Absence of interaction corrections in graphene conductivity

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The exact vanishing of the interaction corrections to the zero temperature and zero frequency conductivity of graphene in the presence of weak short range interactions is rigorously established.

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Graphene [1] has several peculiar properties originating from its perfect two dimensionality and from the Dirac-like nature of its charge carriers at half-filling. In particular, recent optical measurements [2] show that at half-filling and small temperatures, if the frequency is in a range between the temperature and the band-width, the conductivity is essentially constant and equal, up to a few percent, to $\sigma_0 = \frac{e^2}{h}\frac{\pi}{2}$; such value only depends on the fundamental von Klitzing constant h/e^2 and not on the material parameters, like the Fermi velocity. These experimental results confirm the theoretical predictions [3] based on the description of graphene in terms of massless non-interacting Dirac particles [4, 5]; lattice effects have been taken into account in [6]. Since truly universal phenomena are quite rare in condensed matter (an example is provided by the Quantum Hall effect), it is important to understand whether this apparently *universal value* is just an artifact of the idealized description in terms of non-interacting fermions or rather it is a robust property still valid in the presence of electron-electron interactions, which are certainly present and expected to play a role in real graphene. This question is entirely analogous to the one concerning universality in the quantum Hall effect [7], a notoriously difficult and still open problem.

The effects of the electron-electron interactions on the graphene conductivity have been investigated in the Dirac approximation by perturbation theory both in the presence of long- and of short-ranged interactions; however, lowest order explicit computations have produced *different* results [8–10], depending on the regularization scheme (momentum cut-off or dimensional regularization) chosen to cure the spurious ultraviolet divergences introduced by the Dirac approximation. In [9] it was predicted that in the presence of electrostatic interactions the zero frequency conductivity tends to zero, while in [8, 10] it was argued that it converges to the free Dirac one, as a consequence of the *divergence* of the Fermi velocity; however, if screening or retardation effects are taken into account, the Fermi velocity is known to saturate at low frequency [11–13], in which case it is unclear what to expect. The extreme sensitivity of the conductivity computation to approximations or regularizations (see also [14]) calls for a rigorous analysis.

In this paper we consider the Hubbard model on the honeycomb lattice, as a model of monolayer graphene with screened interactions. While in general the understanding of the low temperature behavior of the Hubbard model is a formidable challenge for theoreticians, in the case of the honeycomb lattice at half filling the methods introduced in [12] and based on constructive Renormalization Group have proved to be quite effective. Using these techniques, we rigorously establish *the exact (non-perturbative) vanishing of the interaction corrections to the conducivity* in the zero frequency limit. All Feynman graphs contributing to the conductivity cancel out exactly in the limit, a statement analogous to the Adler-Bardeen theorem in quantum electrodynamics [7, 15].

We introduce creation and annihilation fermionic operators $\psi^{\pm}_{\vec x,\sigma} = (a^{\pm}_{\vec x,\sigma}, b^{\pm}_{\vec x+\vec\delta_1,\sigma}) = L^{-2}\sum_{\vec k\in\mathcal{B}_\Lambda}\psi^{\pm}_{\vec k,\sigma}e^{\pm i\vec k\vec x}$ for electrons with spin index $\sigma=\uparrow\downarrow$ sitting at the sites of the two triangular sublattices Λ_A and Λ_B of a periodic honeycomb lattice of side L; we assume that $\Lambda_A=\Lambda$ has basis vectors $\vec l_{1,2} = \frac12(3,\pm\sqrt3)$ and that $\Lambda_B = \Lambda_A + \vec\delta_j$, with $\vec\delta_1 = (1,0)$ and $\vec\delta_{2,3} = \frac12(-1,\pm\sqrt3)$ the nearest neighbor vectors; $\mathcal{B}_\Lambda = \{\vec k = n_1\vec G_1/L + n_2\vec G_2/L \,:\, 0\le n_i < L\}$ with $\vec G_{1,2} = \frac{2\pi}{3}(1,\pm\sqrt3)$ is the first Brillouin zone (note that in the thermodynamic limit $L^{-2}\sum_{\vec k\in\mathcal{B}_\Lambda}\to|\mathcal{B}|^{-1}\int_{\mathcal{B}}d\vec k$, with $|\mathcal{B}| = 8\pi^2/(3\sqrt3)$). The grand-canonical Hamiltonian at half-filling is $H_\Lambda = H^0_\Lambda + UV_\Lambda$, where H_0 is the free Hamiltonian, describing nearest neighbor hopping (t is the hopping parameter):

$$H^0_\Lambda(t) = -t\sum_{\substack{\vec x\in\Lambda_A\\ j=1,2,3}}\sum_{\sigma=\uparrow\downarrow}(a^+_{\vec x,\sigma}b^-_{\vec x+\vec\delta_j,\sigma} + b^+_{\vec x+\vec\delta_j,\sigma}a^-_{\vec x,\sigma}) \quad (1)$$

and V_Λ is the local Hubbard interaction:

$$V_\Lambda = \sum_{\vec x\in\Lambda_A}\prod_{\sigma=\uparrow\downarrow}(a^+_{\vec x,\sigma}a^-_{\vec x,\sigma}-\frac12) + \sum_{\vec x\in\Lambda_B}\prod_{\sigma=\uparrow\downarrow}(b^+_{\vec x,\sigma}b^-_{\vec x,\sigma}-\frac12)\,. \quad (2)$$

In order to define the lattice current and the conductivity, we modify the hopping parameter along the bond $(\vec x,\vec x+\vec\delta_j)$ as $t\to t_{\vec x,j}(\vec A) = t\exp\{ie\int_0^1\vec A(\vec x+s\vec\delta_j)\cdot\vec\delta_j\,ds\}$, where $\vec A(\vec x)\in\mathbb{R}^2$ is a periodic continuum field on $\mathcal{S}_\Lambda = \{\vec x = L\xi_1\vec l_1 + L\xi_2\vec l_2 \,:\, \xi_i\in[0,1)\}$; its Fourier transform is defined as $\vec A(\vec x) = |\mathcal{S}_\Lambda|^{-1}\sum_{\vec p\in\mathcal{D}_\Lambda}\vec A_{\vec p}e^{-i\vec p\vec x}$, where $|\mathcal{S}_\Lambda| = \frac{3\sqrt3}{2}L^2$ and $\mathcal{D}_\Lambda = \{\vec p = n_1\vec G_1/L +$

$n_2\vec{G}_2/L\ :\ n_i \in \mathbb{Z}\}$ (note that in the thermodynamic limit $|\mathcal{S}_\Lambda|^{-1}\sum_{\vec{p}\in\mathcal{D}_\Lambda} \to (2\pi)^{-2}\int_{\mathbb{R}^2} d\vec{p}$). If we denote by $H(A) = H^0_\Lambda(\{t_{\vec{x},j}(\vec{A})\}) + UV_\Lambda$ the modified Hamiltonian with the new hopping parameters, the lattice current is defined as $\vec{\jmath}_{\vec{p}} = -\frac{\partial H(A)}{\partial \vec{A}_{\vec{p}}}$, which gives, at first order in $\vec{A}$,

$$\vec{\jmath}_{\vec{p}} = \vec{\jmath}^{(P)}_{\vec{p}} + \int \frac{d\vec{q}}{(2\pi)^2}\hat{\jmath}^{(D)}_{\vec{p},\vec{q}}\vec{A}_{\vec{q}}\ , \tag{3}$$

where $\int \frac{d\vec{q}}{(2\pi)^2}$ is a shorthand for $|\mathcal{S}_\Lambda|^{-1}\sum_{\vec{q}\in\mathcal{D}_\Lambda}$ and, if $\int_{\mathcal{B}}\frac{d\vec{k}}{|\mathcal{B}|}$ is a shorthand for $L^{-2}\sum_{\vec{k}\in\mathcal{B}_\Lambda}$ and $\eta^j_{\vec{p}} = \frac{1-e^{-i\vec{p}\vec{\delta}_j}}{i\vec{p}\vec{\delta}_j}$,

$$\begin{aligned}\vec{\jmath}^{(P)}_{\vec{p}} &= -iet\sum_{\sigma,j}\int_{\mathcal{B}}\frac{d\vec{k}}{|\mathcal{B}|}\big(a^+_{\vec{k}+\vec{p},\sigma}b^-_{\vec{k},\sigma}\vec{\delta}_j\eta^j_{\vec{p}}e^{-i\vec{k}(\vec{\delta}_j-\vec{\delta}_1)}\\ &\quad -b^+_{\vec{k}+\vec{p},\sigma}a^-_{\vec{k},\sigma}\vec{\delta}_j\eta^j_{\vec{p}}e^{+i(\vec{k}+\vec{p})(\vec{\delta}_j-\vec{\delta}_1)}\big)\end{aligned} \tag{4}$$

is the *paramagnetic current* and

$$\begin{aligned}\big[\hat{\jmath}^{(D)}_{\vec{p},\vec{q}}\big]_{lm} &= e^2t\sum_{\sigma,j}(\vec{\delta}_j)_l(\vec{\delta}_j)_m\eta^j_{\vec{p}}\eta^j_{\vec{q}}\int_{\mathcal{B}}\frac{d\vec{k}}{|\mathcal{B}|}\big(a^+_{\vec{k}+\vec{p}+\vec{q},\sigma}b^-_{\vec{k},\sigma}\\ &\cdot e^{-i\vec{k}(\vec{\delta}_j-\vec{\delta}_1)} + b^+_{\vec{k}+\vec{p}+\vec{q},\sigma}a^-_{\vec{k},\sigma}e^{i(\vec{k}+\vec{p}+\vec{q})(\vec{\delta}_j-\vec{\delta}_1)}\big)\end{aligned} \tag{5}$$

is the *diamagnetic tensor*. The conductivity, at Matsubara frequency $p_0 \in 2\pi\beta^{-1}(\mathbb{Z}+\frac{1}{2})$ and in units such that $\hbar = 1$, is defined via Kubo formula as [6]

$$\sigma^{\beta,\Lambda}_{lm}(p_0) = -\frac{K^{\beta,\Lambda}_{lm}(p_0,\vec{0})}{p_0|\mathcal{S}_\Lambda|} \tag{6}$$

where, if $\Xi = \mathrm{Tr}\{e^{-\beta H_\Lambda}\}$, $\langle\cdot\rangle = \Xi^{-1}\mathrm{Tr}\{e^{-\beta H_\Lambda}\cdot\}$ and $O_{x_0} = e^{H_\Lambda x_0}Oe^{-H_\Lambda x_0}$ for a generic operator O,

$$K^{\beta,\Lambda}_{lm}(p_0,\vec{p}) = \int_0^\beta dx_0\, e^{-ip_0x_0}\langle j^{(P)}_{x_0,\vec{p},l}\, j^{(P)}_{0,-\vec{p},m}\rangle + \langle\big[\hat{\jmath}^{(D)}_{\vec{p},-\vec{p}}\big]_{lm}\rangle \tag{7}$$

It is known that in general the interaction *modifies* the values of the physical quantities; for instance, the Fermi velocity v_F, the wave function renormalization Z and the vertex functions are known to depend explicitly on the interaction [12]; moreover, it was proven in [12] that v_F, Z and the vertex functions are *analytic* functions of U for $|U|$ small enough, uniformly as $\beta, |\Lambda| \to \infty$. In this Letter we prove a similar result for the conductivity. Moreover, we prove that in the thermodynamic, zero temperature and zero frequency limit, the conductivity is *universal*, i.e., it is exactly independent of U.

Theorem. *There exists a constant $U_0 > 0$ such that, for $|U| \le U_0$ and any fixed p_0, $\sigma^{\beta,\Lambda}_{lm}(p_0)$ is analytic in U uniformly in β, Λ as $\beta, |\Lambda| \to \infty$. Moreover,*

$$\lim_{p_0\to 0^+}\lim_{\beta\to\infty}\lim_{|\Lambda|\to\infty}\sigma^{\beta,\Lambda}_{lm}(p_0) = \frac{e^2}{h}\frac{\pi}{2}\delta_{lm}\ . \tag{8}$$

Note that the limit $\beta \to \infty$ is taken before the limit $p_0 \to 0^+$. In other words, the theorem says that the interaction corrections to the conductivity are negligible at frequencies $\beta^{-1} \ll p_0 \ll t$.

Proof. The idea of the proof is based on the two main ingredients: (i) *exact lattice* Ward Identities (WI) relating the current-current, vertex and 2-point functions; (ii) the fact that the interaction-dependent corrections to the Fourier transform of the current-current correlations are *differentiable* with continuous derivative (in contrast, the free part is continuous and not differentiable at zero frequency). This last property follows from the non-perturbative estimates found in [12], which we now briefly recall. The generating functional for correlations can be written in terms of a Grassmann integral:

$$e^{W(A,\lambda)} = \int P(d\psi)e^{\mathcal{V}(\psi)+(\psi,\lambda)+B(A,\psi)} \tag{9}$$

where, if $\mathbf{k} = (k_0,\vec{k})$ with k_0 the Matsubara frequency, $P(d\psi)$ is the fermionic gaussian integration for $\psi^\pm_{\mathbf{k},\sigma} = (a^\pm_{\mathbf{k},\sigma}, b^\pm_{\mathbf{k},\sigma})$, with inverse propagator

$$g^{-1}(\mathbf{k}) = -Z_0\begin{pmatrix} ik_0 & v_0\Omega^*(\vec{k})\\ v_0\Omega(\vec{k}) & ik_0\end{pmatrix}\ , \tag{10}$$

with $Z_0 = 1$, $v_0 = \frac{3}{2}t$ and $\Omega(\vec{k}) = \frac{2}{3}\sum_{j=1,2,3}e^{i\vec{k}(\vec{\delta}_j-\vec{\delta}_1)}$ (note that $g(\mathbf{k})$ is singular only at the Fermi points $\mathbf{k} = \mathbf{k}^\pm_F = (0, \frac{2\pi}{3}, \pm\frac{2\pi}{3\sqrt{3}})$). Moreover, $B(A,\psi) =$

$$\begin{aligned}&= \sum_\sigma\int_0^\beta dx_0\sum_{\vec{x}\in\Lambda}\Big[-ie\psi^+_{\mathbf{x},\sigma}\begin{pmatrix}A_0(\mathbf{x}) & 0\\ 0 & A_0(\mathbf{x}+\delta_1)\end{pmatrix}\psi^-_{\mathbf{x},\sigma}\\ &+\sum_j\big((t_{\vec{x},j}(\vec{A})-t)a^+_{(x_0,\vec{x}),\sigma}b^-_{(x_0,\vec{x}+\vec{\delta}_j),\sigma} + c.c.\big)\Big]\end{aligned} \tag{11}$$

and $(\psi,\lambda) = \int_0^\beta dx_0\sum_{\vec{x}\in\Lambda}[\psi^+_{\mathbf{x}}\lambda^-_{\mathbf{x}} + \lambda^+_{\mathbf{x}}\psi^-_{\mathbf{x}}]$. The response function $K^{\beta,\Lambda}(\mathbf{p})$ corresponds to the spatial components of the tensor $\widehat{K}_{\mu\nu}(\mathbf{p}) = \frac{\delta^2}{\delta A_\mu(\mathbf{p})\delta A_\nu(-\mathbf{p})}W(A,0)\big|_{A=0}$, with $\mu,\nu = 0,1,2$. Performing the phase transformation $\psi^\pm_{\mathbf{x}} \to e^{\pm ie\alpha_{\mathbf{x}}}\psi^\pm_{\mathbf{x}}$ in Eq.(9), we find

$$W(A+\partial\alpha, \lambda e^{ie\alpha}) = W(A,\lambda)\ , \tag{12}$$

which implies the following lattice Ward Identity [16]

$$\sum_{\mu=0}^2 p_\mu\widehat{K}_{\mu\nu}(\mathbf{p}) = 0\ , \tag{13}$$

for all $\nu \in \{0,1,2\}$. On the other hand, the functional integral Eq.(9) can be evaluated in terms of an exact Renormalization Group (RG) analysis, described in full detail in [12]. We decompose the field ψ as a sum of fields $\psi^{(k)}$, living on momentum scales $|\mathbf{k}-\mathbf{k}^\pm_F| \simeq 2^h$, with $h \le 0$ a scale label; the iterative integration of the fields on scales $h < h' \le 0$ leads to an effective theory similar to Eq.(9) with a cut-off around the Fermi points of width 2^h and with a scale dependent propagator $g^{(\le h)}(\mathbf{k})$ with the same singularity structure as Eq.(10), with Z_0 and v_0 replaced by Z_h and v_h, respectively (the effective wave function renormalization and Fermi velocity on scale h). Moreover, setting for simplicity $\lambda = 0$, at scale h the interaction $\mathcal{V}(\psi) + B(A,\psi)$ is replaced by an effective interaction $\mathcal{V}^{(\le h)}(\psi^{(\le h)}) + B^{(h)}(A,\psi^{(\le h)})$, with the *effective potential* $\mathcal{V}^{(\le h)}(\psi^{(\le h)})$ a sum of monomials in $\psi^{(\le h)}$ of arbitrary order, characterized at order n by kernels $W^{(h)}_{n,0}(\mathbf{x}_1,\dots,\mathbf{x}_n)$ that are analytic in U and decay super-polynomially in the relative distances $|\mathbf{x}_i-\mathbf{x}_j|$ on scale 2^{-h}; moreover the *effective source* is given by

$$B^{(h)}(A,\psi) = \sum_{\mu=0}^{2} Z_{\mu,h}\int \frac{d\mathbf{p}}{(2\pi)^3} A_\mu(\mathbf{p}) j_\mu(\mathbf{p}) + \bar{B}^{(h)} \quad (14)$$

where $j_0(\mathbf{p}) = -ie\sum_\sigma \int \frac{d\mathbf{k}}{(2\pi)|\mathcal{B}|}\psi^+_{\mathbf{k}+\mathbf{p},\sigma}\Gamma_0(\vec{k},\vec{p})\psi^-_{\mathbf{k},\sigma}$, $\vec{\jmath}(\mathbf{p}) = -ie\sum_\sigma \int \frac{d\mathbf{k}}{(2\pi)|\mathcal{B}|}\psi^+_{\mathbf{k}+\mathbf{p},\sigma}\vec{\Gamma}(\vec{k},\vec{p})\psi^-_{\mathbf{k},\sigma}$, $[\Gamma_0(\vec{k},\vec{p})]_{ij} = \delta_{ij}\exp\{-ip_1\delta_{i2}\}$ and

$$\vec{\Gamma}(\vec{k},\vec{p}) = \frac{2}{3}\sum_j \vec{\delta}_j \eta^j_{\vec{p}} \begin{pmatrix} 0 & -e^{-i\vec{k}(\vec{\delta}_j-\vec{\delta}_1)} \\ e^{+i(\vec{k}+\vec{p})(\vec{\delta}_j-\vec{\delta}_1)} & 0 \end{pmatrix} . \quad (15)$$

Finally, $\bar{B}^{(h)}$ is a sum of monomials in (A,ψ) of arbitrary order, characterized at order n in ψ and m in A by kernels $W^{(h)}_{n,m}(\mathbf{x}_1,\dots,\mathbf{x}_n;\mathbf{y}_1,\dots,\mathbf{y}_m)$ that are analytic in U, decay super-polynomially in the relative distances on scale 2^{-h} and are non-zero only if $m \ge 1, n \ge 0$ and $m+n \ge 3$; in particular, for all $0<\theta<1$, they satisfy the bounds (proved in [12]),

$$\int d\mathbf{x}_2\cdots d\mathbf{x}_n d\mathbf{y}_1\dots d\mathbf{y}_m\, |W^{(h)}_{n,m}(\mathbf{x}_1,\dots,\mathbf{x}_n;\mathbf{y}_1,\dots,\mathbf{y}_m)| \le (\text{const.})|e|^m 2^{(3-n-m)h}\big(1-\delta_{m,0}+|U|2^{\theta h}\big) . \quad (16)$$

The bounds Eq.(16) are *non-perturbative* (i.e., they are based on the *convergence* of the expansion for the kernels $W^{(h)}$). They are obtained by exploiting the anticommutativity properties of the Grassmann variables, via a determinant expansion and the use of the *Gram-Hadamard inequality* for determinants, see [12]. The factor $2^{\theta h}$ in the bound will play a crucial role in the following and reflects the fact that the *scaling dimension* $3-n-m$ is always negative for $n>2$. The *running coupling constants* $Z_h, v_h, Z_{\mu,h}$ satisfy recursive equations (*beta function equations*) that, due to the bound Eq.(16), lead to bounded and controlled flows, i.e., $Z(U) = \lim_{h\to-\infty} Z_h$, $Z_\mu(U) = \lim_{h\to-\infty} Z_{\mu,h}$ and $v_F(U) = \lim_{h\to-\infty} v_h$ are analytic functions of U, analytically close to their unperturbed values $Z_0 = Z_{0,0} = 1$ and $Z_{1,0} = Z_{2,0} = v_0 = \frac{3}{2}t$, see [12]. The analyticity of the kernels of the effective potential and of the $h\to-\infty$ limits of the running coupling constants implies the analyticity of the imaginary-time correlation functions (see [12]) and, similarly, the analyticity of $\sigma^{\beta,\Lambda}(p_0)$ claimed in the main theorem.

We are left with proving the universality result Eq.(8). To this aim, it is important to notice that $Z_h, v_h, Z_{\mu,h}$ are related by Ward Identities. Indeed, proceeding as in [13], we consider a reference model defined in a way similar to Eq.(9), with the important difference that the Grassmann integration $P(d\psi)$ is modified into $P_{\ge h}(d\psi)$, whose propagator differs from the original one by the presence of a smooth infrared cutoff selecting the momenta $\ge 2^h$; performing the phase transformation $\psi^\pm_{\mathbf{x}} \to e^{\pm i\alpha_{\mathbf{x}}}\psi^\pm_{\mathbf{x}}$ in this functional integral, we find the analogue of Eq.(12), which implies

$$\frac{Z_{0,h}}{Z_h} = 1 + O(U2^{\theta h}) , \qquad \frac{Z_{1,h}}{Z_h v_h} = \frac{Z_{2,h}}{Z_h v_h} = 1 + O(U2^{\theta h}) \quad (17)$$

where the corrections $O(U2^{\theta h})$ come from the symmetry breaking terms due to the infrared cut-off function. *Therefore, the effective parameters are related by exact identities*; the vertex density renormalization $Z^{(0)}_h$ is equal, up to negligible terms, to the wave function renormalization, and the current renormalization $Z^{(1)}_h$ is equal to the product of the effective velocity and the wave function renormalization [17].

We can write $K_{\mu\nu} = K^{(P)}_{\mu\nu} + K^{(D)}_{\mu\nu}$, where the two terms in the right hand side correspond to the paramagnetic and diamagnetic contributions to $K_{\mu\nu}$, see Eq.(7). Note that $\widehat{K}^{(D)}_{\mu\nu}(p_0,\vec{0})$ is independent of p_0; using Eq.(16), we find that $|\widehat{K}^{(D)}_{\mu\nu}(p_0,\vec{0})| \le (\text{const.})|e|^2\sum_{h=-\infty}^{0} 2^h$, which is finite. On the other hand $K^{(P)}_{\mu\nu}(\mathbf{x}) =$

$$= \sum_{h=-\infty}^{0}\Big[2e^2\frac{Z_{\mu,h}Z_{\nu,h}}{(Z_h)^2}\int\frac{d\mathbf{k}d\mathbf{p}}{(2\pi)^2|\mathcal{B}|^2}e^{i\mathbf{p}\mathbf{x}}F_h(\mathbf{k},\mathbf{p})\cdot \quad (18)$$
$$\cdot\mathrm{Tr}\{\Gamma_\mu(\vec{k},\vec{p})C_h(\mathbf{k})\Gamma_\nu(\vec{k}+\vec{p},-\vec{p})C_h(\mathbf{k}+\mathbf{p})\} + H^{(h)}_{\mu\nu}(\mathbf{x})\Big] ,$$

where the first term corresponds to the zero-th order in U in renormalized perturbation theory ($F_h(\mathbf{k},\mathbf{p})$ is a suitable smooth cutoff function constraining $|\mathbf{k}-\mathbf{k}^\pm_F|$ and $|\mathbf{k}-\mathbf{k}^\pm_F+\mathbf{p}|$ to be $\simeq 2^h$ and such that $\sum_{h=-\infty}^{0}F_h(\mathbf{k},\mathbf{p}) = 1$; moreover, $Z_hC_h^{-1}(\mathbf{k})$ is given by Eq.(10) with Z_0, v_0 replaced by Z_h, v_h) and, for all $N\ge 0$ and suitable constants C_N,

$$|H^{(h)}_{\mu\nu}(\mathbf{x})| \le C_N|U|\frac{2^{(4+\theta)h}}{1+(2^h|\mathbf{x}|)^N} . \quad (19)$$

As compared to the zero-th order contribution to $K^{(P)}_{\mu\nu}$, the dimensional bound on $H^{(h)}_{\mu\nu}$ has an extra factor $2^{\theta h}$, following again from Eq.(16). From Eq.(19),

$$|K^{(P)}_{\mu\nu}(\mathbf{x}-\mathbf{y})| \le (\text{const.})\frac{1}{1+|\mathbf{x}-\mathbf{y}|^4} , \quad (20)$$

that is, $K_{\mu\nu}(\mathbf{x})$ is absolutely integrable and, therefore, its Fourier transform in the thermodynamic and zero temperature limit is continuous at $\mathbf{p}=\mathbf{0}$. Combining this remark with the WI Eq.(13), we find that $\widehat{K}_{\mu\nu}(\mathbf{0}) = 0$. In fact, setting, e.g., $p_2 = 0$, $\widehat{K}_{11}(p_0, p_1, 0) = (-p_0/p_1)\widehat{K}_{01}(p_0, p_1, 0)$; taking first the limit $p_0 \to 0$ and then $p_1 \to 0$ in the right hand side, we get $\widehat{K}_{11}(\mathbf{0}) = 0$; proceeding analogously, we find that $\widehat{K}_{\mu\nu}(\mathbf{0}) = 0$ for all $\mu, \nu \in \{0, 1, 2\}$.

On the other hand Eq.(17) implies that $\frac{Z_{1,h}}{Z_h} = v_F(U) + O(U2^{\theta h})$, so that $K^{(P)}_{\mu\nu}(\mathbf{x}) = K^{(P;0)}_{\mu\nu}(\mathbf{x}) + K^{(P;1)}_{\mu\nu}(\mathbf{x})$ where $K^{(P;0)}_{\mu\nu}(\mathbf{x})$ is the paramagnetic response function for the model with Hamiltonian $H^0_\Lambda(\frac{2}{3}v_F(U))$ and $|K^{(P;1)}_{\mu\nu}(\mathbf{x})| \leq (\text{const.})|U|(1 + |\mathbf{x}|^{4+\theta})^{-1}$, with $0 < \theta < 1$. Therefore, the Fourier transform of $K^{(P;1)}_{\mu\nu}$ is *differentiable* in $\mathbf{p}$ and its derivative is continuous at $\mathbf{p} = \mathbf{0}$. A similar decomposition can be performed in the diamagnetic term, so that, defining $K^{(1)}_{\mu\nu} = K^{(P;1)}_{\mu\nu} + K^{(D;1)}_{\mu\nu}$ and using the WI Eq.(13), $\sum_{\mu=0}^{2} p_\mu \widehat{K}^{(1)}_{\mu\nu}(\mathbf{p}) = 0$ from which, setting, e.g., $p_2 = 0$, we obtain $\widehat{K}^{(1)}_{11}(p_0, p_1, 0) = (p_0/p_1)^2 \widehat{K}^{(1)}_{00}(p_0, p_1, 0)$; deriving with respect to p_0 both sides and taking first the limit $p_0 \to 0$ and next $p_1 \to 0$ in the right hand side, we get $\partial_{p_0}\widehat{K}^{(1)}_{11}(\mathbf{0}) = 0$; proceeding analogously, we find that $\partial_{p_\rho}\widehat{K}^{(1)}_{\mu\nu}(\mathbf{0}) = 0$ for all $\rho, \mu, \nu \in \{0, 1, 2\}$. *Note the crucial role played by the continuity of the derivatives of $\hat{K}^{(1)}_{\mu\nu}$, which allowed us to exchange the zero frequency and zero momentum limits*, as compared to the order in Eq.(8).

In order to compute the conductivity, we are left with the contribution associated to a free theory with Fermi velocity equal to $v_F(U)$ that, for the 11 component, setting $v = v_F(U)$, reads:

$$\sigma_{11} = \lim_{p_0 \to 0^+} \lim_{\beta \to \infty} \lim_{|\Lambda| \to \infty} \sigma^{\beta,\Lambda}_{11}(p_0) = 2e^2 v^2 \lim_{p_0 \to 0^+} \int \frac{dk_0}{2\pi} \cdot$$

$$\cdot \int_{\mathcal{B}} \frac{d\vec{k}}{|\mathcal{B}|} \mathrm{Tr}\Big\{ \frac{S(\mathbf{k}) - S(\mathbf{k} + p_0)}{p_0} \Gamma_1(\vec{k}, \vec{0}) S(\mathbf{k}) \Gamma_1(\vec{k}, \vec{0}) \Big\} \; .$$

The latter integral is not uniformly convergent in p_0; in particular, it is well known that one cannot exchange the limit with the integral [14]. The integral can be evaluated explicitly (using residues to compute the integral over k_0) and leads to Eq.(13). A similar computation shows that $\sigma_{22} = \sigma_{11}$ and that the off-diagonal terms vanish. ∎

The above analysis can be extended to the case of long range electromagnetic interactions; in such case the wave function, density and current renormalizations have a strong (anomalous) power law dependence on the momentum and the Fermi velocity increases up to the speed of light [11, 13]. WIs similar to Eq.(17) are still valid and imply that, even if the effective parameters are strongly momentum dependent, *the conductivity only depends weakly on the frequency* in the optical range.

In conclusion, we rigorously proved the non existence of corrections to the zero temperature and zero frequency limit of the graphene conductivity due to weak short range interactions. The proof is based on a combination of constructive Renormalization Group methods with exact lattice Ward identities. Remarkably, this is one of the very few examples of universality in condensed matter that can be established on firm mathematical grounds.

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