Universal Relations for Non Solvable Statistical Models

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We present the first rigorous derivation of a number of universal relations for a class of models with continuously varying indices (among which are interacting planar Ising models, quantum spin chains and 1D Fermi systems), for which an exact solution is not known, except in a few special cases. Most of these formulas were conjectured by Luther and Peschel, Kadanoff, Haldane, but only checked in the special solvable models; one of them, related to the anisotropic Ashkin-Teller model, is novel.

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It has long been conjectured, mainly by Kadanoff [1–3], Luther and Peschel [4] and Haldane [5], that a number of *universal relations* between critical exponents and other observables hold in a wide class of models, including planar Ising-like models with quartic interactions, vertex or Ashkin-Teller models, quantum spin chains and 1D fermionic systems. Such relations express how the *universality principle* works in models with continuously varying indices: the critical exponents are model dependent (non-universal) but satisfy model independent formulas, so allowing, for instance, *to express all the exponents in terms of a single one.* The universal relations have been verified only in certain special exactly solvable models, but the conjecture is that they are generally valid in a larger class of models, for which an exact solution is not available.

The interest in this kind of universal relations has been renewed by recent experiments on materials described by models in this class, like quantum spin chain models (KCuF3) [6], carbon nanotubes [7], layered structures [8] or even 1D Bose systems [9]. In such systems the critical exponents depend on the extraordinarily complex and largely unknown microscopic details of the compounds, but the universal relations allow concrete and testable predictions for them in terms of a few measurable parameters.

Several attempts in the last thirty years have been devoted to the proof of the universal relations [10], by taking as a starting point the formal continuum limit (identical for all the models considered here), where extra Lorentz and Gauge symmetries are verified and make it solvable. Of course, lattice effects destroy such symmetries and change the exponents; however, this problem has never been analyzed. On the other hand, not all the relations which are valid in the special solvable models are generically true; this happens, for example, for the exponents involved in the dynamic correlations [9] and another example will be shown below. It is therefore important to determine rigorously, and therefore unambiguously, under which conditions and which one among the relations valid in the solvable models are generically true.

Aim of this letter is to report the first rigorous derivation of several of such universal relations in a wide class of models, including *non solvable* models; in addition we will also prove a relation which is totally new.

The simplest class of models in the class of universality we are considering is coupled Ising models. A configuration (σ, σ') is the product of two configurations of spins $\sigma = \{\sigma_{\mathbf{x}} = \pm 1\}_{\mathbf{x}\in\Lambda}$ and $\sigma' = \{\sigma'_{\mathbf{x}} = \pm 1\}_{\mathbf{x}\in\Lambda}$. For a finite lattice Λ, the energy $H(\sigma, \sigma')$ is a function of the parameters J, J', λ

$$H = -J \sum_{\substack{\mathbf{x}\in\Lambda \\ j=0,1}} \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j} - J' \sum_{\substack{\mathbf{x}\in\Lambda \\ j=0,1}} \sigma'_{\mathbf{x}}\sigma'_{\mathbf{x}+\mathbf{e}_j} - \lambda V(\sigma, \sigma') \quad (1)$$

where $\mathbf{e}_0$ and $\mathbf{e}_1$ are the horizontal and vertical unit bond. $V(\sigma, \sigma')$ is a quartic interaction, short ranged and symmetric in the exchange $\sigma \to \sigma'$; for instance

$$V(\sigma, \sigma') = \sum_{j=0,1} \sum_{\mathbf{x},\mathbf{y}\in\Lambda} v(\mathbf{x}-\mathbf{y})\sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j}\sigma'_{\mathbf{y}}\sigma'_{\mathbf{y}+\mathbf{e}_j}$$

with $v(\mathbf{x})$ a short range potential. It is well known that several models in Statistical Mechanics can be rewritten as coupled Ising models. In particular the *Ashkin-Teller* model [11], a natural generalization of the Ising model to four states spins, can be rewritten in the form (1) with $v(\mathbf{x}) = \delta_{\mathbf{x},0}$. Another example is provided by the *Eight Vertex* model, in which $J = J'$ and $V(\sigma, \sigma') = \sum_{j=0,1}\sum_{\mathbf{x}\in\Lambda} \sigma_{\mathbf{x}+j(\mathbf{e}_0+\mathbf{e}_1)}\sigma_{\mathbf{x}+\mathbf{e}_0}\sigma'_{\mathbf{x}+j(\mathbf{e}_0+\mathbf{e}_1)}\sigma'_{\mathbf{x}+\mathbf{e}_1}$. An exact solution [11] exists only in the case of the 8V model and *not* for the generic Hamiltonian (1); even in the case of the 8V model, the correlations have not been computed and only a few indices can be obtained.

Recently new methods have been introduced in [12] and [13] to study 2D statistical mechanics models, which can be considered as a perturbation of the Ising model. These methods take advantage of the fact that such systems can be mapped in systems of interacting fermions in $d = 1 + 1$ dimensions. This mapping was known since a long time [14], but only in recent years a great progress has been achieved in the evaluation of the Grassmann integrals involved in the analysis of the interacting models, in the context of Quantum Field Theory and Solid State Physics, so that one can take advantage of this new technology to get information about statistical mechanics models. At the moment, when an exact solution is lacking, this is the only way to get rigorous quantitative information on the properties of such systems. The algorithm is based on multiscale analysis and allows us to prove convergence of several thermodynamic functions and correlations up to the critical temperature; essential ingredients of the analysis are compensations due to the anticommutativity of Grassmann variables and asymptotic Ward Identities (WI).

By using such methods, it has been proved in [13], in the case $J = J'$ and λ small, that the model is critical in the thermodynamic limit at the inverse temperature $\beta_c = T_c^{-1} = \arctan(\sqrt{2}-1)/|J| + O(\lambda)$; for T near T_c, the specific heat behaves as

$$C_v \sim \alpha^{-1}[|T-T_c|^{-\alpha} - 1] \tag{2}$$

with α a continuous non trivial function of λ. Moreover, if $G^\varepsilon(\mathbf{x}-\mathbf{y})$, $\varepsilon = \pm$, are the correlation functions of the two quadratic observables $O^\varepsilon_{\mathbf{x}} = \sum_{j=0,1} \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j} + \varepsilon \sum_{j=0,1} \sigma'_{\mathbf{x}}\sigma'_{\mathbf{x}+\mathbf{e}_j}$ (which are called *energy*, if $\varepsilon = +$, and *crossover*, if $\varepsilon = -$, in the AT model, while the names are exchanged in the 8V model), in [13] it has been also proved that the large distance decay of $G^\varepsilon(\mathbf{x}-\mathbf{y})$ is faster than any power of $\xi^{-1}|\mathbf{x}-\mathbf{y}|$, with correlation length

$$\xi \sim |T-T_c|^{-\nu}\ , \quad \text{as } T \to T_c$$

while at $T = T_c$, the decay of $G^\varepsilon(\mathbf{x}-\mathbf{y})$ is power law:

$$G^\varepsilon(\mathbf{x}-\mathbf{y}) \sim |\mathbf{x}-\mathbf{y}|^{-2x_\varepsilon}\ .$$

In the Ashkin-Teller model with $J \neq J'$, it has been proved in [15] that there are two critical temperatures, $T_{c,1}$ and $T_{c,2}$, such that

$$C_v \sim -\Delta^{-\alpha}\log[\Delta^{-2}|T-T_{c,1}|\cdot|T-T_{c,2}|] \tag{3}$$

where $2\Delta^2 = (T-T_{c,1})^2 + (T-T_{c,2})^2$ (the index α in (3) is the same as in (2)). While in the isotropic AT the logarithmic singularity of C_v is turned by the interaction in a power law, in the anisotropic AT C_v has still a logarithmic singularity; however, $T_{1,c} - T_{2,c}$ scales with a *transition index* $x_T = 1 + O(\lambda)$ in the isotropic limit:

$$|T_{1,c} - T_{2,c}| \sim |J-J'|^{x_T} \tag{4}$$

The existence of x_T was overlooked in the literature. The indices $x_+, x_-, \nu, \alpha, x_T$ are expressed by expansions which are *convergent* for λ small enough. Hence, the indices can be computed in principle with arbitrary precision by an explicit computation of the first orders and a rigorous bound for the rest; moreover, in this way one can prove that the indices depend on λ and on all details of the model. On the other hand, the complexity of the expansions makes essentially impossible to prove the universal relations directly from them.

Another important class of models whose correlations can be analyzed by similar methods are models of interacting fermions on a 1D lattice or quantum spin chains; they are all described by the Hamiltonian $H =$

$$-\frac{1}{2}\sum_{x=1}^{L-1}[a^+_x a^-_{x+1} + a^+_{x+1}a^-_x] - u[a^+_x a^+_{x+1} + a^-_{x+1}a^-_x] \tag{5}$$
$$+h\sum_{x=1}^{L}(\rho_x - \frac{1}{2}) + \lambda \sum_{1\le x,y\le L} v(x-y)(\rho_x - \frac{1}{2})(\rho_y - \frac{1}{2})$$

where $a^\pm_x$ are the fermion creation or annihilation operators, $\rho_x = a^+_x a^-_x$ and $v(x)$ is a short range potential. By using the Jordan-Wigner transformation, the Hamiltonian of the *Heisenberg quantum spin chains* can be written in this way, if $J_1 + J_2 = 2$, $u = (J_1 - J_2)/2$ and $J_3 = -\lambda$; in particular, if $v(x-y) = \delta_{|x-y|,1}/2$ and $h = 0$, we have the *XYZ model*. Let us define $\mathbf{x} = (x, x_0)$, $O_{\mathbf{x}} = e^{Hx_0}O_x e^{-Hx_0}$ and, if $A = O_{\mathbf{x}_1}...O_{\mathbf{x}_n}$, $\langle A\rangle = \lim_{L\to\infty}\frac{\mathrm{Tr}e^{-\beta H}\mathbf{T}(A)}{\mathrm{Tr}e^{-\beta H}}$, $\mathbf{T}$ being the time order product. If $u = 0$, it was shown in [16] that, for λ small enough, if T denotes the truncated expectation, $\langle S^{(3)}_{\mathbf{x}} S^{(3)}_{\mathbf{0}}\rangle_T \sim$

$$\cos(2p_F x)\frac{1+O(\lambda)}{2\pi^2[x^2 + (v_s x_0)^2]^{x_+}} + \frac{1+O(\lambda)}{2\pi^2[x^2+(v_s x_0)^2]} \tag{6}$$

where $S^{(3)}_{\mathbf{x}} = \rho_{\mathbf{x}} - 1/2$, $p_F = \cos^{-1}(h+\lambda) + O(\lambda)$ is the Fermi momentum (if $h = 0$, $p_F = \pi/2$ by symmetry) and $v_s = \sin p_F + O(\lambda)$ is the Fermi (or sound) velocity, which is modified by the interaction, since, contrary to the previous Ising case, there is no symmetry between space and time. Finally x_+ is a critical index, expressed by a convergent expansion; it depends on all details of the model, as it is apparent from the explicit computation of its first order contribution, which gives $x_+ = 1 - a_1\lambda + O(\lambda^2)$, with $a_1 = [\hat v(0) - \hat v(2p_F)]/(\pi \sin p_F)$ (we denote by $\hat f(\mathbf{k})$, $\mathbf{k} = (k_0, k)$, the Fourier transform of $f(\mathbf{x})$ and by $\hat f(k)$ the Fourier transform of $f(x)$). In the special case of the XXZ model ($v(x-y) = \delta_{|x-y|,1}/2$), (6) agrees with algebraic Bethe ansatz results [17]. When $J_1 \neq J_2$, that is $u \neq 0$, $\langle S^{(3)}_{\mathbf{x}} S^{(3)}_{\mathbf{0}}\rangle_T$ decays exponentially, with correlation length $\xi \sim |J_1 - J_2|^{-\bar\nu}$, with $\bar\nu = 1 + a_1\lambda + O(\lambda^2)$, a_1 being the same constant as before. If $J_1 = J_2$, $\langle a^-_{\mathbf{x}} a^+_{\mathbf{y}}\rangle_T \sim |\mathbf{x}-\mathbf{y}|^{-1-\eta}$, $\eta = O(\lambda^2) > 0$, and the correlation of the Cooper pair operator $\rho^c_{\mathbf{x}} = a^+_{\mathbf{x}} a^+_{\mathbf{x}'} + a^-_{\mathbf{x}} a^-_{\mathbf{x}'}$, $\mathbf{x}' = (x+1, x_0)$, decays as $|\mathbf{x}-\mathbf{y}|^{-2x_-}$, $x_- = 1 + a_1\lambda + O(\lambda^2)$.

If $u = 0$ and $j_{\mathbf{x}} = (2i\sin p_F)^{-1}[a^+_{\mathbf{x}'} a^-_{\mathbf{x}} - a^+_{\mathbf{x}} a^-_{\mathbf{x}'}]$, the following WI, for $\mathbf{k}, \mathbf{k}+\mathbf{p}$ close to $\mathbf{p}^\omega_F \equiv (\omega p_F, 0)$, $\omega = \pm$, are true:

$$-ip_0\langle\hat\rho_{\mathbf{p}}\hat a^+_{\mathbf{k}}\hat a^-_{\mathbf{k}+\mathbf{p}}\rangle + \omega p\tilde v_J\langle\hat j_{\mathbf{p}}\hat a^+_{\mathbf{k}}\hat a^-_{\mathbf{k}+\mathbf{p}}\rangle \sim BG$$
$$-ip_0\langle\hat j_{\mathbf{p}}\hat a^+_{\mathbf{k}}\hat a^-_{\mathbf{k}+\mathbf{p}}\rangle + \omega p\tilde v_N\langle\hat\rho_{\mathbf{p}}\hat a^+_{\mathbf{k}}\hat a^-_{\mathbf{k}+\mathbf{p}}\rangle \sim \bar B G \tag{7}$$

with $G \equiv G(\mathbf{k}, \mathbf{k}+\mathbf{p}) = [\langle\hat a^+_{\mathbf{k}}\hat a^-_{\mathbf{k}}\rangle - \langle\hat a^+_{\mathbf{k}+\mathbf{p}}\hat a^-_{\mathbf{k}+\mathbf{p}}\rangle]$, $B = 1$, $\bar B = 1 + O(\lambda)$ and $\tilde v_J, \tilde v_N = v_s(1+O(\lambda))$; in particular $\tilde v_N/\tilde v_J = 1 + 2a_1\lambda + O(\lambda^2)$, with a_1 the constant defined above, after (6). When $\lambda = 0$, the continuity equations for $\rho_{\mathbf{x}}$ and $j_{\mathbf{x}}$ imply WI similar to (7) with $\bar B = 1$ and

$\tilde v_N = \tilde v_J = v_s$; the interaction has the effect that the normalization $\bar B$ is not 1 ($[H,\rho_x]=0$ but $[H,j_x]\neq 0$) and two different velocities, the charge $\tilde v_J$ and the current velocity $\tilde v_N$, appear. The presence of the lattice, which breaks the Lorentz symmetry of the continuum limit, causes the presence of three distinct velocities, $\tilde v_N$, $\tilde v_J$, v_s. Finally, we recall that the *susceptibility* is defined as $\kappa = \lim_{p\to 0}\hat\Omega(0,p)$, where $\hat\Omega(p_0,p)$ is the Fourier transform of $\langle S^{(3)}_{\mathbf x} S^{(3)}_{\mathbf 0}\rangle_T$; $\kappa\rho^{-2}$ is the *compressibility* if ρ is the fermionic density. Our results are contained in the following Theorem.

Theorem *Given the models with hamiltonian (1), (5), at small coupling all the indices defined above can be uniquely expressed in terms of one of them:*

$$x_- = x_+^{-1} \quad , \quad \alpha = 2(1-x_+)(2-x_+)^{-1} \tag{8}$$

$$\nu^{-1} = 2-x_+ \quad , \quad \bar\nu^{-1} = 2-x_+^{-1} \tag{9}$$

$$2\eta = x_+ + x_+^{-1} - 2 \tag{10}$$

$$x_T = (2-x_+)(2-x_+^{-1})^{-1} \tag{11}$$

Moreover, in the model (5) the velocities appearing in (7) verify $\tilde v_N \tilde v_J = v_s^2$ and $\tilde v_J = \sin p_F$, while the susceptibility κ verifies

$$\kappa = x_+(\pi v_s)^{-1} \ . \tag{12}$$

(11) is a new relation for the Ashkin-Teller model, never proposed before; the first relation in (8) was conjectured in [1] and (9), (10) in [3, 4]. (12) is part of the Haldane *Luttinger liquid* conjecture [5] for fermionic systems or quantum spin chains. Some of the above relations were checked in certain solvable case: the second of (8), which is equivalent, by using the first of (9), to the *hyper-scaling relation* $2\nu = 2-\alpha$, in the case of the Eight Vertex model [11]; (10), (12) in the case of the Luttinger model [18]; (12) in the XYZ spin chain [5]. The above theorem provides the first proof of the validity of such relations for generic non solvable models. Note that, in the notation of [5], $v_N \equiv (\pi\kappa)^{-1}$ should not be confused with $\tilde v_N$ appearing in the WI (7); they are coinciding *only* in the special case of the Luttinger model. Therefore $\tilde v_N = v_N$ is an example of relation true in the Luttinger model but not in the presence of a lattice.

Outline of the proof. The technical details are long and will appear elsewhere [19, 20]; here we just outline the proof. The partition function and some of the correlations of the spin model (1) can be exactly rewritten as sums of *Grassmann integrals* describing $d=1+1$ Dirac fermions on a lattice and with quartic non local (but short ranged) interactions, by using the classical representation of the Ising model in terms of Grassmann integrals [21]. The Grassmann variables are written as $\psi_{\mathbf k} = \sum_{h=-\infty}^{0}\psi^{(h)}_{\mathbf k}$, with $\psi^{(h)}_{\mathbf k}$ living at momentum scale $\mathbf k = O(2^h)$. After the integration of the fields $\psi^{(0)},\ldots,\psi^{(h+1)}$, the partition function can be written [13] as

$$\int P_{Z_h,\mu_h}(d\psi^{(\le h)}) e^{V^{(h)}(\sqrt{Z_h}\psi^{(\le h)})} \tag{13}$$

where $P_{Z_h,\mu_h}(d\psi^{(\le h)})$ is the *Gaussian Grassmann integration* with propagator $g^{(\le h)}(\mathbf k) = \frac{\chi_h(\mathbf k')}{Z_h}\times$

$$\begin{pmatrix} -i\sin k_0 + \sin k + \mu_{++} & -\mu_h - \mu_{-+} \\ -\mu_h - \mu_{+-} & -i\sin k_0 - \sin k_1 + \mu_{--}\end{pmatrix}^{-1} \tag{14}$$

where $\chi_h(\mathbf k)$ is a smooth compact support function nonvanishing only for $|\mathbf k|\le 2^h$, Z_h and μ_h are the effective wave function renormalization and the effective mass, $\mu_{\pm\pm}$ are $O(\mathbf k^2)$ and non vanishing at $\mathbf k = (\pm\pi,\pm\pi)$ (there is no fermion doubling problem); moreover, $\mathcal V^{(h)} = \lambda_h \sum_{\mathbf x}\psi^+_{\mathbf x,+}\psi^-_{\mathbf x,+}\psi^+_{\mathbf x,-}\psi^-_{\mathbf x,-} + R_h$, where λ_h is the effective coupling and R_h is a sum of *irrelevant terms*, represented as space-time integrals of field monomials, multiplied by kernels which are *analytic functions* of λ_k, $k>h$. Analyticity is a very non trivial property, obtained via *tree expansions* [22] and exploiting anticommutativity properties of Grassmann variables, via *Gram inequality* for determinants (which takes into account compensations between different graphs of different signs at a given order). It is important to stress that (13) is *exact*, in the sense that the irrelevant terms and the lattice are fully kept into account (in standard RG applications they are instead neglected). The effective coupling λ_h converges, as $h\to-\infty$, to a function $\lambda_{-\infty}$ (analytic function of λ), thanks to the asymptotic vanishing of the beta function, which is a consequence of Ward Identities. A similar analysis can be repeated in the case of the fermionic model (5), the main (but trivial) difference being that (14) is replaced by a similar expression, taking into account that x_0 is a continuous variable; such asymmetry has the effect that, contrary to what happens in the spin case, the velocity is *renormalized* by the interaction. In order to exploit the asymptotic symmetries of the model, it is convenient to introduce the following Grassmann integral

$$\int P^{th}_Z(d\psi^{(\le N)}) e^{V^{(N)}(\sqrt{Z_N}\psi^{(\le N)})} \tag{15}$$

where, if $\psi = (\psi_+,\psi_-)$ and $\bar\psi = \psi^+\gamma_0$ are Euclidean $d=1+1$ spinors, $P^{th}_Z(d\psi^{(\le N)})$ is the fermionic gaussian integration with propagator $g^{(\le N)}(\mathbf k) = \chi_N(\mathbf k)(\gamma_\mu \mathbf k_\mu)^{-1}$, and $V^{(N)}(\psi^{(\le N)}) = \int d\mathbf x d\mathbf y v(\mathbf x-\mathbf y) j_\mu(\mathbf x) j_\mu(\mathbf y)$, with $j_\mu(\mathbf x) = \bar\psi_{\mathbf x}\gamma_\mu\psi_{\mathbf x}$ and $v(\mathbf x-\mathbf y)$ a short range symmetric interaction. A multiscale integration is now necessary also in the ultraviolet region to perform the limit $N\to\infty$, while in the integration of the infrared scales an expression similar to (13) is found; the effective coupling is denoted by $\tilde\lambda_h$. The crucial point is that it is possible to choose, by a fixed point argument, the values of $\tilde\lambda_\infty$ (fixed $c=1$ in the model (1), while $c=v_s$ in the model (5)) so that $\lambda_{-\infty} = \tilde\lambda_{-\infty}$. This implies that *the critical exponents of the two models are the same*, because the exponents are expressed by series in $\tilde\lambda_{-\infty}/c$ with universal coefficients. Of course $\tilde\lambda_\infty$ is a convergent series in λ depending on all details of the models (1) or (5). On the other hand, the continuum Grassmann integral (15) verifies extra Lorentz and Gauge symmetries, implying exact Ward Identities when the ultraviolet cut-off is removed; by the transformation $\psi \to e^{i\alpha_{\mathbf{x}}}\psi_{\mathbf{x}}$ one finds $-i\mathbf{p}_\mu\langle j_{\mu,\mathbf{p}}\psi_{\mathbf{k}}\bar{\psi}_{\mathbf{k}+\mathbf{p}}\rangle_{th} =$

$$\langle\psi_{\mathbf{k}}\bar{\psi}_{\mathbf{k}}\rangle_{th} - \langle\psi_{\mathbf{k}+\mathbf{p}}\bar{\psi}^-_{\mathbf{k}+\mathbf{p}}\rangle_{th} + \Delta_N(\mathbf{k},\mathbf{p}) \qquad (16)$$

where $\Delta_N = \langle\delta j_{\mathbf{p}}\psi_{\mathbf{k}}\bar{\psi}_{\mathbf{k}+\mathbf{p}}\rangle_{th}$, with $\delta j_{\mathbf{p}} = \int d\mathbf{k}[(\chi_N^{-1}(\mathbf{k}+\mathbf{p})-1)(\gamma_\mu\mathbf{k}_\mu+\gamma_\mu\mathbf{p}_\mu) - (\chi_N^{-1}(\mathbf{k})-1)\gamma_\mu\mathbf{k}_\mu]\bar{\psi}_{\mathbf{k}}\psi_{\mathbf{k}+\mathbf{p}}$; an analogous expression is obtained for the axial current $\bar{\psi}\gamma_\mu\gamma_5\psi$. By a multiscale analysis it can be proved that $\lim_{N\to\infty}\Delta_N(\mathbf{k},\mathbf{p}) =$

$$-i\tau\hat{v}(\mathbf{p})\mathbf{p}_\mu\langle j_{\mu,\mathbf{p}}\psi_{\mathbf{k},\omega}\bar{\psi}_{\mathbf{k}+\mathbf{p},\omega}\rangle_{th} \quad , \quad \tau = \tilde{\lambda}_\infty/(4\pi c) \quad (17)$$

A similar expression holds for the chiral WI; the fact that $\Delta_N(\mathbf{k},\mathbf{p})$ is not vanishing in the limit $N\to\infty$ is a manifestation of a *quantum anomaly*. The anomaly coefficient τ is *linear* in $\tilde{\lambda}_\infty$; this is the non-perturbative analogue of the *anomaly non renormalization* in QED in 4D. Such crucial property depends on our assumption about the interaction in (15); it would not be true, for instance, if we replace $v(\mathbf{x}-\mathbf{y})$ with a delta function [23]. By combining the WI with the Schwinger-Dyson equations, one gets some equations for the correlations, from which the indices can be computed as functions of τ. One can find, for example, that $x_+ = (1-\tau)(1+\tau)^{-1}$, $x_- = (1+\tau)(1-\tau)^{-1}$, so that $x_+x_- = 1$; the other relations between the indices follow by similar arguments. Note that the indices we consider have a simple expression in terms of $\tilde{\lambda}_\infty$, but $\tilde{\lambda}_\infty$ is of course rather complex and model dependent as a function of λ.

A similar RG analysis can be repeated for the model (5); it turns out that the vertex functions in the first line of (7) are asymptotically coinciding with $Z^{(3)}\langle j^0_{\mathbf{p}}\hat{\psi}^+_{\mathbf{k},\omega}\hat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega}\rangle_{th}$ and $i\tilde{Z}^{(3)}\langle j^1_{\mathbf{p}}\hat{\psi}^+_{\mathbf{k},\omega}\hat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega}\rangle_{th}$, with $\tilde{Z}^{(3)}/Z^{(3)} = 1 + a_1\lambda + O(\lambda^2)$; therefore, by using the WI for the model (15), we derive (7) with $B = Z^{(3)}Z^{-1}(1-\tau)^{-1} = 1$, $\bar{B} = \tilde{Z}^{(3)}Z^{-1}(1+\tau)^{-1}$, $\tilde{v}_N = v_s Z^{(3)}/\tilde{Z}^{(3)}$, $\tilde{v}_J = v_s\tilde{Z}^{(3)}/Z^{(3)}$; on the other hand, the equations of motion related to the lattice Hamiltonian impose the constraints $v_J = \sin p_F$ and $B = 1$. Finally a WI for the density correlation can be also derived; if $D_\omega(\mathbf{p}) = -ip_0 + \omega v_s p$ and $\hat{\Omega}(\mathbf{p}) = \langle\hat{\rho}_{\mathbf{p}}\hat{\rho}_{\mathbf{p}}\rangle =$, we get

$$\hat{\Omega}(\mathbf{p}) = \frac{1}{4\pi v_s Z^2}\frac{(Z^{(3)})^2}{1-\tau^2}\left[2 - \frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} - \frac{D_+(\mathbf{p})}{D_-(\mathbf{p})}\right]$$

which implies (12), by using $\kappa = \lim_{p\to 0}\hat{\Omega}(0,p)$.

In conclusion, we have established for the first time the validity of a number of universal relations between critical exponents and other quantities in a wide class of generally non solvable lattice models. They are true in special continuum solvable models and we have proven that the lattice symmetry breaking effects produce different velocities in the model (5) and change the critical exponents, but do not destroy the validity of several universal relations (on the other hand, not all the relations valid in the solvable models are generically true, like the relation $\tilde{v}_N = v_N$ or the relations for the dynamic exponents [9]). Some of the universal relations are used for the analysis of experiments in carbon nanotubes or spin chains, but we believe that their interest goes much beyond this, as they provide one of the very cases in which the *universality principle*, a general belief in Statistical Physics and beyond, can be rigorously verified. Extensions of our methods will hopefully allow to prove universal relations in an even wider class of models, as well as other relations between spin or dynamic exponents.

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