ISING MODELS, UNIVERSALITY AND THE NON RENORMALIZATION OF THE QUANTUM ANOMALIES

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A number of universal relations (proposed by Kadanoff, Luther, Peschel and Haldane) are believed to be true in a wide class of systems with continuously varying indices, among which are interacting planar Ising models, vertex or Ashkin-Teller models, quantum spin chains and 1D Fermi systems; by such relations one can predict several quantities in terms of a few measurable parameters without relying on the specific microscopic details. The validity of such relations can be checked in special solvable models but, despite several attempts, the proof of their general validity was up to now an open problem. A rigorous derivation of several of such relations (for solvable and not solvable models and without any use of exact solutions) has been recently obtained in [8] and [11] through Renormalization Group methods. The proof is based on the representation in terms of Grassmann integrals and the validity of the Adler-Bardeen property of the non renormalization of the quantum anomalies in the asymptotic *Ward identities*. Gauge invariance is exact only in the scaling limit but the lattice corrections can be rigorously taken into account.

Keywords: Universality, critical exponents, Renormalization Group

1. Introduction

The principle of *Universality* in statistical physics is central in the modern understanding of critical phenomena. Universality means that the critical properties close to phase transitions are *insensitive* to the microscopic details inside a certain *universality class* of systems. For models in the class of universality of the Ising model, whose exponents are pure numbers, this principle says that the exponents are *exactly the same*. Experimentally this happens in systems like carbon dioxide [22]; even if they are described by very complex Hamiltonians, depending on a number of microscopic details, their critical exponents *coincide* with the 3D Ising model ones. Despite widely accepted, rigorous results on universality in the Ising classs have only been proven in dimension 4 and above.

There are however systems in which the indices are not pure numbers but *depend on all the microscopic details*; this happens in a wide class of models, including planar Ising-like models with quartic interactions, vertex or Ashkin-Teller models, quantum spin chains and 1D fermionic systems. It has long been conjectured, mainly by Kadanoff [17–19], Luther and Peschel [29] and Haldane [21], that a number of *universal relations* between critical exponents and other observables are valid in this class of models. Such relations express how the *universality principle* works in such systems: the critical exponents are model dependent (non-universal) but satisfy model independent formulas, so allowing, for instance, *to express all the exponents in terms of a single one*. The universal relations have been verified only in certain special exactly solvable models, but the conjecture is that they are generally valid in a larger class of models, for which an exact solution is not available.

The interest in these universal relations has been renewed by recent experiments on materials described by models in this class, like quantum spin chain models (KCuF3) [28], carbon nanotubes [3] or even 1D Bose systems [26]. In such systems the critical exponents depend on the extraordinary complex and largely unknown microscopic details of the compounds, but the universal relations allow concrete and testable predictions for them in terms of a few measurable parameters.

A mathematical proof of such universal relations has shown to be a rather challenging problem. Several attempts in the last thirty years have been devoted to their proof [36], [35], [41] , [38], using a variety of methods ranging from operator product expansions, perturbation theory, Renormalization Group, bosonization and several others. It is common to all such approaches to start from a formal continuum limit in which extra Lorentz and Gauge symmetries are verified. However strictly speaking the formal continuum limit is singular, as it is plagued by ultraviolet divergences which were absent in the original lattice model. Moreover lattice effects destroy such symmetries and change the exponents, and it is not clear at all while the relations between exponents should be true also when such symmetries are violated.

On the other hand, not all the relations which are valid in the special solvable models are generically true; a counterexample is found for the exponents involved in the dynamic correlations [26] and another one will be shown below. It is therefore important to known rigorously, and therefore unambiguously, under which conditions and which one of the relations valid in the solvable models are generically true.

The proof of several of such universal relations has been achieved in [8, 11], and in this paper we will outline such results. Up to recent times, the exponents were known only for a very small number of models where an exact solution was known. New methods have been introduced in [39] and [31] to study 2D statistical mechanics models, which can be considered as a perturbation of the Ising model. These methods take advantage of the fact that such systems can be mapped in systems of interacting fermions in $d = 1 + 1$ dimensions. This mapping was known since a long time [23, 40], but only in recent years a great progress has been achieved in the evaluation of the Grassmann integrals involved in the analysis of the interacting models (see *e.g.* [33] for a review). The outcome is that the exponents and other physical observables can be written as *convergent series*; this means that they can be computed with arbitrary precision with an explicit computation of lowest orders and with a rigorous bound on the rest. The complexity of the expansions make however essentially impossible to prove the universal relations directly from them. It is then shown that such models have the same exponents to the ones of an *effective continuum fermionic model*, provided that the bare parameters of such model are chosen properly. The effective model is expressed in terms of Grassmann integrals which are identical to the ones appearing in certain Quantum Field Theory models; we take advantage of the Gauge symmetry and of a property called *anomaly non-renormalization* to get exact expressions of the critical indices and other quantities in terms of the bare parameters; by them, several of the universal relations can be proved.

2. Ising, Vertex and Ashkin-Teller models

The paradigmatic model for statistical mechanics is the $2D$ Ising model with Hamiltonian

$$H = -J \sum_{j=0,1} \sum_{\mathbf{x}\in\Lambda} \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j} \tag{1}$$

$\sigma_{\mathbf{x}} = \pm 1$, Λ is a square subset of $\mathbb{Z}^2$, $\mathbf{x} = (x_0, x) \in \Lambda$, $\mathbf{e}_0 = (0,1), \mathbf{e}_1 = (1,0)$. By the exact solution, due to Onsager, the critical exponents can be computed and they appear to be J-independent. The universality hypothesis says that a small next to nearest neighbor or a quartic short ranged perturbation *do not change* the exponents, and for a set of exponents this has been recently proved in [39].

The simplest model displaying indices which are non trivial functions of the parameters is obtained considering *two* Ising models coupled by a quartic interaction; the Hamiltonian is

$$H(\sigma,\sigma') = H_J(\sigma) + H_{J'}(\sigma') - \lambda V(\sigma,\sigma') \tag{2}$$

with V is a short ranged, quartic interaction in the spin and invariant in the spin exchange, like

$$V = \sum_{j=0,1} \sum_{\mathbf{x},\mathbf{y}\in\Lambda} v(\mathbf{x}-\mathbf{y})\sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j}\sigma'_{\mathbf{y}}\sigma'_{\mathbf{y}+\mathbf{e}_j} \tag{3}$$

with $v(\mathbf{x})$ a short range potential.

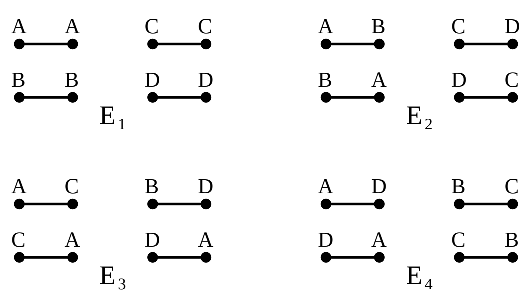

Fig. 1. Interactions between neighbor spins in the Ashkin-Teller models

The interest in the above model is increased by the fact that several systems in statistical mechanics, like the *Ashkin-Teller* and the *Eight Vertex* models, can be rewritten as coupled Ising models.

The Askhin-Teller [2] model is a generalization of the Ising model in which the spin has four values A, B, C, D, and to two neighbor spins is associated an energy E_1 for AA, BB, CC, DD, E_2 for AB, CD, E_3 for AC, BD, E_4 for AD, BC.

The Eight vertex model [5] is a generalization of the Ice model for the hydrogen bounding in which at each point is associated one among eight vertices with four different energies.

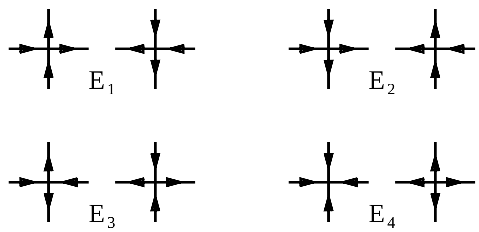

Fig. 2. The vertices in the Eight Vertex model

Both models can be rewritten in the form of coupled Ising models with Hamiltonian (2), see [5]. In the case of the Ashkin-Teller model this is straightforward as it is sufficient to associate to each lattice point $\mathbf{x}$ a couple of spins $(\sigma_{\mathbf{x}}, \sigma'_{\mathbf{x}})$ and to associate the four couples $(\pm 1, \pm 1)$ with the four states A, B, C, D. One can then immediately verify that the Ashkin-Teller hamiltonian is equivalent to (2), up to an additive constant $-J_0$, provided that we choose

$$\begin{aligned} -J &= (E_1 + E_2 - E_3 - E_4)/4 \quad & -J' &= (E_1 + E_3 - E_4 - E_2)/4 \\ -\lambda &= (E_1 + E_4 - E_2 - E_3)/4 \quad & -J_0 &= (E_1 + E_2 + E_3 + E_4)/4 \end{aligned} \tag{4}$$

and

$$V = \sum_{j=0,1} \sum_{\mathbf{x} \in \Lambda} \sigma_{\mathbf{x}} \sigma_{\mathbf{x}+\mathbf{e}_j} \sigma'_{\mathbf{x}} \sigma'_{\mathbf{x}+\mathbf{e}_j} \tag{5}$$

For a choice of parameters such that $J = J'$ the Ashkin-Teller model is called *isotropic*, while for $J \neq J'$ is called *anisotropic*. When $\lambda = 0$ the model is exactly solvable as its hamiltonian is the sum of two independent Ising models, and two *critical temperatures* are present if $J \neq J'$ which reduce to one in the $J = J'$ case.

Analogously also the Eight vertex model can be mapped in (2) with a suitable identification of the parameters; in such a case $J = J'$ and

$$V = \sum_{j=0,1} \sum_{\mathbf{x} \in \Lambda} \sigma_{\mathbf{x}+j(\mathbf{e}_0+\mathbf{e}_1)} \sigma_{\mathbf{x}+\mathbf{e}_0} \sigma'_{\mathbf{x}+j(\mathbf{e}_0+\mathbf{e}_1)} \sigma'_{\mathbf{x}+\mathbf{e}_1} \tag{6}$$

Despite the similarity between the Eight Vertex and the Ashkin-Teller model, quite apparent from (5),(6), an exact solution is known *only* in the case of the Eight Vertex model, due to Baxter ([4]), and, even in that case, only a few indices can

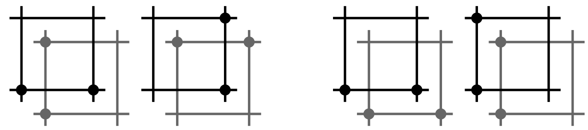

Fig. 3. Graphical representation of the interactions (5) and (6).

be computed. This shows how much the solvability is a delicate property depending on all details of the Hamiltonian; even if the two interactions appear equivalent, as far as the long distance properties are considered, only for a one of them an exact solution is known. By the exact solution of the Eight vertex model ([4]) one sees that the exponents are non trivial functions of λ; it is *not* in the Ising universality class.

3. Quantum spin chain and 1D Fermi systems

A model very related to the previous ones is the Heisenberg spin chain, physically realized in several compounds like $KCuF_3$. Its Hamiltonian is given by

$$H = -\sum_{x=1}^{L-1}[J_1 S^1_x S^1_{x+1} + J_2 S^2_x S^2_{x+1} - hS^3_x] + \lambda \sum_{1\le x,y\le L} v(x-y) S^3_x S^3_y \qquad (7)$$

where $S^\alpha_x = \sigma^\alpha_x/2$ for $i = 1, 2, \ldots, L$ and $\alpha = 1, 2, 3$, σ^α_x being the Pauli matrices and $|v(x-y)| \le Ce^{-\kappa|x-y|}$. In the special case ($\lambda = -J_3$)

$$v(x-y) = \delta_{|x-y|,1}/2, \qquad h = 0 \qquad (8)$$

the model is known as *XYZ chain* and it is exactly solvable. Despite it looks very diffferent, it is related to the previous models: the Hamiltonian of the XYZ model *commutes* with the transfer matrix of the 8V model ([5]). For a generic short range interaction, the solvability is lost.

The spin chain (7) can be exactly mapped in a system of interacting fermions through the *Jordan-Wigner* transformation. It is indeed well known that the operators

$$a^\pm_x = \prod_{y=1}^{x-1}(-\sigma^3_y)\sigma^\pm_x \qquad (9)$$

are a set of anticommuting fermionic operators and that, if $\sigma^\pm_x = (\sigma^1_x \pm i\sigma^2_x)/2$, we can write

$$\sigma^-_x = e^{-i\pi\sum_{y=1}^{x-1} a^+_y a^-_y} a^-_x\ , \quad \sigma^+_x = a^+_x e^{i\pi\sum_{y=1}^{x-1} a^+_y a^-_y}\ , \quad \sigma^3_x = 2a^+_x a^-_x - 1\ . \qquad (10)$$

Hence, if we fix the units so that $J_1 + J_2 = 2$ we get

$$H = -\frac{1}{2}\sum_{x=1}^{L-1}[a_x^+ a_{x+1}^- + a_{x+1}^+ a_x^-] - u\sum_{x=1}^{L-1}[a_x^+ a_{x+1}^+ + a_{x+1}^- a_x^-] \tag{11}$$
$$+h\sum_{x=1}^{L}(a_x^+ a_x^- - \frac{1}{2}) + \lambda \sum_{1\le x,y\le L} v(x-y)(a_x^+ a_x^- - \frac{1}{2})(a_y^+ a_y^- - \frac{1}{2})$$

where $\rho_x = a_x^+ a_x^-$, $u = (J_1 - J_2)/2$. In this form, the model describes interacting non relativistic 1D fermions on a lattice with a short range interaction and a BCS-like term (in the anisotropic case $J_1 \neq J_2$), and it can be used to understand the properties of the conduction electrons of one-dimensional metals.

It is convenient to introduce the density and the current operators:

$$\rho_x = S_x^3 + \frac{1}{2} = a_x^+ a_x^- \ ,$$
$$J_x = S_x^1 S_{x+1}^2 - S_x^2 S_{x+1}^1 = \frac{1}{2i}[a_{x+1}^+ a_x^- - a_x^+ a_{x+1}^-] \ . \tag{12}$$

If $\rho_{\mathbf{x}} = e^{x_0 H}\rho_x e^{-x_0 H}$, $a_{\mathbf{x}}^{\pm} = e^{x_0 H} a_x^{\pm} e^{-x_0 H}$ the above definition of the current is justified by the (imaginary time) conservation equation

$$\frac{\partial \rho_{\mathbf{x}}}{\partial x_0} = e^{Hx_0}[H, \rho_x]e^{-Hx_0} = -i\partial_x^{(1)} J_{\mathbf{x}} = -i[J_{x,x_0} - J_{x-1,x_0}] \tag{13}$$

where we have used that ρ_x commutes with the quartic part of H. Note that J_x does not verify a simple equation like (13), as J_x does not commute with the quartic part of H. Note also that

$$[H_0, \hat{J}_p] = \frac{1}{L}\sum_k \sin k(\cos(k+p) - \cos k)\hat{a}_{k+p}^+ \hat{a}_k \tag{14}$$

a relation which will be useful in the following.

4. Universal relations

It has been conjectured that the above models are in the same *universality class*; this does not mean that the exponents are the same (on the contrary, the indices depend on all the details of the Hamiltonian), rather it means that (for instance) there are *universal relations* between them, such that all *the indices of a single model can be expressed in terms of any one of them.* As an example of such relations, in the coupled Ising model the following relations have been conjectured

$$X_- = \frac{1}{X_+} \qquad \nu = \frac{1}{2 - X_+} \qquad \alpha = \frac{2 - 2X_+}{2 - X_+} = 2 - 2\nu \tag{15}$$

where $X_\pm$ are the exponents of the energy or crossover correlations, ν is the exponents of the correlation length, α the exponent of the specific heat (see below for their exact definition).

The first of above relations was proposed by Kadanoff [17], the second by Kadanoff and Wegner [19] and the third is the hyperscaling relation. In the case of the Eight vertex model, the index ν and α can be computed from the exact solution, and one can check the validity of the last of (15); the indices $X_\pm$ cannot be computed from the solution even in the Eight vertex case. In the spin chains or 1D fermions, Luther and Peschel [29] proposed similar relations, with a different identification of the exponents. Except from a partial verification in the Eight or XYZ models, there were no proof (until now) of the validity of (15) in non solvable models.

In general even the knowledge of a single exponent can be lacking; in the case of spin chains or 1D fermions, Haldane [21] conjectured other relations *allowing the determination of the exponents in terms of thermodynamic quantities* (*Luttinger liquid conjecture*). In particular if v_s is the Fermi velocity and κ is the susceptibility, calling $v_N = (\pi\kappa)^{-1}$ the following universal relation was conjectured in [21]

$$\frac{v_s}{v_N} = X_+ \tag{16}$$

The validity of this relation can be checked in the XYZ case; the correlation length exponent index $\bar{\nu}$ is, if $\cos\bar{\mu} = -J_3/J_1$

$$\bar{\nu} = \frac{\pi}{2\bar{\mu}} \tag{17}$$

from Baxter solution [4]. Moreover $X_- = 2(1 - \frac{\bar{\mu}}{\pi})$ from the relation $\bar{\nu} = \frac{1}{2-X_-}$ (conjectured in [29]) and from Bethe ansatz

$$v_s = \frac{\pi}{\bar{\mu}}\sin\bar{\mu} \qquad \kappa = \frac{1}{[2\pi(\pi/\bar{\mu} - 1)\sin\bar{\mu}]} \tag{18}$$

from which (16) follows. *The conjecture is that this relation is true generically*, for instance in the model (7) with $h \neq 0$ and any short range interaction $v(x-y)$.

5. Renormalization Group analysis for coupled Ising models

It happens sometimes that problems which appear rather hard in certain variables appear simpler in others. This is the case of the planar Ising model: it is a system of *interacting* spins which can be mapped in a system of *non interacting* fermions. Such a mapping is particularly useful because models which are perturbation of the Ising model, like the Eight Vertex or Ashkin-Teller models, can be mapped in models of interacting fermions, which can be analyzed via the methods developed in Quantum Field Theory.

The starting point is the following representation of the Ising model partition function:

$$Z_I = (\cosh\beta J)^B 2^S \sum_\gamma (\tanh\beta J)^{|\gamma|} \ . \tag{19}$$

where B is the total number of bonds, S is the total number of sites and the sum is over all the multipolygon γ with length $|\gamma|$. If open boundary conditions are assumed, only multipolygons *not* winding up the lattice are allowed. In the case of periodic boundary conditions the representation is the same, but the polygons are allowed to wind up the lattice. It was proved in [20], [37], [23] that Z_I can be written as the sum of four *Pfaffians* which can be written as four *Grassmann integrals* with different boundary conditions

$$Z_I = -Z_{+,+} + Z_{+,-} + Z_{-,+} + Z_{-,-} \tag{20}$$

$$Z_{\epsilon,\epsilon'} = (\cosh \beta J)^B 2^S \frac{1}{2} \int \prod_{\mathbf{x}\in\Lambda} dH_{\mathbf{x}} d\bar{H}_{\mathbf{x}} dV_{\mathbf{x}} d\bar{V}_{\mathbf{x}} e^{S_{\epsilon,\epsilon'}} \tag{21}$$

where

$$S_{\epsilon,\epsilon'} = \sum_{\mathbf{x}\in\Lambda} \tanh \beta J [\bar{H}_{x,x_0} H_{x+1,x_0} + \bar{V}_{x,x_0} V_{x,x_0+1}] + \tag{22}$$
$$\sum_{\mathbf{x}\in\Lambda} [\bar{H}_{x,x_0} H_{x,x_0} + \bar{V}_{x,x_0} V_{x,x_0} + \bar{V}_{x,x_0} \bar{H}_{x,x_0} + V_{x,x_0} \bar{H}_{x,x_0} + H_{x,x_0} \bar{V}_{x,x_0} + V_{x,x_0} H_{x,x_0}]$$

and $H_{\mathbf{x}}, \bar{H}_{\mathbf{x}}, V_{\mathbf{x}}, \bar{V}_{\mathbf{x}}$ are anticommuting variables such that

$$\begin{aligned} \bar{H}_{x,x_0+L} &= \epsilon \bar{H}_{x,x_0} \qquad \bar{H}_{x+L,x_0} = \epsilon' \bar{H}_{x,x_0} \\ H_{x,x_0+L} &= \epsilon H_{x,x_0} \quad H_{x+L,x_0} = \epsilon' H_{x,x_0} \end{aligned} \tag{23}$$

and identical relations hold for the variables $V, \bar{V}$. Note that periodic or antiperiodic boundary conditions are imposed in the x_0, x variables depending on $\epsilon, \epsilon' = \pm$. The Grassmann integral is a linear operation such that $\int d\eta_{\mathbf{x}} = 0$, $\int d\eta_{\mathbf{x}} \eta_{\mathbf{x}} = 1$ if $\eta_{\mathbf{x}}$ is any of $(H_{\mathbf{x}}, \bar{H}_{\mathbf{x}}, V_{\mathbf{x}}, \bar{V}_{\mathbf{x}})$.

Let us consider now the coupled Ising model (2); we will be interested in particular in the specific heat C_v and the energy $\epsilon = +$ and cross-over ($\epsilon = -$) correlations, defined as

$$G^\epsilon_\beta(\mathbf{x}-\mathbf{y}) = \lim_{\Lambda\to\infty} \langle O^\epsilon_{\mathbf{x}} O^\epsilon_{\mathbf{y}} \rangle_\Lambda - \langle O^\epsilon_{\mathbf{x}} \rangle_\Lambda \langle O^\epsilon_{\mathbf{y}} \rangle_\Lambda \qquad \epsilon = \pm \tag{24}$$

where

$$O^\epsilon_{\mathbf{x}} = \sum_{j=0,1} \sigma_{\mathbf{x}} \sigma_{\mathbf{x}+\mathbf{e}_j} + \epsilon \sum_{j=0,1} \sigma'_{\mathbf{x}} \sigma'_{\mathbf{x}+\mathbf{e}_j} \tag{25}$$

where $\langle \cdot \rangle_\Lambda$ is the average over all configurations of the spins with statistical weight $e^{-\beta H}$, H given by (2). Starting from (21) and performing suitable change of variables, such correlations can be written as sums of functional derivatives (with respect to A^ϵ, $\epsilon = +$ for the energy and $\epsilon = -$ for the crossover) of Grassmann integrals with different boundary conditions; in the thermodynamic limit it is sufficient to consider only one of them, for instance

$$Z(A) = \int P_{Z_1,\mu_1}(d\psi) e^{L^2 \mathcal{N} + \mathcal{V}^{(1)}(\sqrt{Z_1}\psi) + \mathcal{B}^{(1)}(\sqrt{Z_1}\psi, A)} , \tag{26}$$

where $\mathcal{N}$ is a constant, $\psi^\pm_{\mathbf{k},\omega}$ is a finite set of Grassman variables, $\mathbf{k} = (k_0, k)$, $k_0 = \frac{2\pi}{L}(n_0 + \frac{1}{2})$, $k = \frac{2\pi}{L}(n + \frac{1}{2})$, $n_0, n_1 = -L/2, ...L/2 - 1$, $P_{Z_1,\mu_1}(d\psi)$ is a gaussian

Grassman integration with propagator, in the case $J = J'$

$$g(\mathbf{k}) = \frac{1}{Z_1} \begin{pmatrix} -i\sin k_0 + \sin k + \mu_{++}(\mathbf{k}) & -\mu_1 - \mu_{-+}(\mathbf{k}) \\ -\mu_1 - \mu_{+-}(\mathbf{k}) & -i\sin k_0 - \sin k_1 + \mu_{++}(\mathbf{k}) \end{pmatrix}^{-1} \tag{27}$$

with $\mu_1 = O(|t - t_c|)$, $t = \tanh \beta J$, $t_c = \tanh \beta_c J = \sqrt{2} - 1 - \nu$, $Z_1 = O(1)$; finally $\mu_{\epsilon,\epsilon} = O(\mathbf{k}^3)$ and $\mu_{\epsilon,-\epsilon} = O(\mathbf{k}^2)$ are such that the determinant vanishes only when $\mu_1 = 0$ and $\mathbf{k} = (0,0)$. Moreover (27) is given by

$$\mathcal{V}^{(1)}(\psi) = \nu \sum_{\mathbf{x},\omega=\pm} \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} + \lambda_1 \sum_{\mathbf{x}} \psi^+_{\mathbf{x},+}\psi^-_{\mathbf{x},+}\psi^+_{\mathbf{x},-}\psi^-_{\mathbf{x},-} + R_1(\psi) \tag{28}$$

$$\mathcal{B}^{(1)}(\psi, A) = \sum A_{\epsilon,\mathbf{x}} O^\epsilon_{\mathbf{x}} + R_2(A, \psi) \tag{29}$$

with $\nu_1 = O(\nu), \lambda_1 = O(\lambda)$; R_1 is a sum of monomials in ψ more than quartic in ψ or quartic with at least a derivatives and R_2 is a sum of monomials in A, ψ more than quadratic in ψ or quadratic with at least a derivative; finally

$$O^+_{\mathbf{x}} = \psi^+_{\mathbf{x},+}\psi^-_{\mathbf{x},-} + \psi_{\mathbf{x},-}\psi^-_{\mathbf{x},+} \qquad O^-_{\mathbf{x}} = i[\psi^+_{\mathbf{x},+}\psi^+_{\mathbf{x},-} + \psi^-_{\mathbf{x},+}\psi^-_{\mathbf{x},-}] \tag{30}$$

Note that ν is a *counterterm* to be chosen so that β_c is the critical temperature.

Grassmann integrals appear in the analysis of the fermionic sector of Quantum Field Theory models; in particular, (26) strongly resembles the generating functions of the current correlations of a interacting 2D *Dirac fermions* with mass μ_1 and wave function renormalization Z_1 on a lattice. Zero mass $\mu_1 = 0$ corresponds to criticality. Note also that the lattice regularization of massless Dirac fermions suffer by a well known problem called *fermion doubling*, which is usually solved by the introduction of the so-called *Wilson terms*; in (26) such Wilson terms appear naturally (they are the $\mu_{\epsilon,\epsilon'}(\mathbf{k})$ terms in (27)). In the case $J \neq J'$, the correlations are expressed in terms of Dirac fermions with two different kinds of mass terms.

In the case of the Ising model, the Grassmann integrals are quadratic, and this corresponds to *free* Dirac particles on a lattice; on the contrary, coupled Ising models correspond to *interacting* Dirac particles. The fact that such models can be mapped in interacting fermions was known since a long time, but this was not really used in rigorous analysis for the lack of methods to control the perturbative expansions appearing in interacting quantum field theories. However in recent years a great progress has been achieved in the analysis of non quadratic Grassmann integrals; the key observation of Caianiello [12], saying that fermionic perturbative expansions have much better convergence properties due to the relative signs, has been substantiated in the rigorous construction of 2D fermionic quantum field theories, either renormalizable and asymptotically free [14, 15] and superrenormalizable [30]; the key tool of such analysis are the Gram bound for determinants and the Battle-Brydges-Federbush formula for truncated expectations. The Grassmann integral in (26) is much more difficult to analyze, as the theory is renormalizable but *not* asymptotically free; it belongs to a class of models with *vanishing* beta function (see below for details). Such kinds of models can be constructed only exploiting non trivial cancellations in the expansions. The first example of rigorous construction of a model in this class was in [7], relying on the *exact solution* of the Luttinger model [34]. Later on, models with vanishing Beta function were constructed using a technique capable of combining Ward Identities based on local symmetries with Renormalization Group methods which was developed in [10] (and without any use of exact solutions); the main problem to face is that the momentum cut-off breaks local symmetries producing additional terms in the Ward Identities which can be however rigorously taken into account. By using such methods, the following Theorem can be proved.

Theorem 5.1. *(Mastropietro [31]) The coupled Ising model (2) with $J = J'$ and λ small enough is critical at $\tanh \beta_c J = \sqrt{2} - 1 + O(\lambda)$ and the specific heat is*

$$C_v \sim -\frac{1}{\alpha}[1 - |\beta - \beta_c|^{-\alpha}] \tag{31}$$

with $\alpha = O(\lambda)$. If $\beta \neq \beta_c$ the density and crossover correlations $G^\epsilon_\beta(\mathbf{x} - \mathbf{y})$, $\epsilon = \pm$ (24),(25) decay faster than any power of $\xi^{-1}|\mathbf{x} - \mathbf{y}|$, with $\xi^{-1} \sim C\,|\beta - \beta_c|^\nu$ with $\nu = 1 + O(\lambda)$ and C is a constant. Moreover

$$G^\epsilon_{\beta_c}(\mathbf{x} - \mathbf{y}) \sim \frac{C_\epsilon}{|\mathbf{x} - \mathbf{y}|^{2X_\epsilon}}\ , \text{ as } |\mathbf{x} - \mathbf{y}| \to \infty\ , \tag{32}$$

with $X_\pm = 1 + O(\lambda)$ and $C_\pm$ constants .

From the above theorem we see that the interaction has two main effects. The first one is to change the value of the critical temperature. The second and more dramatic one is to modify qualitatively the critical properties. The logarithmic singularity in the specific heat of the Ising model is changed in a power law singularity when $\lambda > 0$; on the contrary for $\lambda < 0$ the specific heat is a continuous function. Moreover, the exponents $X_\pm, \nu$, which were equal to 1, are now continuous non trivial functions of λ.

The above result gives the first proof of the fact that the critical exponents are non trivial functions of the interaction in coupled Ising models with a generic quartic interaction (2); previously this was known only in the Eight Vertex case thanks to the exact solution. The series for X_+, X_-, ν, X_T are *convergent* for small λ, and the indices can be computed with arbitrary precision by an explicit computation of the first orders. In the case of a single perturbed Ising model, it was proved by Pinson and Spencer [39] that the indices $\nu = 1, X_\pm = 1$, that is they are the same as the Ising ones.

A similar Renormalization Group analysis can be performed in the anisotropic Ashkin-Teller model when $J \neq J'$.

Theorem 5.2. *(Giuliani,Mastropietro [16]) In the case of the anisotropic Ashkin-Teller model (2),(5) ($J \neq J'$) there are two critical temperatures, T_c^+ and T_c^- such that*

$$|T_c^- - T_c^+| \sim |J - J'|^{X_T} \tag{33}$$

with $X_T = 1 + O(\lambda)$ *and*

$$C_v \sim -\Delta^\alpha \log \frac{|T - T_c^-| \cdot |T - T_c^+|}{\Delta^2} \tag{34}$$

where $2\Delta^2 = (T - T_c^-)^2 + (T - T_c^+)^2$.

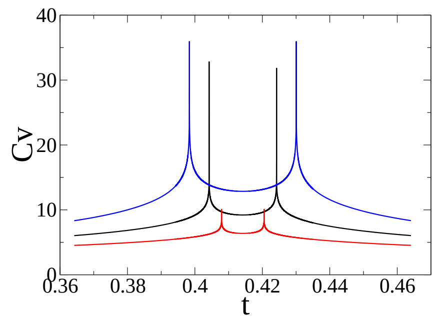

Fig. 4. The specific heat (34), in the case $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$.

In this case the specific heat has the same logarithmic singularity as in the Ising model; however, even if we are in the universality class of the Ising model, the difference between the two critical temperatures rescale with an anomalous exponent in the isotropic limit $|T_{1,c} - T_{2,c}| \sim |J - J'|^{X_T}$; the existence of such a *transition index* was overlooked in the physical literature.

Let us sketch the proof of Theorem 5.1 for the case $J = J'$. The Grassmann variables are written as $\psi_{\mathbf{k}} = \sum_{h=-\infty}^{0} \psi_{\mathbf{k}}^{(h)}$, and after the integration of the fields $\psi^{(0)}, \ldots, \psi^{(h+1)}$, the partition function can be written as

$$Z(A) = e^{S^{(h)}(A)} \int P_{\bar{Z}_{h-1},\mu_{h-1}}(d\psi^{(\leq h)}) e^{\mathcal{V}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)}) + \mathcal{B}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)}, A)} , \tag{35}$$

where $\psi^{(\leq h)} = \sum_{j=-\infty}^{h} \psi^{(j)}$ and $P_{Z_h,\mu_h}(d\psi^{(\leq h)})$ is the *Gaussian Grassmann integration* with propagator

$$g^{(\leq h)}(\mathbf{k}) = \frac{\chi_h(\mathbf{k})}{Z_h} \begin{pmatrix} -i\sin k_0 + \sin k + \mu_{++} & -\mu_h - \mu_{-+} \\ -\mu_h - \mu_{+-} & -i\sin k_0 - \sin k_1 + \mu_{--} \end{pmatrix}^{-1} \tag{36}$$

with $\chi_h(\mathbf{k})$ a smooth compact support function non vanishing for $|\mathbf{k}| \leq 2^h$. The *effective interaction* $\mathcal{V}^{(h)}(\psi)$ is a sum over monomials in the Grassmann variables

$$\mathcal{V}^{(h)}(\psi) = \gamma^h \nu_h F_\nu^{(h)} + \lambda_h F_\lambda^{(h)} + R_h , \tag{37}$$

where

$$F_\nu^{(h)} = \frac{1}{L^2}\sum_{\omega=\pm}\sum_{\mathbf{k}} \hat\psi^{(\le h)+}_{\mathbf{k},\omega}\hat\psi^{(\le h)-}_{\mathbf{k},-\omega} \;, \tag{38}$$

$$F_\lambda^{(\le h)} = \frac{1}{L^8}\sum_{\mathbf{k}_1,\dots,\mathbf{k}_4} \hat\psi^{(\le h)+}_{\mathbf{k}_1,+}\hat\psi^{(\le h)+}_{\mathbf{k}_3,-}\hat\psi^{(\le h)-}_{\mathbf{k}_2,+}\hat\psi^{(\le h)-}_{\mathbf{k}_4,-}\delta(\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}_4)\;.$$

and R^h contains sum of monomials with more than four fields, or quartic with at least a derivative, or bilinear with at least two derivatives. In the same way

$$\mathcal{B}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)}, A) = \sum_{\epsilon=\pm,\mathbf{x}} Z^{(\epsilon)}_{h-1} A^\epsilon_{\mathbf{x}} O^{(\le h)\epsilon}_{\mathbf{x}} + \bar R_h \;, \tag{39}$$

where $O^{\pm}$ is given by (30) and $\bar R_h$ contains terms more than quadratic, or quadratic with a derivative. It is natural the interpretation of λ_h in (37) as the *effective coupling* of the model at momentum scales $O(2^h)$.

Notice that the propagator of the field $\psi^{(h)}$ can be written, for $h \le 0$, as

$$g^{(h)}(\mathbf{x},\mathbf{y}) = g^{(h)}_T(\mathbf{x},\mathbf{y}) + r^{(\le h)}(\mathbf{x},\mathbf{y})\;, \tag{40}$$

where

$$g^{(h)}_T(\mathbf{x},\mathbf{y}) = \frac{1}{L^2}\sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{1}{Z_h}T_h^{-1}(\mathbf{k})\;, \tag{41}$$

$$T_h(\mathbf{k}) = f_h(\mathbf{k})\begin{pmatrix} -ik_0+k & -\mu_h \\ \mu_h & -ik_0-k\end{pmatrix} \tag{42}$$

with $f_h(\mathbf{k})$ a smooth compact support function non vanishing for $2^{h-1}\le|\mathbf{k}|\le 2^{h+1}$ and, for any positive integer M,

$$|r^{(h)}(\mathbf{x},\mathbf{y})| \le C_M\frac{2^{2h}}{1+(2^h|\mathbf{x}-\mathbf{y}|^M)}\;. \tag{43}$$

On the other hand, $g^{(h)}_T(\mathbf{x},\mathbf{y})$ verifies a bound similar to (43) with 2^h replacing 2^{2h}.

The running couplings λ_j (which, by construction, are the same in the massless $\mu_1 = 0$ or in the massive $\mu_1 \neq 0$) satisfy a recursive equation of the form

$$\lambda_{j-1} = \lambda_j + \beta^{(j)}_\lambda(\lambda_j,\dots,\lambda_0) + \bar\beta^{(j)}_\lambda(\lambda_j,\nu_j;\dots;\lambda_0,\nu_0)\;, \tag{44}$$

where $\beta^{(j)}_\lambda$, $\bar\beta^{(j)}_\lambda$ are μ_1-independent and expressed by a *convergent* expansion in $\lambda_j,\nu_j..,\lambda_0,\nu_0$ (convergence follows from Gram bounds); moreover $\bar\beta^{(j)}_\lambda$ vanishes if at least one of the ν_k is zero. The running coupling λ_j stays close to λ_1 for any j as a consequence of the following property, called *vanishing of the Beta function*, which was proved in Theorem 2 of [9]; for suitable positive constants C and $\theta < 1$:

$$|\beta^{(j)}_\lambda(\lambda_j,\dots,\lambda_j)| \le C|\lambda_j|^2 2^{\theta j}\;. \tag{45}$$

It is possible to prove that, for a suitable choice of $\nu_1 = O(\lambda)$, $\nu_j = O(2^{\theta j}\lambda)$, and this implies $\bar\beta_\lambda^{(j)} = O(2^{\theta j}\lambda^2)$ so that the sequence λ_j converges, as $j \to -\infty$, to a smooth function $\lambda_{-\infty}(\lambda) = \lambda_1 + O(\lambda^2)$, such that

$$|\lambda_j - \lambda_{-\infty}| \le C\lambda^2 2^{\theta j} \; . \tag{46}$$

Moreover

$$\frac{Z_{j-1}}{Z_j} = 1 + \beta_z^{(j)}(\lambda_j, ..., \lambda_0) + \bar\beta_z^{(j)}(\lambda_j, \nu_j; .., \lambda_0, \nu_0) \; , \tag{47}$$

with $\bar\beta_z^{(j)}$ vanishing if at least one of the ν_k is zero so that $\bar\beta_z^{(j)} = O(\lambda 2^{\theta j})$. Finally

$$\beta_z(\lambda_j, ..., \lambda_0) = \beta_z(\lambda_{-\infty}, ..., \lambda_{-\infty}) + O(\lambda 2^{\theta h}) \; , \tag{48}$$

where the last identity follows from (46) and the function $\beta_z(\lambda_{-\infty}, ..., \lambda_{-\infty})$ is by definition sum of terms in which only the propagators $g_T^{(h)}$ (41) appear (the terms containing $r^{(j)}$ are included in the second term in (48)). Similar equations hold for $Z_h^{(\pm)}, \mu_h$, with

$$\beta_\pm(\lambda_j, ..., \lambda_0) = \beta_\pm(\lambda_{-\infty}, ..., \lambda_{-\infty}) + O(\lambda 2^{\theta h}) \; . \tag{49}$$

By an explicit computation and (48), (49), there exist $\eta_+(\lambda_{-\infty}) = c_1\lambda_{-\infty} + O(\lambda_{-\infty}^2)$, $\eta_-(\lambda_{-\infty}) = -c_1\lambda_{-\infty} + O(\lambda_{-\infty}^2)$, $\eta_\mu(\lambda_{-\infty}) = c_1\lambda_{-\infty} + O(\lambda_{-\infty}^2)$ and

$\eta_z(\lambda_{-\infty}) = c_2\lambda_{-\infty}^2 + O(\lambda_{-\infty}^3)$, with c_1 and c_2 strictly positive, such that, for any $j \le 0$,

$$\begin{aligned} &|\log_2(Z_{j-1}/Z_j) - \eta_z(\lambda_{-\infty})| \le C\lambda^2 2^{\theta j} \; , \\ &|\log_2(\mu_{j-1}/\mu_j) - \eta_\mu(\lambda_{-\infty})| \le C|\lambda| 2\theta j, \\ &|\log_2(Z_{j-1}^{(\pm)}/Z_j^{(\pm)}) - \eta_\pm(\lambda_{-\infty})| \le C\lambda^2 2^{\theta j} \; . \end{aligned} \tag{50}$$

The critical indices in Theorem 5.1 are functions of $\lambda_{-\infty}$ only, as it is clear from (48); moreover the exponents appearing there are such that

$$X_\pm = 1 - \eta_\pm + \eta_z \qquad \mu = \eta_+ - \eta_z = 1 - x_+ \; . \tag{51}$$

If $\mu_1 \neq 0$ (that is, if the temperature is not the critical one), the correlations decay faster than any power with rate proportional to μ_{h^*}, where, if $[x]$ denotes the largest integer $\le x$, h^* is given by $h^* = \left[\frac{\log_2|\mu_1|}{1+\eta_\mu}\right]$.

In conclusion, the exponents are written as convergent series so that they can be computed with arbitrary precision; the complexity of the expansions makes however essentially impossible to prove the universal relations directly from the expansions and new ideas are necessary.

6. Proof of universality and anomaly non renormalization

Some of the universal relations for coupled Ising models (2) with a generic quartic interaction have been recently proved.

Theorem 6.1. *(Benfatto,Falco,Mastropietro [6]). Given the coupled Ising model with quartic interaction (2), with the same definitions as in Theorems 5.1 and 5.2 and λ small enough the following relations are true*

$$X_- = \frac{1}{X_+} \qquad \alpha = \frac{2-2X_+}{2-X_+}\,, \tag{52}$$
$$\nu = \frac{1}{2-X_+} \qquad X_T = \frac{1-X_+}{1-X_+^{-1}}$$

Despite the exponents depend from all the details of the model, they verify universal, model independent relations allowing to express all the exponents (which we can compute) in terms of any one of them. The first two of the above relations are the ones conjectured by Kadanoff [17] and the second by Kadanoff and Wegner [19] while the last one is completely new.

In order to prove such theorem we introduce a continuum fermionic theory defined as the formal scaling limit of the original one plus an ultraviolet regularization; more exactly, we prove that the critical indices X_+, X_-, ν and X_T of the spin model (2) are *equal* to the indices of a fermionic theory provided that the bare coupling λ_∞ of the new theory is properly chosen as a suitable function of the parameter λ in (2). The correlations of this continuum fermionic theory are the functional derivatives of the following Grassmann integral

$$\int P_Z(d\psi^{(\leq N)}) e^{V^{(N)}(\sqrt{Z}\psi^{(\leq N)})+\sum_{\omega=\pm}\int d\mathbf{x}[\psi^+_{\mathbf{x},\omega}\phi^-_{\mathbf{x},\omega}+\psi^-_{\mathbf{x},\omega}\phi^+_{\mathbf{x},\omega}]} \tag{53}$$

with $\psi, \bar\psi$ are Euclidean $d = 1+1$ spinors ($\psi^\pm = (\psi^\pm_+, \psi^\pm_-)$), $P_Z(d\psi^{(\leq N)})$ is the fermionic gaussian integration with propagator $g^{(\leq N)} = \frac{\chi_N(\mathbf{k})}{\not{\mathbf{k}}}$, $\not{\mathbf{k}} = \gamma_0 k_0 + c\gamma_1 k_1$ (which in components appear to be equal to $g_T^{(\leq N)}(\mathbf{x})$ (41)), and

$$V^{(N)}(\psi^{(\leq N)}) = \lambda_\infty \int d\mathbf{x} d\mathbf{y} v(\mathbf{x}-\mathbf{y}) j_\mu(\mathbf{x}) j_\mu(\mathbf{y}) \tag{54}$$

with $j_\mu(\mathbf{x}) = \bar\psi_{\mathbf{x}}\gamma_\mu\psi_{\mathbf{x}}$ and $v(\mathbf{x}-\mathbf{y})$ a short range symmetric interaction; moreover

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \gamma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{55}$$

A multiscale integration is now necessary also in the ultraviolet region to perform the limit $N \to \infty$; no ultraviolet divergences are present due to the non locality of the interaction in (54). The multiscale integration for the infrared scales can be done exactly as described in the previous section, with the only difference that, by the oddness of the free propagator, $\nu_j = 0$ and

$$\lambda_{j-1} = \lambda_j + \hat\beta^{(j)}_\lambda(\lambda_j, ...\lambda_0)\,, \tag{56}$$

where

$$\hat\beta^{(j)}_\lambda(\lambda_j, ...\lambda_0) = \beta^{(j)}_\lambda(\lambda_j, ...\lambda_0) + O(\lambda_\infty^2 2^{\theta j})\,, \tag{57}$$

$\beta^{(j)}_\lambda(\lambda_j,...\lambda_j)$ being the function appearing in (45), so that we can prove that $\lambda_{-\infty} = \lambda_0 + O(\lambda_0^2)$; since $\lambda_0 = \lambda_\infty + O(\lambda_\infty^2)$, we have

$$\lambda_{-\infty} = h(\lambda_\infty) = \lambda_\infty + O(\lambda_\infty^2) \; , \tag{58}$$

for some analytic function $h(\lambda_\infty)$, invertible for λ_∞ small enough. Moreover

$$\frac{Z^\pm_{j-1}}{Z^\pm_j} = 1 + \hat\beta^{(j)}_\pm(\lambda_j,...,\lambda_0) \; , \tag{59}$$

with

$$\hat\beta^{(j)}_\pm(\lambda_j,...,\lambda_0) = \beta^{(j)}_\pm(\lambda_j,...\lambda_0) + O(\lambda_\infty^2 2^{\theta j}) \; , \tag{60}$$

$\beta^{(j)}_\pm$ being the functions appearing in (49) (as consequence of (40)). This implies that

$$\eta_\pm = \log_2[1 + \beta^{(-\infty)}_\pm(\lambda_{-\infty},...\lambda_{-\infty})] \; , \tag{61}$$

that is *the critical indices in the AT or 8V or in the model (53) are the same as functions of* $\lambda_{-\infty}$.

If we call $\lambda'_j(\lambda)$ the effective couplings of the lattice model (2) appearing in the previous section, the invertibility of $h(\lambda_\infty)$ implies that we can choose λ_∞ so that (chosen $c = 1$)

$$h(\lambda_\infty) = \lambda'_{-\infty}(\lambda) \; . \tag{62}$$

With the above choice of λ_∞ *the exponents in the models (2) and (53) are the same.*

What have we gained by this? The point is that the continuum fermionic theory (53) has correlations expressed by Grassmann integrals which are identical to the ones appearing in certain Quantum Field Theory models; in particular it verifies extra *Gauge symmetries* with respect to the original spin Hamiltonian. If $< ... >_{th}$ are the correlations with respect to $P(d\psi^{(\leq N)})e^{V^{(N)}}$ in (54), by the transformation $\psi \to e^{i\alpha_{\mathbf{x}}}\psi_{\mathbf{x}}$ one finds

$$-i\mathbf{p}_\mu < j_{\mu,\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}+\mathbf{p}} >_{th} = < \psi_{\mathbf{k}}\bar\psi_{\mathbf{k}} >_{th} - < \psi_{\mathbf{k}+\mathbf{p}}\bar\psi^-_{\mathbf{k}+\mathbf{p}} >_{th} + \Delta_N(\mathbf{k},\mathbf{p}) \tag{63}$$

where

$$\Delta_N = < \delta j_{\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}+\mathbf{p}} >_{th} \tag{64}$$

with

$$\delta j_{\mathbf{p}} = \int d\mathbf{k}[(\chi_N^{-1}(\mathbf{k}+\mathbf{p}) - 1)(\not{\mathbf{k}} + \not{\mathbf{p}}) - (\chi_N^{-1}(\mathbf{k}) - 1)\not{\mathbf{k}}]\bar\psi_{\mathbf{k}}\psi_{\mathbf{k}+\mathbf{p}} \tag{65}$$

An analogous expression is obtained for the axial current $\bar\psi\gamma_\mu\gamma_5\psi$. The term Δ_N is due to the momentum regularization which is necessary to have a well defined Grassmann integral but which breaks the continuum phase symmetries. By a multiscale analysis it is found, in the limit of removed cut-off [32]

$$\lim_{N\to\infty} \Delta_N(\mathbf{k},\mathbf{p}) = -i\tau\hat v(\mathbf{p})\mathbf{p}_\mu < j_{\mu,\mathbf{p}}\psi_{\mathbf{k},\omega}\bar\psi_{\mathbf{k}+\mathbf{p},\omega} >_{th} \qquad \tau = \frac{\lambda_\infty}{4\pi c} \tag{66}$$

A similar expression holds for the axial Ward Identity, with j_μ replaced by j^5_μ and τ replaced by $-\tau$. The fact that $\Delta_N(\mathbf{k},\mathbf{p})$ is non vanishing removing the ultraviolet cut-off $N \to \infty$ is a manifestation of a *quantum anomaly.* The anomaly coefficients τ is *linear* in λ_∞: this is the non-perturbative analogue of the *anomaly non renormalization* in 4D Quantum Electrodynamics [1]. In the proof of the validity of such property a crucial role is played by the non locality of the interaction (53); highers orders in λ_∞ are present in τ with other choices of $V^{(N)}$, for instance replacing $v(\mathbf{x}-\mathbf{y})$ with a delta function, as shown in [6].

The fact that τ has no contributions from higher orders implies that also the exponents can be exactly computed in terms of λ_∞. By combining the Ward Identities (63), (66) with the Schwinger-Dyson equation one gets equations for the correlations from which the indices can be computed as functions of λ_∞: it is found

$$X_+ = 1 - \frac{1}{1+\tau}\frac{\lambda_\infty}{2\pi c} \quad X_- = 1 + \frac{1}{1-\tau}\frac{\lambda_\infty}{2\pi c} \tag{67}$$

from which the relation $X_+X_- = 1$ follows (the other relations between indices follow by similar arguments).

The indices have simple expressions in λ_∞, as consequence of the linearity of τ; all the model dependence is included in the function λ_∞, which is given by a convergent non trivial series with coefficients depending from all the details of the spin model (2). The simple and universal expressions of the exponents in terms of λ_∞ allow to check the validity of the universal relations.

7. Renormalization Group for Quantum spin chain and 1D Fermi systems

We now come to the quantum spin chain model or 1D interacting Fermi system (11). If O_x is a local monomial in the S^α_x or $a^\pm_x$ operators, we call $O_\mathbf{x} = e^{Hx_0}O_x e^{-Hx_0}$ where $\mathbf{x} = (x, x_0)$; moreover, if $A = O_{\mathbf{x}_1}\cdots O_{\mathbf{x}_n}$,

$$< A >_{L,\beta} = \frac{\mathrm{Tr}[e^{-\beta H}\mathbf{T}(A)]}{Tr[e^{-\beta H}]} \tag{68}$$

$\mathbf{T}$ being the time order product, denotes its expectation in the grand canonical ensemble, while $< A >_{T;L,\beta}$ denotes the corresponding truncated expectation. We will use also the notation $< A >_T = \lim_{L,\beta\to\infty} < A >_{T;L,\beta}$.

For λ small enough [9] if $\mathbf{x} = (x, x_0)$, x_0 imaginary time, when $J_1 = J_2$, one can prove that, for small λ, $J_1 = J_2 = 1$ and large $\mathbf{x}$,

$$\langle a^-_\mathbf{x} a^+_\mathbf{0}\rangle_T \sim g_0(\mathbf{x})\frac{1+\lambda f(\lambda)}{(x_0^2 + v_s^2 x^2)^{(\eta/2)}} , \tag{69}$$

where $f(\lambda)$ is a bounded function, $\eta = a_0\lambda^2 + O(\lambda^3)$, with $a_0 > 0$, and

$$g_0(\mathbf{x}) = \sum_{\omega=\pm} \frac{e^{i\omega p_F x}}{-ix_0 + \omega v_s x} , \tag{70}$$

$$v_s = v_F + O(\lambda) \qquad p_F = \cos^{-1}(h+\lambda) + O(\lambda) \qquad v_F = \sin p_F \ . \tag{71}$$

From (69) we see that the interaction has two main effects. The first one is to change the value of the Fermi momentum from $\cos^{-1}(h)$ to p_F and the Fermi velocity velocity from v_F in the non interacting case to v_s. The second effect is that the power law decay is changed; the 2-point function is asymptotically given by the product of the non-interacting one (with a different velocity) times an extra power law decay factor with non-universal index η.

It was also proved in [9] that the spin-spin correlation in the direction of the 3-axis (or, equivalently, the fermionic density-density correlation) is given, for large $\mathbf{x}$, by

$$\langle S^{(3)}_{\mathbf{x}} S^{(3)}_{\mathbf{0}} \rangle_T \sim \cos(2p_F x)\Omega^{3,a}(\mathbf{x}) + \Omega^{3,b}(\mathbf{x}) \ , \tag{72}$$

$$\Omega^{3,a}(\mathbf{x}) = \frac{1+A_1(\mathbf{x})}{2\pi^2[x^2+(v_s x_0)^2]^{X_+}} \ , \tag{73}$$

$$\Omega^{3,b}(\mathbf{x}) = \frac{1}{2\pi^2[x^2+(v_s x_0)^2]}\left\{\frac{x_0^2-(x/v_s)^2}{x^2+(v_s x_0)^2} + A_2(\mathbf{x})\right\} \ , \tag{74}$$

with $|A_1(\mathbf{x})|, |A_2(\mathbf{x})| \le C|\lambda|$ and $X_+ = 1 - a_1\lambda + O(\lambda^2)$ with

$$a_1 = [\hat v(0) - \hat v(2p_F)]/(\pi \sin p_F) \tag{75}$$

Finally the Cooper pair density correlation, that is the correlation of the operator $\rho^c_{\mathbf{x}} = a^+_{\mathbf{x}} a^+_{\mathbf{x}'} + a^-_{\mathbf{x}} a^-_{\mathbf{x}'}$, $\mathbf{x}' = (x+1, x_0)$, behaves as

$$\langle \rho^c_{\mathbf{x}} \rho^c_{\mathbf{0}} \rangle_T \sim \frac{1+A_3(\mathbf{x})}{2\pi^2(x^2+v_s^2 x_0^2)^{X_-}} \ , \tag{76}$$

with $X_- = 1 + a_1\lambda + O(\lambda^2)$, a_1 being the same constant appearing in the first order of X_+. In the case $J_1 \neq J_2$ the correlations decay faster than any power with rate @ such that $\xi \sim C|J_1 - J_2|^{\bar\nu}$ with $\bar\nu = 1 + a_1\lambda + O(\lambda^2)$, a_1 given by (75).

If $J_{\mathbf{x}} = v_F j_{\mathbf{x}}$ ($J_{\mathbf{x}}$ was defined in (12)), in the $\lambda = 0$ the commutation relations (13),(14) imply the following Ward Identities, for $\mathbf{k}, \mathbf{k}+\mathbf{p}$ close to $(0, \omega p_F)$, $\omega = \pm$

$$\begin{aligned} -ip_0\langle \hat\rho_{\mathbf{p}} \hat a^+_{\mathbf{k}} \hat a^-_{\mathbf{k}+\mathbf{p}}\rangle + \omega p v_F \langle \hat j_{\mathbf{p}} \hat a^+_{\mathbf{k}} \hat a^-_{\mathbf{k}+\mathbf{p}}\rangle &\sim [\langle \hat a^+_{\mathbf{k}} \hat a^-_{\mathbf{k}}\rangle - \langle \hat a^+_{\mathbf{k}+\mathbf{p}} \hat a^-_{\mathbf{k}+\mathbf{p}}\rangle] \\ -ip_0\langle \hat j_{\mathbf{p}} \hat a^+_{\mathbf{k}} \hat a^-_{\mathbf{k}+\mathbf{p}}\rangle + \omega p v_F \langle \hat \rho_{\mathbf{p}} \hat a^+_{\mathbf{k}} \hat a^-_{\mathbf{k}+\mathbf{p}}\rangle &\sim [\langle \hat a^+_{\mathbf{k}} \hat a^-_{\mathbf{k}}\rangle - \langle \hat a^+_{\mathbf{k}+\mathbf{p}} \hat a^-_{\mathbf{k}+\mathbf{p}}\rangle] \end{aligned} \tag{77}$$

In the presence of the interaction $\lambda \neq 0$, the Ward Identites have the form [11]

$$\begin{aligned} -ip_0\langle \hat\rho_{\mathbf{p}} \hat a^+_{\mathbf{k}} \hat a^-_{\mathbf{k}+\mathbf{p}}\rangle + \omega p \tilde v_J \langle \hat j_{\mathbf{p}} \hat a^+_{\mathbf{k}} \hat a^-_{\mathbf{k}+\mathbf{p}}\rangle &\sim B[\langle \hat a^+_{\mathbf{k}} \hat a^-_{\mathbf{k}}\rangle - \langle \hat a^+_{\mathbf{k}+\mathbf{p}} \hat a^-_{\mathbf{k}+\mathbf{p}}\rangle] \\ -ip_0\langle \hat j_{\mathbf{p}} \hat a^+_{\mathbf{k}} \hat a^-_{\mathbf{k}+\mathbf{p}}\rangle + \omega p \tilde v_N \langle \hat \rho_{\mathbf{p}} \hat a^+_{\mathbf{k}} \hat a^-_{\mathbf{k}+\mathbf{p}}\rangle &\sim \bar B[\langle \hat a^+_{\mathbf{k}} \hat a^-_{\mathbf{k}}\rangle - \langle \hat a^+_{\mathbf{k}+\mathbf{p}} \hat a^-_{\mathbf{k}+\mathbf{p}}\rangle] \end{aligned} \tag{78}$$

with $B = 1$, $\bar B = 1 + O(\lambda)$ and $\tilde v_J, \tilde v_N = v_s(1 + O(\lambda))$; in particular $\frac{\tilde v_N}{\tilde v_J} = 1 + 2a_1\lambda + O(\lambda^2)$ with a_1 the constant defined above in (75). The interaction has the effect that the normalization $\bar B$ is not 1 (the current does not commute with the quartic part of the Hamiltonian) and two different velocities , the charge $\tilde v_J$ and the current velocity $\tilde v_N$, appear. The presence of the lattice, breaking the Lorentz symmetry valid in the continuum limit, causes the presence of three distinct velocities, $\tilde v_N, \tilde v_J, v_s$.

8. Universality for quantum spin chains and 1D Fermi systems

The same strategy followed for proving the universal relations in the coupled Ising models (2) allows to derive the same relations between the indices appearing in the correlations of the spin chain; again all the indices can be expressed in terms of a single one. There is in this case also an extra relation connecting the indices with the Fermi velocity v_s and the susceptibility, defined as

$$\kappa = \lim_{p\to 0} \hat{\Omega}(0,p) \tag{79}$$

where $\hat{\Omega}(0,p)$ is the bidimensional Fourier transform of $\langle S^{(3)}_{\mathbf{x}} S^{(3)}_{\mathbf{0}}\rangle_T$. In the fermionic interpretation, $\kappa\rho^{-2}$ is the compressibility (ρ is the fermionic density).

Theorem 8.1. *(Benfatto,Mastropietro [11]) In the model (7) for λ small enough the exponents in (69),(72),(76) verify*

$$X_+ X_- = 1 \quad \bar{\nu} = \frac{1}{2 - X_+^{-1}} \qquad 2\eta = X_+ + X_+^{-1} - 2 \tag{80}$$

Moreover, the velocities in the Ward Identity (78) are such that

$$\tilde{v}_N \tilde{v}_J = v_s^2 \qquad \tilde{v}_J = v_F \tag{81}$$

and the susceptibility κ verifies

$$\kappa = \frac{1}{\pi}\frac{X_+}{v_s} \tag{82}$$

The relation (82) has been proposed in [21] as a part of the *Luttinger liquid conjecture* and checked previously only in the case of the XYZ chain using the explicit exact formulas, see (17),(18).

Note that, in the notation of [21], $v_N \equiv (\pi\kappa)^{-1}$ should not be confused with $\tilde{v}_N$ appearing in the WI (77); they are coinciding only in the special case of the Luttinger model. Therefore $\tilde{v}_N = v_N$ is an example of relation true in the (exactly solvable) Luttinger model but not in the generic 1D Fermi system model (7).

The Ward Identities of the relativistic model (53) with $c = v_s$ and a suitable choice of λ_∞ and the following relations, valid for $\mathbf{k}\,\mathbf{k}, \mathbf{k}+\mathbf{p}$ small, if $\mathbf{p}_F = (0, \omega p_F)$

$$\begin{aligned} &< \hat{\rho}_{\mathbf{p}} \hat{a}^+_{\mathbf{k}+\mathbf{p}_F^\omega} \hat{a}^-_{\mathbf{k}+\mathbf{p}+\mathbf{p}_F^\omega} > \sim Z^{(3)} < j^0_{\mathbf{p}} \hat{\psi}^+_{\mathbf{k},\omega} \hat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega} >_{th} \\ &< \hat{j}_{\mathbf{p}} \hat{a}^+_{\mathbf{k}+\mathbf{p}_F^\omega} \hat{a}^-_{\mathbf{k}+\mathbf{p}+\mathbf{p}_F^\omega} > \sim \tilde{Z}^{(3)} < j^1_{\mathbf{p}} \hat{\psi}^+_{\mathbf{k},\omega} \hat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega} >_{th} \end{aligned} \tag{83}$$

imply the Ward Identities (78) with, if τ is given by (66)

$$B = \frac{Z^{(3)}}{Z}(1-\tau)^{-1}, \quad \bar{B} = \frac{\tilde{Z}^{(3)}}{Z}(1+\tau)^{-1} \tag{84}$$

and

$$\tilde{v}_N = v_s \frac{Z^{(3)}}{\tilde{Z}^{(3)}} \quad \tilde{v}_J = v_s \frac{\tilde{Z}^{(3)}}{Z^{(3)}} \tag{85}$$

On the other hand, by comparing the first of (78) with the Ward Identity obtained from (13) we get

$$\frac{Z^{(3)}}{(1-\tau)Z} = 1 \qquad \tilde{v}_J = v_F \tag{86}$$

In the same way we can derive from the Ward Identity for the densities the following expressions

$$< \rho_{\mathbf{p}} \rho_{\mathbf{p}} >= \frac{1}{4\pi v_s Z^2} \frac{(Z^{(3)})^2}{1-(\lambda_\infty/4\pi v_s)^2} \left[2 - \frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} - \frac{D_+(\mathbf{p})}{D_-(\mathbf{p})}\right] + O(\mathbf{p}) \tag{87}$$

where $D_\omega(\mathbf{p}) = -ip_0 + \omega v_s p$, so that from (86) and (79)

$$\kappa = \frac{1}{\pi v_s} \frac{1}{Z^2} \frac{(Z^{(3)})^2}{1-(\tilde{\lambda}_\infty/4\pi v_s)^2} = \frac{1}{\pi v_s} \frac{1-(\lambda_\infty/4\pi v_s)}{1+(\lambda_\infty/4\pi v_s)} = \frac{X_+}{\pi v_s} \tag{88}$$

so that (82) is proved.

9. Conclusions

We have established for the first time the validity of a number of universal relations between exponents and other quantities in a wide class models, including solvable and *not solvable* models. Several of such relations were conjectured by Kadanoff [17–19], Luther and Peschel [29] and Haldane [21]. These universal relations are used for the analysis of experiments in carbon nanotubes or spin chains, but their interest goes much beyond this, as they provide one of the very cases in which the *universality* hypothesis can be *rigorously* verified. Extensions of our methods will allow hopefully to prove universal relations in an even wider class of models and to prove other relations between exponents; for instance the ones between the exponents appearing in the spin-spin correlations in coupled Ising models (2) or the dynamic exponents in 1D Fermi systems (7).

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