}\int d\mathbf{x}[\psi^{+[l,N]}_{\mathbf{x},\omega}\phi^-_{\mathbf{x},\omega}+\phi^+_{\mathbf{x},\omega}\psi^{[l,N]}]} , \end{aligned} \tag{29}$$

where

$$\rho_{\mathbf{x},\omega} = \psi^{[l,N]+}_{\mathbf{x},\omega}\psi^{[l,N]-}_{\mathbf{x},\omega} , \tag{30}$$

$\mathbf{x} \in \widetilde\Lambda$ and $\widetilde\Lambda$ is a square subset of $\mathbb{R}^2$ of size γ^{-l}, say $\gamma^{-l}/2 \le |\widetilde\Lambda| \le \gamma^{-l}$, $P_Z(d\psi^{[l,N]})$ is the fermionic measure with propagator

$$\frac{1}{Z} g^{[l,N]}_{th,\omega}(\mathbf{x}-\mathbf{y}) = \frac{1}{Z}\frac{1}{L^2}\sum_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}} \frac{\chi_{l,N}(\mathbf{k})}{-ik_0+\omega c k} , \tag{31}$$

where Z and c are two parameters, to be fixed later, and $\chi_{l,N}(\mathbf{k})$ is the cutoff function. Moreover, the interaction is

$$\mathcal{V}^{(N)}(\psi) = \frac{\lambda_\infty}{2}\sum_\omega \int d\mathbf{x}\int d\mathbf{y} v_0(\mathbf{x}-\mathbf{y})\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega}\psi^+_{\mathbf{y},-\omega}\psi^-_{\mathbf{y},-\omega} , \tag{32}$$

where $v_0(\mathbf{x}-\mathbf{y})$ is a rotational invariant potential, of the form

$$v_0(\mathbf{x}-\mathbf{y}) = \frac{1}{L^2}\sum_{\mathbf{p}} \widehat v_0(\mathbf{p}) e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} , \tag{33}$$

with $|\widehat{v}_0(\mathbf{p})| \le Ce^{-\mu|\mathbf{p}|}$, for some constants C, μ, and $\widehat{v}_0(0)=1$. We shall use the following definitions, analogous to the definitions (27) of the quantum spin chain:

$$\begin{aligned}
G^{2,1}_{th,\rho;\omega}(\mathbf{x},\mathbf{y},\mathbf{z}) &= \lim_{-l,N\to\infty} \frac{\partial}{\partial J_{\mathbf{x}}} \frac{\partial^2}{\partial \phi^+_{\mathbf{y},\omega}\partial\phi^-_{\mathbf{z},\omega}} \mathcal{W}_{l,N}(J,\widetilde{J},\phi)|_{J=\widetilde{J}=\phi=0} \;, \\
G^{2,1}_{th,j;\omega}(\mathbf{x},\mathbf{y},\mathbf{z}) &= \lim_{-l,N\to\infty} \frac{\partial}{\partial \widetilde{J}_{\mathbf{x}}} \frac{\partial^2}{\partial \phi^+_{\mathbf{y},\omega}\partial\phi^-_{\mathbf{z},\omega}} \mathcal{W}_{l,N}(J,\widetilde{J},\phi)|_{J=\widetilde{J}=\phi=0} \;, \\
G^{2}_{th;\omega}(\mathbf{y},\mathbf{z}) &= \lim_{-l,N\to\infty} \frac{\partial^2}{\partial \phi^+_{\mathbf{y},\omega}\partial\phi^-_{\mathbf{z},\omega}} \mathcal{W}_{l,N}(J,\widetilde{J},\phi)|_{J=\widetilde{J}=\phi=0} \;, \qquad (34) \\
G^{0,2}_{th,\rho,\rho}(\mathbf{x},\mathbf{y}) &= \lim_{-l,N\to\infty} \frac{\partial^2}{\partial J_{\mathbf{x}}\partial J_{\mathbf{y}}} \mathcal{W}_{l,N}(J,\widetilde{J},\phi)|_{J=\widetilde{J}=\phi=0} \;.
\end{aligned}$$

The Fourier transforms $\widehat{G}^2_{th;\omega}(\mathbf{k})$ and $\widehat{G}^{0,2}_{th,\rho,\rho}(\mathbf{p})$ of $G^2_{th;\omega}(\mathbf{y},\mathbf{z})$ and $G^{2,0}_{th,\rho,\rho}(\mathbf{x},\mathbf{y})$ are defined in a way analogous to the definition of $\widehat{\Omega}(\mathbf{p})$ in (12). Moreover, we define the Fourier transforms of $G^{2,1}_{th,\alpha;\omega}$, $\alpha=\rho,j$, as in (28).

In §3 of [19] (see also §4 of [15]) it has been proved that, for small $\widetilde{\lambda}_\infty$ and for *non-exceptional momenta* (that is $\mathbf{k}$, $\mathbf{p}$ and $\mathbf{k}-\mathbf{p}$ different from 0),

$$\begin{aligned}
&Z[-ip_0\frac{1}{Z^{(3)}}\widehat{G}^{2,1}_{th,\rho;\omega}(\mathbf{k},\mathbf{k}+\mathbf{p}) + \omega p\, c\frac{1}{\widetilde{Z}^{(3)}}\widehat{G}^{2,1}_{th,j;\omega}(\mathbf{k},\mathbf{k}+\mathbf{p})] = \\
&\quad = A[\widehat{G}^2_{th;\omega}(\mathbf{k}) - \widehat{G}^2_{th;\omega}(\mathbf{k}+\mathbf{p})] \;, \qquad (35)\\
&Z[-ip_0\frac{1}{\widetilde{Z}^{(3)}}\widehat{G}^{2,1}_{th,j;\omega}(\mathbf{k},\mathbf{k}+\mathbf{p}) + \omega p\, c\frac{1}{Z^{(3)}}\widehat{G}^{2,1}_{th,\rho;\omega}(\mathbf{k},\mathbf{k}+\mathbf{p})] = \\
&\quad = \omega\bar{A}[\widehat{G}^2_{th;\omega}(\mathbf{k}) - \widehat{G}^2_{th;\omega}(\mathbf{k}+\mathbf{p})] \;,
\end{aligned}$$

with

$$A^{-1} = 1-\tau \quad , \quad \bar{A}^{-1} = 1+\tau \quad , \quad \tau = \frac{\lambda_\infty}{4\pi c} \;. \qquad (36)$$

Equations (35) are the Ward Identities associated to the invariance of the formal lagrangian with respect to local and local chiral Gauge transformations. The fact that $A, \bar{A}$ are not equal to 1 is a well known manifestation of the *anomalies* in quantum field theory; naively, by a gauge transformation in the non regularized ill defined Grassmann integrals, one would get similar expressions with $A=\bar{A}=1$. Finally, the linearity of $A^{-1}, \bar{A}^{-1}$ in terms of λ_∞ is a property called *anomaly non-renormalization* and it depends crucially on the regularizations used; with different regularizations such a property could be violated, see [17].

An easy extension of the results given in [19] allows us to deduce also a set of Ward Identities for the continuum model correlations of the *density operator* $\rho_{\mathbf{x},\omega}$ defined in (30). To be more precise, let us consider the functional

$$e^{\widetilde{\mathcal{W}}(J)} = \int P_Z(d\psi) e^{-\mathcal{V}^{(N)}(\sqrt{Z}\psi) + \sum_\omega \int d\mathbf{x} J_{\mathbf{x},\omega}\rho_{\mathbf{x},\omega}} \;, \qquad (37)$$

and let us define

$$G_{\omega,\omega'}(\mathbf{x},\mathbf{y}) = \lim_{-l,N\to\infty} \frac{\partial^2}{\partial J_{\mathbf{x},\omega}\partial J_{\mathbf{y},\omega'}} \widetilde{\mathcal{W}}(J)|_{J=0} \; . \tag{38}$$

In App. A we shall prove that, in the limit $-l, N \to \infty$,

$$\begin{aligned} &D_\omega(\mathbf{p})\widehat{G}_{\omega,\omega}(\mathbf{p}) - \tau\; \widehat{v}_0(\mathbf{p}) D_{-\omega}(\mathbf{p})\widehat{G}_{-\omega,\omega}(\mathbf{p}) + \frac{1}{4\pi c Z^2} D_{-\omega}(\mathbf{p}) = 0 \; , \\ &D_{-\omega}(\mathbf{p})\widehat{G}_{-\omega,\omega}(\mathbf{p}) - \tau\; \widehat{v}_0(\mathbf{p}) D_{\omega}(\mathbf{p})\widehat{G}_{\omega,\omega}(\mathbf{p}) = 0 \; , \end{aligned} \tag{39}$$

where

$$D_\omega(\mathbf{p}) = -ip_0 + \omega c p \; . \tag{40}$$

By using (39) and $\widehat{v}_0(\mathbf{p}) = 1 + O(\mathbf{p})$, we get:

$$\begin{aligned} \widehat{G}_{\omega,\omega}(\mathbf{p}) &= -\frac{1}{Z^2} \frac{1}{4\pi c(1-\tau^2)} \frac{D_{-\omega}(\mathbf{p})}{D_\omega(\mathbf{p})} + O(\mathbf{p}) \; , \\ \widehat{G}_{-\omega,\omega}(\mathbf{p}) &= -\frac{1}{Z^2} \frac{\tau}{4\pi c(1-\tau^2)} + O(\mathbf{p}) \; , \end{aligned} \tag{41}$$

which implies, after a few simple calculations, that

$$\widehat{G}^{0,2}_{th,\rho,\rho} = -\frac{1}{4\pi c Z^2} \frac{(Z^{(3)})^2}{1-\tau^2} \left[\frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} + \frac{D_+(\mathbf{p})}{D_-(\mathbf{p})} + 2\tau \right] + O(\mathbf{p}) \; . \tag{42}$$

The crucial point is that it is possible to choose the parameters of the continuum model so that the correlations in the two models are the same, up to small corrections, for small momenta.

Lemma 2.1 *Given λ small enough, there are constants Z, $Z^{(3)}$, $\widetilde{Z}^{(3)}$, λ_∞, depending analytically on λ, such that, if we put $c = v_s$, the critical indices of the two models coincide. Moreover, if $\kappa \le 1$ and $|\mathbf{p}| \le \kappa$,*

$$\widehat{G}^{0,2}_{\rho,\rho}(\mathbf{p}) = \widehat{G}^{0,2}_{th,\rho,\rho}(\mathbf{p}) + A_{\rho,\rho}(\mathbf{p}) \; , \tag{43}$$

with $A_{\rho,\rho}(\mathbf{p})$ continuous in $\mathbf{p}$ and $O(\lambda)$. Finally, if we put $\mathbf{p}_F^\omega = (0, \omega p_F)$ and we suppose that $0 < \kappa \le |\mathbf{p}|, |\mathbf{k}'|, |\mathbf{k}'-\mathbf{p}| \le 2\kappa$, $0 < \vartheta < 1$, then

$$\begin{aligned} &\widehat{G}^{2,1}_{\rho}(\mathbf{k}'+\mathbf{p}_F^\omega, \mathbf{k}'+\mathbf{p}+\mathbf{p}_F^\omega) = \widehat{G}^{2,1}_{th,\rho;\omega}(\mathbf{k}', \mathbf{k}'+\mathbf{p})[1+O(\kappa^\vartheta)] \; , \\ &\widehat{G}^{2,1}_{j}(\mathbf{k}'+\mathbf{p}_F^\omega, \mathbf{k}'+\mathbf{p}+\mathbf{p}_F^\omega) = \widehat{G}^{2,1}_{th,j;\omega}(\mathbf{k}', \mathbf{k}'+\mathbf{p})[1+O(\kappa^\vartheta)] \; , \\ &\widehat{G}^{2}(\mathbf{k}'+\mathbf{p}_F^\omega) = \widehat{G}^{2}_{th,\omega}(\mathbf{k}')[1+O(\kappa^\vartheta)] \; . \end{aligned} \tag{44}$$

This Lemma will be proved in the next section; we now exploit its implications.

By combining (44) and (35) we find that

$$-ip_0\ \widehat{G}^{2,1}_\rho(\mathbf{k}'+\mathbf{p}^\omega_F,\mathbf{k}'+\mathbf{p}+\mathbf{p}^\omega_F)+\omega p\ \widetilde{v}_J\widehat{G}^{2,1}_j(\mathbf{k}'+\mathbf{p}^\omega_F,\mathbf{k}'+\mathbf{p}+\mathbf{p}^\omega_F)=$$
$$=\frac{Z^{(3)}}{(1-\tau)Z}\left[\widehat{G}^2(\mathbf{k}'+\mathbf{p}^\omega_F)-\widehat{G}^2(\mathbf{k}'+\mathbf{p}+\mathbf{p}^\omega_F)\right][1+O(\kappa^\vartheta)] \tag{45}$$

and

$$-ip_0\ \widehat{G}^{2,1}_j(\mathbf{k}'+\mathbf{p}^\omega_F,\mathbf{k}'+\mathbf{p}+\mathbf{p}^\omega_F)+\omega p\ \widetilde{v}_N\widehat{G}^{2,1}_\rho(\mathbf{k}'+\mathbf{p}^\omega_F,\mathbf{k}'+\mathbf{p}+\mathbf{p}^\omega_F)=$$
$$=\frac{\widetilde{Z}^{(3)}}{(1+\tau)Z}\left[\widehat{G}^2(\mathbf{k}'+\mathbf{p}^\omega_F)-\widehat{G}^2(\mathbf{k}'+\mathbf{p}+\mathbf{p}^\omega_F)\right][1+O(\kappa^\vartheta)]\ , \tag{46}$$

with

$$\widetilde{v}_N=v_s\frac{Z^{(3)}}{\widetilde{Z}^{(3)}}\quad,\quad\widetilde{v}_J=v_s\frac{\widetilde{Z}^{(3)}}{Z^{(3)}}\ . \tag{47}$$

On the other hand, a WI for the model (3) can be derived directly from the commutation relations, see App. B; one gets

$$-ip_0\ \widehat{G}^{2,1}_\rho(\mathbf{k}'+\mathbf{p}^\omega_F,\mathbf{k}'+\mathbf{p}+\mathbf{p}^\omega_F)+\omega p\ v_F\widehat{G}^{2,1}_j(\mathbf{k}'+\mathbf{p}^\omega_F,\mathbf{k}'+\mathbf{p}+\mathbf{p}^\omega_F)=$$
$$=\left[\widehat{G}^2(\mathbf{k}'+\mathbf{p}^\omega_F)-\widehat{G}^2(\mathbf{k}'+\mathbf{p}+\mathbf{p}^\omega_F)\right][1+O(\kappa^\vartheta)]\ . \tag{48}$$

Hence, if we compare (48) with (45), we get the identities

$$\frac{Z^{(3)}}{(1-\tau)Z}=1\quad,\quad\widetilde{v}_J=v_F\ . \tag{49}$$

Moreover, in App. B we also show that

$$\widehat{G}^{0,2}_{\rho,\rho}(\mathbf{p})=0\quad,\quad\text{if }\mathbf{p}=(0,p_0)\ , \tag{50}$$

and this fixes the value of $A_{\rho\rho}(0)$ so that

$$\widehat{G}^{0,2}_{\rho,\rho}(\mathbf{p})=\frac{1}{4\pi v_s Z^2}\frac{(Z^{(3)})^2}{1-(\lambda_\infty/4\pi v_s)^2}\left[2-\frac{D_-(\mathbf{p})}{D_+(\mathbf{p})}-\frac{D_+(\mathbf{p})}{D_-(\mathbf{p})}\right]+R(\mathbf{p})\ , \tag{51}$$

with $R(\mathbf{0})=0$. By using (49), we get (17), with

$$K=\frac{1}{Z^2}\frac{(Z^{(3)})^2}{1-(\lambda_\infty/4\pi v_s)^2}=\frac{1-(\lambda_\infty/4\pi v_s)}{1+(\lambda_\infty/4\pi v_s)}\ . \tag{52}$$

It has been proved in Theorem 4.1 of [15] (where we used $c=1$) that the critical indices of the model (29) have a simple expressions in terms of λ_∞; if we take eq. (4.26) of [15] and we put $\tau=\lambda_\infty/4\pi v_s$, we get:

$$X_+=1-\frac{(\lambda_\infty/2\pi v_s)}{1+(\lambda_\infty/4\pi v_s)}\quad,\quad X_-=1+\frac{(\lambda_\infty/2\pi v_s)}{1-(\lambda_\infty/4\pi v_s)}\ ; \tag{53}$$

this implies the relations (14), with K given by (52). Eq. (16) follows from the remark that, at the first order, $\lambda_\infty = \lambda_{-\infty}$, while $\lambda_{-\infty}$, which was imposed to be equal in the two models, is related to λ (always at the first order) by the relation $\lambda_{-\infty} = 2\lambda[\widehat{v}(0) - \widehat{v}(2p_F)]$. The first identity in (15) is proved as eq. (1.11) of [15]; note that $\bar{\nu}$ is different from the index ν appearing in [15], but one can see that this difference only implies that one has to replace, in eq. (1.11) of [15], x_+ with x_-. Finally, by using the identity (4.21) of [15] (where η is denoted η_z), we get also the second identity in (15). The proof of Theorem 1.1 is completed.

Remark 1 - Note that in the WI (45), (46) for the model (21) *three* different velocities appear. This is due to the fact that the *irrelevant* operators (in the RG sense) break the relativistic symmetries present in the model in the scaling limit and produce different renormalization of the velocities. Note also that the velocities $\widetilde{v}_N, \widetilde{v}_J$ defined in (47) verify the universal relation

$$\widetilde{v}_N \widetilde{v}_J = v_s^2 \; . \tag{54}$$

Remark 2 - The constraints (49) and (50) on the renormalization parameters of the continuum model, which describes the large distance behavior, are a consequence of the existence of a well defined lattice hamiltonian.

3 Proof of Lemma 2.1

The proof of the lemma is based on the RG analysis of the Grassmann integrals (24) and (29), described in [12, 13] and [17, 19], respectively.

Let us recall briefly the analysis of 24. Let T^1 be the one dimensional torus, $||k-k'||_{T^1}$ the usual distance between k and k' in T^1 and $||k|| = ||k-0||$. We introduce a *scaling parameter* $\gamma > 1$ and a positive function $\chi(\mathbf{k}') \in C^\infty(T^1 \times R)$, $\mathbf{k}' = (k', k_0)$, such that $\chi(\mathbf{k}') = \chi(-\mathbf{k}') = 1$ if $|\mathbf{k}'| < t_0 = a_0 v_s/\gamma$ and $= 0$ if $|\mathbf{k}'| > a_0$ where $a_0 = \min\{\frac{p_F}{2}, \frac{\pi - p_F}{2}\}$ and $|\mathbf{k}'| = \sqrt{k_0^2 + (v_s||k'||_{T^1})^2}$. The above definition is such that the supports of $\chi(k - p_F, k_0)$ and $\chi(k + p_F, k_0)$ are disjoint and the C^∞ function on $T^1 \times R$

$$\widehat{f}_1(\mathbf{k}) \equiv 1 - \chi(k - p_F, k_0) - \chi(k + p_F, k_0) \tag{55}$$

is equal to 0, if $[||v_s(|k| - p_F)||_{T^1}]^2 + k_0^2 < t_0^2$.

We define also, for any integer $h \le 0$,

$$f_h(\mathbf{k}') = \chi(\gamma^{-h}\mathbf{k}') - \chi(\gamma^{-h+1}\mathbf{k}') \; . \tag{56}$$

We have

$$\chi(\mathbf{k}') = \sum_{h=h_{L,\beta}}^{0} f_h(\mathbf{k}') \; , \tag{57}$$

where

$$h_{L,\beta} = \min\{h : t_0\gamma^{h+1} > \sqrt{(\pi\beta^{-1})^2 + (v_s\pi L^{-1})^2}\} \; . \tag{58}$$

Note that, if $h \leq 0$, $f_h(\mathbf{k}') = 0$ for $|\mathbf{k}'| < t_0\gamma^{h-1}$ or $|\mathbf{k}'| > t_0\gamma^{h+1}$, and $f_h(\mathbf{k}') = 1$, if $|\mathbf{k}'| = t_0\gamma^h$. Let us now define:

$$\widehat{f}_h(\mathbf{k}) = f_h(k - p_F, k_0) + f_h(k + p_F, k_0) \; . \tag{59}$$

This definition implies that, if $h \leq 0$, the support of $\widehat{f}_h(\mathbf{k})$ is the union of two disjoint sets, A_h^+ and A_h^-. In A_h^+, k is strictly positive and $||k - p_F||_{T^1} \leq t_0\gamma^h \leq t_0$, while, in A_h^-, k is strictly negative and $||k + p_F||_{T^1} \leq t_0\gamma^h$. The label h is called the *scale* or *frequency* label. Note that

$$1 = \sum_{h=h_{L,\beta}}^{1} \widehat{f}_h(\mathbf{k}) \; ; \tag{60}$$

hence, if we approximate p_F by $(2\pi/L)(n_F + 1/2)$, n_F equal to the integer part of $Lp_F/(2\pi)$, and we define $\mathcal{D}'_L = \{k' = 2(n+1/2)\pi/L, n \in \mathbb{Z}, -[L/2] \leq n \leq [(L-1)/2]\}$ and $\mathcal{D}'_{L,\beta} = \mathcal{D}'_L \times \mathcal{D}_\beta$, we can write:

$$\begin{aligned}
g(\mathbf{x}-\mathbf{y}) &= g^{(1)}(\mathbf{x}-\mathbf{y}) + \sum_{\omega=\pm}\sum_{h=h_{L,\beta}}^{0} e^{-ip_F(x-y)} g_\omega^{(h)}(\mathbf{x}-\mathbf{y}) \; , \\
g^{(1)}(\mathbf{x}-\mathbf{y}) &= \frac{1}{\beta L}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} \frac{\widehat{f}_1(\mathbf{k})}{-ik_0 + (v_s/v_F)(\cos p_F - \cos k)} \; , \\
g_\omega^{(h)}(\mathbf{x}-\mathbf{y}) &= \frac{1}{\beta L}\sum_{\mathbf{k}'\in\mathcal{D}'_{L,\beta}} e^{-i\mathbf{k}'(\mathbf{x}-\mathbf{y})} \frac{f_h(\mathbf{k}')}{-ik_0 + E_\omega(k')} \; ,
\end{aligned} \tag{61}$$

where

$$E_\omega(k') = \omega v_s \sin k' + (1+\delta)\cos p_F(1 - \cos k') \; . \tag{62}$$

Let us now describe the perturbative expansion of $\mathcal{W}$; for simplicity we shall consider only the case $\phi = 0$. We can write:

$$\begin{aligned}
e^{\mathcal{W}(J,\widetilde{J},0)} &= \int P(d\psi^{\leq 0}) \int P(d\psi^{(1)}) e^{-\mathcal{V}(\psi) + \int d\mathbf{x}[J_\mathbf{x}\rho_\mathbf{x} + \widetilde{J}_\mathbf{x} j_\mathbf{x}]} = \\
&= e^{-L\beta E_0} \int P(d\psi^{\leq 0}) e^{-\mathcal{V}^{(0)}(\psi^{\leq 0}) + \mathcal{B}^{(0)}(\psi^{\leq 0}, J, \widetilde{J})} \; ,
\end{aligned} \tag{63}$$

where, if we put $\underline{\mathbf{x}} = (\mathbf{x}_1, \ldots, \mathbf{x}_{2n})$, $\underline{\omega} = (\omega_1, \ldots, \omega_{2n})$ and $\psi_{\underline{\mathbf{x}},\underline{\omega}} = \prod_{i=1}^n \psi^+_{\mathbf{x}_i,\omega_i} \prod_{i=n+1}^{2n} \psi^-_{\mathbf{x}_i,\omega_i}$, the *effective potential* $\widetilde{\mathcal{V}}^{(0)}(\psi)$ can be represented as

$$\mathcal{V}^{(0)}(\psi) = \sum_{n\geq 1}\sum_{\underline{\omega}} \int d\underline{\mathbf{x}} W^{(0)}_{\underline{\omega},2n}(\underline{\mathbf{x}})\psi_{\underline{\mathbf{x}},\underline{\omega}} \; , \tag{64}$$

the kernels $W^{(0)}_{\underline{\omega},2n}(\underline{\mathbf{x}})$ being analytic functions of λ and ν near the origin; if $|\nu| \le C|\lambda|$ and we put $\underline{\mathbf{k}} = (\mathbf{k}_1, \dots, \mathbf{k}_{2n-1})$, their Fourier transforms satisfy, for any $n \ge 1$, the bounds, see §2.4 of [12],

$$|\widehat{W}^{(0)}_{\underline{\omega},2n}(\underline{\mathbf{k}})| \le C^n |\lambda|^{\max\{1,n-1\}} . \tag{65}$$

A similar representation can be written for the functional $\mathcal{B}^{(0)}(\psi^{\le 0}, J, \widetilde{J})$, containing all terms which are at least of order one in the external fields, including those which are independent on $\psi^{\le 0}$.

The integration of the scales $h \le 0$ is done iteratively in the following way. Suppose that we have integrated the scale $0, -1, -2, .., j$, obtaining

$$e^{\mathcal{W}(J,\widetilde{J},0)} = e^{-L\beta E_j} \int P_{Z_j,C_j}(d\psi^{\le j}) e^{-\mathcal{V}^{(j)}(\sqrt{Z_j}\psi^{\le j}) + \mathcal{B}^{(j)}(\sqrt{Z_j}\psi^{\le j}, J, \widetilde{J})} , \tag{66}$$

where, if we put $C_j(\mathbf{k}')^{-1} = \sum_{h=h_{L,\beta}}^{j} f_h(\mathbf{k}')$, P_{Z_j,C_j} is the Grassmann integration with propagator

$$\frac{1}{Z_j} g^{(\le j)}_\omega(\mathbf{x}-\mathbf{y}) = \frac{1}{Z_j}\frac{1}{\beta L} \sum_{\mathbf{k}\in\mathcal{D}'_{L,\beta}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \frac{C_j^{-1}(\mathbf{k})}{-ik_0 + E_\omega(k')} , \tag{67}$$

$\mathcal{V}^{(j)}(\psi)$ is of the form

$$\mathcal{V}^{(j)}(\psi) = \sum_{n\ge 1}\sum_{\underline{\omega}} \int d\underline{\mathbf{x}} W^{(j)}_{\underline{\omega},2n}(\underline{\mathbf{x}}) \psi_{\underline{\mathbf{x}},\underline{\omega}} , \tag{68}$$

and $\mathcal{B}^{(j)}(\psi^{\le j}, J, \widetilde{J})$ contains all terms which are at least of order one in the external fields, including those which are independent on $\psi^{\le j}$. For $j = 0$, $Z_0 = 1$ and the functional $\mathcal{V}^{(0)}$ and $\mathcal{B}^{(0)}$ are exactly those appearing in (63).

First of all, we define a localization operator (see [12, 13] for details) in the following way:

$$\mathcal{L}\mathcal{V}^{(j)}(\sqrt{Z_j}\psi) = \gamma^j n_j \frac{Z_j}{\beta L}\sum_{\mathbf{k}} \psi^+_{\mathbf{k},\omega}\psi^-_{\mathbf{k},\omega} + a_j \frac{Z_j}{\beta L}\sum_{\mathbf{k}} E_\omega(\mathbf{k})\psi^+_{\mathbf{k},\omega}\psi^-_{\mathbf{k},\omega} + \tag{69}$$
$$z_j \frac{Z_j}{\beta L}\sum_{\mathbf{k}} (-ik_0)\psi^+_{\mathbf{k},\omega}\psi^-_{\mathbf{k},\omega} + l_j \frac{Z_j^2}{(\beta L)^4}\sum_{\mathbf{k}_1,\mathbf{k}',\mathbf{p}} \psi^+_{\mathbf{k},+}\psi^-_{\mathbf{k}-\mathbf{p},+}\psi^+_{\mathbf{k}',-}\psi^-_{\mathbf{k}'+\mathbf{p},-} ,$$

$$\mathcal{L}\mathcal{B}^{(j)}(\sqrt{Z_j}\psi) = \frac{Z^{(1)}_j}{(\beta L)^2}\sum_{\mathbf{k},\mathbf{p}} J_{\mathbf{p}}\Big[\sum_\omega \psi^+_{\mathbf{k},\omega}\psi^-_{\mathbf{k}-\mathbf{p},\omega}\Big] + \tag{70}$$
$$+\frac{Z^{(2)}_j}{(\beta L)^2}\sum_{\mathbf{k},\mathbf{p}} J_{\mathbf{p}+2\omega\mathbf{p}_F}\Big[\sum_\omega \psi^+_{\mathbf{k},\omega}\psi^-_{\mathbf{k}-\mathbf{p},-\omega}\Big] + \frac{\widetilde{Z}^{(1)}_j}{(\beta L)^2}\sum_{\mathbf{k},\mathbf{p}} J_{\mathbf{p}}\Big[\sum_\omega \omega\, \psi^+_{\mathbf{k},\omega}\psi^-_{\mathbf{k}-\mathbf{p},\omega}\Big]$$
$$+\frac{\widetilde{Z}^{(2)}_j}{(\beta L)^2}\sum_{\mathbf{k},\mathbf{p}} \widetilde{J}_{\mathbf{p}+2\omega\mathbf{p}_F}\Big[\sum_\omega \omega\, \psi^+_{\mathbf{k},\omega}\psi^-_{\mathbf{k}-\mathbf{p},-\omega}\Big] ,$$

where $\mathbf{p}_F = (p_F, 0)$. This definitions are such that the difference between $-\mathcal{V}^{(j)} + \mathcal{B}^{(j)}$ and $-\mathcal{L}\mathcal{V}^{(j)} + \mathcal{L}\mathcal{B}^{(j)}$ is made of irrelevant terms.

The constants appearing in (69) and (70) are evaluated in terms of the values of the corresponding kernels at zero external momenta. Since the space momentum k of $\psi^+_{\mathbf{k},\omega}$ is measured from the Fermi surface, this means that the external momenta corresponding to the fermion variables are put equal to $(\omega p_F, 0)$, while $\mathbf{p}$ is put equal to $(0,0)$. On the other hand, it is easy to see that the kernel multiplying $J\psi^+\psi^-$ is *even* in the exchange $k \to -k$ (k is here the true space momentum, not the momentum measured from the Fermi surface), since both the propagator and the interaction are even, while the kernel multiplying $\widetilde{J}\psi^+\psi^-$ is *odd* in the exchange $k \to -k$, because of the parity properties of the current $j_{\mathbf{x}}$. These considerations are used in the definition of the constants in (70).

We then renormalize the integration measure, by moving to it some of the quadratic terms in the r.h.s. of (69), that is $z_j(\beta L)^{-1}\sum_{\mathbf{k}}[-ik_0 + E_\omega(\mathbf{k})]\psi^+_{\mathbf{k},\omega}\psi^-_{\mathbf{k},\omega}$; the Grassmann integral in the r.h.s. of (66) takes the form:

$$\int P_{\widetilde{Z}_{j-1},C_j}(d\psi^{(\le j)}) e^{-\widetilde{\mathcal{V}}^{(j)}(\sqrt{Z_j}\psi^{\le j}) + \mathcal{B}^{(j)}(\sqrt{Z_j}\psi^{\le j}, J, \widetilde{J})} , \tag{71}$$

where $\widetilde{\mathcal{V}}^{(j)}$ is the remaining part of the effective interaction and $P_{\widetilde{Z}_{j-1},C_j}(d\psi^{\le j})$ is the measure whose propagator is obtained by substituting in (67) Z_j with

$$\widetilde{Z}_{j-1}(\mathbf{k}) = Z_j[1 + z_j C_j(\mathbf{k})^{-1}] . \tag{72}$$

It is easy to see that we can decompose the fermion field as $\psi^{\le j} = \psi^{\le j-1} + \psi^{(j)}$, so that

$$P_{\widetilde{Z}_{j-1},C_j}(d\psi^{\le j}) = P_{Z_{j-1},C_{j-1}}(d\psi^{(\le j-1)}) P_{Z_{j-1},\widetilde{f}_j^{-1}}(d\psi^{(j)}) , \tag{73}$$

where $\widetilde{f}_j(\mathbf{k})$ (see eq. (2.90) of [12]) has the same support and scaling properties as $f_j(\mathbf{k})$. Hence, if make the field rescaling $\psi \to [\sqrt{Z_{j-1}}/\sqrt{Z_j}]\psi$ and we call $\widehat{\mathcal{V}}^{(j)}(\sqrt{Z_{j-1}}\psi^{\le j})$ the new effective potential, we can write (71) in the form

$$\int P_{Z_{j-1},C_{j-1}}(d\psi^{(\le j-1)}) \int P_{Z_{j-1},\widetilde{f}_j^{-1}}(d\psi^{(j)}) \cdot \tag{74}$$
$$\cdot e^{-\widehat{\mathcal{V}}^{(j)}(\sqrt{Z_{j-1}}\psi^{\le j}) + \widehat{\mathcal{B}}^{(j)}(\sqrt{Z_{j-1}}\psi^{\le j}, J, \widetilde{J})} .$$

By performing the integration over $\psi^{(j)}$, we finally get (66), with $j-1$ in place of j.

In order to analyze the result of this iterative procedure, we note that $\mathcal{L}\widehat{\mathcal{V}}^{(j)}(\psi)$ can be written as

$$\mathcal{L}\widehat{\mathcal{V}}^{(j)}(\psi) = \gamma^j \nu_j F_\nu(\psi) + \delta_j F_\alpha(\psi) + \lambda_j F_\lambda(\psi) , \tag{75}$$

where $F_\nu(\psi)$, $F_\alpha(\psi)$ and $F_\lambda(\psi)$ are the functions of ψ, which appear in (69) in the terms proportional to n_j, a_j and l_j, respectively. $\nu_j = (\sqrt{Z_j}/\sqrt{Z_{j-1}})n_j$, $\delta_j = (\sqrt{Z_j}/\sqrt{Z_{j-1}})(a_j - z_j)$ and $\lambda_j = (\sqrt{Z_j}/\sqrt{Z_{j-1}})^2 l_j$ are called the *running couplings* (r.c.) on scale j. In Theorem (3.12) of [12] it is proved that the kernels of $\widehat{\mathcal{V}}^{(j)}$ and $\widehat{\mathcal{B}}^{(j)}$ are *analytic* as functions of the r.c., provided that they are small enough. One has then to analyze the flow of the r.c. (the *beta function*) as $j \to -\infty$. We shall now summarize the results, explained in detail in [12, 14].

The propagator $\widetilde{g}^{(j)}_\omega(\mathbf{x}-\mathbf{y})$ of the single scale measure $P_{Z_{j-1},\widetilde{f}_j^{-1}}$, can be decomposed as

$$\widetilde{g}^{(j)}_\omega(\mathbf{x}-\mathbf{y}) = \frac{1}{Z_j}\, g^{(j)}_{th,\omega}(\mathbf{x}-\mathbf{y}) + r_j(\mathbf{x}-\mathbf{y}) \;, \tag{76}$$

where

$$\frac{1}{Z_j}\, g^{(j)}_{th,\omega}(\mathbf{x}-\mathbf{y}) = \frac{1}{Z_j}\frac{1}{\beta L}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{f_j(\mathbf{k})}{-ik_0+\omega v_s k} \tag{77}$$

describes the leading asymptotic behavior, while the remainder $r_j(\mathbf{x}-\mathbf{y})$ satisfies, for any $M>0$ and $\vartheta<1$, the bound

$$|r_j(\mathbf{x}-\mathbf{y})| \le \frac{\gamma^{(1+\vartheta)j}}{Z_j}\frac{C_{M,\vartheta}}{1+(\gamma^j|\mathbf{x}-\mathbf{y}|^M)} \;. \tag{78}$$

We call $Z_j^{(th)}$ the values of Z_j one would obtain by substituting $\mathcal{V}^{(0)}$ with $\mathcal{L}\mathcal{V}^{(0)}$ and by putting $r_h = 0$ for any $h \ge j$ and we observe that, by (4.50) of [12],

$$\left|\frac{Z_j}{Z_{j-1}} - \frac{Z_j^{(th)}}{Z_{j-1}^{(th)}}\right| \le C_\vartheta \lambda^2 \gamma^{\vartheta j} \;. \tag{79}$$

(76) and (79) imply that, see §4.6 of [12], the r.c. satisfy recursive equations of the form:

$$\begin{aligned}
\lambda_{j-1} &= \lambda_j + \beta^{(j)}_\lambda(\lambda_j,...,\lambda_0) + \bar\beta^{(j)}_\lambda(\lambda_j,\delta_j,\nu_j;...;\lambda_0,\delta_0,\nu_0) \;,\\
\delta_{j-1} &= \delta_j + \beta^{(j)}_\delta(\lambda_j,\delta_j,\nu_j;...;\lambda_0,\delta_0,\nu_0) \;,\\
\nu_{j-1} &= \gamma\nu_j + \beta^{(j)}_\nu(\lambda_j,\delta_j,\nu_j;...;\lambda_0,\delta_0,\nu_0) \;,
\end{aligned} \tag{80}$$

where $\beta^{(j)}_\lambda$, $\bar\beta^{(j)}_\lambda$, $\beta^{(j)}_\delta$, $\beta^{(j)}_\nu$ can be written as *convergent* expansions in their arguments, if $\varepsilon_j = \max_{j\le h\le 0}\max\{|\lambda_h|,|\delta_h|,|\nu_h|\}$ is small enough. By definition, $\beta^{(j)}_\lambda$ is given by a sum of multiscale graphs (collected in trees; their definition is in §3 of [12]), containing only λ-vertices with scale ≤ 0 and in which the propagators $g^{(h)}_\omega$ and the wave function renormalizations Z_h, $0\ge h\ge j$, are replaced by $g^{(h)}_{th,\omega}$ and $Z_h^{(th)}$, $0\ge h\ge j$; $\bar\beta^{(j)}_\lambda$ contains the correction terms together with the remainder of the expansion.

The following crucial property, called *vanishing of the Beta function*, was proved by means of Ward Identities in [14]; for any $\vartheta < 1$,

$$|\beta_\lambda^{(j)}(\lambda_j, ..., \lambda_j)| \le C_\vartheta |\lambda_j|^2 \gamma^{\vartheta j} \; . \qquad (81)$$

It is also possible to prove that, for a suitable choice of $\delta, \nu = O(\lambda)$, $\delta_j, \nu_j = O(\gamma^{\vartheta j}\bar\lambda_j)$, if $\bar\lambda_j = \sup_{k\ge j}|\lambda_k|$, and this implies, by the *short memory property* (exponential decreasing contribution of the graphs with propagators of scale $h > j$, as $h-j$ grows, see the remark after (4.31) of [12]), that $\bar\beta_\lambda^{(j)} = O(\gamma^{\vartheta j}\bar\lambda_j^2)$, so that the sequence λ_j converges, as $j \to -\infty$, to a smooth function $\lambda_{-\infty}(\lambda) = \lambda + O(\lambda^2)$, such that

$$|\lambda_j - \lambda_{-\infty}| \le C_\vartheta \lambda^2 \gamma^{\vartheta j} \; . \qquad (82)$$

In a similar way we can also analyze the *renormalization constants* $Z_j^{(\alpha)}$ and $\widetilde Z_j^{(\alpha)}$, $\alpha = 1, 2$, defined in (70), and the field strength renormalization Z_j; we can write:

$$\frac{Z_{j-1}}{Z_j} = 1 + \beta_z^{(j)}(\lambda_j, ..., \lambda_0) + \bar\beta_z^{(j)}(\lambda_j, \delta_j; .., \lambda_0, \delta_0) \; , \qquad (83)$$

$$\frac{Z_{j-1}^{(\alpha)}}{Z_j^{(\alpha)}} = 1 + \beta_{(\rho,\alpha)}^{(j)}(\lambda_j, ..., \lambda_0) + \bar\beta_{(\rho,\alpha)}^{(j)}(\lambda_j, \delta_j; .., \lambda_0, \delta_0) \; , \qquad (84)$$

$$\frac{\widetilde Z_{j-1}^{(\alpha)}}{\widetilde Z_j^{(\alpha)}} = 1 + \beta_{(J,\alpha)}^{(j)}(\lambda_j, ..., \lambda_0) + \bar\beta_{J,\alpha}^{(j)}(\lambda_j, \delta_j; .., \lambda_0, \delta_0) \; , \qquad (85)$$

where, by definition, the $\beta_t^{(j)}$ functions (with $t = z$, (ρ, α) or (J, α)) are given by a sum of multiscale graphs, containing only λ-vertices with scale ≤ 0 and in which the the propagators $g_\omega^{(h)}$ and the renormalization constants Z_h, $Z_h^{(\alpha)}$, $\widetilde Z_h^{(\alpha)}$, $0 \ge h \ge j$, are replaced by $g_{th,\omega}^{(h)}$, $Z_h^{(th)}$, $Z_h^{(th,\alpha)}$ and $\widetilde Z_h^{(th,\alpha)}$ (the definition of $Z_h^{(th,\alpha)}$, $\widetilde Z_h^{(th,\alpha)}$ is analogue to the one of $Z_h^{(th)}$); the $\bar\beta_t^{(j)}$ functions contain the correction terms together the remainder of the expansion. Note that, by definition, the constants $Z_j^{(th)}$ are exactly those generated by (83) with $\bar\beta_z^{(j)} = 0$. Note that $\bar\beta_t^{(j)} = O(\lambda_j\gamma^{\vartheta j})$ and, by using (82) and the short memory property (see *e.g.* §4.9 of [12])

$$\beta_t^{(j)}(\lambda_j, ..., \lambda_0) = \beta_t^{(j)}(\lambda_{-\infty}, ..., \lambda_{-\infty}) + O(\lambda\gamma^{\vartheta h}) \; . \qquad (86)$$

This implies that there exist, if w is small enough, analytic functions $\eta_t(w)$, $t = z, (\rho, \alpha), (J, \alpha)$, of order λ^2 for $t = z, (\rho, 1), (J, 1)$ and order λ for $t = (\rho, 2), (J, 2)$, such that

$$\begin{aligned} &|\log_\gamma(Z_{j-1}/Z_j) - \eta_z(\lambda_{-\infty}/v_s)| \le C_\vartheta \lambda^2 \gamma^{\vartheta j} \; , \\ &|\log_\gamma(Z_{j-1}^{(\alpha)}/Z_j^{(\alpha)}) - \eta_{\rho,\alpha}(\lambda_{-\infty}/v_s)| \le C_\vartheta \lambda^2 \gamma^{\vartheta j} \; , \\ &|\log_\gamma(\widetilde Z_{j-1}^{(\alpha)}/\widetilde Z_j^{(\alpha)}) - \eta_{J,\alpha}(\lambda_{-\infty}/v_s)| \le C_\vartheta \lambda^2 \gamma^{\vartheta j} \; . \end{aligned} \qquad (87)$$

The fact that the *critical indices* η_t are functions of $\lambda_{-\infty}/v_s$ (not of $\lambda_{-\infty}$ and v_s separately) is not stressed in [12, 13], but follows very easily from dimensional arguments. It is also easy to see that (see [13], §3.4), since the propagator (77) satisfies the symmetry property

$$\widehat{g}^{(j)}_{th,\omega}(k,k_0) = -i\omega \widehat{g}^{(j)}_{th,\omega}(-k_0/v_s, v_s k) , \tag{88}$$

then $\eta_{\rho,\alpha}(w) = \eta_{J,\alpha}(w)$, $\alpha = 1,2$. Moreover, by using the approximate Ward identities associated to the linearity in $\mathbf{k}$ of $\widehat{g}^{(j)}_{th,\omega}(\mathbf{k})^{-1}$, one can show (see Theorem 5.6 of [12]) that $\eta_z = \eta_{\rho,1}$.

The analysis of the functional (29) can be done in a similar way. Even in this case, we shall only sketch the main results, by referring to [19] and [15] for more details. Again we perform a multiscale integration, but now we have to consider two different regimes: the first regime, called *ultraviolet*, contains the scales $0 \le h \le N$, while the second one contains the scales $h < 0$, and is called *infrared*.

After the integration of the ultraviolet scales, see [19, 15] (where the external fields $J, \widetilde{J}$ are substituted by two equivalent fields $J_\omega, \omega = \pm 1$), we can write the r.h.s. of (29), with $\phi = 0$, as

$$\lim_{l\to-\infty}\lim_{N\to\infty} \int P_Z(d\psi^{(\le 0)}) e^{-\bar{\mathcal{V}}^{(0)}(\psi^{(\le 0)}) + \bar{\mathcal{B}}^{(0)}(\psi^{(\le 0)}, J, \widetilde{J})} , \tag{89}$$

where the integration measure has a propagator $Z^{-1} g^{(\le 0)}_{th,\omega}(\mathbf{x}-\mathbf{y})$, given by (31) with $N = 0$; moreover, $\bar{\mathcal{V}}^{(0)}$ and $\bar{\mathcal{B}}^{(0)}$ are functionals similar to the functionals $\mathcal{V}^{(0)}$ and $\mathcal{B}^{(0)}$ of (63), with the following main differences. First of all, $\mathcal{L}\bar{\mathcal{V}}^{(0)}$ can be written as in (69), with $E_\omega(\mathbf{k}) = c\omega k$, $n_0 = 0$, $a_0 = z_0$ (these two properties easily from the symmetries of the propagator) and λ_0 replaced by a new constant $\widetilde{\lambda}_0$; moreover, $\mathcal{L}\bar{\mathcal{B}}^{(0)}$ can be written as in (70), with $Z_0^{(2)} = \widetilde{Z}_0^{(2)} = 0$ (since no term proportional to $\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},-\omega}$ can be present) and $Z_0^{(1)}$, $\widetilde{Z}_0^{(1)}$ replaced by two new constants $Z_0^{(3)}$, $\widetilde{Z}_0^{(3)}$. Hence, we can analyze (89) as we did for (63), but now we have only one r.c., to be called $\widetilde{\lambda}_j$, and three renormalization constants, $\widetilde{Z}_j$, $Z_j^{(3)}$ and $\widetilde{Z}_j^{(3)}$, taking the place of Z_j, $Z_j^{(1)}$ and $\widetilde{Z}_j^{(1)}$, respectively. It follows that $\widetilde{\lambda}_j \to \widetilde{\lambda}_{-\infty}$, as $j \to -\infty$, with $\widetilde{\lambda}_{-\infty}$ an analytic function of $\widetilde{\lambda}_0$, such that $\widetilde{\lambda}_{-\infty} = \widetilde{\lambda}_0 + O(\widetilde{\lambda}_0^2)$. On the other hand, $\widetilde{\lambda}_0$ is an analytic function of λ_∞ and $\widetilde{\lambda}_0 = \lambda_\infty + O(\lambda_\infty^2)$, see [19]; hence there exists an analytic function $h(w)$, such that, if λ_∞ is small enough,

$$\widetilde{\lambda}_{-\infty} = h(\lambda_\infty) . \tag{90}$$

Moreover, the flow equations of the new renormalization constants can be written as in (83), (84), (85), with different functions $\beta_t^{(j)}$ and $\bar{\beta}_t^{(j)}$, $t = z, (\rho,3), (J,3)$. However, if we put

$$c = v_s , \tag{91}$$

the functions $\beta_t^{(j)}$ are the same as before, as a consequence of the definitions (77) and (31). It is then an immediate consequence of (76), (79) and (86) that

$$\begin{aligned}
&|\log_\gamma(\widetilde{Z}_{j-1}/\widetilde{Z}_j) - \eta_z(\widetilde{\lambda}_{-\infty}/v_s)| \le C_\vartheta \lambda^2 \gamma^{\vartheta j} \ , \\
&|\log_\gamma(Z^{(3)}_{j-1}/Z^{(3)}_j) - \eta_{\rho,1}(\widetilde{\lambda}_{-\infty}/v_s)| \le C_\vartheta \lambda^2 \gamma^{\vartheta j} \ , \\
&|\log_\gamma(\widetilde{Z}^{(3)}_{j-1}/\widetilde{Z}^{(3)}_j) - \eta_{J,1}(\widetilde{\lambda}_{-\infty}/v_s)| \le C_\vartheta \lambda^2 \gamma^{\vartheta j} \ ,
\end{aligned} \tag{92}$$

where $\eta_z(w)$, $\eta_{\rho,1}(w)$ and $\eta_{J,1}(w)$ are *exactly* the same functions appearing in (87). Hence, if we choose λ_∞, given λ, so that

$$\widetilde{\lambda}_{-\infty} = \lambda_{-\infty} \ , \tag{93}$$

which is possible if λ is small enough, the critical indices in the spin or in the continuum model are the same.

We have now to show that the parameters Z, $Z^{(3)}$ and $\widetilde{Z}^{(3)}$ of the continuum model (with $c = v_s$) can be chosen so that (43) is true. To begin with, we prove that they can fixed so that, for any $j \le 0$,

$$\begin{aligned}
&|Z_j - \widetilde{Z}_j| \le C_\vartheta |\lambda| \gamma^{\frac{\vartheta}{2} j} \ , \\
&|Z^{(1)}_j - Z^{(3)}_j| \le C_\vartheta |\lambda| \gamma^{\frac{\vartheta}{2} j} \quad , \quad |\widetilde{Z}^{(1)}_j - \widetilde{Z}^{(3)}_j| \le C_\vartheta |\lambda| \gamma^{\frac{\vartheta}{2} j} \ .
\end{aligned} \tag{94}$$

Let us prove the first bound. By using (87) and (92), we see that there exist $b_j(\lambda)$, b, $\widetilde{b}_j(\lambda)$ and $\widetilde{b}$, such that

$$Z_j = b_j(\lambda)\gamma^{-j\eta_z} \quad , \quad \widetilde{Z}_j = Z\widetilde{b}_j(\lambda)\gamma^{-j\eta_z} \ , \tag{95}$$

with $|b_j(\lambda) - b| \le C_\vartheta |\lambda| \gamma^{\vartheta j}$ and $|\widetilde{b}_j(\lambda) - \widetilde{b}| \le C_\vartheta |\lambda| \gamma^{\vartheta j}$. Hence, since $\vartheta - \eta_z \ge \vartheta/2$, for λ small enough,

$$|Z_j - \widetilde{Z}_j| = Z_j \left| 1 - \frac{Z\widetilde{b}_j(\lambda)}{b_j(\lambda)} \right| \le C_\vartheta |\lambda| \gamma^{\frac{\vartheta}{2} j} \ , \tag{96}$$

provided that we choose $Z = b/\widetilde{b}$. In the same way we can choose the values of $Z^{(3)}$ and $\widetilde{Z}^{(3)}$.

Note that the values of $Z^{(3)}$ and $\widetilde{Z}^{(3)}$ are expected to be different, even if the asymptotic behavior, as $j \to -\infty$, of $Z^{(3)}_j$ and $\widetilde{Z}^{(3)}_j$ is the same. This follows from the fact that the "remainder" r_j in the representation (76) of the propagator breaks the symmetry (88), which the relation $\eta_z = \eta_{\rho,1}$ is based on. This expectation is confirmed by an explicit first order calculation, see Appendix C; we see that $Z^{(3)} = 1 - a\lambda + O(\lambda^2)$ and $\widetilde{Z}^{(3)} = 1 + a\lambda + O(\lambda^2)$, with

$$a = \frac{1}{2\pi v_s}[\widehat{v}(0) - \widehat{v}(2p_F)] \ . \tag{97}$$

Note that this expression is in agreement with the identity (52), since, at first order $\lambda_{-\infty} = \lambda_\infty = 2\lambda[\widehat{v}(0) - \widehat{v}(2p_F)]$.

In order to complete the proof of (43), we use the representation of $\langle S^{(3)}_{\mathbf{x}} S^{(3)}_{\mathbf{0}} \rangle_T$, given in [12], eq. (1.13), that is

$$\langle S^{(3)}_{\mathbf{x}} S^{(3)}_{\mathbf{0}} \rangle_T = \cos(2p_F x)\Omega^a(\mathbf{x}) + \Omega^b(\mathbf{x}) + \Omega^c(\mathbf{x}) \ , \tag{98}$$

where the first two terms represent the leading asymptotic behavior, while $\Omega^c(\mathbf{x})$ is the remainder. In [12] we proved that, if $\vartheta < 1$ and n is a positive integer, then

$$|\partial^n \Omega^a(\mathbf{x})| \le \frac{C_n}{|\mathbf{x}|^{2X_+ + n}} \quad , \quad |\Omega^c(\mathbf{x})| \le \frac{C_\vartheta}{|\mathbf{x}|^{2+\vartheta}} \ , \tag{99}$$

where $X_+ = K$ is the critical index (16). Moreover, by definition (see §5.9 of [12]), $\Omega^b_{\mathbf{x}}$ is a sum of multiscale graphs containing only λ-vertices with scale ≤ 0 and in which the the propagators $g^{(h)}_\omega$ and the renormalization constants Z_h, $Z^{(1)}_h$, $0 \ge h \ge j$, are replaced by $g^{(h)}_{th,\omega}$ and $Z^{(th)}_h$, $Z^{(th,1)}_h$. It can be written (see (5.39) and (5.43) of [12]), as

$$\Omega^b(\mathbf{x}) = \sum_{h=-\infty}^{0} \sum_{\omega=\pm} \left[\frac{Z^{(1)}_h}{Z_h} \right]^2 [g^{(h)}_{th,\omega}(\mathbf{x}) g^{(h)}_{th,\omega}(-\mathbf{x}) + G^{(h)}(\mathbf{x})] \ , \tag{100}$$

where $G^{(h)}(\mathbf{x})$ is a function satisfying, for any $N > 0$, the bound

$$|G^{(h)}(\mathbf{x})| \le C_N \frac{\gamma^{2h}}{1 + [\gamma^h |\mathbf{x}|^N]} \ . \tag{101}$$

The Fourier transform of $\Omega^c(\mathbf{x})$ is continuous; the same is true for $\cos(2p_F x)$ $\Omega^a(\mathbf{x})$, around $\mathbf{p} = 0$, thanks to the bound (6.45) of [12] (where $\mathbf{k} = \mathbf{p} - 2\mathbf{p}_F$ is bounded for $\mathbf{p}$ small).

On the other hand we can write

$$G^{0,2}_{th,\rho,\rho}(\mathbf{x}) = \sum_{h=-\infty}^{0} \sum_{\omega=\pm} \left[\frac{Z^{(3)}_h}{\bar{Z}_h} \right]^2 [g^{(h)}_{th,\omega}(\mathbf{x}) g^{(h)}_{th,\omega}(-\mathbf{x}) + \bar{G}^{(h)(\mathbf{x})}] + G_1(\mathbf{x}) \ , \tag{102}$$

where $\bar{G}^{(h)}(\mathbf{x})$ satisfies a bound similar to (101), as well as $G_1(\mathbf{x})$, which is given by graphs with at least one propagator of scale ≥ 1. Using (76), (79) and (94), we get

$$\left| \int d\mathbf{x} e^{i\mathbf{p}\mathbf{x}} [\Omega^b(\mathbf{x}) - G^{0,2}_{th,\rho,\rho}(\mathbf{x})] \right| \le \sum_{h=-\infty}^{0} \gamma^{(2+\vartheta)h} \int d\mathbf{x} \frac{C_N}{1 + (\gamma^h |\mathbf{x}|)} \le C_1 \ , \tag{103}$$

which proves (43).

It remains to prove the three equations (44); let us consider the first. If $0 < \kappa \le |\mathbf{p}|, |\mathbf{k}'|, |\mathbf{k}' - \mathbf{p}| \le 2\kappa$, in §2.4 of [17] (see (2.63) of [17]) the following bound was proved,

$$\left| \widehat{G}^{2,1}_{\rho}(\mathbf{k}' + \mathbf{p}_F^{\omega}, \mathbf{k}' + \mathbf{p} + \mathbf{p}_F^{\omega}) \right| \le \frac{C}{\kappa^{2-2\eta}} \ , \tag{104}$$

which is of course valid even for $G^{2,1}_{\rho,th}(\mathbf{k}', \mathbf{k}' + \mathbf{p})$. Moreover, if we choose the parameters of the continuum model as before, we can show, by using again (76), (79) and (94), that the difference $R(\mathbf{k}', \mathbf{k}' + \mathbf{p})$ between $\widehat{G}^{2,1}_{\rho}(\mathbf{k}' + \mathbf{p}_F^{\omega}, \mathbf{k}' + \mathbf{p} + \mathbf{p}_F^{\omega})$ and $G^{2,1}_{\rho,th}(\mathbf{k}', \mathbf{k}' + \mathbf{p})$ is given by a summable sum of terms, each bounded by the r.h.s. of (104) times a factor $\gamma^{\vartheta j}$. On the other hand, if $h_\kappa \equiv \log_\gamma(\kappa)$ is the scale of the external fermion propagators, each term of the expansion must have at least one propagator of scale $h_0 \le h_\kappa$; see (2.61), (2.62) of [17] for a more detailed description of the expansion. Hence, we can write, for $j \ge h_k$, $\gamma^{\vartheta j} = \kappa^{\vartheta} \gamma^{\vartheta(j-h_\kappa)}$ and we can absorb the factor $\gamma^{\vartheta(j-h_\kappa)}$ in the bound, thanks to the short memory property. It follows that

$$\left| \widehat{G}^{2,1}_{\rho}(\mathbf{k}' + \mathbf{p}_F^{\omega}, \mathbf{k}' + \mathbf{p} + \mathbf{p}_F^{\omega}) - G^{2,1}_{\rho,th}(\mathbf{k}', \mathbf{k}' + \mathbf{p}) \right| \le C_\vartheta \frac{\kappa^{\vartheta}}{\kappa^{2-2\eta}} \ , \tag{105}$$

from which the first of (44) is obtained; the second and the third of (44) are proved by similar arguments.

A Derivation of the Ward Identities (39)

Let us define $\psi^{\pm}_{\mathbf{x},\omega} = \psi^{[l,N]\pm}_{\mathbf{x},\omega}$, $\rho_{\mathbf{x},\omega} = \psi^{+}_{\mathbf{x},\omega}\psi^{-}_{\mathbf{x},\omega}$ and let us consider the functional (37). By proceeding as in §2.2 of [13], we can show that, by performing in (37) the change of the variables $\psi^{\pm}_{\mathbf{x},\omega} \to e^{\pm i\alpha_{\mathbf{x},\omega}}\psi^{\pm}_{\mathbf{x},\omega}$, the following identity is obtained:

$$0 = \frac{1}{Z(J)} \int P_Z(d\psi)[-ZD_{\bar\omega}\rho_{\mathbf{x},\bar\omega} + Z\delta T_{\mathbf{x},\bar\omega}] e^{-\mathcal{V}^{(N)}(\sqrt{Z}\psi) + \sum_\omega \int d\mathbf{x} J_{\mathbf{x},\omega}\rho_{\mathbf{x},\omega}} \ , \tag{106}$$

where $D_\omega = \partial_0 + i\omega\partial_1$, $Z(J) = \exp[\mathcal{W}(J)]$ and

$$\delta T_{\mathbf{x},\omega} = \frac{1}{(L\beta)^2} \sum_{\mathbf{k}^+ \neq \mathbf{k}^-} e^{i(\mathbf{k}^+ - \mathbf{k}^-)\mathbf{x}} C_\omega(\mathbf{k}^+, \mathbf{k}^-) \widehat{\psi}^{+}_{\mathbf{k}^+,\omega} \widehat{\psi}^{-}_{\mathbf{k}^-,\omega} \ , \tag{107}$$

$$C_\omega(\mathbf{q}, \mathbf{p}) = [\chi^{-1}_{l,N}(\mathbf{p}) - 1] D_\omega(\mathbf{p}) - [\chi^{-1}_{l,N}(\mathbf{q}) - 1] D_\omega(\mathbf{q}) \ . \tag{108}$$

We now perform one functional derivative with respect to $J_{\mathbf{y},\omega}$ in the r.h.s. of (106), then we put $J = 0$ and we take the Fourier transform. By some trivial algebra, we get the two identities, valid for $\mathbf{p} \neq 0$ and for any τ:

$$\begin{aligned} D_\omega(\mathbf{p}) G_{\omega,\omega}(\mathbf{p}) - \tau\, \widehat{v}_0(\mathbf{p}) D_{-\omega}(\mathbf{p}) G_{-\omega,\omega}(\mathbf{p}) &= R_{N,1}(\mathbf{p}) \ , \\ D_{-\omega}(\mathbf{p}) G_{-\omega,\omega}(\mathbf{p}) - \tau\, \widehat{v}_0(\mathbf{p}) D_{\omega}(\mathbf{p}) G_{\omega,\omega}(\mathbf{p}) &= R_{N,2}(\mathbf{p}) \ , \end{aligned} \tag{109}$$

where

$$R_{N,1}(\mathbf{p}) = \frac{\partial^2 \mathcal{W}_A}{\partial \alpha_{\mathbf{p},\omega} \partial J_{-\mathbf{p},\omega}}\Big|_{J=\alpha=0} \; , \quad R_{N,2}(\mathbf{p}) = \frac{\partial^2 \mathcal{W}_A}{\partial \alpha_{\mathbf{p},-\omega} \partial J_{-\mathbf{p},\omega}}\Big|_{J=\alpha=0} \tag{110}$$

and

$$e^{\mathcal{W}_A(\alpha,\eta,J)} = \int P_Z(d\psi) e^{-\mathcal{V}^{(N)}(\sqrt{Z}\psi) + \sum_\omega \int d\mathbf{x}\; J_{\mathbf{x},\omega}\rho_{\mathbf{x},\omega}} e^{[A_0 - \tau A_-](\alpha,\psi)} \; , \tag{111}$$

with

$$A_0(\alpha,\psi) \;=\; \sum_{\omega=\pm} \int \frac{d\mathbf{q}\, d\mathbf{p}}{(2\pi)^4}\; C_\omega(\mathbf{q},\mathbf{p}) \widehat{\alpha}_{\mathbf{q}-\mathbf{p},\omega} \widehat{\psi}^+_{\mathbf{q},\omega} \widehat{\psi}^-_{\mathbf{p},\omega} \; , \tag{112}$$

$$A_-(\alpha,\psi) \;=\; \sum_{\omega=\pm} \int \frac{d\mathbf{q}\, d\mathbf{p}}{(2\pi)^4}\; D_{-\omega}(\mathbf{p}-\mathbf{q}) \widehat{v}_0(\mathbf{p}-\mathbf{q}) \widehat{\alpha}_{\mathbf{q}-\mathbf{p},\omega} \widehat{\psi}^+_{\mathbf{q},-\omega} \widehat{\psi}^-_{\mathbf{p},-\omega} . \tag{113}$$

Note that the terms proportional to τ in (109) are obtained by adding and subtracting them to the identities one really gets; they are in some sense two counterterms, introduced to erase the local marginal parts of the terms in the effective potential proportional to $\alpha_{\mathbf{x},\omega}\rho_{\mathbf{x},\omega}$, produced by contracting the vertex A_0 with one or more λ vertices. As shown in [17, 19], the introduction of a non local interaction (still gauge invariant) in the continuum model, makes it possible to calculate them explicitly. Hence, the proof of (39) is equivalent to the proof that, if $\tau = \lambda_\infty/4\pi c$ and $\mathbf{p} \neq 0$, then

$$\lim_{-l,N\to\infty} R_{N,1}(\mathbf{p}) = -\frac{1}{4\pi c Z^2} D_{-\omega}(\mathbf{p}) \quad , \quad \lim_{-l,N\to\infty} R_{N,2}(\mathbf{p}) = 0 \; . \tag{114}$$

This result is achieved by using the technique explained in §4 of [15], that we shall now briefly explain.

The functional $\mathcal{W}_A$ is analyzed, as always, by a multiscale integration and a tree expansion; we get

$$R_{N,1}(\mathbf{p}) = -\frac{1}{Z^2} \int \frac{d\mathbf{k}}{(2\pi)^2} C_\omega(\mathbf{k},\mathbf{k}-\mathbf{p}) \widehat{g}^{[l,N]}_{\omega,th}(\mathbf{k}) \widehat{g}^{[l,N]}_{\omega,th}(\mathbf{k}-\mathbf{p}) + \bar{R}_N(\mathbf{p}) \; , \tag{115}$$

where $\bar{R}_N(\mathbf{p})$ is given by the sum over all graphs with at least one λ vertex, while the first term in (115) is the 0 order contribution, coming from the contraction of the vertex $\delta T_{\mathbf{x},\omega}$ with the vertex $\rho_{\mathbf{y},\omega}$. It is easy to show that, if $\mathbf{p} \neq 0$,

$$\lim_{-l,N\to\infty} \int \frac{d\mathbf{k}}{(2\pi)^2} C_\omega(\mathbf{k},\mathbf{k}-\mathbf{p}) \widehat{g}^{(l,N)}_{\omega,th}(\mathbf{k}) \widehat{g}^{(l,N)}_{\omega,\vartheta}(\mathbf{k}-\mathbf{p}) = \frac{1}{4\pi c} D_{-\omega}(\mathbf{p}) \; . \tag{116}$$

Hence, to complete the proof, we have to show that, if $\mathbf{p} \neq 0$, $\bar{R}_N(\mathbf{p})$ and $R_{N,2}(\mathbf{p})$ vanish in the removed cutoffs limit, thanks to the choice of the counterterm τA_-. This result is obtained by a slight extension of the analysis given in §4 of [15] for a similar problem; we shall give some details, for people who have read that paper.

First of all, the sum over the graphs, such that one of the fermionic fields in A_0 or A_- is contracted at scale l, can be bounded by $C\gamma^l|\mathbf{p}|^{-1}$, hence it vanishes as $l \to -\infty$, if $\mathbf{p}$ is kept fixed at a value different from 0. Moreover, the sum over the other graphs, called $\widetilde{R}_{1,N}(\mathbf{p})$, can be written as

$$\widetilde{R}_{1,N}(\mathbf{p}) = \sum_{k=0}^{N} \widehat{K}_{\Delta}^{(1;0;1)(k)} + O(\gamma^{-\vartheta N}) , \tag{117}$$

where $\widehat{K}_{\Delta}^{(1;2m;s)(k)}$ are the kernels of the monomials with one α field, $2m$ ψ fields and s J-fields in the effective potential, after the integration of the scales $N, N-1, ...k$, while the last contribution comes from the trees with the root at a negative scale. The kernel $\widehat{K}_{\Delta}^{(1;2m;s)(k)}$ can be decomposed as in Fig. 4.1 of [15] (with the analogue of the terms d and e missing and a wiggling line in place of the two fermion external lines). By proceeding as in the proof of (4.33)-(4.41) of [15], we can see that

$$|\widehat{K}_{\Delta}^{(1;0;1)(k)}| \leq C|\lambda_\infty|\gamma^{-k}\gamma^{-\vartheta(N-k)} . \tag{118}$$

It follows that $\bar{R}_N(\mathbf{p}) = 0$ vanishes in the removed cutoffs limit; the same is true for $R_{2,N}(\mathbf{p}) = 0$, as we can prove in a similar way.

B Commutation rules and Ward Identities

Let us consider the model (21) and let us introduce the density and the current operators (see *e.g.* [10]):

$$\begin{aligned} \rho_x &= S_x^3 + \frac{1}{2} = a_x^+ a_x^- \quad , \quad x \in Z , \\ J_x &= S_x^1 S_{x+1}^2 - S_x^2 S_{x+1}^1 = \frac{1}{2i}[a_{x+1}^+ a_x^- - a_x^+ a_{x+1}^-] \equiv v_F j_x . \end{aligned} \tag{119}$$

As it is well known, the functions $G_\rho^{2,1}(\mathbf{x},\mathbf{y},\mathbf{z})$ and $G_j^{2,1}(\mathbf{x},\mathbf{y},\mathbf{z})$ can be written as

$$\begin{aligned} G_\rho^{2,1}(\mathbf{x},\mathbf{y},\mathbf{z}) &= < T[\rho_{\mathbf{x}} a_{\mathbf{y}}^- a_{\mathbf{z}}^+] >_{L,\beta} , \\ G_j^{2,1}(\mathbf{x},\mathbf{y},\mathbf{z}) &= < T[j_{\mathbf{x}} a_{\mathbf{y}}^- a_{\mathbf{z}}^+] >_{L,\beta} , \end{aligned} \tag{120}$$

where $< \cdot >_{L,\beta}$ denotes the expectation in the Grand Canonical Ensemble, T is the time-ordered product and

$$\rho_{\mathbf{x}} = e^{x_0 H}\rho_x e^{-x_0 H} \quad , \quad a_{\mathbf{x}}^{\pm} = e^{x_0 H} a_x^{\pm} e^{-x_0 H} . \tag{121}$$

The above definition of the current is justified by the (imaginary time) conservation equation

$$\frac{\partial \rho_{\mathbf{x}}}{\partial x_0} = e^{Hx_0}[H, \rho_x]e^{-Hx_0} = -i\partial_x^{(1)} J_{\mathbf{x}} \equiv -i[J_{x,x_0} - J_{x-1,x_0}] \;, \tag{122}$$

where an important role plays the fact that

$$[H, \rho_x] = [H_T, \rho_x] \quad , \quad H_T = -\frac{1}{2}[a_x^+ a_{x+1}^- + a_{x+1}^+ a_x^-] \;, \tag{123}$$

a property which is not true for J_x.

By using (122) and some trivial calculation, one gets the identity

$$\frac{\partial}{\partial x_0} G_\rho^{2,1}(\mathbf{x},\mathbf{y},\mathbf{z}) = -iv_F \partial_x^{(1)} G_j^{2,1}(\mathbf{x},\mathbf{y},\mathbf{z}) + \\ +\delta(x_0 - z_0)\delta_{x,z} G^2(\mathbf{y}-\mathbf{x}) - \delta(x_0 - y_0)\delta_{x,y} G^2(\mathbf{x}-\mathbf{z}) \;. \tag{124}$$

Let us now take the Fourier transform of the two sides of this equations. The renormalization group analysis described in this paper implies that we can safely take the limit $L, \beta \to \infty$ of $\widehat{G}_\rho^{2,1}(\mathbf{k}, \mathbf{k}+\mathbf{p})$, if $\mathbf{p}$ and $\mathbf{k} - \mathbf{p}_F^\omega$ are different from zero. Hence we get the identity (48), under the conditions on the momenta of Lemma 2.1, for any value of κ.

In the same way we derive a WI for the density-density correlations. First we observe that $G_{\rho,\rho}^{0,2}(\mathbf{x},\mathbf{y}) =< T[\rho_{\mathbf{x}}\rho_{\mathbf{y}}] >_{L,\beta}$; then, by using (122), we get

$$\frac{\partial}{\partial x_0} G_{\rho,\rho}^{0,2}(\mathbf{x},\mathbf{y}) = -iv_F \partial_x^{(1)} G_{j,\rho}^{0,2}(\mathbf{x},\mathbf{y}) + \delta(x_0 - y_0) < [\rho_{(x,x_0)}, \rho_{(y,x_0)}] >_{L,\beta} \;, \tag{125}$$

where $G_{j,\rho}^{0,2}(\mathbf{x},\mathbf{y})$ is defined in a way similar to $G_{\rho,\rho}^{0,2}(\mathbf{x},\mathbf{y})$, that is by using the definition in the last line of (27), with $\widetilde{J}_{\mathbf{x}}$ in place of $J_{\mathbf{x}}$. Let us now take the Fourier Transform; since $[\rho_{(x,x_0)}, \rho_{(y,x_0)}] = 0$, we get, in the limit $L, \beta \to \infty$, under the conditions on the momenta of Lemma 2.1, the identity:

$$-ip_0 G_{\rho,\rho}^{0,2}(\mathbf{p}) - i(1 - e^{-ip}) v_F G_{j,\rho}^{0,2}(\mathbf{p}) = 0 \;, \tag{126}$$

which implies (50).

Remark - The WI (48) and (126) could also be obtained by doing in (24) the change of variables $\psi_{\mathbf{x}}^\pm \to e^{\pm i\alpha_{\mathbf{x}}}\psi_{\mathbf{x}}^\pm$ and by proceeding as in App. A for (37). However, in this case the analysis of the corrections is much easier, since the ultraviolet problem involves only the k_0 variable; it is indeed very easy to prove that the corrections vanish in the $M \to \infty$ limit.

C First order calculation of $Z^{(3)}$ and $\widetilde{Z}^{(3)}$

$Z^{(3)}$ is defined so that $\lim_{h\to-\infty} Z_h^{(3)}/Z_h^{(1)} = 1$, see (94). On the other hand, at the first order, $Z_h^{(1)} = 1 + \alpha_h$, where α_h is the sum of the values of the two Feynmann graphs of Fig. 1, calculated at $\mathbf{p} = 0$ and $\widetilde{\mathbf{k}} = \mathbf{p}_F^\omega = (0, \omega p_F)$ (the result is independent of ω).

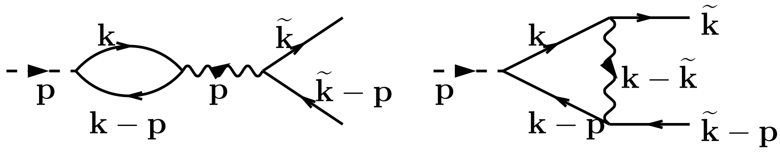

Figure 1: The first order contributions to the renormalization constants.

By a simple calculation, we get, in the limit $M, L, \beta \to \infty$,

$$\alpha_h = -2\lambda \int \frac{d\mathbf{k}}{(2\pi)^2} \widehat{g}^{(\geq h)}(\mathbf{k})^2 [-\widehat{v}(0) + \widehat{v}(k - \omega p_F)] = \tag{127}$$
$$= -2\lambda \int_0^\pi \frac{dk}{(2\pi)} \int_{-\infty}^{+\infty} \frac{dk_0}{(2\pi)} \widehat{g}^{(\geq h)}(\mathbf{k})^2 [-2\widehat{v}(0) + \widehat{v}(k - p_F) + \widehat{v}(k + p_F)] \;,$$

where $\widehat{g}^{(\geq h)}(\mathbf{k}) = \widehat{g}^{(1)}(\mathbf{k}) + \sum_{\omega'} \sum_{j=h}^0 \widehat{g}_{\omega'}^{(j)}(\mathbf{k} - \mathbf{p}_F^{\omega'})$ is the propagator with infrared cutoff at scale h, see (61). Note that, if $|k - \omega' p_F| \geq \gamma^{h-1}$, $\widehat{g}^{(\geq h)}(\mathbf{k}) = [-ik_0 + (v_s/v_F)(\cos p_F - \cos k)]^{-1}$ and that, if $e_0 \neq 0$, $\int dk_0 [-ik_0 + e_0]^{-2} = 0$. It follows that, if $\varepsilon = \gamma^h$,

$$\alpha_h = -\frac{\lambda[\widehat{v}(0) - \widehat{v}(2p_F)]}{2\pi^2 v_s} \int_{-\varepsilon}^{\varepsilon} dt \int_{-\sqrt{\varepsilon^2 - t^2}}^{\sqrt{\varepsilon^2 - t^2}} dk_0 \frac{1}{(-ik_0 + t)^2} + O(\varepsilon) \;, \tag{128}$$

so that

$$\alpha_{-\infty} = \lim_{h \to -\infty} \alpha_h = -\frac{\lambda[\widehat{v}(0) - \widehat{v}(2p_F)]}{2\pi v_s} \;. \tag{129}$$

A similar calculation can be done for $Z_h^{(3)}$; in fact, in this case, there is no term corresponding to the second graph in Fig. 1, while the contribution corresponding to the first one, with the external fermion propagators of index ω, is given by

$$\lambda \int \frac{d\mathbf{k}}{(2\pi)^2} \widehat{g}_{th,-\omega}^{[h,N]}(\mathbf{k})^2 \;, \tag{130}$$

with $g_{th,\omega}^{[h,N]}(\mathbf{k})$ defined as in (31). However, by the symmetry (88), the integral above vanishes for any N; hence, at the first order, $Z_h^{(3)} = Z^{(3)}$, which implies that $Z^{(3)} = 1 + \alpha_{-\infty} + O(\lambda^2)$.

A similar procedure can be followed for the first order calculation of $\widetilde{Z}^{(3)}$. Let us consider first $\widetilde{Z}_h^{(1)}$; since $v_F \widehat{j}(\mathbf{k}) = \sin k\, a_{\mathbf{k}}^+ a_{\mathbf{k}}^-$, we see immediately that $\widetilde{Z}_h^{(1)} = 1 + \lim_{h \to -\infty} \omega \beta_{h,\omega}$, where $\beta_{h,\omega}$ is obtained from (127) by inserting in the integrand a factor $\sin k/v_F$. It follows that

$$\beta_{h,\omega} = -\frac{2\lambda}{v_F}\int_0^\pi \frac{dk}{(2\pi)}\sin k\int_{-\infty}^{+\infty}\frac{dk_0}{(2\pi)}\widehat{g}^{(\geq h)}(\mathbf{k})^2[\widehat{v}(k-\omega p_F)-\widehat{v}(k+\omega p_F)] \ , \tag{131}$$

so that

$$\lim_{h\to-\infty}\omega\beta_{h,\omega} = \frac{\lambda[\widehat{v}(0)-\widehat{v}(2p_F)]}{2\pi v_s} = -\alpha_{-\infty} \ . \tag{132}$$

On the other hand, we get as before that, at the first order, $\widetilde{Z}_h^{(3)} = \widetilde{Z}^{(3)}$; hence $\widetilde{Z}^{(3)} = 1-\alpha_{-\infty}+O(\lambda^2)$.

D Comparison with the Luttinger model

In the case of the Luttinger model, we can repeat the analysis leading to Lemma 2.1 and we can deduce two WI for the Luttinger model, which are similar in the form to (45), (46). If we call $G^{2,1}_{L,\alpha,\omega}$, $\alpha=\rho,j$, and $G^2_{L,\omega}$ the correlation functions analogous to $G^{2,1}_{th,\alpha,\omega}$ and $G^2_{th,\omega}$, we get the identities

$$\begin{aligned}
&-ip_0\ \widehat{G}^{2,1}_{L,\rho,\omega}(\mathbf{k},\mathbf{k}+\mathbf{p})+\omega p\ \widetilde{v}_J\widehat{G}^{2,1}_{L,j,\omega}(\mathbf{k},\mathbf{k}+\mathbf{p}) = \\
&=\frac{Z^{(3)}}{(1-\tau)Z}\left[\widehat{G}^2_{L,\omega}(\mathbf{k})-\widehat{G}^2_{L,\omega}(\mathbf{k}+\mathbf{p})\right][1+O(\kappa^\vartheta)] \ , \\
&-ip_0\ \widehat{G}^{2,1}_{L,j,\omega}(\mathbf{k},\mathbf{k}+\mathbf{p})+\omega p\ \widetilde{v}_N\widehat{G}^{2,1}_{L,\rho,\omega}(\mathbf{k},\mathbf{k}+\mathbf{p}) = \\
&=\frac{\widetilde{Z}^{(3)}}{(1+\tau)Z}\left[\widehat{G}^2_{L,\omega}(\mathbf{k})-\widehat{G}^2_{L,\omega}(\mathbf{k}+\mathbf{p})\right][1+O(\kappa^\vartheta)] \ ,
\end{aligned} \tag{133}$$

$\widetilde{v}_J$ and $\widetilde{v}_N$ being defined as in (47). On the other hand, exact WI for the Luttinger model can be obtained from the anomalous commutation relations, see *e.g.* [10]. In our notation, we can write, if $\sigma=\lambda_L/(2\pi v_F)$ and λ_L is the Luttinger coupling,

$$\begin{aligned}
&-ip_0\widehat{G}^{2,1}_{L,\rho;\omega}(\mathbf{k},\mathbf{k}+\mathbf{p})+\omega v_F p(1-\sigma)\widehat{G}^{2,1}_{L,j;\omega}(\mathbf{k},\mathbf{k}+\mathbf{p})] = \\
&\qquad = \widehat{G}^2_{th;\omega}(\mathbf{k})-\widehat{G}^2_{th;\omega}(\mathbf{k}+\mathbf{p}) \ , \\
&-ip_0\widehat{G}^{2,1}_{th,j;\omega}(\mathbf{k},\mathbf{k}+\mathbf{p})+\omega v_F p(1+\sigma)\widehat{G}^{2,1}_{L,\rho;\omega}(\mathbf{k},\mathbf{k}+\mathbf{p})] = \\
&\qquad = \widehat{G}^2_{th;\omega}(\mathbf{k})-\widehat{G}^2_{th;\omega}(\mathbf{k}+\mathbf{p}) \ .
\end{aligned} \tag{134}$$

By comparing (133) with (134), we get:

$$\widetilde{v}_J = v_s\frac{Z^{(3)}}{\widetilde{Z}^{(3)}} = v_F(1-\sigma) \quad , \quad \widetilde{v}_N = v_s\frac{\widetilde{Z}^{(3)}}{Z^{(3)}} = v_F(1+\sigma) \ , \tag{135}$$

and

$$\frac{Z^{(3)}}{(1-\tau)Z} = 1 \quad , \quad \frac{\widetilde{Z}^{(3)}}{(1+\tau)Z} = 1 \ . \tag{136}$$

The first identity in (136) implies, as in the quantum spin chain case, that $\kappa = K/(\pi v_s)$. Moreover, the identities (135) imply that

$$v_s = v_F\sqrt{(1-\sigma^2)}\ , \tag{137}$$

while (136) and (52) imply that

$$\frac{Z^{(3)}}{\widetilde{Z}^{(3)}} = \frac{1-\tau}{1+\tau} = K\ , \tag{138}$$

the relation between K and $\tau = \lambda_\infty/(4\pi v_s)$ being the same as in the quantum spin model. On the other hand, (135) and (137) imply also that $\frac{Z^{(3)}}{\widetilde{Z}^{(3)}} = \sqrt{\frac{1-\sigma}{1+\sigma}}$; hence we have an explicit expression of K in terms of σ, that is:

$$K = \sqrt{\frac{1-\sigma}{1+\sigma}}\ . \tag{139}$$

Note that (137) and (139) allow us to represent explicitly v_s and K, which depend only on the large distance behavior of the model, in terms of the "bare" quantities λ_L and v_F. This result is strictly related to the second identity in (136), which is missing in the spin model, where it is replaced by the identity $\widetilde{v}_J = v_F$, see (49). For the same reasons, the above equations imply also that, in the Luttinger model, the following identities are true,

$$\widetilde{v}_N = v_s K^{-1} \quad , \quad \widetilde{v}_J = v_s K\ . \tag{140}$$

Note that these relations are also verified by the quantities v_J and v_N, introduced by Haldane in [6], but they are certainly *not true* in the spin model model (3). In fact, in the XYZ case one has, from the second of (49), that $\widetilde{v}_J$ is λ-independent, while $v_s K$ is is not, as it is evident from (19) and (20). The relation (54) is however valid also for the lattice model (3).

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