Anomalous behavior in an effective model of graphene with Coulomb interactions

A. Giuliani

Dipartimento di Matematica, Università di Roma Tre,
L.go S. L. Murialdo 1, 00146 Roma - Italy

V. Mastropietro

Dipartimento di Matematica, Università di Roma Tor Vergata,
V.le della Ricerca Scientifica, 00133 Roma - Italy

M. Porta

Dipartimento di Fisica, Università di Roma La Sapienza,
P.le Aldo Moro 2, 00185 Roma - Italy

We perform a Renormalization Group analysis for an effective model of graphene with long range non-instantaneous interactions. We show that, at all orders in the renormalized expansion, the wave function renormalization diverges in the infrared with an anomalous critical exponent and the interacting Fermi velocity flows to a limit value generically smaller than the speed of light. Ward Identities imply that the effective charges have a bounded but non trivial flow.

1. INTRODUCTION

Graphene, a monoatomic graphitic film that has been recently experimentally realized [16, 17], has attracted an enormous interest in recent years. Its properties are quite different from most conventional quasi–two dimensional electron gases, because of the peculiar quasi–particles dispersion relation, which closely resembles the one of massless Dirac fermions in $2+1$ dimensions [20]. Already in the absence of interactions, the system displays highly unusual features, and several of them have been experimentally observed.

Most of the theoretical research is then focused on the role of the electron-electron interaction. The effect of a weak *short* range interaction in graphene is quite well understood: it turns out that the behavior is qualitatively similar to the free one, except that the Fermi velocity and the wave function renormalization are renormalized by a finite amount. This was expected on the basis of a power counting analysis performed in an effective continuum model [12]; it has been recently rigorously proven in [9, 10] by establishing the *convergence* of the perturbative series using methods coming from constructive Quantum Field Theory and taking into full account the lattice effects (*i.e.* by considering the Hubbard model on the honeycomb lattice).

The situation in the presence of *long* range interactions is much more unclear. Early works considered an effective model of massless Dirac fermions (with velocity $v \neq c$) interacting with a photon field; the model is not Lorentz invariant. In the case of *instantaneous* interactions (*i.e.* for $c = \infty$), it was predicted that the Fermi velocity is logarithmically divergent, see [13, 15], while the wave function renormalization remains finite. On the other hand, in the case of a *retarded* interaction (corresponding to c finite), it was predicted in [11] that, at very small momenta, the wave function renormalization diverges as a power law with a non-universal critical exponent (there is an *anomalous dimension*) and that the interacting Fermi velocity tends to the speed of light. In both cases, the Fermi velocity increases at low momenta up to its *maximal possible value* (respectively, infinity or c); therefore, in the retarded case, *Lorentz symmetry is dynamically restored.* The above two features have been confirmed in all the subsequent analyses, mostly performed on the istantaneous case (see *e.g.* [5] for a review).

In the present paper we analyze the retarded case. The model we consider is very similar to the one in [11], the main difference being the choice of the ultraviolet cut-off (see Remark 1 after (4.17)). We perform an exact RG analysis of the retarded case, at all orders in the renormalized expansion, using methods developed in the framework of constructive QFT (see [14] for an updated introduction). By taking into account the irrelevant terms and the effects due to the momentum cut-offs, we can confirm at all orders the prediction of anomalous behavior for the 2-point function (the wave function renormalization diverges with an anomalous critical exponent), but we find that the Fermi velocity tends to a limit value *smaller* than the speed of light and interaction-dependent, unless a fine tuning of the bare parameters is made.

The paper is organized as follows: in Section 2 we introduce the model and describe our main results; in Section 3 we describe how to evaluate the functional integrals defining the partition function and the correlations of our model in terms of an exact RG scheme (details are discussed in Appendices A and C); in Section 4 we describe the infrared flow of the effective couplings and prove the emergence of an effective Fermi velocity different from the speed of light (the explicit lowest order computations of the beta function are presented in Appendix B); in Section 5 we derive the Ward Identity allowing us to control the flow of the effective charges (details are discussed in Appendix D); finally, in Section 6 we draw the conclusions.

2. THE MODEL

The model we consider describes Dirac fermions in $2+1$ dimensions propagating with velocity $v < c$, and interacting with a $3+1$ dimensional photon field in the Feynman gauge. We will not be concerned with the instantaneous case ($c \to \infty$); therefore, from now on, for notational simplicity, we shall fix units such that $\hbar = c = 1$. The correlations can be computed in terms of derivatives of the following euclidean functional integral:

$$e^{\mathcal{W}(J,\phi)} = \int P(d\psi)P(dA)e^{V(A,\psi)+B(J,\phi)} \tag{2.1}$$

with, setting $\mathbf{x} = (x_0, \vec{x})$ and $\vec{x} = (x_1, x_2)$ (repeated indexes are summed; greek and latin labels run respectively from 0 to 2, 1 to 2),

$$\begin{aligned} V(A,\psi) &:= \int_\Lambda d\mathbf{x}\, \left[e\, j_{\mu,\mathbf{x}} A_{\mu,\mathbf{x}} - \nu_\mu A_{\mu,\mathbf{x}} A_{\mu,\mathbf{x}}\right] \\ B(J,\phi) &:= \int_\Lambda d\mathbf{x}\, \left[j_{\mu,\mathbf{x}} J_{\mu,\mathbf{x}} + \phi_\mathbf{x} \overline{\psi}_\mathbf{x} + \overline{\phi}_\mathbf{x} \psi_\mathbf{x}\right] , \end{aligned} \tag{2.2}$$

where Λ is a three dimensional box of volume $|\Lambda| = L^3$ with periodic boundary conditions (playing the role of an infrared cutoff, to be eventually removed), the couplings e, ν_μ are real and $\nu_1 = \nu_2$; the couplings ν_μ are *counterterms* to be fixed so that the photon mass is vanishing in the deep infrared: it is indeed well known that in quantum electrodynamics with momentum regularization the bare photon mass must be chosen to be > 0, in order to ensure that the dressed mass computed in the framework of Polchinski's [18] or Gallavotti's [6] RG is zero, see e.g. [4]. Moreover, $\overline{\psi}_\mathbf{x}$, $\psi_\mathbf{x}$ are 4-components *Grassmann spinors*, and the μ-th component $j_{\mu,\mathbf{x}}$ of the current is defined as:

$$j_{0,\mathbf{x}} = i\overline{\psi}_\mathbf{x}\gamma_0\psi_\mathbf{x} , \qquad \vec{j}_\mathbf{x} = iv\, \overline{\psi}_\mathbf{x}\vec{\gamma}\psi_\mathbf{x} , \tag{2.3}$$

where γ_μ are euclidean gamma matrices, satisfying the anticommutation relations $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu,\nu}$. The symbol $P(d\psi)$ denotes a Grassmann integration with propagator

$$g^{(\le 0)}(\mathbf{x}) := \int \frac{d\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\mathbf{x}} \frac{ik_0\gamma_0 + iv\vec{k}\cdot\vec{\gamma}}{k_0^2 + v^2|\vec{k}|^2} \chi_0(\mathbf{k}) . \tag{2.4}$$

where $(2\pi)^{-3}\int d\mathbf{k}$ is a shorthand for $|\Lambda|^{-1}\sum_{\mathbf{k}=2\pi\mathbf{n}/L}$ with $\mathbf{n}\in\mathbb{Z}^3$, and $\chi_0(\mathbf{k})=\chi(|\mathbf{k}|)$ plays the role of a prefixed ultraviolet cutoff (here $\chi(t)$ is a non increasing C^∞ function from $\mathbb{R}^+$ to $[0,1]$ such that $\chi(t)=1$ if $t\le 1$ and $\chi(t)=0$ if $t\ge M>1$). Finally, $A_{\mu,\mathbf{x}}$ are gaussian variables and $P(dA)$ is a gaussian integration with propagator

$$w^{(\le 0)}(\mathbf{x}):=\int\frac{d\mathbf{p}}{(2\pi)^3}\,e^{i\mathbf{p}\mathbf{x}}\frac{\chi_0(\mathbf{p})}{2|\mathbf{p}|}=\int\frac{d\mathbf{p}\,dp_3}{(2\pi)^4}\,e^{i\mathbf{p}\mathbf{x}}\frac{\chi_0(\mathbf{p})}{\mathbf{p}^2+p_3^2}\,. \tag{2.5}$$

In the following we will describe a RG analysis of $\mathcal{W}(J,\phi)$ that will allow us to define a renormalized expansion for the correlations of our model, well defined at all orders. Let us denote by $\langle\ldots\rangle=\lim_{|\Lambda|\to\infty}\langle\ldots\rangle_\Lambda$ the expectation value with respect to the interaction (2.2) in the infinite volume limit; our main result can be summarized as follows.

Main result. *There exists a choice of the counterterms $\nu_0>0$ and $\nu_1=\nu_2>0$,*

$$\nu_\mu=\sum_{n\ge1}c_{\mu,n}\,e^{2n}\;,\qquad |c_{\mu,n}|\le(\text{const.})^n n!\;, \tag{2.6}$$

such that, for $\mathbf{k}$ small,

$$\langle\psi_{\mathbf{k}}\overline{\psi}_{\mathbf{k}}\rangle=\frac{1}{Z(\mathbf{k})}\frac{ik_0\gamma_0+iv(\mathbf{k})\vec{k}\cdot\vec{\gamma}}{k_0^2+v(\mathbf{k})^2|\vec{k}|^2}(1+B(\mathbf{k}))\;, \tag{2.7}$$

with:

$$Z(\mathbf{k})\sim|\mathbf{k}|^{-\eta}\;,\qquad v(\mathbf{k})-v_{eff}\sim(1-v)|\mathbf{k}|^{\tilde\eta} \tag{2.8}$$

and

$$\eta=\frac{e^2}{12\pi^2}+e^4H_1(e)\;,\qquad \tilde\eta=\frac{2e^2}{5\pi^2}+e^4H_2(e)\;,$$
$$v_{eff}=1-F(v)\Big(\frac{e^2}{6\pi^2}+e^4H_3(e)\Big)\;, \tag{2.9}$$

where:

$$F(v)=\frac58\Big[\Big(\frac{1}{2v^2}-2\Big)\frac{\xi_0-\arctan\xi_0}{\xi_0^3}+\frac{1}{2v^2}\frac{\arctan\xi_0}{\xi_0}\Big]\;,\qquad \xi_0:=\frac{\sqrt{1-v^2}}{v}\;, \tag{2.10}$$

and $H_i=\sum_{n\ge0}a_{n,i}e^{2n}$ are formal power series in e, with coefficients of order n bounded as $|a_{i,n}|\le(\text{const.})^n n!$. Moreover, the error term $B(\mathbf{k})=\sum_{n\ge1}b_n(\mathbf{k})e^{2n}$ is a formal (renormalized) series in $e, Z(\mathbf{k}), v(\mathbf{k})$, with coefficients of order n bounded as $|b_n(\mathbf{k})|\le$ $(\text{const.})^n n!|\mathbf{k}|^{1/2}$.

Remarks.

1. The function $F(v)$ in (2.10) is positive for $0 < v < 1$ and vanishes linearly as $v \to 1^-$, in such a way that $\lim_{v\to 1^-} F(v)(1-v)^{-1} = 1$.

2. The bound on $B(\mathbf{k})$ (and in particular the factor $|\mathbf{k}|^{1/2}$) is not expected to be optimal: it just tells us that $B(\mathbf{k})$ vanishes for $\mathbf{k} \to \mathbf{0}$ and is not necessarily analytic in $\mathbf{k}$ around $\mathbf{k} = \mathbf{0}$.

3. The coefficients $c_{\mu,n}$, $a_{i,n}$ and $b_n(\mathbf{k})$ of the resummed power series for ν_μ, H_i and $B(\mathbf{k})$ depend on e and v. They admit bounds that are uniform for v close to 1, but not for v close to 0: in particular our analysis breaks down for $v \to 0$.

4. It turns out that the counterterms ν_μ are positive (see Section 4 and Appendix B); this allows us to interpret them as *bare* photon mass terms, fixed in such a way that the dressed photon mass is vanishing.

5. A similar result is valid even in the case that the bare interaction involves two different charges, e_0 and e_1, describing the couplings of the photon field with the temporal and spatial components of the current. If $e = (e_0 + e_1)/2$ and $e_0 - e_1 = O(e^3)$, we find that $v_{eff} = 1 - (e^2/6\pi^2)F(v) + (5/6)(e_0 - e_1)/e + O(e^4)$ and it is of course possible to fine tune the bare parameters e_0 and e_1 in such a way that $v_{eff} = 1$.

6. The effective continuum model we considered is obviously not fundamental. A more realistic model for single layer graphene could be obtained by considering tight binding electrons hopping on the honeycomb lattice, whose lattice currents are coupled to a 3D photon field. We expect that, in the weak coupling regime, such tight binding model is asymptotic in the infrared to the model (2.1) provided that the bare parameters e_0, e_1, v (see previous item) are properly chosen.

From (2.8), (2.9) we see that the wave function renormalization diverges with a critical exponent, which is expressed by a power series with finite coefficients at all orders in perturbation theory. At lowest order, our result coincides with the prediction of [11]. The Fermi velocity approaches its limiting value v_{eff} with an anomalous power law behavior, with an anomalous exponent that is expressed by a power series with finite coefficients at all orders in perturbation theory. Finally, we find that in general the Fermi velocity *does not* flow to the speed of light, but to a smaller value $v_{eff} < 1$. Therefore, Lorentz symmetry does not dynamically emerge.

3. RENORMALIZATION GROUP ANALYSIS

In this section we show how to evaluate the functional integral (2.1); the integration will be performed in an iterative way, starting from the momenta "close" to the ultraviolet cutoff moving towards smaller momentum scales. At the n-th step of the iteration the functional integral (2.1) is rewritten as an integral involving only the momenta smaller than a certain value, proportional to M^{-n}, with $M > 1$ a constant (to be chosen sufficiently close to 1), and both the propagators and the interaction will be replaced by "effective" ones; they differ from their "bare" counterparts because the physical parameters appearing in their definitions (the Fermi velocity v, the charge e, and the "photon mass" ν_μ) are *renormalized* by the integration of the momenta on higher scales.

We start from the following identity:

$$\chi_0(\mathbf{k}) = \sum_{h=-\infty}^{0} f_h(\mathbf{k}) \qquad f_h(\mathbf{k}) := \chi_h(\mathbf{k}) - \chi_{h-1}(\mathbf{k}) \;, \qquad \chi_h(\mathbf{k}) := \chi(M^{-h}|\mathbf{k}|) \;; \tag{3.1}$$

let $\psi = \sum_{h=-\infty}^{0} \psi^{(h)}$ and $A = \sum_{h=-\infty}^{0} A^{(h)}$, where $\{\psi^{(h)}\}_{h\leq 0}$, $\{A^{(h)}\}_{h\leq 0}$ are independent free fields with the same support of the functions f_h introduced above.

We evaluate the functional integral (2.1) by integrating the fields in an iterative way starting from $\psi^{(0)}$, $A^{(0)}$; for simplicity, we start by treating the case $J = \phi = 0$. We define $\mathcal{V}^{(0)}(A,\psi) := V(A,\psi)$ and we want to inductively prove that after the integration of $\psi^{(0)}, A^{(0)}, \ldots, \psi^{(h+1)}, A^{(h+1)}$ we can rewrite:

$$e^{\mathcal{W}(0,0)} = e^{|\Lambda| E_h} \int P(d\psi^{(\leq h)}) P(dA^{(\leq h)}) e^{\mathcal{V}^{(h)}(A^{(\leq h)}, \sqrt{Z_h}\psi^{(\leq h)})} \;, \tag{3.2}$$

where $P(d\psi^{(\leq h)})$ and $P(dA^{(\leq h)})$ have propagators

$$g^{(\leq h)}(\mathbf{k}) = \frac{\chi_h(\mathbf{k})}{\tilde{Z}_h(\mathbf{k})} \frac{i\gamma_0 k_0 + i\tilde{v}_h(\mathbf{k})\vec{k}\cdot\vec{\gamma}}{k_0^2 + \tilde{v}_h(\mathbf{k})^2|\vec{k}|^2} \;, \qquad w^{(\leq h)}(\mathbf{p}) = \frac{\chi_h(\mathbf{p})}{2|\mathbf{p}|} \;, \tag{3.3}$$

$\mathcal{V}^{(h)}$ has the form

$$\mathcal{V}^{(h)}(\psi, A) = \sum_{\substack{n,m\geq 0\\ n+m\geq 1}} \sum_{\underline{\rho},\underline{\mu}} \int \frac{d\mathbf{k}_1}{(2\pi)^3} \cdots \frac{d\mathbf{k}_{2n}}{(2\pi)^3} \frac{d\mathbf{p}_1}{(2\pi)^3} \cdots \frac{d\mathbf{p}_m}{(2\pi)^3} \prod_{i=1}^{n} \overline{\psi}_{\mathbf{k}_{2i-1},\rho_{2i-1}} \psi_{\mathbf{k}_{2i},\rho_{2i}} \cdot$$
$$\cdot \prod_{i=1}^{m} A_{\mu_i,\mathbf{p}_1} W^{(h)}_{m,n,\underline{\rho},\underline{\mu}}(\{\mathbf{k}_i\},\{\mathbf{p}_j\}) \delta\left(\sum_{j=1}^{m} \mathbf{p}_j + \sum_{i=1}^{2n} (-1)^i \mathbf{k}_i\right) \;, \tag{3.4}$$

and E_h, $\tilde{Z}_h(\mathbf{k})$, $\tilde{v}_h(\mathbf{k})$ and the kernels $W^{(h)}_{m,n,\underline{\rho},\underline{\mu}}$ will be defined recursively.

In order to inductively prove (3.2), we split $\mathcal{V}^{(h)}$ as $\mathcal{L}\mathcal{V}^{(h)}+\mathcal{R}\mathcal{V}^{(h)}$, where $\mathcal{R}=1-\mathcal{L}$ and $\mathcal{L}$, the *localization operator*, is a linear operator on functions of the form (3.4), defined by its action on the kernels $W^{(h)}_{m,n,\underline{\rho},\underline{\mu}}$ in the following way:

$$\begin{aligned}
&\mathcal{L}W^{(h)}_{0,1,\underline{\rho}}(\mathbf{k}) := W^{(h)}_{0,1,\underline{\rho}}(\mathbf{0})+\mathbf{k}\partial_{\mathbf{k}}W^{(h)}_{0,1,\underline{\rho}}(\mathbf{0})\ ,\\
&\mathcal{L}W^{(h)}_{1,1,\underline{\rho},\mu}(\mathbf{p},\mathbf{k}) := W^{(h)}_{1,1,\underline{\rho},\mu}(\mathbf{0},\mathbf{0})\ ,\\
&\mathcal{L}W^{(h)}_{2,0,\underline{\mu}}(\mathbf{p}) := W^{(h)}_{2,0,\underline{\mu}}(\mathbf{0})+\mathbf{p}\partial_{\mathbf{p}}W^{(h)}_{2,0,\underline{\mu}}(\mathbf{0})\ ,\qquad \mathcal{L}W^{(h)}_{3,0,\underline{\mu}}(\mathbf{p}_1,\mathbf{p}_2) := W^{(h)}_{3,0,\underline{\mu}}(\mathbf{0},\mathbf{0})\ ,
\end{aligned}\tag{3.5}$$

and $\mathcal{L}W^{(h)}_P := 0$ otherwise. As a consequence of the symmetries of our model it turns out that

$$\begin{aligned}
&W^{(h)}_{1,0,\mu}(\mathbf{0})=0\ ,\qquad W^{(h)}_{3,0,\underline{\mu}}(\mathbf{0},\mathbf{0})=0\ ,\qquad W^{(h)}_{0,1,\underline{\rho}}(\mathbf{0})=0\ ,\\
&\hat{W}^{(h)}_{2,0,\underline{\mu}}(\mathbf{0})=-\delta_{\mu_1,\mu_2}M^h\nu_{\mu_1,h}\ ,\qquad \partial_{\mathbf{p}}\hat{W}^{(h)}_{2,0,\underline{\mu}}(\mathbf{0})=0\ ,
\end{aligned}\tag{3.6}$$

and, moreover, that

$$\begin{aligned}
&\overline{\psi}_{\mathbf{k}}\,\mathbf{k}\partial_{\mathbf{k}}W^{(h)}_{0,1}(\mathbf{0})\psi_{\mathbf{k}}=-iz_{\mu,h}k_\mu\overline{\psi}_{\mathbf{k}}\gamma_\mu\psi_{\mathbf{k}}\\
&\overline{\psi}_{\mathbf{k}+\mathbf{p}}W^{(h)}_{1,1,\mu}(\mathbf{0},\mathbf{0})\psi_{\mathbf{k}}A_{\mu,\mathbf{p}}=i\lambda_{\mu,h}\overline{\psi}_{\mathbf{k}+\mathbf{p}}\gamma^\mu\psi_{\mathbf{k}}A_{\mu,\mathbf{p}}\ ,
\end{aligned}\tag{3.7}$$

with $z_{\mu,h}$, $\lambda_{\mu,h}$ real, and $z_{1,h}=z_{2,h}$, $\lambda_{1,h}=\lambda_{2,h}$. We can *renormalize* $P(d\psi^{(\le h)})$ by adding to the exponent of its gaussian weight the local part of the quadratic terms in the fermionic fields; we get that

$$\int P(d\psi^{(\le h)})P(dA^{(\le h)})e^{\mathcal{V}^{(h)}(A,\sqrt{Z_h}\psi)}=e^{|\Lambda|t_h}\int \widetilde{P}(d\psi^{(\le h)})P(dA^{(\le h)})e^{\widetilde{\mathcal{V}}^{(h)}(A,\sqrt{Z_h}\psi)}\ ,\tag{3.8}$$

where t_h takes into account the different normalization of the two functional integrals, $\widetilde{\mathcal{V}}^{(h)}$ is given by

$$\widetilde{\mathcal{V}}^{(h)}(A,\psi)=\mathcal{V}^{(h)}(A,\psi)+\int\frac{d\mathbf{k}}{(2\pi)^3}\,iz_{\mu,h}k_\mu\overline{\psi}_{\mathbf{k}}\gamma_\mu\psi_{\mathbf{k}}=:\mathcal{V}^{(h)}(A,\psi)-\mathcal{L}_\psi\mathcal{V}^{(h)}(A,\psi)\ ,\tag{3.9}$$

and $\widetilde{P}(d\psi^{(\le h)})$ has propagator equal to

$$\tilde{g}^{(\le h)}(\mathbf{k})=\frac{\chi_h(\mathbf{k})}{\tilde{Z}_{h-1}(\mathbf{k})}\frac{i\gamma_0k_0+i\tilde{v}_{h-1}(\mathbf{k})\vec{k}\cdot\vec{\gamma}}{k_0^2+\tilde{v}_{h-1}(\mathbf{k})^2|\vec{k}|^2}\ ,\tag{3.10}$$

with

$$\tilde{Z}_{h-1}(\mathbf{k})=\tilde{Z}_h(\mathbf{k})+Z_hz_{0,h}\chi_h(\mathbf{k})\ ,\qquad \tilde{Z}_{h-1}(\mathbf{k})\tilde{v}_{h-1}(\mathbf{k})=\tilde{Z}_h(\mathbf{k})\tilde{v}_h(\mathbf{k})+Z_hz_{1,h}\chi_h(\mathbf{k})\ .\tag{3.11}$$

After this, defining $Z_{h-1} := \tilde{Z}_{h-1}(\mathbf{0})$, we *rescale* the fermionic field so that

$$\widetilde{\mathcal{V}}^{(h)}(A, \sqrt{Z_h}\psi) = \hat{\mathcal{V}}^{(h)}(A, \sqrt{Z_{h-1}}\psi) \; ; \tag{3.12}$$

therefore, setting

$$v_{h-1} := \tilde{v}_{h-1}(\mathbf{0}) \; , \qquad e_{0,h} := \frac{Z_h}{Z_{h-1}}\lambda_{0,h} \; , \qquad e_{1,h}v_{h-1} = e_{2,h}v_{h-1} := \frac{Z_h}{Z_{h-1}}\lambda_{1,h} \; , \tag{3.13}$$

we have that:

$$\mathcal{L}\hat{\mathcal{V}}^{(h)}(A^{(\le h)}, \sqrt{Z_{h-1}}\,\psi^{(\le h)}) = \int_\Lambda d\mathbf{x} \left(Z_{h-1}e_{\mu,h}j^{(\le h)}_{\mu,\mathbf{x}}A^{(\le h)}_{\mu,\mathbf{x}} - M^h\nu_{\mu,h}A^{(\le h)}_{\mu,\mathbf{x}}A^{(\le h)}_{\mu,\mathbf{x}} \right) \; , \tag{3.14}$$

where

$$j^{(\le h)}_{0,\mathbf{x}} := i\overline{\psi}^{(\le h)}_{\mathbf{x}}\gamma_0\psi^{(\le h)}_{\mathbf{x}} \; , \qquad \vec{j}^{(\le h)}_{\mathbf{x}} := iv_{h-1}\,\overline{\psi}^{(\le h)}_{\mathbf{x}}\vec{\gamma}\psi^{(\le h)}_{\mathbf{x}} \; . \tag{3.15}$$

After this rescaling, we can rewrite (3.8) as

$$\int P(d\psi^{(\le h)})P(dA^{(\le h)})e^{\mathcal{V}^{(h)}(A,\sqrt{Z_h}\psi)} = e^{|\Lambda|t_h} \int P(d\psi^{(\le h-1)})P(dA^{(\le h-1)}) \cdot \tag{3.16}$$
$$\cdot \int P(d\psi^{(h)})P(dA^{(h)})e^{\hat{\mathcal{V}}^{(h)}(A^{(\le h-1)}+A^{(h)},\sqrt{Z_{h-1}}(\psi^{(\le h-1)}+\psi^{(h)}))} \; ,$$

where $\psi^{(\le h-1)}, A^{(\le h-1)}$ have propagators given by (3.3) (with h replaced by $h-1$) and $\psi^{(h)}, A^{(h)}$ have propagators given by

$$\frac{g^{(h)}(\mathbf{k})}{Z_{h-1}} = \frac{\tilde{f}_h(\mathbf{k})}{Z_{h-1}}\frac{i\gamma_0k_0 + i\tilde{v}_{h-1}(\mathbf{k})\vec{k}\cdot\vec{\gamma}}{k_0^2 + \tilde{v}_{h-1}(\mathbf{k})^2|\vec{k}|^2}, \qquad w^{(h)}(\mathbf{p}) = \frac{f_h(\mathbf{p})}{2|\mathbf{p}|}$$
$$f_h(\mathbf{k}) = \chi_h(\mathbf{k}) - \chi_{h-1}(\mathbf{k}) \; , \qquad \tilde{f}_h(\mathbf{k}) = \frac{Z_{h-1}}{\tilde{Z}_{h-1}(\mathbf{k})}f_h(\mathbf{k}) \; . \tag{3.17}$$

At this point, we can integrate the scale h and, defining

$$e^{\mathcal{V}^{(h-1)}(A,\sqrt{Z_{h-1}}\psi)+|\Lambda|\tilde{E}_h} := \int P(d\psi^{(h)})P(dA^{(h)})e^{\hat{\mathcal{V}}^{(h)}(A+A^{(h)},\sqrt{Z_{h-1}}(\psi+\psi^{(h)}))} \; , \tag{3.18}$$

our inductive assumption (3.2) is reproduced at the scale $h-1$ with $E_{h-1} := E_h + t_h + \tilde{E}_h$.

The integration in (3.18) is performed by expanding in series the exponential in the r.h.s. (which involves interactions of any order in ψ and A, as apparent from (3.4)), and integrating term by term with respect to the gaussian integration $P(d\psi^{(h)})P(dA^{(h)})$. This procedure gives rise to an expansion for the effective potentials $\mathcal{V}^{(h)}$ (and to an analogous expansion for the correlations) in terms of the renormalized parameters $\{e_{\mu,k}, \nu_{\mu,k}, Z_{k-1}, v_{k-1}\}_{h<k\le 0}$, which can be conveniently represented as a sum over *Feynman graphs* according to rules that will

be explained in Appendix A. Note that such *renormalized* expansion is significantly different from the power series expansion in the bare couplings e, ν_μ; while the latter is plagued by *logarithmic divergences*, the former is *order by order finite*. Let $W^{N;(h)}_{m,n,\underline{\rho},\underline{\mu}}$ be the N-th order contribution in renormalized perturbation theory to the kernel $W^{(h)}_{m,n,\underline{\rho},\underline{\mu}}$ in (3.4); as proved in Appendix A, the following result is valid.

Theorem 3.1 **($N!$ bound)** *Let* $\bar\varepsilon_h = \max_{h<k\le 0}\{|e_{\mu,k}|, |\nu_{\mu,k}|\}$*; if* $Z_k/Z_{k-1} \le e^{C\bar\varepsilon_h^2}$*,* $C^{-1} \le v_{k-1} \le 1$*, for all* $h < k \le 0$ *and a suitable constant* $C > 0$*, then*

$$||W^{N;(h)}_{m,n,\underline{\rho},\underline{\mu}}|| \le (\text{const.})^N \bar\varepsilon_h^N \left(\frac{N}{2}\right)!\; M^{h(3-m-2n)}\;. \tag{3.19}$$

The factor $3-2n-m$ in (3.19) is referred to as the *scaling dimension* of the kernel with $2n$ external fermionic fields and m external bosonic fields; according to the usual RG therminology, kernels with positive, vanishing or negative scaling dimensions are called *relevant, marginal* or *irrelevant* operators, respectively. Notice that, if we tried to expand the effective potential in terms of the bare couplings e, ν_μ, the N-th order contributions in this "naive" perturbation series could not be bounded as in (3.19), but rather by the r.h.s. of (3.19) times $|h|^N$, an estimate which blows up order by order as $h \to -\infty$.

By comparing (2.1) and (2.2) with (3.2), (3.4) and (3.14), we see that the integration of the fields living on momentum scales $\ge M^h$ produces an *effective theory* very similar to the original one, modulo the presence of a new propagator, involving a renormalized velocity v_h and a renormalized wave function Z_h, and the presence of a modified interaction $\mathcal{V}^{(h)}$. The lack of Lorentz symmetry in our model (implied by the fact that $v \ne 1$) has two main effects: (1) the Fermi velocity has a non trivial flow; (2) the marginal terms in the effective potential are defined in terms of *two* charges, namely $e_{0,h}$ and $e_{1,h} = e_{2,h}$, which are *different*, in general.

A similar analysis can be performed for the 2-point function; as discussed in Appendix C, this object can be evaluated following a procedure similar to the one discussed in this section. It follows that the 2-point function can be written as pertubative series in terms of the effective couplings $\{e_{\mu,k}, \nu_{\mu,k}\}_{k\le 0}$, where the coefficients of the renormalized expansions can be represented as sums of Feynman graphs uniformly bounded as $|\Lambda| \to \infty$; in contrast, the graphs forming the naive expansion in e, ν_μ are plagued by logarithmic infrared divergences.

More explicitly, if $|\mathbf{k}| \simeq M^h$,

$$\langle \psi_{\mathbf{k}} \overline{\psi}_{\mathbf{k}} \rangle = \frac{g^{(h)}(\mathbf{k})}{Z_{h-1}} \Big(1 + \tilde B(\mathbf{k})\Big)\;, \tag{3.20}$$

where $\tilde{B}(\mathbf{k})$ is given by a formal power series in $\{e_{\mu,k},\nu_{\mu,k}\}_{k\le 0}$ with coefficients depending on $\{Z_k, v_k\}_{k\le 0}$, and starting from second order; under the same hypothesis of Theorem 3.1, the N-th order contribution to $\tilde{B}(\mathbf{k})$ is bounded by $(\mathrm{const.})^N(\bar{\varepsilon}_{-\infty})^N(N/2)!$ uniformly in $\mathbf{k}$.

To prove our main result we need to control the flow of the effective charges at all orders in perturbation theory, and to do this we shall use Ward Identities, see Section 5. These are nontrivial relations for the three point functions, which can be related to the renormalized charges in the following way. Consider a theory with a bosonic infrared cutoff M^{h_*}, that is assume that the bare bosonic propagator is given by (2.5) with $\chi_0(\mathbf{p})$ replaced by $\chi_{[h^*,0]}(\mathbf{p}) := \chi_0(\mathbf{p}) - \chi_0(M^{-h^*}\mathbf{p})$, which is vanishing for $|\mathbf{p}| \le M^{h_*}$ and it is equal to $\chi_0(\mathbf{p})$ for $|\mathbf{p}| \ge M^{h^*+1}$; denote by $\langle \ldots \rangle_{h^*}$ the expectation value in the presence of the bosonic infrared cutoff. As shown in Appendix C, setting $\bar{e}_{0,h} := e_{0,h}$, $\bar{e}_{1,h} = \bar{e}_{2,h} := v_{h-1}e_{1,h}$, and taking $|\mathbf{q}+\mathbf{p}|, |\mathbf{q}| \simeq M^{h^*}$, $|\mathbf{p}| \ll M^{h^*}$, the following result holds:

$$\langle j_{\mu,\mathbf{p}}; \psi_{\mathbf{q}+\mathbf{p}}\overline{\psi}_{\mathbf{q}}\rangle_{h^*} = iZ_{h^*-1}\frac{\bar{e}_{\mu,h^*}}{e}\langle \psi_{\mathbf{q}+\mathbf{p}}\overline{\psi}_{\mathbf{q}+\mathbf{p}}\rangle_{h^*}\Big(\gamma_\mu + \bar{B}_{\mu,h^*}(\mathbf{p},\mathbf{q})\Big)\langle \psi_{\mathbf{q}}\overline{\psi}_{\mathbf{q}}\rangle_{h^*}\ , \tag{3.21}$$

where $\bar{B}_{\mu,h^*}$ is given by a formal power series in $\{e_{\mu,k},\nu_{\mu,k}\}_{h^*<k\le 0}$, with the N-th order of the series admitting a bound proportional to $(\bar{\varepsilon}_{h^*})^N(N/2)!$, uniformly in $\mathbf{k}$. Eq.(3.21) is one of the two desired equations relating the 3-point function to the 2-point function and the effective charge e_{μ,h^*}. A second independent equation expressing the 3-point function in terms of the 2-point function and of the *bare* charge e will be derived in Section 5, see (5.4), using the (approximate) gauge invariance of the theory. Combining the two equations we will be able to relate e_{μ,h^*} to the bare charge e, for all $h^* < 0$, and this will allows us to control the flow of the effective couplings on all infrared scales. This procedure will be described in detail in the next two sections.

4. THE FLOW OF THE EFFECTIVE COUPLINGS

A crucial point for the consistency of our approach is that the running coupling constants $e_{\mu,h}, \nu_{\mu,h}$ are small for all $h \le 0$, that the ratios Z_h/Z_{h-1} are close to 1, and the effective Fermi velocity v_h does not approach zero. Even if we do not prove the convergence of the series but only $N!$ bounds, we expect that our series gives meaningful information only as long as the running coupling constants satisfy these conditions. In this section we describe how to control their flow. We shall proceed by induction: we will first assume that $\bar\varepsilon = \max_{k\le 0}\{|e_{\mu,k}|\}$ is small, that $Z_h/Z_{h-1} \le e^{C\bar\varepsilon^2}$ and $C^{-1} \le v_h \le 1$ for all $h \le 0$ and a suitable constant $C > 0$, and we will show that, by properly choosing the values of the counterterms ν_μ in (2.2), the constants $\nu_{\mu,h}$ remain small: $\max_{h\le 0}\{|\nu_{\mu,h}|\} \le (\text{const.})\,\bar\varepsilon^2$. Next, once that the flow of $\nu_{\mu,h}$ is controlled, we will show that the constants $e_{\mu,h}$ remain bounded and small for all $h \le 0$, thanks to remarkable cancellations following from a *Ward Identity*. Finally, we will study the flow of Z_h and v_h and show that, asymptotically as $h \to -\infty$, $Z_h \sim M^{-\eta h}$, with $\eta = O(e^2)$ a positive exponent, while v_h grows, approaching a limiting value v_{eff} close to the speed of light.

The renormalized parameters obey to recursive equations induced by the previous construction; i.e., (3.6), (3.7), (3.11), (3.13) imply the flow equations:

$$\frac{Z_{h-1}}{Z_h} = 1 + z_{0,h} := 1 + \beta^z_h\ , \quad v_{h-1} = \frac{Z_h}{Z_{h-1}}(v_h + z_{1,h}) := v_h + \beta^v_h\ , \tag{4.1}$$

$$\nu_{\mu,h} = -M^{-h}\, W^{(h)}_{2,0,\mu,\mu}(\mathbf{0}) := M\nu_{\mu,h+1} + \beta^\nu_{\mu,h+1}\ , \tag{4.2}$$

$$e_{0,h} = \frac{Z_h}{Z_{h-1}}\lambda_{0,h} := e_{0,h+1} + \beta^e_{0,h+1}\ , \tag{4.3}$$

$$e_{1,h} = \frac{Z_h}{Z_{h-1}}\frac{\lambda_{1,h}}{v_{h-1}} := e_{1,h+1} + \beta^e_{1,h+1}\ , \tag{4.4}$$

and $e_{2,h} = e_{1,h}$. The *beta functions* appearing in the r.h.s. of flow equations are related to the kernels $W^{N;(h)}_{m,n,\underline{\rho},\underline{\mu}}$, so that they are expressed by series in the running coupling constants admitting the bound (3.19). For the explicit expressions of the one-loop contributions to the beta function, see below.

The flow of $\nu_{\mu,h}$. Let us assume that $\bar\varepsilon = \max_{k\le 0}\{|e_{\mu,k}|\}$ is small, that $Z_h/Z_{h-1} \le e^{(\text{const})\,\bar\varepsilon^2}$, $v_{h-1} \ge v_h$ and $0 < v_0 \le v_h \le 1$ for all $h \le 0$. Under these assumptions, the flow of $\nu_{\mu,h}$ can be controlled by suitably choosing the counterterms ν_μ; in fact, if ν_μ is chosen as

$$\nu_\mu = -\sum_{k=-\infty}^{0} M^{k-1}\beta^\nu_{\mu,k}\ , \tag{4.5}$$

then the effective coupling $\nu_{\mu,h}$ is

$$\nu_{\mu,h} = -\sum_{k=-\infty}^{h} M^{-h-1+k}\beta^\nu_{\mu,k}\ , \tag{4.6}$$

which gives, by using a standard fixed point argument:

$$|\nu_{\mu,h}| \le (\text{const.})\,\bar\varepsilon^2\ . \tag{4.7}$$

In particular, the effective photon mass on scale h, $M^h\nu_{\mu,h}$, goes to zero as $h \to -\infty$. The inequality (4.7) must be understood as an order by order inequality, that is, $\nu_{\mu,h}$ is expressed by a series in $\{e_{\mu,k}\}_{k\le 0}$, starting at second order, whose N-th order coefficient is bounded by (const.)$^N (N/2)!$. From now on, all the inequalities we will write have to be understood in this sense. At lowest order, if $h < 0$ and setting $\xi_h := \frac{\sqrt{1-v_h^2}}{v_h}$ (see Appendix B):

$$\beta^{\nu,(2)}_{0,h} = -(M-1)\frac{e^2_{0,h}v_h^{-2}}{\pi^2}\Big[\frac{\xi_h - \arctan\xi_h}{\xi_h^3}\Big]\int_0^\infty dt\,\big(2\chi(t)-\chi^2(t)\big)\;, \tag{4.8}$$

$$\beta^{\nu,(2)}_{1,h} = -(M-1)\frac{e^2_{1,h}}{2\pi^2}\Big[\frac{\arctan\xi_h}{\xi_h} - \frac{\xi_h - \arctan\xi_h}{\xi_h^3}\Big]\int_0^\infty dt\,\big(2\chi(t)-\chi^2(t)\big)\;. \tag{4.9}$$

By the above equations we see that lowest order contributions to ν_μ are positive, that is ν_μ can be interpreted as *bare* photon masses. By using the short memory property (see Appendix A), one can show that $\beta^\nu_{0,h} - \beta^\nu_{1,h}$ is sum of terms of order $O(1-v_h)$ or $O(e_{0,h}-e_{1,h})$; therefore, from (4.6):

$$|\nu_{0,h} - \nu_{1,h}| \le C\big[\bar\varepsilon^2(1-v_h) + \bar\varepsilon|e_{0,h}-e_{1,h}|\big]\;. \tag{4.10}$$

The flow of the effective charges. Once that the flow of $\nu_{\mu,h}$ has been controlled, let us prove the following key property:

$$e_{\mu,h} = e + e\alpha^{(2)}_\mu + e^5 F_{\mu,h}(e)\;, \tag{4.11}$$

where $F_{\mu,h}(e)$ is given by a renormalized series with coefficients admitting $N!$-bounds and

$$\alpha^{(2)}_0 = \frac{e^2}{8\pi^2}\big(2-v^{-2}\big)\Big(\frac{\xi_0 - \arctan\xi_0}{\xi_0^3}\Big) + O(e^2(M-1))\;, \tag{4.12}$$

$$\alpha^{(2)}_1 = \frac{e^2}{16\pi^2}\frac{1}{v^2}\Big(\frac{\arctan\xi_0}{\xi_0} - \frac{\xi_0 - \arctan\xi_0}{\xi_0^3}\Big) + O(e^2(M-1))\;, \tag{4.13}$$

where the correction terms $O(e^2(M-1))$ can be made as small as desired, by choosing $0 < M-1 \ll 1$. The h-dependence of (4.11) is very weak: $|F_{\mu,h}(e) - F_{\mu,-\infty}(e)| \le$ (const.) $M^{h/2}$. Notice that, remarkably, in the deep infrared the two effective charges are different:

$$e_{0,-\infty} - e_{1,-\infty} = -\frac{e^3}{5\pi^2}F(v) + O(e^5(1-v))\;, \tag{4.14}$$

with $F(v)$ the function defined in (2.10). At one-loop, Eqs.(4.11)–(4.14) can be understood as follows. Here, for illustrative purposes, we perform the lowest order computation in non-renormalized perturbation theory, *in the presence of an infrared cutoff on the bosonic* *propagator*; at this lowest order, such a "naive" computation gives the same result as the renormalized one; for the full computation, see Appendix D. If $(\bar\gamma_0,\bar\gamma_1,\bar\gamma_2):=(\gamma_0,v\gamma_1,v\gamma_2)$ and $(\bar k_0,\bar k_1,\bar k_2):=(k_0,vk_1,vk_2)$, the effective charges on scale h are given by:

$$\begin{aligned}(e_{\mu,h}-e)\gamma_\mu \;&=\; ie^3\int\frac{d\mathbf{k}}{(2\pi)^3}\,\frac{\chi_{[h,0]}(\mathbf{k})}{2|\mathbf{k}|}\bar\gamma_\nu g^{(\le 0)}(\mathbf{k})i\gamma_\mu g^{(\le 0)}(\mathbf{k})\bar\gamma_\nu + \\ &+\; ie^3\int\frac{d\mathbf{k}}{(2\pi)^3}\,\frac{\chi_{[h,0]}(\mathbf{k})}{2|\mathbf{k}|}\bar\gamma_\nu\partial_{\bar k_\mu}g^{(\le 0)}(\mathbf{k})\bar\gamma_\nu + O(e^3M^{h/2})+O(e^5)\ ,\end{aligned} \tag{4.15}$$

where the first term in the r.h.s. is the vertex renormalization, while the second term is due to the wave function and velocity renormalizations. Note that both integrals are well defined in the ultraviolet (thanks to the presence of an ultraviolet cutoff in the propagators), while for $h\to-\infty$ they are logarithmically divergent in the infrared. However, a remarkable cancellation takes place between the two integrals; in fact:

$$g^{(\le 0)}(\mathbf{k})i\gamma_\mu g^{(\le 0)}(\mathbf{k})+\partial_{\bar k_\mu}g^{(\le 0)}(\mathbf{k})=\frac{\partial_{\bar k_\mu}\chi_0(\mathbf{k})}{i\not{\mathbf{k}}}+\chi_0(\mathbf{k})\big(\chi_0(\mathbf{k})-1\big)\frac{1}{i\not{\mathbf{k}}}i\gamma_\mu\frac{1}{i\not{\mathbf{k}}}\ , \tag{4.16}$$

with $\not{\mathbf{k}}:=k_0\gamma_0+v\vec k\cdot\vec\gamma$, so that

$$\begin{aligned}e_{0,h} \;&=\; e+ie^3\int\frac{d\mathbf{k}}{(2\pi)^3}\,\bar\gamma_\mu\frac{1}{i\not{\mathbf{k}}}\bar\gamma_\mu\frac{k_0}{2|\mathbf{k}|^2}\chi_0'(\mathbf{k})\chi_{[h,0]}(\mathbf{k})+\cdots\ ,\\ e_{1,h} \;&=\; e+\frac{ie^3}{v}\int\frac{d\mathbf{k}}{(2\pi)^3}\,\bar\gamma_\mu\frac{1}{i\not{\mathbf{k}}}\bar\gamma_\mu\frac{k_1}{2|\mathbf{k}|^2}\chi_0'(\mathbf{k})\chi_{[h,0]}(\mathbf{k})+\cdots\ ,\end{aligned} \tag{4.17}$$

where the error terms are of the order $O(e^3(M-1))+O(e^3M^{h/2})+O(e^5)$. An explicit computation of (4.17) leads to (4.11)–(4.14).

Remarks

1. If we used a loop or a dimensional regularization, as in [11], rather than introducing a momentum cut-off in the propagators, then we would get an exact cancellation between the two integrals in (4.15). On the other hand, the regularization that we use mimics the lattice (which is truly present in graphene), in the sense that we expect our effective continuum model (with the momentum regularization that we considered) to be asymptotic in the infrared to the "real" tight binding model describing electrons hopping on the honeycomb lattice and interacting via a retarded 3D Coulomb interactions, provided that its bare parameters e_0, e_1, v are properly chosen (see the last two remarks after the statement of the main result). Moreover, momentum regularization is the only one admitting a real non perturbative analysis, see e.g. [9, 14].

2. The result $e_{0,h} \neq e_{1,h}$ is *independent* of the details of the ultraviolet cutoff (e.g., of its location, or of the shape of the cutoff function). For instance, if we replace $\chi_{[h,0]}(\mathbf{k})$ (resp. $\chi_0(\mathbf{k})$) by $\chi_{[h,N]}(\mathbf{k})$ (resp. $\chi_N(\mathbf{k})$) in the definition of the bosonic (resp. fermionic) propagator, then the dressed charges in the far infrared *do not change* at lowest order. This is ultimately due to the fact that the electric charge is marginal, i.e., that its scaling dimension is zero. Similarly, it is apparent from (4.17) that the value of the integral is independent of the shape of χ.

Of course, the one-loop computation that we just described does not say much: if we could not exclude that a similar cancellation takes place at all orders there would always be the possibility that higher orders produce a completely different behavior, *e.g.* a vanishing or diverging flow for $e_{\mu,h}$, corresponding to completely different physical properties of the system. In order to obtain a control at all orders on $e_{\mu,h}$ one needs to combine the multiscale evaluation of the effective potentials with Ward Identities. This is not a trivial task: Wilsonian RG methods are based on a multiscale momentum decomposition which breaks the local gauge invariance, which Ward identities are based on. In Section 5 below, following a strategy recently proposed and developed by G. Benfatto and V. Mastropietro [3], we will prove (4.11).

The flow of Z_h and v_h. It remains to discuss the flow for the wave function renormalization and the Fermi velocity, which is given by (4.1). At lowest order we get, by an explicit computation (see Appendix B):

$$\beta_h^{z,(2)} = \frac{\log M}{4\pi^2}\,(2e_{1,h}^2 - e_{0,h}^2 v_h^{-2})\frac{\xi_h - \arctan\xi_h}{\xi_h^3}\,, \tag{4.18}$$

$$\beta_h^{v,(2)} = \frac{\log M}{4\pi^2}\left[\frac{e_{0,h}^2 v_h^{-1}}{2}\frac{\arctan\xi_h}{\xi_h} - \left(2e_{1,h}^2 v_h - \frac{e_{0,h}^2 v_h^{-1}}{2}\right)\frac{\xi_h - \arctan\xi_h}{\xi_h^3}\right]. \tag{4.19}$$

Eqs.(4.11)–(4.14), if combined with the flow equation for v_h, implies that in the infrared limit $h \to -\infty$ the Fermi velocity flows to a limiting value v_{eff} smaller than the speed of light, $v_{eff} < 1$. In order to prove this fact, we preliminarily note that at second order in e the beta function (4.19) is positive for all $\xi_h > 0$, and vanishes linearly for $\xi_h \to 0^+$: whatever the initial value of v_0 is, v_h grows in the infrared up to a limiting value close to 1. Therefore, for $h \ll 1$, $(1 - v_h)$ is small, and it is meaningful to linearize (4.19) with respect to $(1 - v_h)$

and $(e_{0,h} - e_{1,h})$, so that:

$$\frac{v_{h-1}}{v_h} = 1 + \frac{\log M}{4\pi^2}\left[\frac{8}{5}e^2(1-v_h)(1+A'_h) + \frac{4}{3}e(1+B'_h)(e_{0,h}-e_{1,h})\right] , \qquad (4.20)$$

with $|A'_h| \le C\big[e^2 + (1-v_h)\big]$ and $|B'_h| \le Ce^2$. If we now plug (4.11)–(4.14) into (4.20), we get

$$\frac{v_{h-1}}{v_h} = 1 + \frac{\log M}{4\pi^2}\left[\frac{8}{5}e^2(1-v_h)(1+A''_h) - \frac{4}{15\pi^2}e^4 F(v)(1+B''_h)\right] , \qquad (4.21)$$

with $|A''_h| \le C\big[e^2 + (1-v_h)\big]$ and $|B''_h| \le C\big[e^2 + (M-1) + M^{h/2}\big]$, so that $\lim_{h\to-\infty} v_h = v_{eff}$ and $v_{eff} - v_h \simeq (1-v)M^{h\tilde\eta}$, with v_{eff} and $\tilde\eta$ given by (2.9).

Finally, plugging this result into (4.1) and (4.18), we find that *at all orders* in renormalized perturbation theory $Z_h \simeq M^{-h\eta}$, where the *anomalous dimension* η is given by (2.9).

5. WARD IDENTITIES

In this section we prove that order by order in perturbation theory the effective charges $e_{\mu,h}$ remain close to their original values $e_{\mu,0} = e$; moreover, we prove that asymptotically as $h \to -\infty$, $e_{0,h} \neq e_{1,h}$, see (4.11)–(4.14). The proof is based on a suitable combination of the RG methods described in the previous sections together with Ward Identities; even though the momentum regularization breaks the local gauge invariance needed to formally derive the WIs, we will be able, following the strategy of [3], to rigorously take into account the effects of cutoffs, and to control the corrections generated by their presence.

As anticipated at the end of Section 3, we consider a sequence of models, to be called *reference models* in what follows, with different infrared bosonic cutoffs on scale h, *i.e.* with bosonic propagator given by:

$$w^{[h,0]}(\mathbf{p}) \equiv \frac{\chi_{[h,0]}(\mathbf{p})}{2|\mathbf{p}|} , \qquad \chi_{[h,0]}(\mathbf{p}) \equiv \chi_0(\mathbf{p}) - \chi_0(M^{-h}\mathbf{p}) \qquad (5.1)$$

(the idea of introducing an infrared cutoff only in the bosonic sector is borrowed from Adler and Bardeen [1], who used a similar regularization scheme in order to understand anomalies in quantum field theory). The generating functional $\mathcal{W}_{[h,0]}(J,\phi)$ of the correlations of the reference model can be evaluated following an iterative procedure similar to the one described in Section 3 (see Appendix C for details), with the important difference that after the integration of the scale h we are left with a purely fermionic theory, which is *superrenormalizable*: in fact, setting $m = 0$ in the formula for the scaling dimension (see lines following (3.19) and recall that for scales smaller than h the reference model has no bosonic lines) we recognize that the scaling dimension of this fermionic theory is $3 - 2n$, which is always negative once that the two-legged subdiagrams have been renormalized, see [9]. Let us denote by $\{e^{(h)}_{\mu,k}\}_{k\geq h}$ the effective couplings of the reference model; of course, if $k \geq h$

$$e^{(h)}_{\mu,k} = e_{\mu,k} \; , \tag{5.2}$$

where $\{e_{\mu,k}\}_{k\leq 0}$ are the running coupling constants of the original model. On the other hand, as proven in Appendix C, the vertex functions $\langle j_{\mu,\mathbf{p}}; \psi_{\mathbf{k}+\mathbf{p}}\overline{\psi}_{\mathbf{k}}\rangle_h$ of the reference model with bosonic cutoff on scale h computed at external momenta $\mathbf{k}$, $\mathbf{k}+\mathbf{p}$ such that $|\mathbf{k}+\mathbf{p}|, |\mathbf{k}| \simeq M^h$ and $|\mathbf{p}| \ll M^h$ are proportional to the charges $e^{(h)}_{\mu,h} = e_{\mu,h}$, see (3.21); therefore, if we get informations on the vertex functions of the reference models, we automatically infer informations on the effective couplings of the original model.

Such informations are provided by Ward Identities; by performing the change of variables $\psi_{\mathbf{x}} \to e^{i\alpha_{\mathbf{x}}}\psi_{\mathbf{x}}$, $\overline{\psi}_{\mathbf{x}} \to e^{-i\alpha_{\mathbf{x}}}\overline{\psi}_{\mathbf{x}}$ in the generating functional $\mathcal{W}_{[h,0]}(J,\phi)$ of the reference model and using that the Jacobian of this transformation is equal to 1, see [3, 14], we get:

$$e^{\mathcal{W}_{[h,0]}(J,\phi)} = \int P(d\psi)P_{[h,0]}(dA)e^{-\int d\mathbf{x}\,\overline{\psi}_{\mathbf{x}}(e^{-i\alpha_{\mathbf{x}}}De^{i\alpha_{\mathbf{x}}}-D)\psi_{\mathbf{x}}+V(A,\psi)+B(J,\phi e^{-i\alpha})} \; , \tag{5.3}$$

where $P_{[h,0]}(dA)$ is the gaussian integration with propagator (5.1) and, if $\not{\mathbf{k}} = \gamma_0 k_0 + v\vec{\gamma}\cdot\vec{k}$, the pseudo-differential operator D is defined by:

$$(D\psi)_{\mathbf{x}} = \int_{\chi(\mathbf{k})>0} \frac{d\mathbf{k}}{(2\pi)^3}\frac{e^{i\mathbf{k}\mathbf{x}}}{\chi_0(\mathbf{k})} i\not{\mathbf{k}}\,\psi_{\mathbf{k}} \; .$$

If we derive (5.3) with respect to α, $\overline{\phi}$ and ϕ and then set $\alpha = \phi = J = 0$, we get the following identity:

$$p_\mu \langle j_{\mu,\mathbf{p}}; \psi_{\mathbf{k}+\mathbf{p}}\overline{\psi}_{\mathbf{k}}\rangle_h = \langle \psi_{\mathbf{k}}\overline{\psi}_{\mathbf{k}}\rangle_h - \langle \psi_{\mathbf{k}+\mathbf{p}}\overline{\psi}_{\mathbf{k}+\mathbf{p}}\rangle_h + \Delta_h(\mathbf{k},\mathbf{p}) \tag{5.4}$$

where

$$\Delta_h(\mathbf{k},\mathbf{p}) = \int \frac{d\mathbf{k}'}{(2\pi)^3} \langle \overline{\psi}_{\mathbf{k}'+\mathbf{p}} C(\mathbf{k}',\mathbf{p})\psi_{\mathbf{k}'}; \psi_{\mathbf{k}+\mathbf{p}}\overline{\psi}_{\mathbf{k}}\rangle_h \tag{5.5}$$

and

$$C(\mathbf{k},\mathbf{p}) = i\not{\mathbf{k}}\Big(\chi_0(\mathbf{k})^{-1} - 1\Big) - i(\not{\mathbf{k}} + \not{\mathbf{p}})\Big(\chi_0(\mathbf{k}+\mathbf{p})^{-1} - 1\Big) \; . \tag{5.6}$$

The correction term $\Delta_h(\mathbf{k},\mathbf{p})$ in (5.4) is due to the presence of the ultraviolet momentum cut-off, and it can be computed by following a strategy analogous to the one used to prove the vanishing of the beta function in one-dimensional Fermi systems [3]. The result (see Appendix D for details) is that, if $|\mathbf{k}|, |\mathbf{k}+\mathbf{p}| \simeq M^h$ and $|\mathbf{p}| \ll M^h$,

$$\Delta_h(\mathbf{k},\mathbf{p}) = \alpha_\mu p_\mu \langle j_{\mu,\mathbf{p}}; \psi_{\mathbf{k}+\mathbf{p}}\overline{\psi}_{\mathbf{k}}\rangle_h + \frac{p_\mu}{Z_h} R_{\mu,h}(\mathbf{k},\mathbf{p}) \;, \tag{5.7}$$

where, setting $\bar{\varepsilon}_h = \max_{k\geq h}\{e_{\mu,h}, \nu_{\mu,h}\}$, the correction $R_{\mu,h}(\mathbf{k},\mathbf{p})$ is dimensionally negligible with respect to the first term, *i.e.*, $|R_{\mu,h}(\mathbf{k},\mathbf{p})| \leq (\text{const.})M^{-2h}M^{\frac{h}{2}}\bar{\varepsilon}_h^2$. Moreover, $|\alpha_\mu| \leq (\text{const.})\bar{\varepsilon}_h^2$ and, if we assume that $e_{\mu,h} = e + O(e^3)$, then $\alpha_\mu = \alpha_\mu^{(2)} + O(e^4)$, with $\alpha_\mu^{(2)}$ defined in (4.12)-(4.13). Let us pick $|\mathbf{k}| = M^h$ and $|\mathbf{p}| \ll M^h$; by using Eqs.(3.20)-(3.21) and the fact that

$$g^{(h)}(\mathbf{k}) - g^{(h)}(\mathbf{k}+\mathbf{p}) = g^{(h)}(\mathbf{k}+\mathbf{p})(ip_0\gamma_0 + iv_{h-1}\vec{p}\cdot\vec{\gamma})g^{(h)}(\mathbf{k}) + O(|\mathbf{p}|^2 M^{-3h}) \;, \tag{5.8}$$

we find that

$$\langle \psi_{\mathbf{k}}\overline{\psi}_{\mathbf{k}}\rangle_h - \langle \psi_{\mathbf{k}+\mathbf{p}}\overline{\psi}_{\mathbf{k}+\mathbf{p}}\rangle_h = \frac{1}{Z_{h-1}} g^{(h)}(\mathbf{k}+\mathbf{p})(ip_0\gamma_0 + iv_{h-1}\vec{p}\cdot\vec{\gamma})g^{(h)}(\mathbf{k}) + \\ + \frac{p_\mu}{Z_{h-1}}\tilde{r}_\mu(\mathbf{k},\mathbf{p}) \;, \tag{5.9}$$

$$p_\mu \langle j_{\mu,\mathbf{p}}; \psi_{\mathbf{k}+\mathbf{p}}\overline{\psi}_{\mathbf{k}}\rangle_h = \frac{1}{eZ_{h-1}} g^{(h)}(\mathbf{k}+\mathbf{p})(ie_{0,h}p_0\gamma_0 + iv_{h-1}e_{1,h}\vec{p}\cdot\vec{\gamma})g^{(h)}(\mathbf{k}) + \\ + \frac{p_\mu}{Z_{h-1}} r_\mu(\mathbf{k},\mathbf{p}) \;, \tag{5.10}$$

with $|r_\mu(\mathbf{k},\mathbf{p})|, |\tilde{r}_\mu(\mathbf{k},\mathbf{p})| \leq (\text{const.})M^{-2h}(\bar{\varepsilon}_h^2 + |\mathbf{p}|M^{-h})$. Now, if we plug (5.7) into the Ward identity (5.4), and we use the relations (5.9)-(5.10), we get an identity that, computed at $\mathbf{k} = \mathbf{k}_0 := (M^h, \vec{0})$ and $\mathbf{p} = \mathbf{p}_0 := (p, \vec{0})$, after taking the limit $p \to 0$, reduces to:

$$\frac{e_{0,h}}{e}(1-\alpha_0) = 1 + iM^{2h}\big[\tilde{r}_0(\mathbf{k}_0,\mathbf{0}) + R_{0,h}(\mathbf{k}_0,\mathbf{0}) - (1-\alpha_0)r_0(\mathbf{k}_0,\mathbf{0})\big]\gamma_0 \equiv 1 + A_{0,h} \;, \tag{5.11}$$

with $|A_{0,h}| \leq (\text{const.})\bar{\varepsilon}_h^2$, as it follows from the estimates on $R_{0,h}, r_0, \tilde{r}_0$. Eq.(5.11) combined with (4.12) implies, as desired, that *the effective charge $e_{0,h}$ remains close to $e_{0,0} = e$ at all orders in renormalized perturbation theory.* Similarly, if $\mathbf{k}_1 := (0, M^h, 0)$, we get:

$$\frac{e_{1,h}}{e}(1-\alpha_1) = \frac{v_{h-1}}{v_h} + iM^{2h}v_{h-1}\big[\tilde{r}_1(\mathbf{k}_1,\mathbf{0}) + R_{1,h}(\mathbf{k}_1,\mathbf{0}) - (1-\alpha_1)r_1(\mathbf{k}_1,\mathbf{0})\big]\gamma_1 \equiv 1 + A_{1,h} \;, \tag{5.12}$$

with $|A_{1,h}| \leq (\text{const.})\bar{\varepsilon}_h^2$, which implies that *the effective charge $e_{1,h}$ remains close to $e_{1,0} = e$ at all orders in renormalized perturbation theory.*

Equations (5.11), (5.12) not only imply the boundedness of the effective charges $e_{\mu,h}$, but they also allow us to compute the difference $e_{0,h} - e_{1,h}$, asymptotically as $h \to -\infty$. In fact, if for simplicity we assume that $0 < M - 1 \ll 1$, then combining (5.11), (5.12), and using (4.12), (4.13), we get

$$\begin{aligned} e_{0,h} - e_{1,h} &= e(\alpha_0 - \alpha_1) + e(A'_{0,h} - A'_{1,h}) \\ &= -\frac{1}{5\pi^2} e^3 F(v)(1 + B_h(e)) + e(A'_{0,h} - A'_{1,h}) \ , \end{aligned} \tag{5.13}$$

where $F(v)$ is defined in (2.10), $|B_h| \le (\text{const.}) \big[e^2 + (M-1)\big]$ and $A'_{\mu,h} := A_{\mu,h} + \alpha_\mu(e_{\mu,h}/e - 1)$. Now, notice that $A'_{0,h} - A'_{1,h}$ can be written in terms of a renormalized expansion involving, in particular, $\{e_{\mu,k}, v_k, \nu_k\}_{h<k\le 0}$. If, in this expansion, we set $e_{1,k} = e_{0,k}$, $v_k = 1$ and $\nu_{0,k} = \nu_{1,k}$ then, by Lorentz symmetry, we get $A'_{0,h} - A'_{1,h} = 0$. This remarkable cancellation, combined with (4.10) and the so-called short memory property (see Appendix A), implies that, for $h \ll 1$,

$$\big|A'_{0,h} - A'_{1,h}\big| \le (\text{const.}) \Big[e^2(1 - v_h) + e\big|e_{0,h} - e_{1,h}\big| + e^2(1 - v)\big(e^2 + M^{\frac{h}{2}}\big)\Big] \ , \tag{5.14}$$

see Appendix C for details. This concludes the proof of (4.11)–(4.14); in particular, it concludes the proof of the consistency at all orders of the renormalized perturbative expansion for $\mathcal{W}(J, \phi)$ in (2.1), as discussed in previous sections.

6. CONCLUSIONS

We considered an effective continuum model for the low energy physics of single-layer graphene, first introduced by Gonzalez et al in [11]. We analyzed it by *Constructive Renormalization Group* methods, which have already been proved effective in the *non perturbative* study of several low-dimensional fermionic models, such as one-dimensional interacting fermions [3], or the Hubbard model on the honeycomb lattice [9]. While in the present case we are not able yet to prove the convergence of the renormalized expansion, we can prove that it is *order by order finite*, see Theorem 3.1 above. Note that, on the contrary, the power series expansion in the bare couplings is plagued by *logarithmic divergences* and, therefore, informations obtained from it by lowest order truncation are quite unreliable. In perspective, the proof of convergence of the renormalized expansion appears to be much more difficult than the one in [3] or [9], due to the simultaneous presence of bosons and fermions, but it should be feasible (by using determinant bounds for the fermionic sector and cluster expansion techniques for the boson sector).

A key point of our analysis is the control at all orders of the flow of the effective couplings: this is obtained via Ward Identities relating three- and two-point functions, by using a technique developed in [3] for the analysis of Luttinger liquids, in cases where bosonization cannot be applied (like in the presence of an underlying lattice or of non-linear bands). The Ward Identities have *corrections* with respect to the formal ones, due to the presence of a fermionic ultraviolet cut-off. Remarkably, these corrections can be rigorously bounded at all orders in renormalized perturbation theory (see Section 5). We stress that in one dimension these corrections are crucial for establishing Luttinger liquid behavior (if one naively neglected them, then no anomalous dimension would emerge, see [3]). In our effective model for graphene with Coulomb interactions, the presence of such corrections produces *two* effective charges associated respectively with the temporal and spatial components of the current, flowing in general towards two different asymptotic values. This difference has the effect that the Fermi velocity, in the infrared, increases toward an asymptotic *non-universal* value, generically smaller than the speed of light, unless a *fine tuning* of the bare charges e_0, e_1 is made, see the last two remarks after the statement of the main result; therefore, Lorentz symmetry does not dynamically emerge. This is the main difference with respect to the lowest order analysis in [11], which predicted a Fermi velocity flowing towards the speed of light and a dynamical emergence of Lorentz symmetry. As discussed in Remark 1 of Section 4, the different conclusion we get is ultimately due to our different choice of the ultraviolet cutoff (momentum cutoff rather than loop or dimensional regularization).

Several questions remain to be understood. First of all, the effective model we considered is clearly not fundamental: a more realistic model for graphene should be obtained by considering electrons on the honeycomb lattice coupled to an electromagnetic field living in the 3D continuum; the "correct" electron-photon interaction term could be obtained via a Peierls' substitution or, even better, by introducing Wilson loops associated to the elementary plaquettes of the honeycomb lattice. Of course, such a model would have a natural ultraviolet cutoff, provided by the honeycomb lattice itself. We believe that a Renormalization Group analysis, similar to the one we performed here, is possible also for the lattice model, by combining the techniques and results of [9] with those of the present paper; we expect that the lattice model is asymptotic to the continuum one considered here, provided that the bare parameters of the continuum model are properly tuned (with, possibly, $e_0 \neq e_1$). Another important open problem is to understand the behavior of the system in the case of an istantaneous Coulomb interaction; this case can be obtained by taking the limit $c \to \infty$ together with a proper rescaling of the electronic charge in the model with retarded interactions. However, as discussed in the Remark at the end of Appendix A, the instantaneous case seems to be much more subtle, since it apparently requires cancellations even to prove renormalizability of the theory at all orders. We plan to come back to this case in a future publication.

Appendix A: Bounds for the renormalized expansion and proof of (3.19)

The iterative integration procedure described in Section 3 leads to a representation of the effective potentials in terms of a sum over connected Feynman diagrams, as explained in the following. The key formula, which we start from, is (3.18), which can be rewritten as

$$|\Lambda|\tilde{E}_h + \mathcal{V}^{(h-1)}(A^{(\leq h-1)}, \sqrt{Z_{h-1}}\,\psi^{(\leq h-1)}) = \sum_{n\geq 1} \frac{1}{n!}\mathcal{E}_h^T\left(\hat{\mathcal{V}}^{(h)}(A^{(\leq h)}, \sqrt{Z_{h-1}}\psi^{(\leq h)}); n\right) , \quad \text{(A.1)}$$

with $\mathcal{E}_h^T$ the truncated expectation on scale h, defined as

$$\mathcal{E}_h^T(X(A^{(h)}, \psi^{(h)}); n) := \frac{\partial^n}{\partial \lambda^n} \log \int P(d\psi^{(h)})P(dA^{(h)})e^{\lambda X(A^{(h)},\psi^{(h)})}\Big|_{\lambda=0} . \quad \text{(A.2)}$$

If X is graphically represented as a vertex with external lines $A^{(h)}$ and $\psi^{(h)}$, the truncated expectation (A.2) can be represented as the sum over the Feynman diagrams obtained by contracting in all possible connected ways the lines exiting from n vertices of type X. Every contraction corresponds to a propagator on scale h, as defined in (3.17). Since $\hat{\mathcal{V}}^{(h)}$ is related to $\mathcal{V}^{(h)}$ by a rescaling and a subtraction, see (3.9) and (3.12), Eq.(A.1) can be iterated until scale 0, and $\mathcal{V}^{(h-1)}$ can be written as a sum over connected Feynman diagrams with lines on all possible scales between h and 0. The iteration of (A.1) induces a natural hierarchical organization of the scale labels of every Feynman diagram, which will be conveniently represented in terms of tree diagrams. In fact, let us rewrite $\hat{\mathcal{V}}^{(h)}$ in the r.h.s. of (A.1) as $\hat{\mathcal{V}}^{(h)}(A, \sqrt{Z_{h-1}}\psi) = \overline{\mathcal{L}}\mathcal{V}^{(h)}(A, \sqrt{Z_h}\psi) + \mathcal{R}\mathcal{V}^{(h)}(A, \sqrt{Z_h}\psi)$, where $\overline{\mathcal{L}} := \mathcal{L} - \mathcal{L}_\psi$, see (3.9). Let us graphically represent $\mathcal{V}^{(h)}$, $\overline{\mathcal{L}}\mathcal{V}^{(h)}$ and $\mathcal{R}\mathcal{V}^{(h)}$ as in the first line of Fig.1, and let us represent Eq.(A.1) as in the second line of Fig.1; in the second line, the node on scale h represents the action of $\mathcal{E}_h^T$. Iterating the graphical equation in Fig.1 up to scale 0, we end

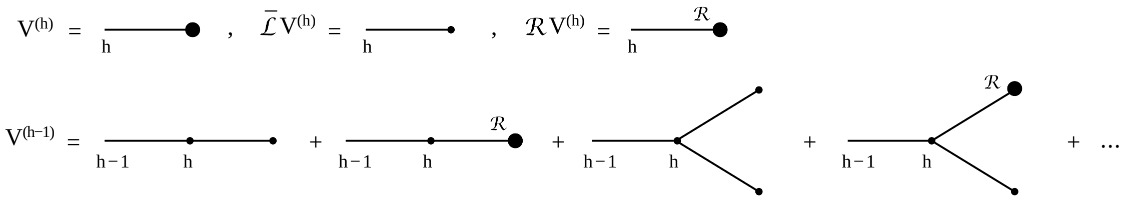

FIG. 1: Graphical interpretation of Eq.(A.1). The graphical equations for $\overline{\mathcal{L}}\mathcal{V}^{(h-1)}$, $\mathcal{R}\mathcal{V}^{(h-1)}$ are obtained from the equation in the second line by putting an $\overline{\mathcal{L}}$, $\mathcal{R}$ label, respectively, over the vertices on scale h.

up with a representation of $\mathcal{V}^{(h)}$ in terms of a sum over *Gallavotti-Nicolò* trees τ [2, 6–8, 14]:

$$\mathcal{V}^{(h)}(A^{(\le h)},\sqrt{Z_h}\,\psi^{(\le h)}) = \sum_{N\ge 1}\sum_{\tau\in\mathcal{T}_{h,N}} \mathcal{V}^{(h)}(\tau)\ , \tag{A.3}$$

where $\mathcal{T}_{h,N}$ is the of rooted trees with *root* r on scale $h_r = h$ and N endpoints, see Fig.2. The

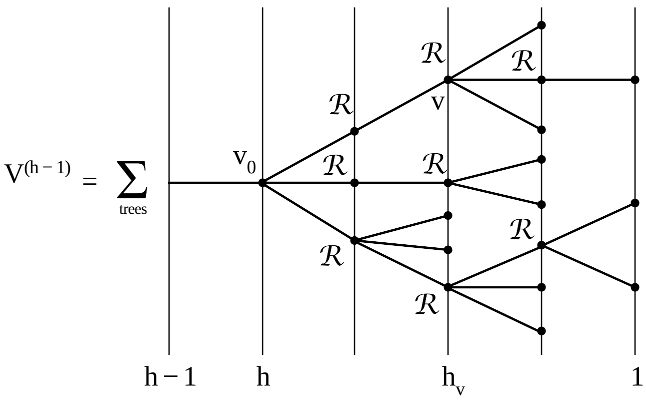

FIG. 2: The effective potential $\mathcal{V}^{(h-1)}$ can be represented as a sum over *Gallavotti – Nicolò* trees. The black dots will be called *vertices* of the tree. All the vertices except the first (*i.e.* the one on scale h) have an $\mathcal{R}$ label attached, which means that they correspond to the action of $\mathcal{R}\mathcal{E}^T_{h_v}$, while the first represents $\mathcal{E}^T_h$. The endpoints correspond to the graph elements in Fig.3 associated to the two terms in (3.14).

tree value $\mathcal{V}^{(h)}(\tau)$ can be evaluated in terms of a sum over connected Feynman diagrams, defined by the following rules.

With each endpoint v of τ we associate a graph element of type e or ν, corresponding to the two terms in the r.h.s. of (3.14), see Fig. 3. We introduce a *field label* f to distinguish the fields associated to the graph elements e and ν (any field label can be either of type A or of type ψ); the set of field labels associated with the endpoint v will be called I_v. Analogously, if v is not an endpoint, we call I_v the set of field labels associated with the endpoints following the vertex v on τ.

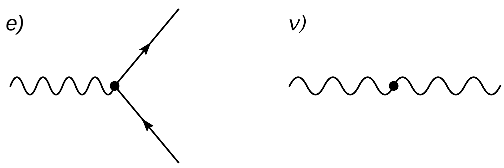

FIG. 3: The two possible graph elements associated to the endpoints of a tree, corresponding to the two terms in the r.h.s. of (3.14).

We start by looking at the graph elements corresponding to endpoints of scale 1: we group them in *clusters*, each cluster G_v being the set of endpoints attached to the same vertex v of scale 0, to be graphically represented by a box enclosing its elements. For any G_v of scale 0 (associated to a vertex v of scale 0 that is not an endpoint), we contract in pairs some of the fields in $\cup_{w\in G_v} I_w$, in such a way that after the contraction the elements of G_v are connected; each contraction produces a propagator $g^{(0)}$ or $w^{(0)}$, depending on whether the two fields are of type ψ or of type A. We denote by $\mathcal{I}_v$ the set of contracted fields inside the box G_v and by $P_v = I_v \setminus \mathcal{I}_v$ the set of external fields of G_v; if v is not the vertex immediatly following the root we attach a label $\mathcal{R}$ over the box G_v, which means that the $\mathcal{R}$ operator, defined after (3.4), acts on the value of the graph contained in G_v. Next, we group together the scale-0 clusters into scale-(-1) clusters, each scale-(-1) cluster G_v being a set of scale-0 clusters attached to the same vertex v of scale -1, to be graphically represented by a box enclosing its elements, see Fig.4.

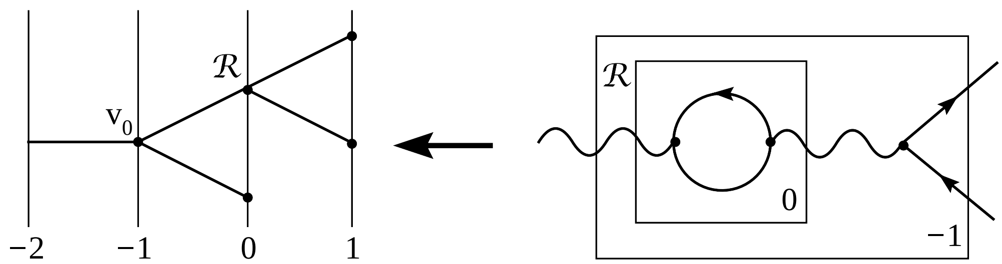

FIG. 4: A possible Feynman diagram contributing to $V^{(-2)}$ and its cluster structure.

Again, for each v of scale -1 that is not an endpoint, if we denote by $v_1,\ldots,v_{s_v}$ the vertices immediately following v on τ, we contract some of the fields of $\cup_{i=1}^{s_v} P_{v_i}$ in pairs, in such a way that after the contraction the boxes associated to the scale-0 clusters contained in G_v are connected; each contraction produces a propagator $g^{(-1)}$ or $w^{(-1)}$. We denote by $\mathcal{I}_v$ the set of fields in $\cup_{i=1}^{s_v} P_{v_i}$ contracted at this second step and by $P_v = \cup_{i=1}^{s_v} P_{v_i} \setminus \mathcal{I}_v$ the set of fields external to G_v; if v is not the vertex immediatly following the root we attach a label $\mathcal{R}$ over the box G_v.

Now, we iterate the construction, producing a sequence of boxes into boxes, hierarchically arranged with the same partial ordering as the tree τ. Each box G_v is associated to many different Feynman (sub-)diagrams, constructed by contracting in pairs some of the lines external to G_{v_i}, with v_i, $i = 1, \ldots, s_v$, the vertices immediately following v on τ; the contractions are made in such a way that the clusters $G_{v_1}, \ldots, G_{v_{s_v}}$ are connected through propagators of scale h_v. We denote by P_v^A and by P_v^ψ the set of fields of type A and ψ, respectively, external to G_v. The set of connected Feynman diagrams compatible with this hierarchical cluster structure will be denoted by $\Gamma(\tau)$. Given these definitions, we can write:

$$\mathcal{V}^{(h)}(\tau) = \sum_{\mathcal{G}\in\Gamma(\tau)} \mathrm{Val}(\mathcal{G}) \;, \qquad \mathrm{Val}(\mathcal{G}) = (-1)^\pi \Big[\prod_{f\in P_{v_0}^A} A_{\mu(f)}^{(\leq h)}\Big]\Big[\prod_{f\in P_v^\psi} \sqrt{Z_{h-1}}\,\widetilde{\psi}_{\rho(f)}^{(\leq h)}\Big]\widehat{\mathrm{Val}}(\mathcal{G}) \;,$$

$$\widehat{\mathrm{Val}}(\mathcal{G}) = \int \prod_{v \text{ not e.p.}} \Big(\frac{Z_{h_v-1}}{Z_{h_v-2}}\Big)^{\frac{|P_v^\psi|}{2}} \cdot \frac{1}{s_v!}\mathcal{R}^{\alpha_v}\Bigg[\Big(\prod_{\ell\in v} g_\ell^{(h_v)}\Big)\Big(\prod_{\substack{v^* \text{ e.p.} \\ v^*>v,\, h_{v^*}=h_v+1}} K_{v^*}^{(h_v)}\Big)\Bigg] \;, \qquad \text{(A.4)}$$

where: $(-1)^\pi$ is the sign of the permutation needed to bring the contracted fermionic fields next to each other; in the product over $f \in P_v^\psi$, $\widetilde{\psi}$ can be either $\overline{\psi}$ or ψ, depending on the specific field label f; s_v is the number of vertices immediately following v on τ; $\mathcal{R} = 1 - \mathcal{L}$ is the operator defined in Sec.3 (see Eq.(3.5) and preceding lines); $\alpha_v = 0$ if $v = v_0$, and otherwise $\alpha_v = 1$; $g_\ell^{(k)}$ is equal to $g^{(k)}$ or to $w^{(k)}$ depending on the fermionic or bosonic nature of the line ℓ, and $\ell \in v$ means that ℓ is contained in the box G_v but not in any other smaller box; finally, $K_{v^*}^{(k)}$ is the matrix associated to the endpoints v^* on scale $k+1$ (given by $ie_{0,k}\gamma_0$ if v^* is of type (a) with label $\rho = 0$, by $ie_{j,k}v_k\gamma_j$ if v^* is of type e with label $\rho = 1, 2$, or by $-M^k\nu_{\mu,k}$ if v^* is of type ν. In (A.4) it is understood that the operators $\mathcal{R}$ act in the order induced by the tree ordering (i.e., starting from the endpoints and moving toward the root); moreover, the matrix structure of $g_\ell^{(k)}$ is neglected, for simplicity of notations.

An example of Feynman graph. To be concrete, let us apply the rules described above in the evaluation of a simple Feynman graph $\mathcal{G}$ arising in the tree expansion of $\mathcal{V}^{(h-1)}$. Let $\mathcal{G}$ be the diagram in Fig.5, associated to the tree τ drawn in the left part of the figure; let us assume that the sets P_v of the external lines associated to the vertices of τ are all assigned. Setting

$$\bar{e}_{0,h} := e_{0,h} \;, \qquad \bar{e}_{j,h} := v_{h-1}e_{j,h} \;, \qquad \text{(A.5)}$$

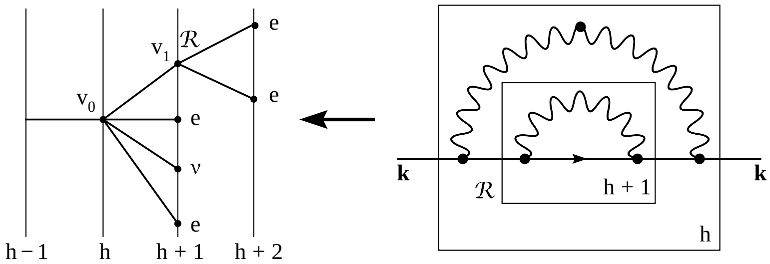

FIG. 5: A possible Feynman diagram contributing to $\mathcal{V}^{(h-1)}$ and its cluster structure.

we can write:

$$\mathrm{Val}(\mathcal{G}) = -\frac{2}{4!2!}\frac{Z_h}{Z_{h-1}}\frac{Z_{h-1}}{Z_{h-2}}\bar{e}^2_{\mu_1,h}\bar{e}^2_{\mu_2,h+1}M^h\nu_{\mu_1,h}\cdot\overline{\psi}_{\mathbf{k}}\Big\{\int\frac{d\mathbf{p}}{(2\pi)^3}\,|w^{(h)}(\mathbf{p})|^2\cdot \tag{A.6}$$

$$\cdot\gamma_{\mu_1}g^{(h)}(\mathbf{k}+\mathbf{p})\mathcal{R}\Big[\int\frac{d\mathbf{q}}{(2\pi)^3}\,\gamma_{\mu_2}g^{(h+1)}(\mathbf{k}+\mathbf{p}+\mathbf{q})\gamma_{\mu_2}w^{(h+1)}(\mathbf{q})\Big]\cdot g^{(h)}(\mathbf{k}+\mathbf{p})\gamma_{\mu_1}\Big\}\psi_{\mathbf{k}}\ ,$$

where $\mathcal{R}\big[F(\mathbf{k}+\mathbf{p})\big] = F(\mathbf{k}+\mathbf{p}) - F(\mathbf{0}) - (\mathbf{k}+\mathbf{p})\cdot\nabla F(\mathbf{0}) \equiv \frac{1}{2}(k_\mu+p_\mu)(k_\nu+p_\nu)\partial_\mu\partial_\nu F(\mathbf{k}^*)$. Notice that the same Feynman graph appears in the evaluation of other trees, which are topologically equivalent to the one represented in the left part of Fig.5 and that can be obtained from it by: (i) relabeling the fields in P_{v_1}, P_{v_0}, (ii) relabeling the endpoints of the tree, (iii) exchanging the relative positions of the topologically different subtrees with root v_0. If one sums over all these trees, the resulting value one obtains is the one in Eq.(A.6) times a combinatorial factor $2^2\cdot 3\cdot 4$ (2^2 is the number of ways for choosing the fields in P_{v_1} and in P_{v_0}; 3 is the number of ways in which one can associate the label ν to one of the endpoints of scale $h+1$; 4 is the number of distinct unlabelled trees that can be obtained by exchanging the positions of the subtrees with root v_0).

Dimensional bounds. We are now ready to derive a general bound for the Feynman graphs produced by the multiscale integration. Let $\bar{\varepsilon}_h := \sup_{h<k\le 0}\{|e_{0,k}|,|e_{1,k}|,|\nu_k|\}$ and let us inductively assume that $C^{-1}\le v_k\le 1$ and $Z_k/Z_{k-1}\le e^{C\bar{\varepsilon}_h^2}$, for a suitable constant C (these inductive bounds are proven in Section 4 at all orders, using the beta function equation). Using the bounds

$$\big\|g^{(h)}(\mathbf{k})\big\|\le \mathrm{const}\cdot M^{-h}\ ,\qquad \int d\mathbf{k}\big\|g^{(h)}(\mathbf{k})\big\|\le\mathrm{const}\cdot M^{2h}\ ,$$

$$\big|w^{(h)}(\mathbf{k})\big|\le \mathrm{const}\cdot M^{-h}\ ,\qquad \int d\mathbf{k}\big|w^{(h)}(\mathbf{k})\big|\le\mathrm{const}\cdot M^{2h}\ , \tag{A.7}$$

into (A.4), we find that, if $\tau \in \mathcal{T}_{h,N}$ and $\mathcal{G} \in \Gamma(\tau)$,

$$|\widehat{\mathrm{Val}}(\mathcal{G})| \le (\mathrm{const.})^N \bar{\varepsilon}_h^N \prod_{v \text{ not e.p.}} M^{-3h_v(s_v-1)} M^{2h_v n_v^0} M^{h_v m_v^\nu} \prod_{\substack{v \text{ not e.p.} \\ v>v_0}} M^{-z_v(h_v-h_{v'})} , \qquad (A.8)$$

where: n_v^0 is the number of propagators $\ell \in v$, i.e., of propagators ℓ contained in the box G_v but not in any smaller cluster; s_v is the number of vertices immediately following v on τ; m_v^ν is the number of end-points of type ν immediately following v on τ (i.e., contained in G_v but not in any smaller cluster); v' is the vertex immediately preceding v on τ and $z_v = 2$ if $|P_v^\psi| = |P_v| = 2$, $z_v = 1$ is $|P_v^\psi| = 2|P_v^A| = 2$ and $z_v = 0$ otherwise. The last product in (A.8) is due to the action of $\mathcal{R}$ on the vertices $v > v_0$ that are not end-points. In fact, the operator $\mathcal{R}$, when acting on a kernel $W_{1,1}^{(h_v)}(\mathbf{p}, \mathbf{k})$ associated to a vertex v with $|P_v^\psi| = 2|P_v^A| = 2$, extracts from $W_{1,1}^{(h_v)}$ the rest of first order in its Taylor expansion around $\mathbf{p} = \mathbf{k} = \mathbf{0}$: if $|W_{1,1}^{(h_v)}(\mathbf{p}, \mathbf{k})| \le C(v)$, then $|\mathcal{R}W_{1,1}^{(h_v)}(\mathbf{p}, \mathbf{k})| = \frac{1}{2}|(\mathbf{p}\partial_{\mathbf{p}} + \mathbf{k}\partial_{\mathbf{k}})W_{1,1}^{(h_v)}(\mathbf{p}^*, \mathbf{k}^*)| \le (\mathrm{const.})M^{-h_v+h_{v'}}C(v)$, where M^{-h_v} is a bound for the derivative with respect to momenta on scale h_v and $M^{h_{v'}}$ is a bound for the external momenta $\mathbf{p}$, $\mathbf{k}$; i.e., $\mathcal{R}$ is dimensionally equivalent to $M^{-(h_v-h_{v'})}$. Similarly, if $\mathcal{R}$ acts on a terms with $|P_v^\psi| = |P_v| = 2$, it extracts the rest of second order in the Taylor expansion around $\mathbf{k} = \mathbf{0}$, and it is dimensionally equivalent to $\mathbf{k}^2\partial_{\mathbf{k}}^2 \sim M^{-2(h_v-h_{v'})}$. As a result, we get (A.8).

Now, let n_v^e (n_v^ν) be the number of vertices of type e (of type ν) following v on τ. If we plug in (A.8) the identities

$$\begin{aligned} \sum_{v \text{ not e.p.}} (h_v - h)(s_v - 1) &= \sum_{v \text{ not e.p.}} (h_v - h_{v'})(n_v^e + n_v^\nu - 1) \\ \sum_{v \text{ not e.p.}} (h_v - h)n_v^0 &= \sum_{v \text{ not e.p.}} (h_v - h_{v'})\Big(\frac{3}{2}n_v^e + n_v^\nu - \frac{|P_v|}{2}\Big) \\ \sum_{v \text{ not e.p.}} (h_v - h)m_v^\nu &= \sum_{v \text{ not e.p.}} (h_v - h_{v'})n_v^\nu \end{aligned} \qquad (A.9)$$

we get the bound

$$|\widehat{\mathrm{Val}}(\mathcal{G})| \le (\mathrm{const.})^N \bar{\varepsilon}_h^N M^{h(3-|P_{v_0}|)} \prod_{\substack{v \text{ not e.p.} \\ v>v_0}} M^{(h_v-h_{v'})(3-|P_v|-z_v)} . \qquad (A.10)$$

In the latter equation, $3 - |P_v|$ is the *scaling dimension* of the cluster G_v, and $3 - |P_v| - z_v$ is its renormalized scaling dimension. Notice that the renormalization operator $\mathcal{R}$ has been introduced precisely to guarantee that $3 - |P_v| - z_v < 0$ for all v, by construction. This fact allows us to sum over the scale labels $h \le h_v \le 1$, and to conclude that the perturbative expansion is well defined at any order N of the renormalized expansion. More precisely, the fact that the renormalized scaling dimensions are all negative implies, via a standard argument (see *e.g.* [2, 8, 14]), the following bound, valid for a suitable constant C:

$$||W^{N;(h)}_{m,n,\underline{\rho},\underline{\mu}}|| \le C^N \bar\varepsilon_h^N M^{h(3-m-2n)} \sum_{\tau\in\mathcal{T}_{h,N}} \sum_{\substack{\mathcal{G}\in\Gamma(\tau)\\ |P^A_{v_0}|=m,\\ |P^\psi_{v_0}|=2n}} \prod_{\substack{v \text{ not e.p.}\\ v>v_0}} M^{(h_v-h_{v'})(3-|P_v|-z_v)} \tag{A.11}$$

from which, after counting the number of Feynman graphs contributing to the sum in (A.11), (3.19) follows.

An immediate corollary of the proof leading to (3.19) is that contributions from trees in $\mathcal{T}_{h,N}$ with a vertex v on scale $h_v = k > h$ admit an improved bound with respect to (3.19), of the form $\le$ (const.)$^N \bar\varepsilon_h^N (N/2)!\, M^{h(3-|P_{v_0}|)} M^{\theta(h-k)}$, for any $0 < \theta < 1$; the factor $M^{\theta(h-k)}$ can be thought of as a dimensional gain with respect to the "basic" dimensional bound in (3.19). This improved bound is usually referred to as the *short memory* property (i.e., long trees are exponentially suppressed).

Remark. All the analysis above is based on the fact that the scaling dimension $3-|P_v|$ in (A.10) is independent of the number of endpoints of the tree τ; i.e., the model is *renormalizable.* A rather different situation is found in the case of instantaneous Coulomb interactions, in which case the bosonic propagator is given by $(2|\vec p|)^{-1}$ rather than by $(2|\mathbf{p}|)^{-1}$. In this case, choosing the bosonic single scale propagator as $w^{(h)}(\mathbf{p}) = \chi_0(\mathbf{p}) f_h(\vec p)(2|\vec p|)^{-1}$, one finds that the last bound in (A.7) is replaced by $\int d\mathbf{p}\big|w^{(h)}(\mathbf{p})\big| \le$ (const.) M^h (dimensionally, this bound has a factor M^h missing). Repeating the steps leading to (A.10), one finds a general bound valid at all orders, in which the new scaling dimension is $3 - |P_v| + n^e_v + n^\nu_v$; this (pessimistic) general bound assumes that at each scale the loop lines of the graph are all bosonic. Perhaps, this bound can be improved, by taking into account the explicit structure of the expansion; however, it shows that the renormalizability of the istantaneous case, *if true*, does not follow from purely dimensional considerations and its proof will require the implementation of suitable cancellations.

Appendix B: Lowest order computations

In this Appendix we reproduce the details of the second order computations leading to (4.8), (4.9), (4.18), (4.19).

1. Computation of $\beta_h^{z,(2)}$

By definition, see (3.7) and (4.1), $\beta_h^z = z_{0,h} = -i\gamma_0\partial_{k_0}W_{0,1}^{(h)}(\mathbf{0})$. At one-loop, defining $\bar e_{0,h} = e_{0,h}$ and $\bar e_{1,h} = v_{h-1}e_{1,h}$, we find:

$$\begin{aligned}\beta_h^{z,(2)} = z_{0,h}^{(2)} &= -i\gamma_0\bar e^2_{\mu,h+1}\int\frac{d\mathbf{p}}{(2\pi)^3}\partial_{p_0}\Big(\frac{f_{h+1}(\mathbf{p})}{2|\mathbf{p}|}\Big)\gamma_\mu g^{(h+1)}(\mathbf{p})\gamma_\mu + \\ &- i\gamma_0\bar e^2_{\mu,h+2}\Big(\frac{Z_{h+1}}{Z_h}\Big)^2\int\frac{d\mathbf{p}}{(2\pi)^3}\partial_{p_0}\Big(\frac{f_{h+2}(\mathbf{p})}{2|\mathbf{p}|}\Big)\gamma_\mu g^{(h+1)}(\mathbf{p})\gamma_\mu + \\ &- i\gamma_0\bar e^2_{\mu,h+2}\frac{Z_{h+1}}{Z_h}\int\frac{d\mathbf{p}}{(2\pi)^3}\partial_{p_0}\Big(\frac{f_{h+1}(\mathbf{p})}{2|\mathbf{p}|}\Big)\gamma_\mu g^{(h+2)}(\mathbf{p})\gamma_\mu\ .\end{aligned}\tag{B.1}$$

Using inductively the beta function equations for $Z_{h+1}, v_{h+1}, e_{\mu,h+2}$, and neglecting higher order terms, we can rewrite (B.1) as

$$\begin{aligned}z_{0,h}^{(2)} &= i\gamma_0\bar e^2_{\mu,h+1}\frac12\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{p_0^2}{|\mathbf{p}|^3}\frac{i\gamma_\mu\gamma_0\gamma_\mu}{p_0^2+v_h^2|\vec p|^2}\Big[(f_{h+1}(\mathbf{p})-|\mathbf{p}|f'_{h+1}(\mathbf{p}))(f_{h+1}(\mathbf{p})+f_{h+2}(\mathbf{p})) + \\ &\qquad +(f_{h+2}(\mathbf{p})-|\mathbf{p}|f'_{h+2}(\mathbf{p}))f_{h+1}(\mathbf{p})\Big]\ .\end{aligned}\tag{B.2}$$

Passing to radial coordinates, $\mathbf{p} = p(\cos\theta, \sin\theta\cos\varphi, \sin\theta\sin\varphi)$, and using the fact that $\int dp(f'_{h+1}f_{h+1} + f'_{h+1}f_{h+2} + f'_{h+2}f_{h+1}) = 0$, we find:

$$z_{0,h}^{(2)} = (2v_h^2e_{1,h}^2 - e_{0,h}^2)\frac{1}{8\pi^2}\Big[\int_0^\infty\frac{dp}{p}(f_{h+1}^2+2f_{h+1}f_{h+2})\Big]\cdot\Big[\int_{-1}^1 d\cos\theta\,\frac{\cos^2\theta}{\cos^2\theta+v_h^2\sin^2\theta}\Big]\ .\tag{B.3}$$

The integral over the radial coordinate p can be computed by using the definition (3.1):

$$\begin{aligned}\int_0^\infty\frac{dp}{p}(f_{h+1}^2+2f_{h+1}f_{h+2}) &= \int_0^\infty\frac{dp}{p}[2(\chi(p)-\chi(Mp))-(\chi^2(p)-\chi^2(Mp))] = \\ &= \lim_{\varepsilon\to0}\int_\varepsilon^{M\varepsilon}\frac{dp}{p} = \log M\ .\end{aligned}\tag{B.4}$$

Finally, an explicit evaluation of the integral over $d\cos\theta$ leads to (4.18).

2. Computation of $z_{1,h}^{(2)}$

By definition, see (3.7) and (4.1), $\beta_h^{v,(2)} = z_{1,h}^{(2)} - v_h z_{0,h}^{(2)}$, with $z_{1,h} = -i\gamma_1 \partial_{k_1} W_{0,1}^{(h)}(\mathbf{0})$. At second order, proceeding as in the derivation of (B.2), we find:

$$\begin{aligned} z_{1,h}^{(2)} &= i\gamma_1 \bar{e}_{\mu,h+1}^2 \frac{1}{2} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{p_1^2}{|\mathbf{p}|^3} \frac{i\gamma_\mu v_h \gamma_1 \gamma_\mu}{p_0^2 + v_h^2 |\vec{p}|^2} \Big[(f_{h+1}(\mathbf{p}) - |\mathbf{p}| f'_{h+1}(\mathbf{p}))(f_{h+1}(\mathbf{p}) + f_{h+2}(\mathbf{p})) + \\ &\qquad + (f_{h+2}(\mathbf{p}) - |\mathbf{p}| f'_{h+2}(\mathbf{p})) f_{h+1}(\mathbf{p}) \Big] = \qquad \text{(B.5)} \\ &= e_{0,h}^2 v_h \frac{1}{16\pi^2} \Big[\int_0^\infty \frac{dp}{p} (f_{h+1}^2 + 2 f_{h+1} f_{h+2}) \Big] \cdot \Big[\int_{-1}^1 d\cos\theta \, \frac{\sin^2\theta}{\cos^2\theta + v_h^2 \sin^2\theta} \Big] \,. \end{aligned}$$

An explicit evaluation of the integral leads to

$$z_{1,h}^{(2)} = e_{0,h}^2 v_h^{-1} \frac{\log M}{8\pi^2} \Big(\frac{\arctan \xi_h}{\xi_h} - \frac{\xi_h - \arctan \xi_h}{\xi_h^3} \Big) \,, \qquad \text{(B.6)}$$

which, combined with $\beta_h^{v,(2)} = z_{1,h}^{(2)} - v_h z_{0,h}^{(2)}$, leads to (4.18).

3. Computation of $\beta_{\mu,h}^{\nu,(2)}$

By definition, see (3.6) and (4.1), $\beta_{\mu,h}^{\nu} = -M^{-h+1} W_{2,0,\mu,\mu}^{(h-1)}(\mathbf{0}) - M\nu_{\mu,h}$. At second order, we find:

$$\begin{aligned} \beta_{\mu,h}^{\nu,(2)} &= -M^{-h+1} \frac{\bar{e}_{\mu,h}^2}{2} \int \frac{d\mathbf{p}}{(2\pi)^3} \operatorname{Tr}\Big(\gamma_\mu g^{(h)}(\mathbf{p}) \gamma_\mu g^{(h)}(\mathbf{p}) \Big) + \\ &\quad - M^{-h+1} \bar{e}_{\mu,h+1}^2 \frac{Z_h}{Z_{h-1}} \int \frac{d\mathbf{p}}{(2\pi)^3} \operatorname{Tr}\Big(\gamma_\mu g^{(h+1)}(\mathbf{p}) \gamma_\mu g^{(h)}(\mathbf{p}) \Big) \,. \qquad \text{(B.7)} \end{aligned}$$

Using inductively the beta function equations for $e_{\mu,h}, Z_{h-1}, v_{h-1}$, and neglecting higher orders, we can rewrite (B.7) as

$$\begin{aligned} \beta_{0,h}^{\nu,(2)} &= -2M^{-h+1} e_{0,h}^2 \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{f_h(\mathbf{p})^2 + 2 f_h(\mathbf{p}) f_{h+1}(\mathbf{p})}{(p_0^2 + v_h^2 |\vec{p}|^2)^2} (-p_0^2 + v_h^2 |\vec{p}|^2) \,, \\ \beta_{1,h}^{\nu,(2)} &= -2M^{-h+1} \bar{e}_{1,h}^2 \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{f_h(\mathbf{p})^2 + 2 f_h(\mathbf{p}) f_{h+1}(\mathbf{p})}{(p_0^2 + v_h^2 |\vec{p}|^2)^2} p_0^2 \,, \qquad \text{(B.8)} \end{aligned}$$

where we used that $\operatorname{Tr}\big(\gamma_\mu \gamma_\alpha \gamma_\mu \gamma_\alpha\big) = -4$ if $\mu \neq \alpha$ and 4 otherwise; passing to radial coordinates we find

$$\begin{aligned} \beta_{0,h}^{\nu,(2)} &= \frac{-2}{(2\pi)^2} M^{-h+1} e_{0,h}^2 \Big[\int_0^\infty dp \, (f_h^2 + 2 f_h f_{h+1}) \Big] \int_{-1}^1 d\cos\theta \, \frac{-\cos^2\theta + v_h^2 \sin^2\theta}{(\cos^2\theta + v_h^2 \sin^2\theta)^2} \\ \beta_{1,h}^{\nu,(2)} &= \frac{-2}{(2\pi)^2} M^{-h+1} \bar{e}_{1,h}^2 \Big[\int_0^\infty dp \, (f_h^2 + 2 f_h f_{h+1}) \Big] \int_{-1}^1 d\cos\theta \, \frac{\cos^2\theta}{(\cos^2\theta + v_h^2 \sin^2\theta)^2} \,. \text{(B.9)} \end{aligned}$$

The integral over the radial coordinate p can be rewritten as, using the definition (3.1):

$$\int_0^\infty dp\left(f_h^2+2f_hf_{h+1}\right)=M^{h-1}(M-1)\int_0^\infty dp\left(2\chi(p)-\chi^2(p)\right)\,. \tag{B.10}$$

Finally, an explicit evaluation of the integral over $d\cos\theta$ leads to (4.8).

Appendix C: Multiscale integration for the correlation functions

The multiscale integration used to compute the partition function $\mathcal{W}(0,0)$, described in Section 3, can be suitably modified in order to compute the two and three-point correlation functions in the reference model with bosonic infrared cutoff on scale h, see (5.1). We start by rewriting the two and three point Schwinger functions in the following way:

$$\begin{aligned}\langle\psi_{\mathbf{k}}\overline{\psi}_{\mathbf{k}}\rangle_{h^*} &= \frac{\partial^2}{\partial\overline{\phi}_{\mathbf{k}}\partial\phi_{\mathbf{k}}}\mathcal{W}_{[h^*,0]}(J,\phi)\big|_{J=\phi=0}\,,\\ \langle j_{\mu,\mathbf{p}};\psi_{\mathbf{k}+\mathbf{p}}\overline{\psi}_{\mathbf{k}}\rangle_{h^*} &= \frac{\partial^3}{\partial J_{\mathbf{p}}\partial\overline{\phi}_{\mathbf{k}+\mathbf{p}}\partial\phi_{\mathbf{k}}}\mathcal{W}_{[h^*,0]}(J,\phi)\big|_{J=\phi=0}\,,\end{aligned} \tag{C.1}$$

where $\mathcal{W}_{[h^*,0]}(J,\phi)$ is the generating function of the reference model.

In order to compute $\mathcal{W}_{[h^*,0]}(J,\phi)$, we proceed in a way analogous to the one described in Appendix A. We iteratively integrate the fields $\psi^{(0)},A^{(0)},\ldots,\psi^{(h+1)},A^{(h+1)},\ldots$, and after the integration of the first $|h|$ scales we are left with a functional integral similar to (3.2), but now involving new terms depending on J,ϕ. Let us first consider the case $h\geq h^*$; the regime $h<h^*$ will be discussed later.

Case $h\geq h^*$. We want to inductively prove that

$$\begin{aligned}e^{\mathcal{W}_{[h^*,0]}(J,\phi)}&=e^{|\Lambda|E_h+\mathcal{S}^{(\geq h)}(J,\phi)}\int P(d\psi^{(\leq h)})P_{[h^*,0]}(dA^{(\leq h)})e^{\mathcal{V}^{(h)}(A^{(\leq h)}+G_AJ,\sqrt{Z_h}\psi^{(\leq h)})}\,\cdot\\ &\cdot e^{\mathcal{B}_\phi^{(h)}(A^{(\leq h)}+G_AJ,\sqrt{Z_h}\psi^{(\leq h)},\phi)+W_R^{(h)}(A^{(\leq h)}+G_AJ,\sqrt{Z_h}\psi^{(\leq h)},\phi)}\,,\end{aligned} \tag{C.2}$$

where: $\mathcal{S}^{(\geq h)}(J,\phi)$ is independent of (A,ψ), $W_R^{(h)}$ contains terms explicitly depending on (A,ψ) and of order ≥ 2 in ϕ, while $\mathcal{B}_\phi^{(h)}$ is given by:

$$\begin{aligned}\mathcal{B}_\phi^{(h)}(A,\sqrt{Z_h}\psi,\phi)&=\int\frac{d\mathbf{k}}{(2\pi)^3}\left[\overline{\phi}_{\mathbf{k}}[Q^{(h+1)}(\mathbf{k})]^\dagger\psi_{\mathbf{k}}+\overline{\psi}_{\mathbf{k}}Q^{(h+1)}(\mathbf{k})\phi_{\mathbf{k}}\right]+\\ &+\int\frac{d\mathbf{k}}{(2\pi)^3}\left[\overline{\phi}_{\mathbf{k}}[G_\psi^{(h+1)}(\mathbf{k})]^\dagger\frac{\partial}{\partial\overline{\psi}_{\mathbf{k}}}\mathcal{V}^{(h)}(A,\sqrt{Z_h}\psi)+\frac{\partial}{\partial\psi_{\mathbf{k}}}\mathcal{V}^{(h)}(A,\sqrt{Z_h}\psi)G_\psi^{(h+1)}(\mathbf{k})\phi_{\mathbf{k}}\right]\,.\end{aligned} \tag{C.3}$$

Moreover, the functions G_A, $Q^{(h)}$, $G_\psi^{(h)}$ are defined by the following relations:

$$
\begin{aligned}
& eG_{A,\mu}(\mathbf{p}) := 1 + \nu_\mu w^{[h^*,0]}(\mathbf{p}) \;, \qquad G_\psi^{(h)}(\mathbf{k}) := \sum_{i=h}^{0} \frac{g^{(i)}(\mathbf{k})}{Z_{i-1}} Q^{(i)}(\mathbf{k}) \;, \\
& Q^{(h)}(\mathbf{k}) := Q^{(h+1)}(\mathbf{k}) - iZ_h z_{\mu,h} k_\mu \gamma_\mu G_\psi^{(h+1)}(\mathbf{k}) \;,
\end{aligned} \tag{C.4}
$$

with $Q^{(1)}(\mathbf{k}) \equiv 1$, $G^{(1)}(\mathbf{k}) \equiv 0$. Note that, if $\mathbf{k}$ is in the support of $g^{(h)}(\mathbf{k})$,

$$
Q^{(h)}(\mathbf{k}) = 1 - iz_{\mu,h} k_\mu \gamma_\mu g^{(h+1)}(\mathbf{k}) \;, \qquad G_\psi^{(h)}(\mathbf{k}) = \frac{g^{(h)}(\mathbf{k})}{Z_{h-1}} Q^{(h)}(\mathbf{k}) + \frac{g^{(h+1)}(\mathbf{k})}{Z_h} \;, \tag{C.5}
$$

that is $||Q^{(h)}(\mathbf{k}) - 1|| \leq (\text{const.})\, \bar\varepsilon_h^2$ and $||G_\psi^{(h)}(\mathbf{k})|| \leq (\text{const.})\, Z_h^{-1} M^{-h}$. Moreover, by the compact support properties of $w^{[h^*,0]}(\mathbf{p})$, $G_{A,\mu}(\mathbf{p}) \equiv e^{-1}$ for all $|\mathbf{p}| \leq M^{h^*}$.

In order to prove (C.2)–(C.4) by induction, let us first check them at the first step. The generating functional of the correlations is defined as (see (2.1)-(2.2))

$$
e^{\mathcal{W}_{[h^*,0]}(J,\phi)} = \int P(d\psi^{(\leq 0)}) P_{[h^*,0]}(dA^{(\leq 0)}) e^{\int \frac{d\mathbf{p}}{(2\pi)^3} (eA^{(\leq 0)}_{\mu,\mathbf{p}} + J_{\mu,\mathbf{p}}) j^{(\leq 0)}_{\mu,\mathbf{p}} - \nu_\mu (A^{(\leq 0)}_\mu, A^{(\leq 0)}_\mu) + B(J,\phi)} \;, \tag{C.6}
$$

which, under the change of variables

$$
A^{(\leq 0)}_{\mu,\mathbf{p}} \to A^{(\leq 0)}_{\mu,\mathbf{p}} + e^{-1} \nu_\mu w^{[h^*,0]}(\mathbf{p}) J_{\mu,\mathbf{p}} \;; \tag{C.7}
$$

can be rewritten in the form (C.2), with $E_h = 0$, $e^2 \mathcal{S}^{(\geq 0)} = \nu_\mu(J_\mu,\, J_\mu) + \nu_\mu^2 (J_\mu,\, w^{[h^*,0]} J_\mu)$, $W_R^{(0)} = 0$, $\mathcal{V}^{(0)} = V$ and $\mathcal{B}_\phi^{(0)} = B$.

Let us now assume that (C.2)–(C.4) are valid at scales $\geq h$, and let us prove that the inductive assumption is reproduced at scale $h-1$. We proceed as in Section 2; first, we renormalize the free measure by reabsorbing into $\widetilde{P}(d\psi^{(\leq h)})$ the term $\exp\{\mathcal{L}_\psi \mathcal{V}^{(h)}\}$, see (3.8)–(3.11), and then we rescale the fields as in (3.12). Similarly, in the definition of $\mathcal{B}_\phi^{(h)}$, Eq.(C.3), we rewrite $\mathcal{V}^{(h)} = \mathcal{L}_\psi \mathcal{V}^{(h)} + \hat{\mathcal{V}}^{(h)}$, combine the terms proportional to $\mathcal{L}_\psi \mathcal{V}^{(h)}$ with those proportional to $Q^{(h+1)}$, and rewrite

$$
\begin{aligned}
& \mathcal{B}^{(h)}(A, \sqrt{Z_h}\psi, \phi) = \hat{\mathcal{B}}^{(h)}(A, \sqrt{Z_{h-1}}\psi, \phi) := \int \frac{d\mathbf{k}}{(2\pi)^3} \Big[\bar\phi_\mathbf{k} [Q^{(h)}(\mathbf{k})]^\dagger \psi_\mathbf{k} + \bar\psi_\mathbf{k} Q^{(h)}(\mathbf{k}) \phi_\mathbf{k}\Big] + \\
& + \int \frac{d\mathbf{k}}{(2\pi)^3} \Big[\bar\phi_\mathbf{k} [G_\psi^{(h+1)}(\mathbf{k})]^\dagger \frac{\partial}{\partial \bar\psi_\mathbf{k}} \hat{\mathcal{V}}^{(h)}(A, \sqrt{Z_{h-1}}\psi) + \frac{\partial}{\partial \psi_\mathbf{k}} \hat{\mathcal{V}}^{(h)}(A, \sqrt{Z_{h-1}}\psi) G_\psi^{(h+1)}(\mathbf{k}) \phi_\mathbf{k}\Big] \;,
\end{aligned}
$$

with $Q^{(h)}$ defined by (C.4). Finally, we rescale $W_R^{(h)}$, by defining $\hat{W}_R^{(h)}(A + G_A \tilde{J}, \sqrt{Z_{h-1}}\psi) := W_R^{(h)}(A + G_A \tilde{J}, \sqrt{Z_h}\psi)$, and perform the integration on scale h:

$$
\begin{aligned}
& \int P(d\psi^{(h)}) P(dA^{(h)}) e^{\hat{\mathcal{V}}^{(h)}(A^{(\leq h)} + G_A J, \sqrt{Z_{h-1}}\psi^{(\leq h)}) + \hat{\mathcal{B}}_\phi^{(h)}(A^{(\leq h)} + G_A J, \sqrt{Z_{h-1}}\psi^{(\leq h)}) + \hat{W}_R^{(h)}} \equiv \\
& \equiv e^{|\Lambda| \tilde{E}_h + \mathcal{S}^{(h-1)}(J,\phi) + \mathcal{V}^{(h-1)}(A^{(\leq h-1)} + G_A J, \sqrt{Z_{h-1}}\psi^{(\leq h-1)}) + \mathcal{B}_\phi^{(h-1)}(A^{(\leq h-1)} + G_A J, \sqrt{Z_{h-1}}\psi^{(\leq h-1)}) + W_R^{(h-1)}} \;,
\end{aligned}
$$

where $\mathcal{S}^{(h-1)}(J,\phi)$ contains terms depending on (J,ϕ) but independent of $(A^{(\leq h-1)},\psi^{(\leq h-1)})$. Defining $\mathcal{S}^{(\geq h-1)} := \mathcal{S}^{(h-1)} + \mathcal{S}^{(\geq h)}$, we immediately see that the inductive assumption is reproduced on scale $h-1$.

Case $h < h^*$. For scales smaller than h^*, there are no more bosonic fields to be integrated out, and we are left with a purely fermionic theory, with scaling dimensions $3-2n$, $2n$ being the number of external fermionic legs, see Theorem 3.1 and following lines. Therefore, once that the two-legged subdiagrams have been renormalized and step by step reabsorbed into the free fermionic measure, we are left with a superrenormalizable theory, as in [9]. In particular, the four fermions interaction is irrelevant, while the wave function renormalization and the Fermi velocity are modified by a finite amount with respect to their values at h^*; that is, if $\bar\varepsilon_{h^*} = \max_{k\geq h^*}\{|e_{\mu,k}|,|\nu_{\mu,k}|\}$:

$$Z_h = Z_{h^*}(1+O(\bar\varepsilon^2_{h^*}))\;, \qquad v_h = v_{h^*}(1+O(\bar\varepsilon^2_{h^*}))\;. \tag{C.8}$$

Tree expansion for the 2*-point function.* As for the partition function, the kernels of the effective potentials produced by the multiscale integration of $\mathcal{W}_{[h^*,0]}(J,\phi)$ can be represented as sums over trees, which in turn can be evaluated as sums over Feynman graphs. Let us consider first the expansion for the 2-point Schwinger function. After having taken functional derivatives with respect to $\phi_{\mathbf{k}}$, $\overline{\phi}_{\mathbf{k}}$ and after having set $J=\phi=0$, we get an expansion in terms of a new class of trees $\tau\in\mathcal{T}^{(h^*)}_{\bar k,\bar h,N}$, with $\bar k\in(-\infty,-1]$ the scale of the root and $\bar h>\bar k$; these trees are similar to the ones described in Appendix A, up to the following differences.
(1) There are $N+2$ end–points and two of them, called v_1,v_2, are special and, respectively, correspond to $\big[Q^{(h_{v_1}-1)}(\mathbf{k})\big]^\dagger\psi^{(\leq h_{v_1}-1)}_{\mathbf{k}}$ or to $\overline{\psi}^{(\leq h_{v_2}-1)}_{\mathbf{k}}Q^{(h_{v_2}-1)}(\mathbf{k})$.
(2) The first vertex whose cluster contains both v_1, v_2, denoted by $\bar v$, is on scale $\bar h$. No $\mathcal{R}$ operation is associated to the vertices on the line joining $\bar v$ to the root.
(3) There are no lines external to the cluster corresponding to the root.
(4) There are no bosonic lines external to clusters on scale $h<h^*$.

In terms of the new trees, we can expand the 2-point Schwinger function as:

$$\langle\psi_{\mathbf{k}}\overline{\psi}_{\mathbf{k}}\rangle_{h^*} = \sum_{j=h_{\mathbf{k}}}^{h_{\mathbf{k}}+1}[Q^{(j)}_\psi(\mathbf{k})]^\dagger\frac{g^{(j)}(\mathbf{k})}{Z_{j-1}}Q^{(j)}(\mathbf{k}) + \sum_{N=2}^{\infty}\sum_{\bar h=-\infty}^{0}\sum_{\bar k=-\infty}^{\bar h-1}\sum_{\tau\in\mathcal{T}^{(h^*)}_{\bar k,\bar h,N}}\mathcal{S}_2(\tau;\mathbf{k})\;, \tag{C.9}$$

where $h_{\mathbf{k}}<0$ is the integer such that $M^{h_{\mathbf{k}}}\leq|\mathbf{k}|<M^{h_{\mathbf{k}}+1}$, and $\mathcal{S}_2(\tau;\mathbf{k})$ is defined in a way similar to $\mathcal{V}^{(h)}(\tau)$ in (A.4), modulo the modifications described in items (1)-(4) above.

Using the bounds described immediately after (C.5), which are valid for $\mathbf{k}$ belonging to the support of $g^{(h)}(\mathbf{k})$, and proceeding as in Appendix A, we get bounds on $\mathcal{S}_2(\tau;\mathbf{k})$, which are the analogues of Theorem 3.1:

$$\sum_{\bar h=-\infty}^{0}\sum_{\bar k=-\infty}^{\bar h-1}\sum_{\tau\in\mathcal{T}^{(h^*)}_{\bar k,\bar h,N}}||\mathcal{S}_2(\tau;\mathbf{k})||\le(\text{const.})^N\bar\varepsilon^N_{h^*}\big(\frac{N}{2}\big)!\frac{\gamma^{-h_{\mathbf{k}}}}{Z_{h_{\mathbf{k}}}}\ . \tag{C.10}$$

In order to understand (C.10), it is enough to notice that, as far as dimensional bounds are concerned, the vertices v_1 and v_2 play the role of two ν vertices with an external line (the ϕ line) and an extra $Z^{-1/2}_{h_{\mathbf{k}}}M^{-h_{\mathbf{k}}}$ factor each. Morever, since the vertices on the path $\mathcal{P}_{r,\bar v}$ connecting the root with $\bar v$ are not associated with any $\mathcal{R}$ operation, we need to multiply the value of the tree $\tau\in\mathcal{T}^{(h^*)}_{\bar k,\bar h,N}$ by $M^{(1/2)(\bar h-\bar k)}M^{(1/2)(\bar k-\bar h)}$, and to exploit the factor $M^{(1/2)(\bar k-\bar h)}$ in order to renormalize all the clusters in $\mathcal{P}_{r,\bar v}$. Therefore,

$$\sum_{\bar h=-\infty}^{0}\sum_{\bar k=-\infty}^{\bar h-1}\sum_{\tau\in\mathcal{T}^{(h^*)}_{\bar k,\bar h,N}}||\mathcal{S}_2(\tau;\mathbf{k})||\le(\text{const.})^N\,\big(\frac{N}{2}\big)!\,\bar\varepsilon^N_{h^*}\frac{1}{Z_{h_{\mathbf{k}}}}\sum_{\bar h\le h_{\mathbf{k}}}\sum_{\bar k\le\bar h}M^{\bar k}M^{\bar h-h_{\mathbf{k}}}M^{(1/2)(\bar h-\bar k)}M^{-2h_{\mathbf{k}}} \tag{C.11}$$

where: the factor $M^{\bar k}$ is due to the fact that graphs associated to the trees $\tau\in\mathcal{T}^{(h^*)}_{\bar k,\bar h,N}$ have two external lines; the factor $M^{\bar h-h_{\mathbf{k}}}$ is given by the product of the two short memory factors associated to the two paths connecting $\bar v$ with v_1 and v_2, respectively; the "bad" factor $M^{(1/2)(\bar h-\bar k)}$ is the price to pay to renormalize the vertices in $\mathcal{P}_{r,\bar v}$; the $Z^{-1}_{h_{\mathbf{k}}}$ and the last $M^{-2h_{\mathbf{k}}}$ are due to the fact that v_1, v_2 behave dimensionally as ν vertices times an extra $Z^{-1/2}_{h_{\mathbf{k}}}M^{-h_{\mathbf{k}}}$ factor. Performing the summation over $\bar k$ and $\bar h$ in (C.11), we get (C.10). Note also that, if $\mathbf{k}$ and $\mathbf{k}+\mathbf{p}$ are on scale $h_{\mathbf{k}}\simeq h^*$, then the derivatives of $||\mathcal{S}_2(\tau;\mathbf{k})||$ can be dimensionally bounded as

$$\sum_{\bar h=-\infty}^{0}\sum_{\bar k=-\infty}^{\bar h-1}\sum_{\tau\in\mathcal{T}^{(h^*)}_{\bar k,\bar h,N}}||\partial^n_{\mathbf{k}}\mathcal{S}_2(\tau;\mathbf{k})||\le(\text{const.})^N\bar\varepsilon^N_{h^*}\big(\frac{N}{2}\big)!\frac{\gamma^{-(1+n)h_{\mathbf{k}}}}{Z_{h_{\mathbf{k}}}}\ , \tag{C.12}$$

from which the bound on $\tilde r_\mu(\mathbf{k},\mathbf{p})$ stated after (5.9)-(5.10) immediately follows.

Tree expansion for the 3*-point function.* Let us pick $|\mathbf{k}|=M^{h^*}$, $|\mathbf{k}+\mathbf{p}|\le M^{h^*}$ and $|\mathbf{p}|\ll M^{h^*}$, which is the condition that we need in order to apply Ward Identities in the form described in Section 5. In this case, the expansion of 3-point function $\langle j_{\mu,\mathbf{p}};\psi_{\mathbf{k}+\mathbf{p}}\overline{\psi}_{\mathbf{k}}\rangle_{h^*}$ is very similar to the one just described for the 2-point function. The result can be written in the form

$$\langle j_{\mu,\mathbf{p}};\psi_{\mathbf{k}+\mathbf{p}}\overline{\psi}_{\mathbf{k}}\rangle_{h^*}=i\frac{\bar{e}_{\mu,h^*}}{e}[G_\psi^{(h^*-1)}(\mathbf{k}+\mathbf{p})]^\dagger\gamma_\mu g^{(h^*)}(\mathbf{k})Q^{(h^*)}(\mathbf{k})+\sum_{\substack{N\geq1,\\ \bar{h}\leq h^*}}\sum_{\substack{\bar{k}<\bar{h},\\ h_{v_3}>h^*}}\sum_{\tau\in\mathcal{T}^{(h^*)}_{\bar{k},\bar{h},h_{v_3},N}}\mathcal{S}_3(\tau;\mathbf{k},\mathbf{p}) \tag{C.13}$$

where $\mathcal{T}^{(h^*)}_{\bar{k},\bar{h},h_{v_3},N}$ is a new class of trees, with $\bar{k}<0$ the scale of the root, similar to the trees in $\mathcal{T}^{(h^*)}_{\bar{k},\bar{h},N}$, up to the fact that they have $N+3$ endpoints rather than $N+2$ (see item (1) in the list preceding (C.9)); three of them are special: v_1 and v_2 are associated to the same contributions described in item (1) above, while v_3 is associated to a contribution $Z_{h_{\bar{v}_3}-1}(e_{\mu,h_{\bar{v}_3}}/e)j^{(\leq h_{\bar{v}_3})}_{\mu,\mathbf{p}}-M^{h_{\bar{v}_3}}(\nu_{\mu,h_{\bar{v}_3}}/e)A_{\mu,\mathbf{p}}$, with $\bar{v}_3$ the vertex immediately preceding v_3 on τ (which the endpoint v_3 is attached to) and $h_{v_3}>h^*$. The value of the tree, $\mathcal{S}_3(\tau;\mathbf{k},\mathbf{p})$, is defined in a way similar to $\mathcal{S}_2(\tau;\mathbf{k})$, modulo the modifications described above. $\mathcal{S}_3(\tau;\mathbf{k},\mathbf{p})$ admits bounds analogous to (C.10)-(C.11); recalling that $|\mathbf{k}|=M^{h^*}$, $|\mathbf{k}+\mathbf{p}|\leq M^{h^*}$ and $|\mathbf{p}|\ll M^{h^*}$, we find:

$$\sum_{\bar{h}=-\infty}^{h^*}\sum_{\bar{k}=-\infty}^{\bar{h}-1}\sum_{h_{v_3}=h^*+1}^{1}\sum_{\tau\in\mathcal{T}^{(h^*)}_{\bar{k},\bar{h},h_{v_3},N}}||\mathcal{S}_3(\tau;\mathbf{k},\mathbf{p})||\leq \tag{C.14}$$
$$\leq(\text{const.})^N\,\big(\frac{N}{2}\big)!\,\bar{\varepsilon}^N_{h^*}\frac{1}{Z_{h^*-1}}\sum_{\substack{\bar{h}\leq h^*\\ \bar{k}<\bar{h}\\ h_{v_3}>h^*}}M^{(1/2)(\bar{k}-\bar{h})}M^{\bar{h}-h^*}M^{(1/2)(h^*-h_{v_3})}M^{-2h^*}\;,$$

where: $M^{(1/2)(\bar{k}-\bar{h})}$ is the short memory factor associated to the path between the root and $\bar{v}$; $M^{\bar{h}-h^*}$ is the product of the two short memory factors associated to the paths connecting $\bar{v}$ with v_1 and v_2, respectively; $M^{(1/2)(h^*-h_{v_3})}$ is the short memory factor associated to a path between h^* and v_3; M^{-2h^*}/Z_{h^*-1} is the product of two factors $M^{-h_{\mathbf{k}}}Z^{-1/2}_{h_{\mathbf{k}}-1}$ associated to the vertices v_1 and v_2 (see the discussion following (C.10) and recall that in this case $h_{\mathbf{k}}=h^*$). We remark that in this case, contrary to the case of the 2-point function, the fact that there is no $\mathcal{R}$ operator acting on the vertices on the path between the root and $\bar{v}$ does not create any problem, since those vertices are automatically irrelevant (they behave as vertices with at least 5 external lines, i.e., J, ϕ, $\bar{\phi}$ and at least two fermionic lines) and, therefore, $\mathcal{R}=1$ on them. Note also that the vertices of type $J\phi\psi$, which have an $\mathcal{R}$ operator acting on, can only be on scale h^*-1 or h^* (by conservation of momentum) and, therefore, the action of the $\mathcal{R}$ operator on such vertices automatically gives the usual dimensional gain of the form const. $M^{h_v-h_{v'}}$, see Appendix A. Performing the summations over $\bar{k},\bar{h},h_{v_3}$ in (C.14), we find the analogue of (C.10):

$$\sum_{\bar h=-\infty}^{h^*}\sum_{\bar k=-\infty}^{\bar h-1}\sum_{h_{v_3}=h^*+1}^{1}\sum_{\tau\in\mathcal{T}^{(h^*)}_{\bar k,\bar h,h_{v_3},N}} ||\mathcal{S}_3(\tau;\mathbf{k},\mathbf{p})|| \le (\text{const.})^N\,\big(\frac{N}{2}\big)!\,\bar\varepsilon^N_{h^*}\frac{M^{-2h^*}}{Z_{h^*-1}}\ , \tag{C.15}$$

from which the bound on $r_\mu(\mathbf{k},\mathbf{p})$ stated after (5.9)-(5.10) immediately follows.

Proof of (5.14). Let $h^*=h$, $\mathbf{k}_0=(M^h,0,0)$, $\mathbf{k}_1=(0,M^h,0)$ and $\mathbf{k}_2=(0,0,M^h)$. Recalling the definition of $A'_{\mu,h}$, i.e., $A'_{\mu,h}:=A_{\mu,h}+\alpha_\mu(e_{\mu,h}/e-1)$, and using the definitions of α_μ and $A_{\mu,h}$, see (4.12)-(4.13) and (5.11)-(5.12), one recognizes that (5.14) follows from analogous bounds on $M^{2h}\tilde r_\mu(\mathbf{k}_\mu,\mathbf{0})$, $M^{2h}r_\mu(\mathbf{k}_\mu,\mathbf{0})$ and $M^{2h}R_{\mu,h}(\mathbf{k}_\mu)$. Let us now prove the relevant bounds for $M^{2h}\tilde r_\mu(\mathbf{k}_\mu,\mathbf{0})$ and $M^{2h}r_\mu(\mathbf{k}_\mu,\mathbf{0})$; the bounds on $M^{2h}R_{\mu,h}(\mathbf{k}_\mu,\mathbf{0})$ will follow from the discussion in Appendix D. In the perturbative expansions for $M^{2h}(\tilde r_0(\mathbf{k}_0,\mathbf{0})-\tilde r_1(\mathbf{k}_1,\mathbf{0}))$ and $M^{2h}(r_0(\mathbf{k}_0,\mathbf{0})-r_1(\mathbf{k}_1,\mathbf{0}))$, we rewrite all the propagators in the form: $g^{(k)}(\mathbf{k})=g^{(k)}_{rel}(\mathbf{k})+\tilde g^{(k)}(\mathbf{k})$, where $g^{(k)}_{rel}(\mathbf{k})$ is the single scale propagator with v_{k-1} replaced by 1, while $\tilde g^{(k)}(\mathbf{k})$ satisfies the same dimensional bounds as $g^{(k)}(\mathbf{k})$ times a factor $(1-v_{k-1})$; moreover, we rewrite $e_{1,k}=e_{2,k}=:e_{0,k}+\delta^e_k$ and $\nu_{1,k}=\nu_{2,k}=:\nu_{0,k}+\delta^\nu_k$. Correspondingly, we expand the trees contributing to r_μ and $\tilde r_\mu$ into a sum of modified labelled trees, similar to the original ones, but with further labels on the fields and the vertices, specifying whether a given fermionic line is associated to $g^{(k)}_{rel}(\mathbf{k})$ or to $\tilde g^{(k)}(\mathbf{k})$, and whether a given vertex is associated to $e_{0,k}$ ($\nu_{0,k}$) or to δ^e_k (δ^ν_k). As already observed in Section 5, the key remark is that, when considering the differences $M^{2h}(\tilde r_0(\mathbf{k}_0,\mathbf{0})-\tilde r_1(\mathbf{k}_1,\mathbf{0}))$ or $M^{2h}(r_0(\mathbf{k}_0,\mathbf{0})-r_1(\mathbf{k}_1,\mathbf{0}))$, the contributions associated to the "relativistic" trees (i.e., the trees whose field lines are all associated to $g^{(k)}_{rel}(\mathbf{k})$ and whose vertices are all associated to $e_{0,h}$ or $\nu_{0,h}$) are exactly vanishing, by relativistic invariance. Therefore, the only non vanishing contributions to these differences involve trees with at least one line of type $\tilde g^{(k)}$ or one vertex of type $\delta^\#_k$. On the other hand, the sum over the trees contributing to $r_0(\mathbf{k}_0,\mathbf{0})-r_1(\mathbf{k}_1,\mathbf{0})$ with at least one line of type $\tilde g^{(k)}$ (resp. one vertex of type $\delta^\#_k$) can be bounded by the r.h.s. of (C.14), times an extra factor $(1-v_k)M^{(1/4)(\bar k-k)}$ (resp. $|\delta^\#_k|M^{(1/4)(\bar k-k)}$) inside the summation, with $M^{(1/4)(\bar k-k)}$ the short memory factor associated to the path between the root and the vertex containing $\tilde g^{(k)}$ (resp. associated to $\delta^\#_k$). Performing the summation over the scale indices, and using (4.10), (4.11), we find:

$$M^{2h}|r_0(\mathbf{k}_0,\mathbf{0})-r_1(\mathbf{k}_1,\mathbf{0})|\le(\text{const.})\Big[e^2(1-v_h)+|e|\big|e_{0,h}-e_{1,h}\big|\Big]\ , \tag{C.16}$$

which is the desired estimate; the bound on $\tilde r_0(\mathbf{k}_0,\mathbf{0})-\tilde r_1(\mathbf{k}_1,\mathbf{0})$ is derived in the same way.

Appendix D: Multiscale integration of the correction term to the WI

In this Appendix we prove (5.7) and the bound on $R_{\mu,h}$ stated right after (5.7). We assume that $h = h^*$, $|\mathbf{k}| = M^h$ and $|\mathbf{p}| \ll M^h$. We start by rewriting

$$\frac{p_\mu}{Z_h} R_{\mu,h^*}(\mathbf{k},\mathbf{p}) = \frac{\partial^3}{\partial \tilde{J}_{\mathbf{p}} \partial \overline{\phi}_{\mathbf{k}+\mathbf{p}} \partial \phi_{\mathbf{k}}} \widetilde{\mathcal{W}}_{[h,0]}(\tilde{J},\phi)\big|_{\tilde{J}=\phi=0} \,, \tag{D.1}$$

with $\widetilde{\mathcal{W}}_{[h,0]}(\tilde{J},\phi)$ defined as:

$$e^{\widetilde{\mathcal{W}}_{[h,0]}(\tilde{J},\phi)} := \int P(d\psi) P_{[h,0]}(dA)\, e^{V(A,\psi)+\widetilde{B}(\tilde{J},\phi)} \,, \tag{D.2}$$

and

$$\widetilde{B}(\tilde{J},\phi) = \int \frac{d\mathbf{p}}{(2\pi)^3} \tilde{J}_{\mathbf{p}} \Big[\int \frac{d\mathbf{k}}{(2\pi)^3} \overline{\psi}_{\mathbf{k}+\mathbf{p}} C(\mathbf{k},\mathbf{p}) \psi_{\mathbf{k}} - \alpha_\mu p_\mu j_{\mu,\mathbf{p}} \Big] + \int \frac{d\mathbf{k}}{(2\pi)^3} \left[\phi_{\mathbf{k}} \overline{\psi}_{\mathbf{k}} + \overline{\phi}_{\mathbf{k}} \psi_{\mathbf{k}} \right] . \tag{D.3}$$

The main difference with respect to the generating functional of the correlation functions is the presence of the correction term proportional to $C(\mathbf{k},\mathbf{p})$, see (5.6) for a definition. Eq.(D.2) can again be studied by RG methods, see [3] for further details. A crucial role is played by the properties of the function $C(\mathbf{k},\mathbf{p})$; it is easy to verify that

$$g^{(i)}(\mathbf{k}+\mathbf{p}) C(\mathbf{k},\mathbf{p}) g^{(j)}(\mathbf{k}) \tag{D.4}$$

is non vanishing only if at least one of the indices i, j is equal to 0; moreover, when it is nonvanishing, it is dimensionally bounded from above by (const.)$|\mathbf{p}| M^{-i-j}$.

We start by integrating the scale 0, and we find:

$$e^{\widetilde{\mathcal{W}}_{[h,0]}(\tilde{J},\phi)} = e^{|\Lambda| E_{-1} + \widetilde{\mathcal{S}}^{(\geq -1)}(\tilde{J},\phi)} \int P(d\psi^{(\leq -1)}) P_{[h,-1]}(dA^{\leq -1})\, e^{\mathcal{V}^{(-1)}(A^{(\leq -1)}, \sqrt{Z_{-1}}\psi^{(\leq -1)}) + \widetilde{\mathcal{B}}^{(-1)}} \,, \tag{D.5}$$

where $\widetilde{\mathcal{S}}^{(\geq -1)}$ collects the terms depending on $\tilde{J}, \phi$ but independent of A, ψ, and

$$\widetilde{\mathcal{B}}^{(-1)}(A,\psi) = \widetilde{\mathcal{B}}_J^{(-1)}(A,\psi,\phi) + \mathcal{B}_\phi^{(-1)}(A,\psi) + \widetilde{W}_R^{(-1)} \,, \tag{D.6}$$

with: $\widetilde{\mathcal{B}}_J^{(-1)}(A,\psi,\phi)$ linear in $\tilde{J}$ and independent of ϕ; $\mathcal{B}_\phi^{(-1)}(A,\psi)$ given by (C.3); $\widetilde{W}_R^{(-1)}$ the rest, which is at least quadratic in $(\tilde{J},\phi)$. With respect to the computation of $\mathcal{W}_{[h,0]}(\tilde{J},\phi)$, we now have new marginal terms of the form $\tilde{J}\,\bar{\psi}\psi$, which are contained in $\widetilde{\mathcal{B}}_J^{(-1)}(A,\psi,\phi)$ and need to be renormalized. Let us simbolically represent by $\widetilde{W}_{m,n}^{(-1)}(\{\mathbf{k}_i\},\{\mathbf{q}_i\},\mathbf{p})$ the generic non-trivial kernel appearing in $\widetilde{\mathcal{B}}_J^{(-1)}(A,\psi,\phi)$; m is the number of bosonic external lines (of either $\tilde{J}$ or A type) while $2n$ is the number of ψ fields; $\{\mathbf{k}_i\}, \{\mathbf{q}_i\}$ are respectively the fermionic/bosonic momenta and $\mathbf{p}$ is the momentum flowing through $\tilde{J}$. As usual, these new kernels can be represented as sums over Feynman graphs. The $\tilde{J}$ external line can be attached to a simple vertex, corresponding to the monomial $-\alpha_\mu p_\mu \tilde{J}_{\mathbf{p}} j^{(\le 0)}_{\mu,\mathbf{p}}$, or to a "thick" vertex, representing $\tilde{J}_{\mathbf{p}}\overline{\psi}_{\mathbf{k}+\mathbf{p}} C(\mathbf{k},\mathbf{p})\psi_{\mathbf{k}}$ (the "small circle" associated to the vertex represents the matrix kernel $C(\mathbf{k},\mathbf{p})$, see Fig.6). Let us denote by $W^{(-1),C}_{m,n}$ the contribution to $\widetilde{W}^{(-1)}_{m,n}$ coming from graphs with the $\tilde{J}$ line attached to a thick vertex, see Figure 6. By the

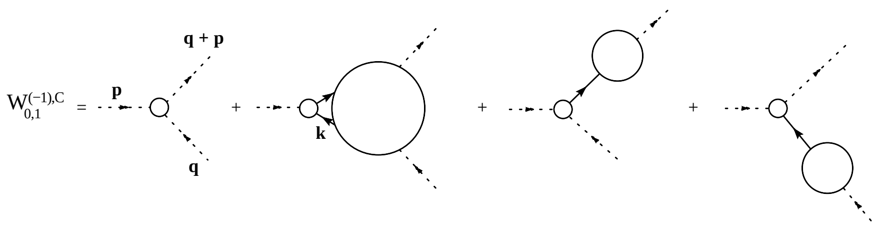

FIG. 6: Schematic representation of the expansion for $W^{(-1),C}_{0,1}$; the small circle represents $C(\mathbf{k},\mathbf{p})$.

properties of the $C(\mathbf{k},\mathbf{p})$ function, see [3] for details, it follows that $\widetilde{W}^{(-1)}_{m,n}(\{\mathbf{k}_i\},\{\mathbf{q}_i\},\mathbf{p}) =: p_\mu \bar{W}^{(-1)}_{m,n,\mu}(\{\mathbf{k}_i\},\{\mathbf{q}_i\},\mathbf{p})$, with $\bar{W}^{(-1)}_{m,n,\mu}$ dimensionally bounded as an $A\overline{\psi}\psi$ kernel, uniformly in $\mathbf{p}$. We define the action of the $\mathcal{R} \equiv 1-\mathcal{L}$ operator on $\bar{W}^{(-1)}_{m,n,\mu}$ in a way similar to (3.5)–(3.7). In particular, $\mathcal{L}\bar{W}^{(-1)}_{0,1,\mu}(\mathbf{k},\mathbf{p}) := \bar{W}^{(-1)}_{0,1,\mu}(\mathbf{0},\mathbf{0})$ and, by symmetry,

$$Z_{-1}\tilde{J}_{\mathbf{p}}\overline{\psi}_{\mathbf{k}}\mathcal{L}\bar{W}^{(-1)}_{0,1,\mu}\psi_{\mathbf{k}+\mathbf{p}} = -Z_{-2}\tilde{J}_{\mathbf{p}}\alpha_{\mu,-1}j^{(\le -1)}_{\mathbf{p}} \ , \tag{D.7}$$

for a real constant $\alpha_{\mu,-1}$, which is by definition the effective α-coupling on scale -1. Note that the last two graphs in Fig.6 do not contribute to $\alpha_{\mu,-1}$ simply because they are one-particle reducible and, therefore, they are vanishing at zero external momenta.

We now iterate the same procedure, and step by step the local parts of the kernels of type $\tilde{J}\bar{\psi}\psi$ are collected together to form a new running coupling constant, $\alpha_{\mu,k}$; in order to show that $R_{\mu,h}$ is dimensionally negligible as $h \to -\infty$, we need to show that it is possible to fix the initial data $\alpha_\mu = \alpha_{\mu,0}$ in such a way that $\alpha_{\mu,h}$ goes exponentially to zero as $\eta \to -\infty$, which is proved in the following.

The flow of $\alpha_{\mu,k}$. The new marginal running coupling constants $\alpha_{\mu,h}$ evolve according to the flow equation: $\alpha_{\mu,k-1} = \alpha_{\mu,k} + \beta^\alpha_{\mu,k}$, where $\alpha_{\mu,0} = \alpha_\mu$ are the counterterms appearing in the bare interaction (D.3). The beta function $\beta^\alpha_{\mu,h}$ can be split as

$$\beta^\alpha_{\mu,k} = \beta^{\alpha,1}_{\mu,k} + \beta^{\alpha,2}_{\mu,k} \ , \tag{D.8}$$

where $\beta^{\alpha,1}_{\mu,k}$ collects the contributions independent of $\alpha_{\mu,k'}$ (which, therefore, are associated to graphs with the $\tilde{J}$ external line emerging from the thick vertex representing $C(\mathbf{k},\mathbf{p})$), and $\beta^{\alpha,2}_{\mu,k}$ collects the terms from graphs with one vertex of type $\alpha_{\mu,k'}$ for some $k' > k$. It is crucial to recall that by the properties of $C(\mathbf{k},\mathbf{p})$, the graphs contributing to $\beta^{\alpha,1}_{\mu,k}$ have at least one propagator on scale 0 or -1; by the short memory property, this means that they can be dimensionally bounded by (const.)$\bar{\varepsilon}^2_k M^{\theta k}$, for any $0 < \theta < 1$. Similarly, the contributions to $\beta^{\alpha,2}_{\mu,k}$ associated to graphs with at least one vertex of type $\alpha_{\mu,k'}$ for some $k' > k$. can be bounded by (const.)$\bar{\varepsilon}^2_k |\alpha_{\mu,k'}| M^{\theta(k-k')}$. The counterterms α_μ are fixed in such a way that $\alpha_{\mu,-\infty} = 0$, i.e., $\alpha_\mu = -\sum_{k=-\infty}^{0}(\beta^{\alpha,1}_{\mu,k} + \beta^{\alpha,2}_{\mu,k})$. Finally, by using the fact that $|\beta^{\alpha,1}_{\mu,k}| \leq$ (const.)$\bar{\varepsilon}^2_k M^{\theta k}$ and $|\beta^{\alpha,2}_{\mu,k}| \leq$ (const.)$\bar{\varepsilon}^2_k |\alpha_{\mu,k'}| M^{\theta(k-k')}$, we find that $|\alpha_{\mu,h}| \leq$ (const.)$\bar{\varepsilon}^2_{h^*} M^{(\theta/2)h}$. This dimensional estimate on $\alpha_{\mu,h}$ easily implies the desired estimate on $R_\mu(\mathbf{k},\mathbf{p})$ stated right after (5.7), as well as the one on $M^{2h}(R_{0,h}(\mathbf{k}_0,\mathbf{0}) - R_{1,h}(\mathbf{k}_1,\mathbf{0}))$, required for concluding the proof of (5.14), and we will not belabor the details here.

Lowest order computation of α_μ. At lowest order, $\alpha^{(2)}_\mu = -\sum_{k\leq 0} \beta^{\alpha,1,(2)}_{\mu,k}$, where $\beta^{\alpha,1,(2)}_{\mu,k}$ is the one-loop contribution to $\beta^{\alpha,1}_{\mu,k}$. Moreover, $\beta^{\alpha,1,(2)}_{\mu,k} = 0$ for all $k \leq -1$. Therefore, neglecting higher order terms, we find $\alpha^{(2)}_\mu = -\beta^{\alpha,1,(2)}_{\mu,0}$, that is (see Fig.7):

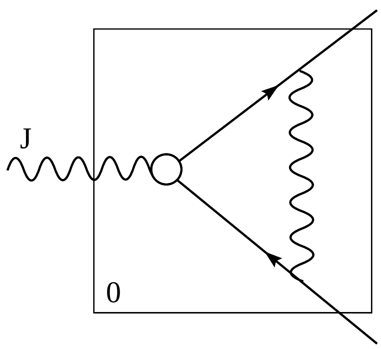

FIG. 7: Lowest order contribution to α_0, α_1.

$$\alpha^{(2)}_0 = -i\gamma_0 \bar{e}^2_{\nu,0} \int \frac{d\mathbf{k}}{(2\pi)^3}\,\gamma_\nu \partial_{p_0}\Big[g^{(0)}(\mathbf{k}+\mathbf{p})C(\mathbf{k},\mathbf{p})g^{(0)}(\mathbf{k})\Big]_{\mathbf{p}=\mathbf{0}} \gamma_\nu w^{(0)}(\mathbf{k})\,, \tag{D.9}$$

$$\alpha^{(2)}_1 = -\frac{i\gamma_1}{v} \bar{e}^2_{\nu,0} \int \frac{d\mathbf{k}}{(2\pi)^3}\,\gamma_\nu \partial_{p_1}\Big[g^{(0)}(\mathbf{k}+\mathbf{p})C(\mathbf{k},\mathbf{p})g^{(0)}(\mathbf{k})\Big]_{\mathbf{p}=\mathbf{0}} \gamma_\nu w^{(0)}(\mathbf{k})\,. \tag{D.10}$$

After a straightforward computation, using the fact that

$$\partial_{p_\mu}\big[g^{(0)}(\mathbf{k}+\mathbf{p})C(\mathbf{k},\mathbf{p})g^{(0)}(\mathbf{k})\big]_{\mathbf{p}=\mathbf{0}} = \frac{1}{i\not{\mathbf{k}}}\big[-i\bar{\gamma}_\mu \chi_0(\mathbf{k})\big(1-\chi_0(\mathbf{k})\big) + i\not{\mathbf{k}}\partial_\mu \chi_0(\mathbf{k})\big]\frac{1}{i\not{\mathbf{k}}}\,,$$

where $\not{k} = k_0\gamma_0 + v\vec{k}\cdot\vec{\gamma}$ and $(\bar{\gamma}_0, \bar{\gamma}_1, \bar{\gamma}_2) = (\gamma_0, v\gamma_1, v\gamma_2)$, we finally get (4.12)-(4.13).

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[1] S. Adler and W. Bardeen, *Phys. Rev.* **182**, 1517-1536 (1969).

[2] G. Benfatto and G. Gallavotti, Jour. Stat. Phys. **59**, 541-664 (1990).

[3] G. Benfatto and V. Mastropietro, Comm. Math. Phys. **258**, 609-655 (2005).

[4] M. Bonini, M. D'Attanasio and G. Marchesini, *Nucl. Phys. B* **418**, 81-112 (1994).

[5] A. H. Castro Neto et al., *Rev. Mod. Phys.* **81**, 109 (2009).

[6] G. Gallavotti, Rev. Mod. Phys. **57**, 471-562 (1985).

[7] G. Gallavotti and F. Nicolò, Comm. Math. Phys. **100**, 545-580 (1985) and **101**, 247-282 (1985).

[8] G. Gentile and V. Mastropietro, Phys. Rep. **352**, 273-437 (2001).

[9] A. Giuliani and V. Mastropietro, *Comm. Math.Phys.* **293**, 301-346 (2010).

[10] A. Giuliani and V. Mastropietro, *Phys. Rev. B* **79**, 201403(R) (2009).

[11] J. González, F. Guinea and M. A. H. Vozmediano, *Nucl. Phys. B* **424**, 595-618 (1994).

[12] J. González, F. Guinea and M. A. H. Vozmediano, *Phys. Rev. B* **63**, 134421 (2001).

[13] J. González, F. Guinea and M. A. H. Vozmediano, *Phys. Rev. B* **59**, R2474 (1999).

[14] V. Mastropietro, *Non-perturbative Renormalization*, World Scientific 2009.

[15] E. G. Mishchenko, *Phys. Rev. Lett.* **98**, 216801 (2007).

[16] K. S. Novoselov et al., *Science* **306**, 666-669 (2004).

[17] K. S. Novoselov et al., Nature (London) **438**, 197 (2005).

[18] J. Polchinski *Nucl. Phys. B* **231**, 269 (1984)

[19] G. W. Semenoff, *Phys. Rev. Lett.* **53**, 2449-2452 (1984).

[20] P. R. Wallace, *Phys. Rev.* **71**, 622-634 (1947).