Developments in the theory of universality

Vieri Mastropietro

Dipartimento di Matematica, Università di Roma Tor Vergata,

V.le della Ricerca Scientifica, 00133 Roma Italy

Recently a rigorous foundation of several aspects of the theory of universality for statistical mechanics models with continuously varying exponents (among which are interacting planar Ising models, quantum spin chains and one dimensional Fermi systems), has been reached; it has its root in the mapping of such systems into fermionic interacting theories, and uses the modern Renormalization Group methods developed in the context of constructive Quantum Field Theory. No use of exact solutions is done and the analysis applies either to solvable or not solvable models. A review of such developments will be given here.

I. INTERACTING ISING MODELS AND QUANTUM SPIN CHAINS

The aim of Statistical mechanics is well explained by the words of Democritus, still actual after 2500 years: "From the ordering and motion of the atoms the properties and modifications of the matter can be understood". A particularly relevant question for statistical mechanics is the understanding of *phase transitions* from one state of matter to another (for instance from liquid to solid) starting from a microscopic description.

Mathematically phase transitions are signaled by singularities of the thermodynamic functions in the *thermodynamic limit* when the number of atoms is sent to infinity. In several cases the singularities have the form of power laws driven by *critical exponents*; given certain microscopic models, one would like to compare the theoretical values with the experiments. However realistic models for matter are extremely complex, as they depend on a number of details like the exact structure of the molecules and the nature of the molecular forces. The computation of exponents in realistic systems is essentially hopeless; even in highly simplified models the exponents cannot be computed, apart in a very small number of *exactly solvable* models.

In this context, the *universality* hypothesis plays a crucial role; it says that the critical properties close to phase transitions should be *insensitive* to microscopic details, at least inside a certain *universality class*. By such hypothesis highly oversimplified models can be used to get information on realistic and complex systems, for which a mathematical analysis would be too difficult. In the case of models in the class of universality of the Ising model, universality simply says that the *critical exponents* are the same as the Ising ones; for instance the experimental value of the exponents of Carbon dioxide or Xenon coincide with the three dimensional Ising model ones [5].

The Ising model in *two* dimensions plays the role of a paradigm for the understanding of critical phenomena. The Hamiltonian is given by

$$H_J = -J \sum_{j=0,1} \sum_{\mathbf{x}\in\Lambda} \sigma_{\mathbf{x}} \sigma_{\mathbf{x}+\mathbf{e}_j} \tag{1.1}$$

where $\sigma_{\mathbf{x}} = \pm$ are spin variables, Λ is a square lattice, $\mathbf{x} \in \Lambda$, $\mathbf{e}_0 = (0,1), \mathbf{e}_1 = (1,0)$. The model is an oversimplified description of a planar magnet in which the dipoles can point in two directions. By the celebrated exact solution by Onsager [45], one can determine the critical temperature $\tanh \beta_c J = \sqrt{2} - 1$ at which the thermodynamic functions have singularities; for instance specific heat is logarithmically diverging at $\beta = \beta_c$, while at the critical temperature the spin-spin correlations decays for large distances with exponent $1/4$ and the energy-energy correlations decays with exponent 2. Note that the indices are *pure numbers*, that is J-independent.

From a physical point of view, there is no reason for which only nearest-neighbor spins should interact; much more reasonable is to assume that the interaction is short ranged, in the sense that it becomes weaker and weaker as more distant spins are considered. In the same way, it is also very natural to include interactions involving four or a greater number of spins. Such considerations suggest to consider a more general Ising model with hamiltonian

$$H = H_J + \lambda V \tag{1.2}$$

where λ is a coupling and V is quartic in the spins, for instance $\sum_j \sum_{\mathbf{x},\mathbf{y}\in\Lambda} v(\mathbf{x} - \mathbf{y}) \sigma_{\mathbf{x}} \sigma_{\mathbf{x}+\mathbf{e}_j} \sigma_{\mathbf{y}} \sigma_{\mathbf{y}+\mathbf{e}_j}$; it belongs to this class a next to nearest neighbor interaction like $\sum_j \sum_{\mathbf{x}\in\Lambda} \sigma_{\mathbf{x}} \sigma_{\mathbf{x}+2\mathbf{e}_j}$. While the λV term in (1.2) appear physically harmless, it has the effect that the exact solvability is completely lost. The universality hypothesis suggests that, while the critical temperature is a non trivial function of λ, the exponents are λ-independent and *coinciding with the Ising ones.*

However, the issue of universality for planar Ising like models is far from trivial. One can consider *two* Ising models coupled by a quartic interaction; the Hamiltonian is

$$H(\sigma,\sigma') = H_J(\sigma) + H_{J'}(\sigma') - \lambda V(\sigma,\sigma') \tag{1.3}$$

with V is a short ranged, quartic interaction in the spin and invariant in the spin exchange, like

$$V = \sum_{j=0,1} \sum_{\mathbf{x},\mathbf{y}\in\Lambda} v(\mathbf{x}-\mathbf{y})\sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j}\sigma'_{\mathbf{y}}\sigma'_{\mathbf{y}+\mathbf{e}_j} \tag{1.4}$$

with $v(\mathbf{x})$ a short range potential.

A system with hamiltonian (1.3) may describe coupled layered magnetic materials. Moreover several systems in statistical mechanics, like the *Ashkin-Teller* and the *Eight Vertex* models, can be rewritten as coupled Ising models. The Ashkin-Teller [2] model is a generalization of the Ising model in which the spin has four values A, B, C, D, and to two neighbor spins is associated an energy E_1 for AA, BB, CC, DD, E_2 for AB, CD, E_3 for AC, BD, E_4 for AD, BC. The Eight Vertex model [5] is a generalization of the Ice model for the hydrogen bounding in which at each point is associated one among eight vertices with four different energies.

Both models can be rewritten in the form of coupled Ising models with Hamiltonian (1.3), see [5]. In the case of the Ashkin-Teller model this is straightforward as it is sufficient to associate to each lattice point $\mathbf{x}$ a couple of spins $(\sigma_{\mathbf{x}}, \sigma'_{\mathbf{x}})$ and to associate the four couples $(\pm 1, \pm 1)$ with the four states A, B, C, D. One can then immediately verify that the Ashkin-Teller hamiltonian is equivalent to (1.3), up to an additive constant $-J_0$, provided that we choose

$$\begin{aligned} -J &= (E_1+E_2-E_3-E_4)/4 \quad & -J' &= (E_1+E_3-E_4-E_2)/4 \\ -\lambda &= (E_1+E_4-E_2-E_3)/4 \quad & -J_0 &= (E_1+E_2+E_3+E_4)/4 \end{aligned} \tag{1.5}$$

and

$$V = \sum_{j=0,1} \sum_{\mathbf{x}\in\Lambda} \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j}\sigma'_{\mathbf{x}}\sigma'_{\mathbf{x}+\mathbf{e}_j} \tag{1.6}$$

For a choice of parameters such that $J = J'$ the Ashkin-Teller model is called *isotropic*, while for $J \neq J'$ is called *anisotropic*. When $\lambda = 0$ the model is exactly solvable as its hamiltonian is the sum of two independent Ising models, and two *critical temperatures* are present if $J \neq J'$ which reduce to one in the $J = J'$ case.

Analogously also the Eight vertex model can be mapped in (1.3) with a suitable identification of the parameters; in such a case $J = J'$ and

$$V = \sum_{j=0,1} \sum_{\mathbf{x}\in\Lambda} \sigma_{\mathbf{x}+j(\mathbf{e}_0+\mathbf{e}_1)}\sigma_{\mathbf{x}+\mathbf{e}_0}\sigma'_{\mathbf{x}+j(\mathbf{e}_0+\mathbf{e}_1)}\sigma'_{\mathbf{x}+\mathbf{e}_1} \tag{1.7}$$

The exact solution for the Eight Vertex model was found by Baxter [4] in the early seventies. For a particular choice of the parameters the Eight vertex model reduces to Six vertex models, previously solved by Lieb [34], Sutherland [51] and Lieb and Wu [36]. From the Baxter solution the specific heat and the correlation length exponent can be computed (see (10.12.22), (10.12.23) of [5]) and it is found that they are *non trivial functions of* λ and different from the Ising ones. This was considered somewhat surprising at that time; coupled Ising models are *not* in the Ising universality class, in contrast to what a too extended application of universality would suggest.

When expressed in terms of Ising spins, the Eight Vertex and the Ashkin-Teller model appear very similar; the interactions (1.6),(1.7) look equivalent as far the long distance properties are considered. Nevertheless, an exact solution is known *only* in the case of the Eight Vertex model and not in the case of Askhin-Teller model or for the generic model (1.3), a fact showing how much exact solvability is a delicate property.

A model very related to the previous ones is the Heisenberg spin chain, physically realized in several compounds like $KCuF_3$. Its Hamiltonian is given by

$$H = -\sum_{x=1}^{L-1}[J_1 S^1_x S^1_{x+1} + J_2 S^2_x S^2_{x+1} - h S^3_x] + \lambda \sum_{1\le x,y\le L} v(x-y) S^3_x S^3_y \tag{1.8}$$

where $S^\alpha_x = \sigma^\alpha_x/2$ for $i = 1, 2, \dots, L$ and $\alpha = 1, 2, 3$, σ^α_x being the Pauli matrices and $|v(x-y)| \le Ce^{-\kappa|x-y|}$. In the case of zero external magnetic field and nearest neighbor interaction ($\lambda = -J_3$)

$$v(x-y) = \delta_{|x-y|,1}/2 \quad h = 0 \tag{1.9}$$

the model is known as XYZ *model* if $J_1 \ne J_2$ (if $J_1 = J_2$ the model is called XXZ model) and it is exactly solvable; remarkably, it appears to be equivalent, with a suitable identification of the parameters, to the Eight Vertex model in the sense that, as it was shown by Sutherland [50], the transfer matrix of the Eight Vertex model commutes with the hamiltonian of the XYZ model. From the exact solution, the index ν of the correlation length of the XYZ model is given by, if $J_1 \ne J_2$ and $h = 0$ (see (10.12.24) of [5])

$$\nu = \frac{\pi}{2\bar\mu} \qquad \cos\bar\mu = -J_3/J_1 \tag{1.10}$$

and, for small J_3/J_1, $\nu = 1 - \frac{2J_3}{\pi J_1} + O((\frac{J_3}{J_1})^2)$. Also in this case the model is not in the Ising universality class.

The spin chain (1.8) can be exactly mapped in a system of interacting fermions through the *Jordan-Wigner* transformation. It is indeed well known that the operators

$$a_x^{\pm} = \prod_{y=1}^{x-1} (-\sigma_y^3) \sigma_x^{\pm} \tag{1.11}$$

are a set of anticommuting fermionic operators and that, if $\sigma_x^{\pm} = (\sigma_x^1 \pm i\sigma_x^2)/2$, we can write

$$\sigma_x^- = e^{-i\pi \sum_{y=1}^{x-1} a_y^+ a_y^-} a_x^- \ , \quad \sigma_x^+ = a_x^+ e^{i\pi \sum_{y=1}^{x-1} a_y^+ a_y^-} \ , \quad \sigma_x^3 = 2a_x^+ a_x^- - 1 \ . \tag{1.12}$$

Hence, if we fix the units so that $J_1 + J_2 = 2$ we get

$$\begin{aligned} H = & -\frac{1}{2} \sum_{x=1}^{L-1} [a_x^+ a_{x+1}^- + a_{x+1}^+ a_x^-] - u \sum_{x=1}^{L-1} [a_x^+ a_{x+1}^+ + a_{x+1}^- a_x^-] \\ & + h \sum_{x=1}^{L} (a_x^+ a_x^- - \frac{1}{2}) + \lambda \sum_{1 \le x,y \le L} v(x-y)(a_x^+ a_x^- - \frac{1}{2})(a_y^+ a_y^- - \frac{1}{2}) \end{aligned} \tag{1.13}$$

where $\rho_x = a_x^+ a_x^-$, $u = (J_1 - J_2)/2$. In this form, the model describes interacting non relativistic 1D fermions on a lattice with a short range interaction and a BCS-like term (in the anisotropic case $J_1 \neq J_2$), and it can be used to describe the properties of the conduction electrons of one-dimensional metals.

II. UNIVERSALITY CONJECTURES

The understanding of the universality issue in models with continously varying exponents grew out from a number of authors, among which Kadanoff and Wegner [31] and Luther and Peschel [35] played a crucial role. It appeared that universality does not mean (contrary to what happens in the Ising case) that all the exponents in the universality class are the same; rather, universality means that the exponents, though model-dependent, verify the same set of *universal relations* allowing to express *every exponents of a single model in terms of any one of them.*

As an example of such universal relations, in the coupled Ising model (1.3) or in Vertex or Ashkin Teller models, the following relations were conjectured

$$X_- = \frac{1}{X_+} \qquad \nu = \frac{1}{2 - X_+} \qquad \alpha = \frac{2 - 2X_+}{2 - X_+} = 2 - 2\nu \tag{2.14}$$

where $X_\pm$ are the exponents of the energy or crossover correlations, ν is the exponents of the correlation length, α the exponent of the specific heat (see below for their exact definition). The exponents depend from the choice of V but the relations are model-independent and, once that an exponent is fixed (say X_+) all the others are determined.

The first of above relations was proposed by Kadanoff [28], the second by Kadanoff and Wegner [31] and the third is the hyperscaling relation. In the case of the Eight vertex model, the index ν and α can be computed from the exact solution in [5], and one can check the validity of the last of (2.14); the indices $X_\pm$ cannot be computed from the solution even in the Eight Vertex case.

In the case of spin chains or 1D fermions Luther and Peschel [35] proposed similar relations, for instance

$$X_- = \frac{1}{X_+} \qquad \bar\nu = \frac{1}{2 - X_-} \tag{2.15}$$

where X_+ is the exponent of the oscillating part of the spin correlation along the third axis, X_- is the exponent of the Cooper pairs correlation and $\bar\nu$ is the correlation length.

In general even the knowledge of a single exponent can be lacking; in the case of spin chains or 1D fermions, Haldane [22] conjectured other relations *allowing the determination of the exponents in terms of thermodynamic quantities*, like the compressibility, which in several cases can be easier to be computed (or measured) . In particular if v_s is the Fermi velocity and κ is the susceptibility, calling $v_N = (\pi\kappa)^{-1}$ the following universal relation was conjectured

$$\frac{v_s}{v_N} = X_+ \tag{2.16}$$

Note that, in the fermion system, $\kappa = \kappa_c \rho^2$, where κ_c is the *fermionic compressibility* and ρ is the fermionic density. The validity of (2.16) can be checked in the XYZ case, using (1.10), the second of (2.15) and the Bethe ansatz results for the Ground state energy, obtained by Yang and Yang [56]

$$v_s = \frac{\pi}{\bar\mu}\sin\bar\mu \qquad \kappa = \frac{1}{[2\pi(\pi/\bar\mu - 1)\sin\bar\mu]} \tag{2.17}$$

The conjecture is that this relation is true generically, for instance in the model (1.8) with $h \neq 0$ and any short range interaction $v(x-y)$.

Eq.(2.16) is part of the *Luttinger liquid* conjecture; the Luttinger model describes interacting fermions in the continuum with a linear dispersion relation, in which all the infinite states with negative energy are filled up. This model was solved by Mattis and Lieb [42] and all the correlations can be computed; the conjecture is that a number of relations valid for this model, like (2.16) , are true for generic spin chains or interacting fermionic models.

There has been a number of attempts of providing a rigorous justifications to the universal relations (2.14),(2.15) and (2.16). Luther and Peschel [35] used bosonization, Kadanoff and Brown [29] used operator product expansion, Den Nijs used a mapping into the Luttinger model, Pruisken and Kadanoff [47] used perturbation expansion.

All such derivations start from a formal continuum limit in which extra Lorentz and Gauge symmetries are verified, which implies the relations between exponents. While the assumption of a continuum scaling limit description of planar lattice models is very powerful, it is well known that a mathematical justification of it is very difficult. Strictly speaking the formal continuum limit is singular, as it is plagued by ultraviolet divergences which were absent in the original lattice model. Moreover lattice effects destroy such symmetries and change the exponents, and it is not clear at all while the relations between exponents should be true also when such symmetries are violated.

The interest in these universal relations has been renewed by recent experiments on materials described by models in this class, like quantum spin chain models (KCuF3) [32], carbon nanotubes [3] or even 1D Bose systems [25]. In such systems the critical exponents depend on the extraordinary complex and largely unknown microscopic details of the compounds, but the universal relations allow concrete and testable predictions for them in terms of a few measurable parameters.

On the other hand, not all the relations which are valid in the special solvable models are generically true; a counterexample is found for the exponents involved in the dynamic correlations [25] and another one will be shown below. The problem of a rigorous derivation of such relation is therefore important both mathematically and physically.

III. FERMIONIZATION FOR THE ISING MODEL

It happens sometimes that problems which appears hard in certain variables looks rather simple in others. This is the case of the Ising model: it is a system of *interacting* spins which can be mapped in a system of *non interacting* fermions. Such a mapping is particularly useful because models which are perturbation of the Ising model, like the Eight Vertex or Ashkin-Teller models, can be mapped in models of interacting fermions, which can be analyzed via the methods developed in Quantum Field Theory.

Schultz, Mattis and Lieb [53] showed that the transfer matrix of the Ising model can be written as the exponential of an hamiltonian of free massive non-relativistic fermions on a chain, which can be diagonalized through a Bogolubov transformation. Another way to see the equivalence of the Ising model with non-interacting fermions was found by Hurst and Green [23] (for an update derivation see Samuel [52]). One starts from the well known formula for the Ising model partition function

$$Z_I = (\cosh \beta J)^B 2^S \sum_\gamma (\tanh \beta J)^{|\gamma|} \, . \tag{3.1}$$

where S is the total number of sites, the sum is over all the multipolygon γ with length $|\gamma|$; if open boundary conditions are assumed, only multipolygons *not* winding up the lattice are allowed. One uses Kasteleyn results [30] to express the Ising model in terms of the dimer covering of a decorated square lattice; the dimers are expressed in terms of Pfaffians which can be naturally expressed in terms of *grassman integrals* so that the Ising partition function can be written as

$$Z_I = -Z_{+,+} + Z_{+,-} + Z_{-,+} + Z_{-,-} \tag{3.2}$$

$$Z_{\epsilon,\epsilon'} = (\cosh \beta J)^B 2^S \frac{1}{2} \int \prod_{\mathbf{x}\in\Lambda} dH_{\mathbf{x}} d\bar{H}_{\mathbf{x}} dV_{\mathbf{x}} d\bar{V}_{\mathbf{x}} e^{S_{\epsilon,\epsilon'}} \tag{3.3}$$

where

$$\begin{aligned} S_{\epsilon,\epsilon'} = & \sum_{\mathbf{x}\in\Lambda} \tanh \beta J [\bar{H}_{x,x_0} H_{x+1,x_0} + \bar{V}_{x,x_0} V_{x,x_0+1}] + \\ & \sum_{\mathbf{x}\in\Lambda} [\bar{H}_{x,x_0} H_{x,x_0} + \bar{V}_{x,x_0} V_{x,x_0} + \bar{V}_{x,x_0} \bar{H}_{x,x_0} + V_{x,x_0} \bar{H}_{x,x_0} + H_{x,x_0} \bar{V}_{x,x_0} + V_{x,x_0} H_{x,x_0}] \end{aligned} \tag{3.4}$$

and $H_{\mathbf{x}}, \bar{H}_{\mathbf{x}}, V_{\mathbf{x}}, \bar{V}_{\mathbf{x}}$ are anticommuting Grassman variables and the Grassmann integral is a linear operation such that $\int d\eta_{\mathbf{x}} = 0$, $\int d\eta_{\mathbf{x}} \eta_{\mathbf{x}} = 1$ if $\eta_{\mathbf{x}}$ is any of $(H_{\mathbf{x}}, \bar{H}_{\mathbf{x}}, V_{\mathbf{x}}, \bar{V}_{\mathbf{x}})$ (see e.g. [52] for an introduction). Note that periodic or antiperiodic boundary conditions are imposed in the x_0, x variables depending on $\epsilon, \epsilon' = \pm$.

It is well known that Grassman integrals appear for the description of the fermionic sector of Quantum Field Theory models [15]. The similarity of the Grassman integrals appearing in (3.2) with the ones appearing in Quantum Field Theory is made clearer through a suitable changes of variables discovered by Itzykson and Drouffe [24]; it is found that the square root of the Z functions appearing in (3.2) can be equivalently written as

$$\int \prod_{\omega=\pm,\mathbf{k}} d\psi^+_{\mathbf{k},\omega} d\psi^-_{\mathbf{k},\omega} e^{-\frac{Z}{L^2} \sum_{\mathbf{k}} \psi^+_{\mathbf{k},\omega} A_{\mathbf{k}} \psi^-_{\mathbf{k},\omega}} = \mathcal{N} \int P_{Z,\mu}(d\psi) \tag{3.5}$$

where $\mathcal{N}$ is a constant, $\psi^{\pm}_{\mathbf{k},\omega}$, $\omega = \pm 1$, $\mathbf{k} = (k_0, k)$ are a finite set of Grassman variables and

$$A_{\mathbf{k}} = \begin{pmatrix} -i\sin k_0 + \sin k + \mu_{11}(\mathbf{k}) & -\mu + \mu_{12}(\mathbf{k}) \\ -\mu + \mu_{21}(\mathbf{k}) & -i\sin k_0 - \sin k_1 + \mu_{22}(\mathbf{k}) \end{pmatrix} \tag{3.6}$$

with $\mu = O(|\beta - \beta_c|)$, $\tanh \beta_c J = \sqrt{2} - 1$, $Z = O(1)$, $\mu_{ij} = O(\mathbf{k}^2)$.

The Grassman integral $\frac{1}{Z}\int P(d\psi)\psi^-_{\omega,\mathbf{k}}\psi^+_{\omega',\mathbf{k}}$ is the *propagator* of a Dirac fermion with a mass $O(|\beta - \beta_c|)$ regularized with a lattice. It is well known that the lattice regularization of Dirac fermions presents the *fermion doubling* problem [24], which means that the propagator (in 2D) has four (unphysical) poles instead than a single one in the massless case; a way to eliminate such unphysical poles is to introduce the so-called *Wilson terms* which are naturally present in (3.6) (they are the $\mu_{ij}(\mathbf{k})$ terms). Indeed the only singularity in (3.6) in the massless case is at $\mathbf{k} = (0, 0)$.

The specific heat and the correlations of the Ising model can be expressed in terms of Grassman integrals, describing non interacting Dirac fermions. The Grassman integrals can be explicitly computed and this representation is just an equivalent way to recover the Ising model solution.

The advantage of this mapping is that coupled Ising models, Vertex or Ashkin-Teller models can be mapped in models of *interacting fermions* and can be analyzed in principle via the perturbative methods developed in Quantum Field Theory. Let us consider now the coupled Ising model (1.3); we will be interested in particular in the specific heat C_v and the energy $\epsilon = +$ and cross-over $(\epsilon = -)$ correlations, defined as

$$G^{\epsilon}_{\beta}(\mathbf{x} - \mathbf{y}) = \lim_{\Lambda\to\infty} \langle O^{\epsilon}_{\mathbf{x}} O^{\epsilon}_{\mathbf{y}}\rangle_{\Lambda} - \langle O^{\epsilon}_{\mathbf{x}}\rangle_{\Lambda}\langle O^{\epsilon}_{\mathbf{y}}\rangle_{\Lambda} \qquad \epsilon = \pm \tag{3.7}$$

where

$$O^{\epsilon}_{\mathbf{x}} = \sum_{j=0,1} \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j} + \epsilon \sum_{j=0,1} \sigma'_{\mathbf{x}}\sigma'_{\mathbf{x}+\mathbf{e}_j} \tag{3.8}$$

where $\langle\cdot\rangle_{\Lambda}$ is the average over all configurations of the spins with statistical weight $e^{-\beta H}$, H given by (1.3). Starting from (3.3) such correlations can be written as sums of functional derivatives (with respect to A^{ϵ}, $\epsilon = +$ for the energy and $\epsilon = -$ for the crossover) of Grassmann integrals with different boundary conditions; in the thermodynamic limit it is sufficient to consider only one of them which is given by, in the case $J = J'$ (for definiteness)

$$Z(A) = \int P_{Z_1,\mu_1}(d\psi) e^{L^2\mathcal{N} + \mathcal{V}^{(1)}(\sqrt{Z_1}\psi) + \mathcal{B}^{(1)}(\sqrt{Z_1}\psi, A)}\ , \tag{3.9}$$

where $\mathcal{N}$ is a constant, $\psi^{\pm}_{\mathbf{k},\omega}$ is a finite set of Grassman variables, $\mathbf{k} = (k_0, k)$, $k_0 = \frac{2\pi}{L}(n_0 + \frac{1}{2})$, $k = \frac{2\pi}{L}(n + \frac{1}{2})$, $n_0, n_1 = -L/2, ...L/2 - 1$, $P_{Z_1,\mu_1}(d\psi)$ is given by (3.5) with $Z = Z_1$ and with $\mu_1 = O(|t - t_c|)$, $t = \tanh \beta J$, $t_c = \tanh \beta_c J = \sqrt{2} - 1 - \zeta$ and

$$\mathcal{V}^{(1)}(\psi) = \zeta_1 \sum_{\mathbf{x},\omega=\pm} \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} + \lambda_1 \sum_{\mathbf{x}} \psi^+_{\mathbf{x},+}\psi^-_{\mathbf{x},+}\psi^+_{\mathbf{x},-}\psi^-_{\mathbf{x},-} + R_1(\psi) \tag{3.10}$$

$$\mathcal{B}^{(1)}(\psi, A) = \sum A_{\epsilon,\mathbf{x}} O^\epsilon_{\mathbf{x}} + R_2(A, \psi) \tag{3.11}$$

with $\zeta_1 = O(\zeta), \lambda_1 = O(\lambda)$; R_1 is a sum of monomials in ψ more than quartic in ψ or quartic with at least a derivatives and R_2 is a sum of monomials in A, ψ more than quadratic in ψ or quadratic with at least a derivative; finally

$$O^+_{\mathbf{x}} = \psi^+_{\mathbf{x},+}\psi^-_{\mathbf{x},-} + \psi^+_{\mathbf{x},-}\psi^-_{\mathbf{x},+} \qquad O^-_{\mathbf{x}} = i[\psi^+_{\mathbf{x},+}\psi^+_{\mathbf{x},-} + \psi^-_{\mathbf{x},+}\psi^-_{\mathbf{x},-}] \tag{3.12}$$

Note that ζ is a *counterterm* to be chosen so that β_c is the critical temperature (in general the critical temperature in (1.3) is different with respect to the Ising one). The above representation is valid in the *isotropic* case $J = J'$; in the anisotropic case $J \neq J'$ a similar expression hold but the Dirac fermions have two different kind of masses.

IV. RENORMALIZATION GROUP ANALYSIS

The fact that Ising, Vertex or Ashkin Teller models can be mapped in interacting fermions, as it is made manifest from (3.9), was known since a long time. This representation opens the way to the use of the perturbative methods developed in Quantum Field Theory in this context; in practice however such a representation has been unutilized for several years for the combinatorial problems associated to such expansions.

Only in recent times (3.9) has been considered a good starting point for the analysis of spin models like (1.3), the reason being that in the last years a great progress has been achieved in the analysis of non quadratic Grassmann integrals, in the context of constructive Quantum Field Theory or solid state physics, which have finally provided the necessary technical tools for analyzing an expression like (3.9).

The starting point is the Renormalization Group approach of Wilson [55] and the treatment of renormalization due to Polchinski [48] and Gallavotti [17]. Fermionic or bosonic functional integrals are analyzed decomposing the fields in a sum of fields living at different momentum scales; the integration is perfomed iteratively starting from the fields with higher scales. After each integration, one obtains a functional integral similar to the initial one, with the difference that the fields have support in a smaller momentum region and have, in general, *renormalized* masses, velocities and wave function renormalization; in addition, the interaction is replaced by an *effective interaction* which is typically sum of monomials of any degree in the fields. To each kernel multiplying the monomials in the effective potential is associated a *scaling dimension*, and the monomials are defined relevant, marginal or irrelevant if their dimensions is positive, vanishing or negative. If only a finite set of monomials have non negative scaling dimension the theory is said renormalizable; the couplings of the non irrelevant monomials in the effective potential are called *effective couplings* and their size depend on the scale.

The outcome of this integration procedure, based on the iterative integration of fields with decreasing momentum scale, is that, in the case of fermionic renormalizable theories, the physical obervables can be written as series in the effective couplings which are *convergent* if the effective couplings are small. This basically follows from the key observation of Caianiello [15], saying that fermionic perturbative expansions have much better convergence properties due to the relative signs; the implementation of such idea pass through the Battle-Bridges-Federbush formula [14] for the truncated expectation together with the Gram bounds for determinants. Historically, the existence of a non trivial Grassman integral was proved for the first time, using such ideas, by Gawedsky and Kupianen [18] and Feldman, Magnen, Rivasseau and Seneor [16]; the Grassman integral they considered represents the generating function for the correlation of the 2D Gross-Neveu model, describing Dirac fermions with $N > 1$ colours, large masses and a quartic interaction. It was proved that the continumm limit could be performed safely, as the theory is asymptotically free, that is the effective couplings become smaller and smaller as the lattice step is sent to zero. By similar methods Lesniewski constructed a superrenormalizable theory, the Yukawa model [33].

The Grassmann integral in (3.9) is somewhat similar to such fermionic models arising in the context of Quantum Field Theory, but there are crucial differences: the first is that one is not interested in the continuum limit (the lattice step is fixed) but rather one has to face the problems represented by small or vanishing fermionic masses. As it is usually said, (3.9) poses an *infrared* and not an *ultraviolet* problem. Another difference is that the fermionic fields have no colour index. The problem posed by (3.9) is much more difficult than the one posed by the Gross-Neveu or Yukawa model; the theory is renormalizable but *not* asymptotically free; it belongs to a class of models with *vanishing* beta function (see below for details). Such kinds of models can be constructed only exploiting non trivial cancellations in the expansions at *all orders* in the perturbative expansion; this is a crucial difference with respect to the asymptotically free models in which a second order computation is enough for establishing the nature of the flow of the effective coupling.

The first example of rigorous construction of a model with vanishing beta function was in [7] and it regards the Jellium model in 1D, describing interacting non relativistic fermions in the continuum. The cancellations were proved using an indirect argument based on comparison with the *exact solution* of the Luttinger model found by Mattis and Lieb [42]. Later on, models with vanishing Beta function were constructed without any use of exact solutions, using a new technique capable of combining Ward Identities based on local symmetries with Renormalization Group methods which was developed in [12]; the main problem to face is that the momentum cut-off breaks local symmetries producing additional terms in the Ward Identities which can be however rigorously taken into account.

Starting from (3.9), the use of the above Renormalization Group techniques combined with Ward Identities (see [41] for a pedagogical introduction), allowed the proof of the following theorem.

Theorem IV.1 *(Mastropietro [37],[38]) The coupled Ising model (1.3) with $J = J'$ and λ small enough is critical at* $\tanh \beta_c J = \sqrt{2} - 1 + O(\lambda)$ *and the specific heat is*

$$C_v \sim -\frac{1}{\alpha}[1 - |\beta - \beta_c|^{-\alpha}] \tag{4.1}$$

with $\alpha = O(\lambda)$. If $\beta \neq \beta_c$ the density and crossover correlations $G^\epsilon_\beta(\mathbf{x}-\mathbf{y})$, $\epsilon = \pm$ (3.7),(3.8) decay faster than any power of $\xi^{-1}|\mathbf{x} - \mathbf{y}|$, with $\xi^{-1} \sim C\,|\beta - \beta_c|^\nu$ with $\nu = 1 + O(\lambda)$ and C is a constant. Moreover

$$G^\epsilon_{\beta_c}(\mathbf{x} - \mathbf{y}) \sim \frac{C_\epsilon}{|\mathbf{x} - \mathbf{y}|^{2X_\epsilon}}\ ,\ as\ |\mathbf{x} - \mathbf{y}| \to \infty\ , \tag{4.2}$$

with $X_\pm = 1 + O(\lambda)$ and $C_\pm$ constants .

From the above theorem we see that the interaction has two main effects. The first one is to change the value of the critical temperature. The second and more dramatic one is to modify qualitatively the critical properties. The logarithmic singularity in the specific heat of the Ising model is changed in a power law singularity when $\lambda > 0$; on the contrary for $\lambda < 0$ the specific heat is a continuous function. Moreover, the exponents $X_\pm, \nu$ are continuous non trivial functions of λ.

The above result gives the first proof of the fact that the critical exponents are non trivial functions of the interaction in coupled Ising models with a generic quartic interaction (2); previously this was known only in the Eight Vertex case thanks to the exact solution. The series for X_+, X_-, ν, X_T are *convergent* for small λ, and the indices can be computed with arbitrary precision by an explicit computation of the first orders.

A similar Renormalization Group analysis can be performed in the anisotropic Ashkin-Teller model when $J \neq J'$.

Theorem IV.2 *(Giuliani,Mastropietro [20],[21]) In the case of the anisotropic Ashkin-Teller model (1.3),(1.6) ($J \neq J'$) there are two critical temperatures, β_c^+ and β_c^- such that*

$$|\beta_c^- - \beta_c^+| \sim |J - J'|^{X_T} \tag{4.3}$$

with $X_T = 1 + O(\lambda)$ and

$$C_v \sim -\Delta^\alpha \log \frac{|\beta - \beta_c^-| \cdot |\beta - \beta_c^+|}{\Delta^2} \tag{4.4}$$

where $2\Delta^2 = (\beta - \beta_c^-)^2 + (\beta - \beta_c^+)^2$.

In this case the specific heat has the same logarithmic singularity as in the Ising model; however, even if we are in the universality class of the Ising model, the difference between the two critical temperatures rescale with an anomalous exponent in the isotropic limit $|\beta_c^+ - \beta_c^-| \sim |J - J'|^{X_T}$; the existence of such a *transition index* was overlooked in the physical literature.

The above theorems were preceded by the following result by Pinson and Spencer for the perturbed Ising model (1.2).

Theorem IV.3 *(Pinson, Spencer [54],[49]) For λ small enough the model (1.2) is critical at $\tanh \beta_c J = \sqrt{2} - 1 + O(\lambda)$ and*

$$C_v \sim -C \log |\beta - \beta_c| \tag{4.5}$$

and, if $\mathbf{x}, \mathbf{x}'$ are nearest neighbor, if $\beta \neq \beta_c$ $< \sigma_{\mathbf{x}}\sigma_{\mathbf{x}'}; \sigma_{\mathbf{y}}\sigma_{\mathbf{y}'} >_{T,\beta}$ decays at large distances faster than any power of $\xi^{-1}|\mathbf{x} - \mathbf{y}|$, with $\xi^{-1} \sim C|\beta - \beta_c|$; moreover at $\beta = \beta_c$ decays with a power law with exponent 2.

The above result establishes a form of universality for the Ising model; the critical indices for the specific heat and the correlation length are insensitive to the perturbation. On the contrary the value of the critical temperature is not universal but it depends from the detail of the perturbation.

Let us sketch the proof of Theorem 5.1 for the case $J = J'$. The Grassmann variables are written as $\psi_{\mathbf{k}} = \sum_{h=-\infty}^{0} \psi_{\mathbf{k}}^{(h)}$, and after the integration of the fields $\psi^{(0)}, \ldots, \psi^{(h+1)}$, the partition function can be written as

$$Z(A) = e^{S^{(h)}(A)} \int P_{Z_{h-1},\mu_{h-1}}(d\psi^{(\leq h)}) e^{\mathcal{V}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)}) + \mathcal{B}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)},A)} , \tag{4.6}$$

where $\psi^{(\leq h)} = \sum_{j=-\infty}^{h} \psi^{(j)}$ and $P_{Z_h,\mu_h}(d\psi^{(\leq h)})$ is the *Gaussian Grassmann integration* with propagator

$$g^{(\leq h)}(\mathbf{k}) = \frac{\chi_h(\mathbf{k})}{Z_h} \begin{pmatrix} -i\sin k_0 + \sin k + \mu_{++} & -\mu_h - \mu_{-+} \\ -\mu_h - \mu_{+-} & -i\sin k_0 - \sin k_1 + \mu_{--} \end{pmatrix}^{-1} \tag{4.7}$$

with $\chi_h(\mathbf{k})$ a smooth compact support function non vanishing for $|\mathbf{k}| \leq 2^h$. The *effective interaction* $\mathcal{V}^{(h)}(\psi)$ is a sum over monomials in the Grassmann variables

$$\mathcal{V}^{(h)}(\psi) = \gamma^h \zeta_h F_\nu^{(h)} + \lambda_h F_\lambda^{(h)} + R_h , \tag{4.8}$$

where

$$\begin{aligned} F_\nu^{(h)} &= \frac{1}{L^2} \sum_{\omega=\pm} \sum_{\mathbf{k}} \hat\psi^{(\leq h)+}_{\mathbf{k},\omega} \hat\psi^{(\leq h)-}_{\mathbf{k},-\omega} , \\ F_\lambda^{(\leq h)} &= \frac{1}{L^8} \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_4} \hat\psi^{(\leq h)+}_{\mathbf{k}_1,+} \hat\psi^{(\leq h)+}_{\mathbf{k}_3,-} \hat\psi^{(\leq h)-}_{\mathbf{k}_2,+} \hat\psi^{(\leq h)-}_{\mathbf{k}_4,-} \delta(\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_4) . \end{aligned} \tag{4.9}$$

and R^h contains sum of monomials with more than four fields, or quartic with at least a derivative, or bilinear with at least two derivatives. In the same way

$$\mathcal{B}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)}, A) = \sum_{\epsilon=\pm,\mathbf{x}} Z_{h-1}^{(\epsilon)} A_{\mathbf{x}}^\epsilon O_{\mathbf{x}}^{(\leq h)\epsilon} + \bar R_h , \tag{4.10}$$

where $O^\pm$ is given by (3.12) and $\bar R_h$ contains terms more than quadratic, or quadratic with a derivative. It is natural the interpretation of λ_h in (37) as the *effective coupling* of the model at momentum scales $O(2^h)$.

The *scaling dimension* is given, for the monomials with n ψ-fields and m A-fields, by

$$D = 2 - \frac{n}{2} - m . \tag{4.11}$$

In the Renormalization Group language, the terms with positive or vanishing dimension are called *relevant* or *marginal* terms, respectively; the terms in R_1 or R_2 are *irrelevant*.

Notice that the propagator of the field $\psi^{(h)}$ can be written, for $h \le 0$, as

$$g^{(h)}(\mathbf{x},\mathbf{y}) = g_T^{(h)}(\mathbf{x},\mathbf{y}) + r^{(\le h)}(\mathbf{x},\mathbf{y}) \ , \tag{4.12}$$

where

$$g_T^{(h)}(\mathbf{x},\mathbf{y}) = \frac{1}{L^2}\sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{1}{Z_h}T_h^{-1}(\mathbf{k}) \ , \tag{4.13}$$

$$T_h(\mathbf{k}) = f_h(\mathbf{k})\begin{pmatrix} -ik_0+k & -\mu_h \\ \mu_h & -ik_0-k \end{pmatrix} \tag{4.14}$$

with $f_h(\mathbf{k})$ a smooth compact support function non vanishing for $2^{h-1} \le |\mathbf{k}| \le 2^{h+1}$ and, for any positive integer M,

$$|r^{(h)}(\mathbf{x},\mathbf{y})| \le C_M \frac{2^{2h}}{1+(2^h|\mathbf{x}-\mathbf{y}|^M)} \ . \tag{4.15}$$

On the other hand, $g_T^{(h)}(\mathbf{x},\mathbf{y})$ verifies a bound similar to (4.15) with 2^h replacing 2^{2h}.

The running couplings λ_j (which, by construction, are the same in the massless $\mu_1 = 0$ or in the massive $\mu_1 \neq 0$) satisfy a recursive equation of the form

$$\lambda_{j-1} = \lambda_j + \beta_\lambda^{(j)}(\lambda_j, ..., \lambda_0) + \bar\beta_\lambda^{(j)}(\lambda_j,\zeta_j; ...; \lambda_0, \zeta_0) \ , \tag{4.16}$$

where $\beta_\lambda^{(j)}$, $\bar\beta_\lambda^{(j)}$ are μ_1-independent and expressed by a *convergent* expansion in $\lambda_j, \zeta_j.., \lambda_0, \zeta_0$ (convergence follows from Gram bounds); moreover $\bar\beta_\lambda^{(j)}$ vanishes if at least one of the ζ_k is zero. The running coupling λ_j stays close to λ_1 for any j as a consequence of the following property, called *vanishing of the Beta function*, which was proved in Theorem 2 of [12]; for suitable positive constants C and $\theta < 1$:

$$|\beta_\lambda^{(j)}(\lambda_j, ..., \lambda_j)| \le C|\lambda_j|^2 2^{\theta j} \ . \tag{4.17}$$

It is possible to prove that, for a suitable choice of $\zeta_1 = O(\lambda)$, $\zeta_j = O(2^{\theta j}\lambda)$, and this implies $\bar\beta_\lambda^{(j)} = O(2^{\theta j}\lambda^2)$ so that the sequence λ_j converges, as $j \to -\infty$, to a smooth function $\lambda_{-\infty}(\lambda) = \lambda_1 + O(\lambda^2)$, such that

$$|\lambda_j - \lambda_{-\infty}| \le C\lambda^2 2^{\theta j} \ . \tag{4.18}$$

Moreover

$$\frac{Z_{j-1}}{Z_j} = 1 + \beta_z^{(j)}(\lambda_j, ..., \lambda_0) + \bar\beta_z^{(j)}(\lambda_j, \nu_j; .., \lambda_0, \nu_0) \ , \tag{4.19}$$

with $\bar\beta_z^{(j)}$ vanishing if at least one of the ν_k is zero so that $\bar\beta_z^{(j)} = O(\lambda 2^{\theta j})$. Finally

$$\beta_z(\lambda_j, ..., \lambda_0) = \beta_z(\lambda_{-\infty}, ..., \lambda_{-\infty}) + O(\lambda 2^{\theta h}) \;, \tag{4.20}$$

where the last identity follows from (4.18) and the function $\beta_z(\lambda_{-\infty}, ..., \lambda_{-\infty})$ is by definition sum of terms in which only the propagators $g_T^{(h)}$ (4.13) appear (the terms containing $r^{(j)}$ are included in the second term in (4.20)). Similar equations hold for $Z_h^{(\pm)}, \mu_h$, with

$$\beta_\pm(\lambda_j, ..., \lambda_0) = \beta_\pm(\lambda_{-\infty}, ..., \lambda_{-\infty}) + O(\lambda 2^{\theta h}) \;. \tag{4.21}$$

By an explicit computation and (4.20), (4.21), there exist $\eta_+(\lambda_{-\infty}) = c_1\lambda_{-\infty} + O(\lambda_{-\infty}^2)$, $\eta_-(\lambda_{-\infty}) = -c_1\lambda_{-\infty} + O(\lambda_{-\infty}^2)$, $\eta_\mu(\lambda_{-\infty}) = c_1\lambda_{-\infty} + O(\lambda_{-\infty}^2)$ and $\eta_z(\lambda_{-\infty}) = c_2\lambda_{-\infty}^2 + O(\lambda_{-\infty}^3)$, with c_1 and c_2 strictly positive, such that, for any $j \leq 0$,

$$\begin{aligned} &|\log_2(Z_{j-1}/Z_j) - \eta_z(\lambda_{-\infty})| \leq C\lambda^2 2^{\theta j} \;, \\ &|\log_2(\mu_{j-1}/\mu_j) - \eta_\mu(\lambda_{-\infty})| \leq C|\lambda| 2\theta j, \\ &|\log_2(Z_{j-1}^{(\pm)}/Z_j^{(\pm)}) - \eta_\pm(\lambda_{-\infty})| \leq C\lambda^2 2^{\theta j} \;. \end{aligned} \tag{4.22}$$

The critical indices in Theorem 5.1 are functions of $\lambda_{-\infty}$ only, as it is clear from (4.20); moreover the exponents appearing there are such that

$$X_\pm = 1 - \eta_\pm + \eta_z \qquad \mu = \eta_+ - \eta_z = 1 - X_+ \;. \tag{4.23}$$

If $\mu_1 \neq 0$ (that is, if the temperature is not the critical one), the correlations decay faster than any power with rate proportional to μ_{h^*}, where, if $[x]$ denotes the largest integer $\leq x$, h^* is given by $h^* = \left[\frac{\log_2 |\mu_1|}{1+\eta_\mu}\right]$.

The above analysis makes also transparent the difference between the model (1.2) and (1.3); in the first case the Grassman variables do not have the ω-index so that quartic local terms are vanishing $\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}} = 0$. The theory is *superrenormalizable* (and not marginally relevant) and, once that a counterterm is introduced to tune the critical temperature, the perturbative expansion converges and the critical exponents are λ-independent.

V. AN UNIVERSALITY RESULT

The exponents of the model (1.3) are written as convergent series so that they can be computed with arbitrary precision; the complexity of the expansions makes however essentially impossible to prove the universal relations directly from the expansions and new ideas are necessary. Recently some of the universal relations have been proved.

Theorem V.1 *(Benfatto,Falco,Mastropietro [8]). Given the coupled Ising model with quartic interaction (2), with the same definitions as in Theorems 5.1 and 5.2 and* λ *small enough the following relations are true*

$$X_- = \frac{1}{X_+} \qquad \alpha = \frac{2-2X_+}{2-X_+}\ , \qquad (5.1)$$
$$\nu = \frac{1}{2-X_+} \qquad X_T = \frac{1-X_+}{1-X_+^{-1}}$$

The first two of the above relations are the ones conjectured by Kadanoff [28] and the second by Kadanoff and Wegner [31] while the last one is completely new.

The proof of the above theorem is based on the introduction of a a continuum fermionic theory, whose correlations are the functional derivatives of the following Grassmann integral

$$\int P_Z(d\psi^{(\leq N)}) e^{V^{(N)}(\sqrt{Z}\psi^{(\leq N)})+\sum_{\omega=\pm}\int d\mathbf{x}[\psi^+_{\mathbf{x},\omega}\phi^-_{\mathbf{x},\omega}+\psi^-_{\mathbf{x},\omega}\phi^+_{\mathbf{x},\omega}]} \qquad (5.2)$$

with $\psi,\bar\psi$ are Euclidean $d=1+1$ spinors ($\psi^\pm = (\psi^\pm_+, \psi^\pm_-)$), $P_Z(d\psi^{(\leq N)})$ is the fermionic gaussian integration with propagator $g^{(\leq N)} = \frac{1}{Z}\frac{\chi_N(\mathbf{k})}{\not{\mathbf{k}}}$, $\not{\mathbf{k}} = \gamma_0 k_0 + c\gamma_1 k_1$ (which in components appear to be equal to $g_T^{(\leq N)}(\mathbf{x})$ (4.13)), and

$$V^{(N)}(\psi^{(\leq N)}) = \lambda_\infty \int d\mathbf{x} d\mathbf{y} v(\mathbf{x}-\mathbf{y}) j_\mu(\mathbf{x}) j_\mu(\mathbf{y}) \qquad (5.3)$$

with $j_\mu(\mathbf{x}) = \bar\psi_{\mathbf{x}}\gamma_\mu\psi_{\mathbf{x}}$ and $v(\mathbf{x}-\mathbf{y})$ a short range symmetric interaction; moreover

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \gamma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad (5.4)$$

Contrary to the spin case, in which the lattice introduce an ultraviolet cut-off on the size of the momenta, here a cut-off N must be introduced in order to have a well defined Grassman integral.

A multiscale integration is now necessary also in the ultraviolet region to perform the limit $N\to\infty$, see [39]; no ultraviolet divergences are present due to the non locality of the interaction in (5.3). The multiscale integration for the infrared scales can be done exactly as described in the previous section, with the only difference that, by the oddness of the free propagator, $\zeta_j = 0$ and

$$\lambda_{j-1} = \lambda_j + \hat\beta^{(j)}_\lambda(\lambda_j, ...\lambda_0)\ , \qquad (5.5)$$

where

$$\hat\beta^{(j)}_\lambda(\lambda_j, ...\lambda_0) = \beta^{(j)}_\lambda(\lambda_j, ...\lambda_0) + O(\lambda^2_\infty 2^{\theta j})\ , \qquad (5.6)$$

$\beta^{(j)}_\lambda(\lambda_j,...\lambda_j)$ being the function appearing in (4.17), so that we can prove that $\lambda_{-\infty} = \lambda_0+O(\lambda_0^2)$; since $\lambda_0=\lambda_\infty+O(\lambda_\infty^2)$, we have

$$\lambda_{-\infty}=h(\lambda_\infty)=\lambda_\infty+O(\lambda_\infty^2)\;, \tag{5.7}$$

for some analytic function $h(\lambda_\infty)$, invertible for λ_∞ small enough. Moreover

$$\frac{Z^\pm_{j-1}}{Z^\pm_j}=1+\hat\beta^{(j)}_\pm(\lambda_j,...,\lambda_0)\;, \tag{5.8}$$

with

$$\hat\beta^{(j)}_\pm(\lambda_j,...,\lambda_0)=\beta^{(j)}_\pm(\lambda_j,...\lambda_0)+O(\lambda_\infty^2 2^{\theta j})\;, \tag{5.9}$$

$\beta^{(j)}_\pm$ being the functions appearing in (4.21) (as consequence of (4.12)). This implies that

$$\eta_\pm=\log_2[1+\beta^{(-\infty)}_\pm(\lambda_{-\infty},...\lambda_{-\infty})]\;, \tag{5.10}$$

that is *the critical indices in the AT or 8V or in the model (5.2) are the same as functions of* $\lambda_{-\infty}$.

If we call $\lambda'_j(\lambda)$ the effective couplings of the lattice model (1.3) appearing in the previous section, the invertibility of $h(\lambda_\infty)$ implies that we can choose λ_∞ so that

$$h(\lambda_\infty)=\lambda'_{-\infty}(\lambda)\;. \tag{5.11}$$

The exponents in the models (1.3) and (5.2) are the same, provided that bare coupling λ_∞ *is chosen properly and* $Z=c=1$. What have we gained by this?

The point is that the continuum fermionic theory (5.2) has correlations expressed by Grassmann integrals which are identical to the ones appearing in certain Quantum Field Theory models, and it verifies extra *Gauge symmetries* with respect to the original spin Hamiltonian. If $<...>_{th}$ are the correlations with respect to $P_Z(d\psi^{(\le N)})e^{\mathcal{V}^{(N)}}$ in (5.2), by the transformation $\psi\to e^{i\alpha_{\mathbf{x}}}\psi_{\mathbf{x}}$ one finds

$$-i\mathbf{p}_\mu<j_{\mu,\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}+\mathbf{p}}>_{th}=<\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}}>_{th}-<\psi_{\mathbf{k}+\mathbf{p}}\bar\psi_{\mathbf{k}+\mathbf{p}}>_{th}+\Delta_N(\mathbf{k},\mathbf{p}) \tag{5.12}$$

where

$$\Delta_N=<\delta j_{\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}+\mathbf{p}}>_{th} \tag{5.13}$$

with

$$\delta_{\mathrm{J}\mathbf{p}}=\int d\mathbf{k}[(\chi_N^{-1}(\mathbf{k}+\mathbf{p})-1)(\not{\mathbf{k}}+\not{\mathbf{p}})-(\chi_N^{-1}(\mathbf{k})-1)\not{\mathbf{k}}]\bar\psi_{\mathbf{k}}\psi_{\mathbf{k}+\mathbf{p}} \tag{5.14}$$

An analogous expression is obtained for the axial current $\bar{\psi}\gamma_\mu\gamma_5\psi$. The term Δ_N is due to the momentum regularization which is necessary to have a well defined Grassmann integral but which breaks the continuum phase symmetries. If Ward Identities are derived from the (formal) theory without cut-off, one would get the same WI with $\Delta_N = 0$; on the contrary by a multiscale analysis it is found, in the limit of removed cut-off [39]

$$\lim_{N\to\infty} \Delta_N(\mathbf{k},\mathbf{p}) = -i\tau\hat{v}(\mathbf{p})\mathbf{p}_\mu < j_{\mu,\mathbf{p}}\psi_{\mathbf{k},\omega}\bar{\psi}_{\mathbf{k}+\mathbf{p},\omega} >_{th} \qquad \tau = \frac{\lambda_\infty}{4\pi c} \tag{5.15}$$

A similar expression holds for the axial Ward Identity, with j_μ replaced by j^5_μ and τ replaced by $-\tau$ so that, in the limit $N\to\infty$

$$-i\mathbf{p}_\mu < j_{\mu,\mathbf{p}}\psi_{\mathbf{k},\omega}\bar{\psi}_{\mathbf{k}+\mathbf{p}} >_{th}= A[< \psi_{\mathbf{k}}\bar{\psi}_{\mathbf{k}} >_{th} - < \psi_{\mathbf{k}+\mathbf{p}}\bar{\psi}^-_{\mathbf{k}+\mathbf{p}} >_{th}] \tag{5.16}$$
$$-i\mathbf{p}_\mu < j_{5,\mu,\mathbf{p}}\psi_{\mathbf{k},\omega}\bar{\psi}_{\mathbf{k}+\mathbf{p}} >= \bar{A}[< \psi_{\mathbf{k}}\bar{\psi}_{\mathbf{k}} >_{th} - < \psi_{\mathbf{k}+\mathbf{p}}\bar{\psi}^-_{\mathbf{k}+\mathbf{p}} >_{th}]$$

with

$$A^{-1} = 1 - \tau v(\mathbf{p}) \qquad A^{-1} = 1 + \tau v(\mathbf{p}) \tag{5.17}$$

The fact that $\Delta_N(\mathbf{k},\mathbf{p})$ is non vanishing removing the ultraviolet cut-off $N\to\infty$ is a manifestation of a *quantum anomaly*. The Ward Identities can be combined with the Schwinger-Dyson equation and it turns out that

$$X_+ = 1 - \frac{1}{1+\tau}(\lambda_\infty/2\pi c) \quad X_- = 1 + \frac{1}{1-\tau}(\lambda_\infty/2\pi c) \tag{5.18}$$

from which, using (4.24), the relations (5.1) follows. Indeed the indices have simple expressions in λ_∞, as consequence of the linearity of τ; all the model dependence is included in the function λ_∞, which is given by a convergent non trivial series with coefficients depending from all the details of the spin model (1.3).

There are then two crucial points in the proof of the universal relations (5.1). The first is that the exponents of the lattice theory (1.3) are equal to the ones of a continuum relativistic quantum field theory, *provided that* the coupling, light velocity and wave function renormalization of the continuum theory are chosen properly. The relativistic quantum field theory is defined thanks to an ultraviolet regularization, and there are several possible choices of it. The second crucial point is that there exists a choice of the ultraviolet regularization so that τ is linear, otherwise the indices have not a simple expressions as functions of λ_∞ and one could not check the validity of the exponents.

The linearity of τ in the bare coupling λ_∞ is the non-perturbative analogue of a property called *anomaly non renormalization* in 4D Quantum Electrodynamics, proved at a perturbative level by Adler and Bardeen [1] with a careful analysis of the perturbative expansion. This analogy can be better understood introducing a gaussian field A_μ and performing in (5.2) an Hubbard-Stratonovich transformation; in this way the second of (5.16) can be written as

$$-i\mathbf{p}_\mu < j_{5,\mu,\mathbf{p}}\psi_{\mathbf{k},\omega}\bar\psi_{\mathbf{k+p}} >_{th}=$$
$$[< \psi_{\mathbf{k}}\bar\psi_{\mathbf{k}} >_{th} - < \psi_{\mathbf{k+p}}\bar\psi^-_{\mathbf{k+p}} >_{th}] + \tau\varepsilon_{\mu,\nu} < A_{\nu,\mathbf{P}}\psi_{\mathbf{k},\omega}\bar\psi_{\mathbf{k+p}} >_{th} \tag{5.19}$$

an expression previously derived order by order in perturbation theory by Georgi and Rawls [19].

On the other hand there are ultraviolet regularizations for which the anomaly non-renormalization is not valid. If we choose (5.2) as a delta function we get a regularized version of the *Thirring model* and it is possible to prove, see [9], that, with a suitable choice of Z as function of N (vanishing with a power law as $N \to \infty$), the limit $N \to \infty$ can be taken; in this case the anomaly is renormalized by higher orders corrections, that is $\tau = \frac{\lambda_\infty}{4\pi c} + b(\lambda_\infty)^2 + O(\lambda_\infty^3)$ with a non vanishing b. On the other hand, another possible regularization can be obtained taking in (5.2) $v(\mathbf{x}-\mathbf{y}) = \delta_M(\mathbf{x}-\mathbf{y})$ with $\delta_M(\mathbf{x}-\mathbf{y}) \to_{M\to\infty} \delta(\mathbf{x}-\mathbf{y})$; in [40] is proved that, for a suitable choice of Z, the limit $M \to \infty, N \to \infty$ can be taken, and in this case (5.17) are true with $\hat v(\mathbf{p}) = 1$, an expression coinciding with the one derived by Johnson [27] by self-consistency arguments.

VI. RENORMALIZATION GROUP FOR QUANTUM SPIN CHAIN AND 1D FERMI SYSTEMS

We consider now the spin chain (1.8) or (1.13). It is convenient to introduce the density and the current operators:

$$\begin{aligned}\rho_x &= S^3_x + \frac{1}{2} = a^+_x a^-_x \ , \\ J_x &= S^1_x S^2_{x+1} - S^2_x S^1_{x+1} = \frac{1}{2i}[a^+_{x+1}a^-_x - a^+_x a^-_{x+1}] \ .\end{aligned} \tag{6.20}$$

If $\rho_{\mathbf{x}} = e^{x_0 H}\rho_x e^{-x_0 H}$, $a^\pm_{\mathbf{x}} = e^{x_0 H} a^\pm_x e^{-x_0 H}$ the above definition of the current is justified by the (imaginary time) conservation equation

$$\frac{\partial \rho_{\mathbf{x}}}{\partial x_0} = e^{Hx_0}[H, \rho_x]e^{-Hx_0} = -i[J_{x,x_0} - J_{x-1,x_0}] \tag{6.21}$$

where we have used that ρ_x commutes with the quartic part of H. Note that J_x does not verify a simple equation like (6.21), as J_x does not commute with the quartic part of H. Note also that

$$[H_0, \hat{J}_p] = \frac{1}{L}\sum_k \sin k(\cos(k+p) - \cos k)\hat{a}^+_{k+p}\hat{a}_k \tag{6.22}$$

If O_x is a local monomial in the S^α_x or $a^\pm_x$ operators, we call $O_{\mathbf{x}} = e^{Hx_0}O_x e^{-Hx_0}$ where $\mathbf{x} = (x, x_0)$; moreover, if $A = O_{\mathbf{x}_1}\cdots O_{\mathbf{x}_n}$,

$$< A >_{L,\beta} = \frac{\mathrm{Tr}[e^{-\beta H}\mathbf{T}(A)]}{Tr[e^{-\beta H}]} \tag{6.23}$$

$\mathbf{T}$ being the time order product, denotes its expectation in the grand canonical ensemble, while $< A >_{T;L,\beta}$ denotes the corresponding truncated expectation. We will use also the notation $< A >_T = \lim_{L,\beta\to\infty} < A >_{T;L,\beta}$.

By Renormalization Group methods it was proved in [10] for small λ, $J_1 = J_2 = 1$ and large $\mathbf{x}$,

$$\langle a^-_{\mathbf{x}} a^+_{\mathbf{0}}\rangle_T \sim g_0(\mathbf{x})\frac{1+\lambda f(\lambda)}{(x_0^2 + v_s^2 x^2)^{(\eta/2)}} \;, \tag{6.24}$$

where $f(\lambda)$ is a bounded function, $\eta = a_0\lambda^2 + O(\lambda^3)$, with $a_0 > 0$, and

$$g_0(\mathbf{x}) = \sum_{\omega=\pm} \frac{e^{i\omega p_F x}}{-ix_0 + \omega v_s x} \;, \tag{6.25}$$

$$v_s = v_F + O(\lambda) \qquad p_F = \cos^{-1}(h+\lambda) + O(\lambda) \qquad v_F = \sin p_F \;. \tag{6.26}$$

From (6.24) we see that the interaction has two main effects. The first one is to change the value of the Fermi momentum from $\cos^{-1}(h)$ to p_F and the Fermi velocity velocity from v_F in the non interacting case to v_s. The second effect is that the power law decay is changed; the 2-point function is asymptotically given by the product of the non-interacting one (with a different velocity) times an extra power law decay factor with non-universal index η.

It was also proved in [10] that the spin-spin correlation in the direction of the 3-axis (or, equivalently, the fermionic density-density correlation) is given, for large $\mathbf{x}$, by

$$\langle S^{(3)}_{\mathbf{x}} S^{(3)}_{\mathbf{0}}\rangle_T \sim \cos(2p_F x)\Omega^{3,a}(\mathbf{x}) + \Omega^{3,b}(\mathbf{x}) \;, \tag{6.27}$$

$$\Omega^{3,a}(\mathbf{x}) = \frac{1+A_1(\mathbf{x})}{2\pi^2[x^2+(v_s x_0)^2]^{X_+}} , \tag{6.28}$$

$$\Omega^{3,b}(\mathbf{x}) = \frac{1}{2\pi^2[x^2+(v_s x_0)^2]}\Big\{\frac{x_0^2-(x/v_s)^2}{x^2+(v_s x_0)^2}+A_2(\mathbf{x})\Big\} , \tag{6.29}$$

with $|A_1(\mathbf{x})|, |A_2(\mathbf{x})| \le C|\lambda|$ and $X_+ = 1 - a_1\lambda + O(\lambda^2)$ with

$$a_1 = [\hat{v}(0) - \hat{v}(2p_F)]/(\pi \sin p_F) \tag{6.30}$$

Finally the Cooper pair density correlation, that is the correlation of the operator $\rho^c_{\mathbf{x}} = a^+_{\mathbf{x}} a^+_{\mathbf{x}'} + a^-_{\mathbf{x}} a^-_{\mathbf{x}'}$, $\mathbf{x}' = (x+1, x_0)$, behaves as

$$\langle \rho^c_{\mathbf{x}} \rho^c_{\mathbf{0}} \rangle_T \sim \frac{1+A_3(\mathbf{x})}{2\pi^2(x^2+v_s^2 x_0^2)^{X_-}} , \tag{6.31}$$

with $X_- = 1 + a_1\lambda + O(\lambda^2)$, a_1 being the same constant appearing in the first order of X_+. In the case $J_1 \neq J_2$ the correlations decay faster than any power with rate ξ such that $\xi \sim C|J_1 - J_2|^{\bar{\nu}}$ with $\bar{\nu} = 1 + a_1\lambda + O(\lambda^2)$, a_1 given by (6.30).

If $J_{\mathbf{x}} = v_F j_{\mathbf{x}}$, in the $\lambda = 0$ the commutation relations (6.21),(6.22) imply the following Ward Identities, for $\mathbf{k}, \mathbf{k}+\mathbf{p}$ close to $(0, \omega p_F)$, $\omega = \pm$

$$\begin{aligned}
&-ip_0\langle \hat{\rho}_{\mathbf{p}} \hat{a}^+_{\mathbf{k}} \hat{a}^-_{\mathbf{k}+\mathbf{p}} \rangle + \omega p v_F \langle \hat{j}_{\mathbf{p}} \hat{a}^+_{\mathbf{k}} \hat{a}^-_{\mathbf{k}+\mathbf{p}} \rangle \sim [\langle \hat{a}^+_{\mathbf{k}} \hat{a}^-_{\mathbf{k}} \rangle - \langle \hat{a}^+_{\mathbf{k}+\mathbf{p}} \hat{a}^-_{\mathbf{k}+\mathbf{p}} \rangle] \\
&-ip_0\langle \hat{j}_{\mathbf{p}} \hat{a}^+_{\mathbf{k}} \hat{a}^-_{\mathbf{k}+\mathbf{p}} \rangle + \omega p v_F \langle \hat{\rho}_{\mathbf{p}} \hat{a}^+_{\mathbf{k}} \hat{a}^-_{\mathbf{k}+\mathbf{p}} \rangle \sim [\langle \hat{a}^+_{\mathbf{k}} \hat{a}^-_{\mathbf{k}} \rangle - \langle \hat{a}^+_{\mathbf{k}+\mathbf{p}} \hat{a}^-_{\mathbf{k}+\mathbf{p}} \rangle]
\end{aligned} \tag{6.32}$$

In the presence of the interaction $\lambda \neq 0$, the Ward Identites have the form [11]

$$\begin{aligned}
&-ip_0\langle \hat{\rho}_{\mathbf{p}} \hat{a}^+_{\mathbf{k}} \hat{a}^-_{\mathbf{k}+\mathbf{p}} \rangle + \omega p \tilde{v}_J \langle \hat{j}_{\mathbf{p}} \hat{a}^+_{\mathbf{k}} \hat{a}^-_{\mathbf{k}+\mathbf{p}} \rangle \sim B[\langle \hat{a}^+_{\mathbf{k}} \hat{a}^-_{\mathbf{k}} \rangle - \langle \hat{a}^+_{\mathbf{k}+\mathbf{p}} \hat{a}^-_{\mathbf{k}+\mathbf{p}} \rangle] \\
&-ip_0\langle \hat{j}_{\mathbf{p}} \hat{a}^+_{\mathbf{k}} \hat{a}^-_{\mathbf{k}+\mathbf{p}} \rangle + \omega p \tilde{v}_N \langle \hat{\rho}_{\mathbf{p}} \hat{a}^+_{\mathbf{k}} \hat{a}^-_{\mathbf{k}+\mathbf{p}} \rangle \sim \bar{B}[\langle \hat{a}^+_{\mathbf{k}} \hat{a}^-_{\mathbf{k}} \rangle - \langle \hat{a}^+_{\mathbf{k}+\mathbf{p}} \hat{a}^-_{\mathbf{k}+\mathbf{p}} \rangle]
\end{aligned} \tag{6.33}$$

with $B = 1$, $\bar{B} = 1+O(\lambda)$ and $\tilde{v}_J, \tilde{v}_N = v_s(1+O(\lambda))$; in particular $\frac{\tilde{v}_N}{\tilde{v}_J} = 1+2a_1\lambda+O(\lambda^2)$ with a_1 the constant defined above in (6.30). The interaction has the effect that the normalization $\bar{B}$ is not 1 (the current does not commute with the quartic part of the Hamiltonian) and two different velocities , the charge $\tilde{v}_J$ and the current velocity $\tilde{v}_N$, appear. The presence of the lattice, breaking the Lorentz symmetry valid in the continuum limit, causes the presence of three distinct velocities, $\tilde{v}_N, \tilde{v}_J, v_s$.

The same strategy followed for proving the universal relations in the coupled Ising models (1.3) allows to derive the same relations between the indices appearing in the correlations of the spin chain; again all the indices can be expressed in terms of a single one. There is in this case also an extra relation connecting the indices with the Fermi velocity v_s and the susceptibility, defined as

$$\kappa = \lim_{p\to 0} \hat{\Omega}(0,p) \tag{6.34}$$

where $\hat{\Omega}(0,p)$ is the bidimensional Fourier transform of $\langle S^{(3)}_{\mathbf{x}} S^{(3)}_{\mathbf{0}}\rangle_T$. In the fermionic interpretation, $\kappa\rho^{-2}$ is the compressibility (ρ is the fermionic density).

Theorem VI.1 *(Benfatto,Mastropietro [11]) In the model (1.8) for λ small enough the exponents in (6.24),(6.27),(6.31) verify*

$$X_+ X_- = 1 \quad \bar{\nu} = \frac{1}{2 - X_+^{-1}} \qquad 2\eta = X_+ + X_+^{-1} - 2 \tag{6.35}$$

Moreover, the velocities in the Ward Identity (6.33) are such that

$$\tilde{v}_N \tilde{v}_J = v_s^2 \qquad \tilde{v}_J = v_F \tag{6.36}$$

and the susceptibility κ verifies

$$\kappa = \frac{1}{\pi}\frac{X_+}{v_s} \tag{6.37}$$

The relation (6.37) has been proposed in [22] as a part of the *Luttinger liquid conjecture* and checked previously only in the case of the XYZ chain using the explicit exact formulas (2.17).

Note that, in the notation of [22], $v_N \equiv (\pi\kappa)^{-1}$ should not be confused with $\tilde{v}_N$ appearing in the WI (6.32); they are coinciding only in the special case of the Luttinger model. Therefore $\tilde{v}_N = v_N$ is an example of relation true in the (exactly solvable) Luttinger model but not in the generic 1D Fermi system model (1.13).

The Ward Identities of the relativistic model (5.2) with $c = v_s$ and a suitable choice of λ_∞ and the following relations, valid for $\mathbf{k}\ \mathbf{k}, \mathbf{k}+\mathbf{p}$ small, if $\mathbf{p}_F = (0, \omega p_F)$

$$\begin{aligned} &< \hat{\rho}_{\mathbf{p}} \hat{a}^+_{\mathbf{k}+\mathbf{p}_F^\omega} \hat{a}^-_{\mathbf{k}+\mathbf{p}+\mathbf{p}_F^\omega} >\sim Z^{(3)} < j^0_{\mathbf{p}} \hat{\psi}^+_{\mathbf{k},\omega} \hat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega} >_{th} \\ &< \hat{j}_{\mathbf{p}} \hat{a}^+_{\mathbf{k}+\mathbf{p}_F^\omega} \hat{a}^-_{\mathbf{k}+\mathbf{p}+\mathbf{p}_F^\omega} >\sim \tilde{Z}^{(3)} < j^1_{\mathbf{p}} \hat{\psi}^+_{\mathbf{k},\omega} \hat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega} >_{th} \end{aligned} \tag{6.38}$$

imply the Ward Identities (6.33) with, if τ is given by (5.15)

$$B = \frac{Z^{(3)}}{Z}(1-\tau)^{-1}, \quad \bar{B} = \frac{\tilde{Z}^{(3)}}{Z}(1+\tau)^{-1} \tag{6.39}$$

and

$$\tilde{v}_N = v_s \frac{Z^{(3)}}{\tilde{Z}^{(3)}} \quad \tilde{v}_J = v_s \frac{\tilde{Z}^{(3)}}{Z^{(3)}} \tag{6.40}$$

On the other hand, by comparison with the WI obtained from the continuity equation (6.21) we get

$$\frac{Z^{(3)}}{(1-\tau)Z} = 1 \qquad \tilde{v}_J = v_F \tag{6.41}$$

that is the renormalizations in (6.38) are not indipendent. We can derive from the Ward Identity for the densities the following expressions

$$< \rho_{\mathbf{p}} \rho_{\mathbf{p}} >= \frac{1}{4\pi v_s Z^2} \frac{(Z^{(3)})^2}{1-(\lambda_\infty/4\pi v_s)^2} \left[2 - \frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} - \frac{D_+(\mathbf{p})}{D_-(\mathbf{p})} \right] + O(\mathbf{p}) \tag{6.42}$$

where $D_\omega(\mathbf{p}) = -ip_0 + \omega v_s p$, so that from (6.34) and the first of (6.41)

$$\kappa = \frac{1}{\pi v_s} \frac{1}{Z^2} \frac{(Z^{(3)})^2}{1-(\tilde{\lambda}_\infty/4\pi v_s)^2} = \frac{1}{\pi v_s} \frac{1-(\lambda_\infty/4\pi v_s)}{1+(\lambda_\infty/4\pi v_s)} = \frac{X_+}{\pi v_s} \tag{6.43}$$

VII. CONCLUSIONS

The validity of a number of universal relations between exponents and other quantities in a wide class models, including solvable and *not solvable* models, has been established. These universal relations are used for the analysis of experiments in carbon nanotubes or spin chains, but their interest goes much beyond this, as they provide one of the very few cases in which the *universality* hypothesis can be *rigorously* verified.

So far, spin-spin correlations in the above models seem to be beyond the reach of the method. Note that already in the integrable case the computation of such correlations is very tricky: it is based on an asymptotic analysis of a Toeplitz determinant or, alternatively, on the derivation of highly non trivial non linear finite difference equations, whose scaling limit is related to the third Painleve' equation. In order to progress on this problem, it is necessary to combine Renormalization Group and Ward Identities techniques with different powerful methods, such as non-linear recursive finite difference equations or bosonization .

Another important direction is to use the above methods to take into account the lattice and mass effects in other two dimensional statistical models which can be mapped in the scaling limit in conformal field theory models.

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