

# Resonances within Chaos

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**Abstract:** *A chaotic system under periodic forcing can develop a periodically visited strange attractor. We discuss simple models in which the phenomenon, quite easy to see in numerical simulations, can be completely studied analytically.*

In chaotic systems synchronization phenomena, i.e. resonances, can occur: the simplest being motions asymptotically developing over a low dimension attractor which is visited in synchrony with a weak time-periodic perturbing force. This is analogous to the synchronization (“phase locking”) occurring when an integrable system, perturbed by a periodic forcing, develops attracting periodic orbits with period being a simple fraction (1, 3/2, 2, . . .) of the forcing period. Formation of an attractor and existence of dissipation, i.e. of phase volume contraction, are concomitant. A chaotic system under a weak volume preserving periodic perturbation can have, instead, a very singular chaotic evolution, [1, 2], with dense orbits. A simple illustration of the resonance phenomenon in a chaotic system is exhibited here and exactly solved by deriving the shape of the attractor: which turns out to be a continuous surface with weak smoothness properties (being a strange attractor with an exponent of Hölder-continuity with a very small lower bound). The peculiarity of the class of systems treated is the possibility of determining analytically an attractor whose existence can be rather easily seen in simulations.

## 1 Resonances and chaos

A chaotic system weakly interacting with a very regular system may generate a fully chaotic dynamics: as a basic example, consider the suspension flow of Arnold’s cat map  $A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$  with a flat floor function (i.e. the flow on  $\mathbb{T}^3 = [0, 2\pi]^3$  where every point  $x \in \mathbb{T}^2$  of the base moves vertically with constant velocity until it reaches the flat floor, at which point it reappears on the base, at the location  $A_2x$ ) coupled with a clock, which moves at the same speed as the suspension flow. If observed at the times when the suspension flow hits the floor, the unperturbed

dynamics takes the form of a linear discrete dynamical system  $A_3 : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ , with  $A_3 \stackrel{def}{=} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ . Consider now a small perturbation of  $A_3$ : it is remarkable that in the case of volume preserving perturbations, some pathological phenomena can take place [1, 2]. Namely, there is an open set  $U$  of  $C^1$  conservative perturbations of  $A_3$  such that the following happens: the perturbed motion  $g \in U$  is ergodic with respect to the volume measure in  $\mathbb{T}^3$  and there exists a foliation of  $\mathbb{T}^3$  into  $C^2$  curves  $\ell_x$  that are invariant under  $g$  (i.e.  $g\ell_x = \ell_{gx}$ ); moreover, the central Lyapunov exponent along the leaves is positive on the set  $\Omega$  of full three-dimensional measure where it is defined; the set  $\Omega$  intersects every leaf at exactly  $k$  points, for some  $k > 0$  [1, 2]. Such results provide an interesting example of an absolutely singular foliation, in the sense of Katok [3].

The situation is destined to change substantially in the presence of friction: if we couple via a dissipative perturbation the suspension flow of  $A_2$  with a clock, the phase between the two (that is arbitrary in the unperturbed case) might lock, making the motion fall onto a lower dimensional attractor, which would naturally be close to the union of  $k$  elements of a “horizontal” foliation of  $\mathbb{T}^3$ . We can say, when this happens, that there is a resonance between the periodic forcing and the chaotic evolution on  $\mathbb{T}^3$ : and unlike the cases quite typical in generic perturbations of periodically forced integrable systems where the resonance may lead to the appearance of stable periodic orbits, in the presence of hyperbolicity and of small dissipation the forcing gives rise to a “periodic strange attractor”, i.e. to an attractor for the time evolution observed at times integer multiples of the forcing period, on which the dynamics is conjugate to a uniformly hyperbolic system.

In this paper we provide examples of the above phenomenon: remark that in the non-dissipative case the perturbations discussed in [1, 2] can be in a set arbitrarily close to the unperturbed system and open in the  $C^1$  topology (not containing the unperturbed system itself). Here we consider a class of analytic perturbations depending on a parameter  $\varepsilon$  and show that for any small  $\varepsilon \neq 0$  the resonance occurs. The phenomenon is reminiscent of certain synchronization phenomena in chaotic systems [4] and the simple case considered below is, as far as we know, one of the first explicit examples of such phenomena that can be fully worked out analytically.

## 2 A simple example

Denoting by  $x = (x_1, x_2)$  a point of  $\mathbb{T}^2$ , consider the evolution equation:

$$\begin{aligned}\dot{x} &= \delta(z) (Sx - x) + \varepsilon f(x, z, w), \\ \dot{w} &= 1 + \varepsilon g(x, z, w), \\ \dot{z} &= 1,\end{aligned}\tag{2.1}$$

with  $(x, w, z) \in \mathbb{T}^2 \times \mathbb{T} \times \mathbb{T}$  and  $\varepsilon \geq 0$ . Here  $\delta$  is the  $2\pi$ -periodic delta function, with the convention that  $\int_0^t dt' \delta(t') = 1$  for all  $t > 0$ . Set  $f = 0$  to simplify as much as possible the problem. The map  $S$  could be any linear hyperbolic automorphism of  $\mathbb{T}^2$ , in particular  $S = A_2$ .

Then we can prove the following result.

**Theorem:** *Let  $S_g^t$  be the flow on  $\mathbb{T}^4$  associated with Eq.(2.1) with  $f = 0$  and  $g$  analytic. Let also*

$$\bar{\gamma}_0(x, w) \stackrel{\text{def}}{=} \int_0^{2\pi} dt g(Sx, t, w+t), \quad \bar{\gamma}_1(x, w) \stackrel{\text{def}}{=} \int_0^{2\pi} dt \frac{\partial}{\partial w} g(Sx, t, w+t),\tag{2.2}$$

and suppose existence of  $w_0$  such that  $\bar{\gamma}_0(x, w_0) = 0$  and  $\bar{\gamma}_1(x, w_0) = \Gamma < 0$ , independent of  $x, w$ . Then there are constants  $\varepsilon_0, C > 0$  such that for  $0 < \varepsilon < \varepsilon_0$  there exists a Hölder-continuous function  $W : \mathbb{T}^2 \rightarrow \mathbb{T}$  of exponent  $\beta \geq C\varepsilon$  and

- the surface  $\mathcal{A} \stackrel{\text{def}}{=} \{(x, W(x), 0) : x \in \mathbb{T}^2\}$  is invariant under the Poincaré map  $S_g^{2\pi}$ ;
- the dynamics of  $S_g^{2\pi}$  on  $\mathcal{A}$  is conjugated to that of  $S$  on  $\mathbb{T}^2$ , i.e.

$$S_g^{2\pi}(x, W(x), 0) = (Sx, W(Sx), 0).\tag{2.3}$$

### Remarks.

(1) It should be clear from the proof below that, for  $\varepsilon > 0$ ,  $\mathcal{A}$  is an attractor for  $S_g^t$  with negative central Lyapunov exponent equal to  $\frac{\varepsilon\Gamma}{2\pi}(1 + O(\varepsilon))$ , see concluding remarks in Section 7. Furthermore the function  $W$  is holomorphic in  $\varepsilon = \rho e^{i\alpha}$  inside the domain which in polar coordinates is  $\rho(\alpha) = \varepsilon_0 \cos \alpha$ .

(2) A simple explicit example of a function  $g$  satisfying the assumptions of the theorem with  $w_0 = \pi$  and  $\Gamma = -1$  is:

$$g(x, z, w) = \sin(w - z) + \sin(x_1 + w + z).\tag{2.4}$$

(3) It would be interesting to estimate the positions of the poles of the power spectrum of the correlations of pairs of smooth observables in the sense of the chaotic resonances theory in [5]. The spectrum is meromorphic in a strip with poles off the real axis for a dense set of observables: the unperturbed system has poles at  $-i \log \lambda_+ + 2\pi k$ ,  $k \in \mathbb{Z}$ , if  $\lambda_+$  is the largest eigenvalue of the matrix  $S$ ; we expect the position of the poles closest to the origin to be modified by order  $\varepsilon$ . Notice that in the unperturbed system the spectrum is an entire function for all analytic observables and the poles really appear with non-zero residue for Hölder continuous functions (see Sec. 4.3 and Fig.4.3.5 in [6]).

Let us now turn to the proof of the theorem. The main ideas and the strategy of the proof are the following: (i) after having written the equation for the invariant manifold, we solve it recursively; (ii) the result of the recursion is conveniently expressed in terms of tree diagrams; (iii) these can be easily shown to give rise to a convergent expansion, provided that the large factors  $\varepsilon^{-1}$  produced under iterations by the small rate of contraction of phase space (of order  $O(\varepsilon)$ ) is compensated by suitable cancellations, stemming from the condition that  $\bar{\gamma}_0(x, w_0) = 0$ . All these steps are discussed in detail in the following Sections 3–6. Finally, in Section 7, we add some comments on the assumptions made and on possible generalizations of our result.

### 3 The equation for the invariant manifold

Let  $W(x) = w_0 + U(x)$ ,  $x(0) = x$  and for  $0 < t \leq 2\pi$  write  $w(t) = w_0 + t + u(x, t)$ , with  $u(x, 0^+) = U(x)$ . Substituting into Eq.(2.1), for all  $0 < t \leq 2\pi$  we find:

$$\dot{u}(x, t) = \varepsilon g(Sx, w_0 + t + u(x, t), t). \quad (3.1)$$

Defining

$$\begin{aligned} \gamma_0(x, t) &= g(Sx, w_0 + t, t), & \gamma_1(x, t) &= \frac{\partial}{\partial w} g(Sx, w_0 + t, t), \\ G(x, t, u) &= g(Sx, w_0 + t + u, t) - \gamma_0(x, t) - u \gamma_1(x, t), \end{aligned} \quad (3.2)$$

Eq.(3.1) can be rewritten for all  $0 < t \leq 2\pi$  as

$$\dot{u}(x, t) = \varepsilon u(x, t) \gamma_1(x, t) + \varepsilon \left( \gamma_0(x, t) + G(x, t, u(x, t)) \right), \quad (3.3)$$

with

$$G(x, t, u) = \sum_{p \geq 2} G_p(x, t) u^p. \quad (3.4)$$

Integrating from 0 to  $t \in (0, 2\pi]$ , the equation for  $u(x, t)$  becomes

$$u(x, t) = e^{\varepsilon\Gamma(x, t, 0)}U(x) + \varepsilon \int_0^t d\tau e^{\varepsilon\Gamma(x, t, \tau)} \left( \gamma_0(x, \tau) + G(x, \tau, u(x, \tau)) \right), \quad (3.5)$$

where

$$\Gamma(x, t, \tau) = \int_\tau^t d\tau' \gamma_1(x, \tau'). \quad (3.6)$$

By construction  $G$  vanishes to second order in  $u$ , hence, if  $\delta$  is fixed once and for all, one has  $u(x, 2\pi) = e^{\Gamma\varepsilon}u(x, 0) + O(\varepsilon^2)$ , so that

$$|u(x, 2\pi)| \leq e^{\Gamma\varepsilon/2}|u(x, 0)|, \quad \text{if } \frac{\delta}{2} \leq |u(x, 0)| \leq \delta, \quad (3.7)$$

provided  $\varepsilon$  is small enough. Therefore the region  $|u| \leq \delta$  is an attracting set for  $\varepsilon$  small (compared to  $\delta$ ).

The condition of invariance for  $W(x)$  reads  $u(x, 2\pi) = U(Sx)$ , which gives

$$U(Sx) - e^{\varepsilon\Gamma}U(x) = \varepsilon \int_0^{2\pi} d\tau e^{\varepsilon\Gamma(x, 2\pi, \tau)} \left( \gamma_0(x, \tau) + G(x, \tau, u(x, \tau)) \right), \quad (3.8)$$

and hence

$$U(x) = \sum_{k=1}^{\infty} e^{\varepsilon\Gamma(k-1)} F(S^{-k}x), \quad (3.9)$$

$$F(x) = \varepsilon \int_0^{2\pi} d\tau e^{\varepsilon\Gamma(x, 2\pi, \tau)} \left( \gamma_0(x, \tau) + G(x, \tau, u(x, \tau)) \right).$$

In order to solve the two coupled equations Eqs.(3.5)-(3.8), we find convenient to consider the two-parameters  $(\mu, \varepsilon)$  equations

$$u(x, t) = e^{\mu\Gamma(x, t, 0)}U(x) + \varepsilon \int_0^t d\tau e^{\mu\Gamma(x, t, \tau)} \left( \gamma_0(x, \tau) + G(x, \tau, u(x, \tau)) \right), \quad (3.10)$$

$$U(Sx) - e^{\mu\Gamma}U(x) = \varepsilon \int_0^{2\pi} d\tau e^{\mu\Gamma(x, 2\pi, \tau)} \left( \gamma_0(x, \tau) + G(x, \tau, u(x, \tau)) \right),$$

first showing existence of a solution analytic in  $\varepsilon$  and Hölder-continuous in  $x$ , at fixed  $\mu$ , and subsequently proving that  $\mu$  can be taken to have values  $\mu \geq \varepsilon$  a little beyond  $\varepsilon$ .

## 4 Recursion equations

Eq.(3.10) can be solved recursively as follows. Let

$$u(x, t) = e^{\mu\Gamma(x, t, 0)} U(x) + \xi(x, t) \quad (4.1)$$

and write  $\xi(x, t) = \varepsilon\xi_1(x, t) + \varepsilon^2\xi_2(x, t) + \dots$  and  $U(x) = \varepsilon U_1(x) + \varepsilon^2 U_2(x) + \dots$ . Plugging the expansions into Eq.(3.10) and setting  $\lambda = e^{\mu\Gamma}$ , we find

$$\begin{aligned} \xi_1(x, t) &= \int_0^t d\tau e^{\mu\Gamma(x, t, \tau)} \gamma_0(x, \tau), & \xi_2(x, t) &= 0, \\ U_1(x) &= \sum_{k=1}^{\infty} \lambda^{k-1} \int_0^{2\pi} d\tau e^{\mu\Gamma(S^{-k}x, 2\pi, \tau)} \gamma_0(S^{-k}x, \tau), & U_2(x) &= 0, \end{aligned} \quad (4.2)$$

and, for  $n \geq 3$ ,

$$\begin{aligned} \xi_n(x, t) &= \int_0^t d\tau \sum_{p=2}^{\infty} \sum_{\substack{n_1, \dots, n_p \geq 1 \\ n_1 + \dots + n_p = n-1}} e^{\mu\Gamma(x, t, \tau)} G_p(x, \tau) \prod_{i=1}^p \left( e^{\mu\Gamma(x, \tau, 0)} U_{n_i}(x) + \xi_{n_i}(x, t) \right), \\ U_n(Sx) - e^{\mu\Gamma} U_n(x) &= \int_0^{2\pi} d\tau \sum_{p=2}^{\infty} \sum_{\substack{n_1, \dots, n_p \geq 1 \\ n_1 + \dots + n_p = n-1}} e^{\mu\Gamma(x, 2\pi, \tau)} G_p(x, \tau) \cdot \\ &\quad \cdot \prod_{i=1}^p \left( e^{\mu\Gamma(x, \tau, 0)} U_{n_i}(x) + \xi_{n_i}(x, t) \right), \end{aligned} \quad (4.3)$$

which give a recursive definition of  $\xi_n, U_n$ . Note that the second of Eq.(4.3) can be solved in a way analogous to Eq.(3.9), so that

$$\begin{aligned} U_n(x) &= \sum_{k=1}^{\infty} \lambda^{k-1} \int_0^{2\pi} d\tau \sum_{p=2}^{\infty} \sum_{\substack{n_1, \dots, n_p \geq 1 \\ n_1 + \dots + n_p = n-1}} e^{\mu\Gamma(S^{-k}x, 2\pi, \tau)} G_p(S^{-k}x, \tau) \cdot \\ &\quad \cdot \prod_{i=1}^p \left( e^{\mu\Gamma(S^{-k}x, \tau, 0)} U_{n_i}(S^{-k}x, \tau) + \xi_{n_i}(S^{-k}x, \tau) \right). \end{aligned} \quad (4.4)$$

For the incoming discussion, it is useful to note that by the analyticity of  $g$ , there exists a constant  $C_0$  such that

$$|G_p(x, t)| \leq C_0^p, \quad e^{\mu\Gamma(x, t, \tau)} \leq C_0, \quad (4.5)$$

uniformly in  $x, t, \tau$ . Moreover, if we consider the Fourier expansion

$$G_p(x, \tau) = \sum_{\nu \in \mathbb{Z}^2} e^{i\nu \cdot x} \hat{G}_p(\nu, \tau), \quad e^{\mu\Gamma(x, t, \tau)} = \sum_{\nu \in \mathbb{Z}^2} e^{i\nu \cdot x} \tilde{\Gamma}(\nu, t, \tau), \quad (4.6)$$

there exists  $\kappa > 0$ , such that

$$|\hat{G}_p(\nu, \tau)| \leq C_0 e^{-\kappa|\nu|}, \quad |\tilde{\Gamma}(\nu, t, \tau)| \leq C_0 e^{-\kappa|\nu|}, \quad (4.7)$$

uniformly in  $t, \tau$ .

## 5 Tree expansion

The recursion for  $\xi_n, U_n$  can be conveniently represented graphically in terms of tree graphs. We start with some basic definitions. A connected graph  $\mathcal{G}$  is a collection of points (nodes) and lines connecting all of them. We denote with  $V(\mathcal{G})$  and  $L(\mathcal{G})$  the set of nodes and the set of lines, respectively. A path between two nodes is the minimal subset of  $L(\mathcal{G})$  connecting the two nodes. A graph is planar if it can be drawn in a plane without graph lines crossing.

A tree is a planar graph  $\mathcal{G}$  containing no closed loops. One can consider a tree  $\mathcal{G}$  with a single special node  $v_0$ : this introduces a natural partial ordering on the set of lines and nodes, and one can imagine that each line carries an arrow pointing toward the node  $v_0$  thus establishing a partial order  $\succ$  on the tree. We can add an extra (oriented) line  $\ell_0$  exiting the special node  $v_0$ ; the added line  $\ell_0$  will be called the root line and the point  $r$  it enters (*which is not a node*) will be called the root of the tree. In this way we obtain a rooted tree  $\theta$  defined by  $V(\theta) = V(\mathcal{G})$  and  $L(\theta) = L(\mathcal{G}) \cup \ell_0$ .

Hence given two nodes  $v, w \in V(\theta)$ , it is  $v \prec w$  (or  $w \succ v$ ) if  $w$  is on the path connecting  $v$  to the root line.

A labeled tree is a rooted tree  $\theta$  together with a label function defined on the sets  $L(\theta)$  and  $V(\theta)$ .

We shall call equivalent two rooted trees which can be transformed into each other by continuously deforming the lines in the plane in such a way that the latter do not cross each other (i.e. without destroying the graph structure). We can extend the notion of equivalence also to labeled trees, simply by considering equivalent two labeled trees if they can be transformed into each other in such a way that also the labels match.

We can identify a line with the nodes it connects; given a line  $\ell = vw$  we say that  $\ell$  enters  $w$  and exits  $v$ , and we shall write also  $\ell = \ell_v$ . Given two comparable nodes  $v$  and  $w$ , with  $v \prec w$ , we denote with  $\mathcal{P}(v, w)$  the path of lines connecting  $w$  to  $v$ , with  $v$  and  $w$  being included. For any node  $v \prec v_0$  we denote by  $v' \succ v$  the node immediately following  $v$ , hence  $\ell_v = v'v$ , and we set  $v'_0 = r$ .

Associate with the nodes and lines of any tree  $\theta$  some labels, according to the following rules.

With each node  $v$  we associate the labels  $(\eta_v, k_v)$ , where  $\eta_v = 0, 1$ , while  $k_v = 0$  if  $\eta_v = 0$  and  $k_v \in \mathbb{N}$  (i.e.  $k_v \geq 1$ ) if  $\eta_v = 1$ ; we call  $\eta_v$  the type label of the node  $v$ . Furthermore we define

$$k(v) = \sum_{v \in \mathcal{P}(v, v_0)} k_v. \quad (5.1)$$

For each node there are  $p_v \geq 0$  entering lines. If  $p_v = 0$  we say that  $v$  is an end-node. If  $p_v > 0$  then  $p_v \geq 2$ . For  $p_v \geq 2$  let  $s_v$  be the number of nodes  $w$  with

$w' = v$  such that  $\eta_w = 1$ .

Represent graphically the node  $v$  as a big white ball (white node) if  $\eta_v = 1$  and as a small black ball (black node) if  $\eta_v = 0$ . Thus  $s_v$  is the number of white nodes immediately preceding  $v$ .

With each node  $v$  we associate a time variable  $\tau_v \in (0, 2\pi]$ , with  $\tau_{v'_0} = t$ , a node factor  $F_v = F_v(x, \tau_{v'}, \tau_v)$  defined as

$$F_v = \begin{cases} e^{\mu\Gamma(S^{-k(v)}x, \tau_{v'}, \tau_v)} G_{p_v}(S^{-k(v)}x, \tau_v) e^{\mu s_v \Gamma(S^{-k(v)}x, \tau_v, 0)}, & \eta_v = 0, \\ \lambda^{k_v-1} e^{\mu\Gamma(S^{-k(v)}x, 2\pi, \tau_v)} G_{p_v}(S^{-k(v)}x, \tau_v) e^{\mu s_v \Gamma(S^{-k(v)}x, \tau_v, 0)}, & \eta_v = 1, \end{cases} \quad (5.2)$$

and an integral  $I_v(\eta_v)$ , with

$$I_v(0) = \int_0^{\tau_{v'}} d\tau_v, \quad I_v(1) = \int_0^{2\pi} d\tau_v. \quad (5.3)$$

With the definitions above, we denote by  $\mathcal{T}_{n,\nu,\eta}$  the set of trees with  $n$  nodes and  $\eta_{v_0} = \eta$ . Then one can check (for instance by induction) that

$$\begin{aligned} U_n(x) &= \sum_{\theta \in \mathcal{T}_{n,1}} \text{Val}(x; \theta), & \xi_n(x, t) &= \sum_{\theta \in \mathcal{T}_{n,0}} \text{Val}(x; \theta), \\ \text{Val}(x; \theta) &= \prod_{v \in V(x; \theta)} I_v(\eta_v) F_v(x, \tau_{v'}, \tau_v), \end{aligned} \quad (5.4)$$

where the integrals are understood to be performed by following the tree ordering (i.e. by starting from the end-nodes and moving toward the root). Note that in the formula for  $\xi_n(x, t)$  in Eq.(5.4), the variable  $\tau_{v'_0} = t$  plays no role, as  $I_{v_0}(1)$  and  $F_{v_0}(t, \tau_{v_0})$  in fact do not depend on  $t$ .

For instance  $\hat{\xi}_1(\nu, t)$  and  $\hat{U}_1(\nu)$  are represented graphically as in Figure 1.

$$\xi_1(x, t) = \text{---} \bullet \quad U_1(x) = \text{---} \bigcirc$$

**Figure 1.** Graphical representation of  $\xi_1(x, t)$  and  $U_1(x)$ .

Since  $\xi_2, U_2 = 0$  the next simplest example is the representation of  $\xi_3, U_3$ . For instance, say,  $U_3(x)$  is described in Figure 2.

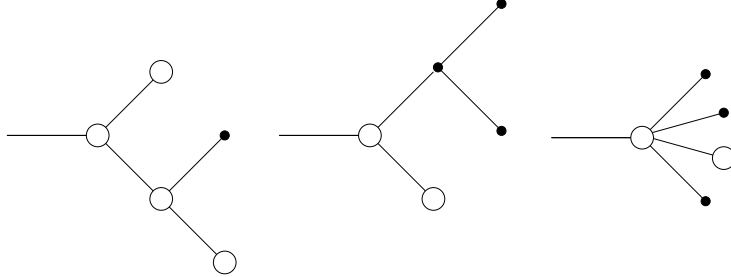
$$U_3(x) = \text{---} \bigcirc \begin{matrix} / \bigcirc \\ \backslash \bigcirc \end{matrix} + \text{---} \bigcirc \begin{matrix} / \bullet \\ \backslash \bigcirc \end{matrix} + \text{---} \bigcirc \begin{matrix} / \bigcirc \\ \backslash \bullet \end{matrix} + \text{---} \bigcirc \begin{matrix} / \bullet \\ \backslash \bullet \end{matrix}$$

**Figure 2.** Graphical representation of  $U_3(x)$ .

In a similar way we can represent graphically the other  $U_n$ 's or  $\xi_n$ 's. For example in Figure 3 we represent some contributions to  $U_5(x)$ : all the other contributions are obtained by replacing some white nodes with black nodes and vice



versa, and possibly permuting the lines entering the nodes (the only node which must be necessarily white is the node  $v_0$ ).



**Figure 3.** Graphical representation of some contributions to  $U_5(x)$ .

In drawing the graphs in Figures 1–3 we have used that  $p_v \geq 2$  whenever  $p_v \neq 0$  and that  $\xi_2(x, t)$  and  $U_2(x)$  vanish identically, as it follows from (4.2).

## 6 Convergence

Given a tree  $\vartheta \in \mathcal{T}_{n,x,\eta}$ ,  $\eta \in \{0, 1\}$ , let  $E(\vartheta)$  be the number of its end-nodes, and let  $E_0(\vartheta)$  and  $E_1(\vartheta)$  be the number of end-nodes of type 0 and 1, respectively. Similarly, let  $N^i(\vartheta)$  be the number of “internal” vertices of  $\vartheta$  (i.e. the number of vertices of  $\vartheta$  with  $p_v > 0$ ) and  $N_\eta^i(\vartheta)$  be the number of internal vertices of type  $\eta$ , with  $\eta = 0, 1$ . Note that, since  $p_v \geq 2$  whenever  $p_v \neq 0$ , we have  $E(\vartheta) \geq (n+1)/2$ . A key remark is that, thanks to the condition that  $\bar{\gamma}_0(x, w_0) = 0$ , the contribution associated to every end-node  $v$  of type 1 is of order  $\mu$ ,

$$\left| \int_0^{2\pi} d\tau_v \lambda^{k_v-1} e^{\mu\Gamma(S^{-k(v)}x, 2\pi, \tau_v)} \gamma_0(S^{-k(v)}x, \tau_v) \right| \leq (\text{const.})\mu. \quad (6.1)$$

Using this cancellation and the bounds Eq.(4.5), we find that, for all  $n \in \mathbb{N}$  and all  $\nu \in \mathbb{Z}^2$ ,

$$|\text{Val}(x; \theta)| \leq C_1^n \mu^{E_1(\vartheta)} \prod_{\substack{v \in V(\theta) \\ \eta_v=1}} \lambda^{k_v-1}, \quad (6.2)$$

for a suitable constant  $C_1$ , uniformly in  $x$ .

Therefore, in order to obtain a bound on  $U_n(x)$  we have to estimate the number of labeled trees in  $\mathcal{T}_{n,1}$ . For each node  $v$  with  $\eta_v = 1$  the corresponding sum over  $k_v$  produces a factor  $C_2\mu^{-1}$ . Finally the number of unlabelled trees with  $n$  nodes is bounded by  $C_3^n$ , with  $C_3 = 4$ .

By collecting together the bounds above we find

$$|U_n(x)| \leq C^n \mu^{-N_1^i(\vartheta)}, \quad C = C_1 C_2 C_3. \quad (6.3)$$

Using  $N_1^i(\vartheta) \leq N^i(\vartheta) = n - E(\vartheta) \leq n - (n+1)/2$ , we find

$$|U_n(x)| \leq C^n \mu^{-(n-1)/2}, \quad (6.4)$$

which yields a radius of convergence in  $\varepsilon$  given by  $\varepsilon_0 = C^{-1} \mu^{1/2}$ , hence we can take  $\mu = \varepsilon$ .

To study the regularity in  $x$  we need to get a similar upper bound for

$$\frac{|U_n(x) - U_n(x')|}{|x - x'|^\beta}, \quad \beta > 0. \quad (6.5)$$

We proceed as follows. Given a tree  $\vartheta$  we introduce an arbitrary ordering of the nodes by setting  $V(\vartheta) = \{v_1, \dots, v_{|V(\vartheta)|}\}$  and define  $V_p^-(\vartheta) = \{v_1, \dots, v_p\}$  and  $V_p^+ = \{v_{p+1}, \dots, v_{|V(\vartheta)|}\}$ , with the convention that  $V_0^-(\vartheta) = V_{|V(\vartheta)|}^+(\vartheta) = \emptyset$  and  $V_{|V(\vartheta)|}^-(\vartheta) = V_0^+(\vartheta) = V(\vartheta)$ . Then we define

$$\text{Val}_p^\pm(x; \vartheta) = \prod_{v \in V_p^\pm(\vartheta)} I_v(\eta_v) F_v(x, \tau_{v'}, \tau_v), \quad (6.6)$$

with  $\text{Val}_0^-(x; \vartheta) = \text{Val}_{|V(\vartheta)|}^+(x; \vartheta) = 1$ , and write

$$\begin{aligned} U_n(x) - U_n(x') &= \sum_{\vartheta \in \mathcal{T}_{n,1}} \left( \text{Val}(x; \vartheta) - \text{Val}(x'; \vartheta) \right) \\ &= \sum_{\vartheta \in \mathcal{T}_{n,1}} \sum_{p=1}^{|V(\vartheta)|} \text{Val}_{p-1}^-(x; \vartheta) \text{Val}_{p+1}^+(x'; \vartheta) \cdot \\ &\quad \cdot \left( I_{v_p}(\eta_{v_p}) \left( F_{v_p}(x, \tau_{v_p'}, \tau_{v_p}) - F_{v_p}(x', \tau_{v_p'}, \tau_{v_p}) \right) \right). \end{aligned} \quad (6.7)$$

Each summand in (6.7) can be studied as in the case  $\beta = 0$ , with the difference that the node factor is  $F_v(x, \tau_{v'}, \tau_v)$  for  $v \in V_{p-1}^-(\vartheta)$ ,  $F_v(x', \tau_{v'}, \tau_v)$  for  $v \in V_{p+1}^+(\vartheta)$  and  $F_v(x, \tau_{v'}, \tau_v) - F_v(x', \tau_{v'}, \tau_v)$  for  $v = v_p$ . Using Eqs.(5.2) and (4.6), we find that

$$\begin{aligned} F_v(x, \tau_{v'}, \tau_v) &= \sum_{\nu \in \mathbb{Z}^2} e^{i\nu \cdot x} \hat{F}_v(\nu, \tau_v, \tau_{v'}), \\ |\hat{F}_v(\nu, \tau_v, \tau_{v'})| &\leq \lambda^{(k_v-1)\delta_{\eta_v,1}} \bar{C}_1 \bar{C}_0^{s_v+2} e^{-\frac{\kappa}{2} |S^{k(v)} \nu|}, \end{aligned} \quad (6.8)$$

for suitable constants  $\bar{C}_0, \bar{C}_1$ , independent of  $\tau_v, \tau_{v'}$  (in deriving the bound on the Fourier coefficients we used that  $F_v(x, \tau_{v'}, \tau_v)$  depends on  $x$  through  $S^{-k(v)}x$ , see Eq.(5.2)). Therefore,

$$\begin{aligned} \frac{|F_v(x, \tau_{v'}, \tau_v) - F_v(x', \tau_{v'}, \tau_v)|}{|x - x'|^\beta} &\leq \lambda^{(k_v-1)\delta_{\eta_v,1}} \bar{C}_1 \bar{C}_0^{s_v+2} \sum_{\nu \in \mathbb{Z}^2} |\nu|^\beta e^{-\frac{\kappa}{2} |\nu|} |\lambda_-|^{-\beta k(v)}, \end{aligned} \quad (6.9)$$

where  $\lambda_-$  is the eigenvalue of the hyperbolic automorphism  $S$  with smallest absolute value; note that, if  $\lambda_+$  is the other eigenvalue of  $S$ , we have  $|\lambda_-| = |\lambda_+|^{-1}$ , with  $|\lambda_+| > 1$ . The factor  $|\lambda_-|^{-\beta k(v)}$  in Eq.(6.9) is bounded by  $\prod_{v \in V(\vartheta): \eta_v=1} |\lambda_+|^{\beta k_v}$ , so that

$$\frac{|U_n(x) - U_n(x')|}{|x - x'|^\beta} \leq \sum_{\vartheta \in \mathcal{T}_{n,1}} n \bar{C}^n \mu^{E_1(\vartheta)} \prod_{\substack{v \in V(\vartheta) \\ \eta_v=1}} \lambda^{k_v-1} |\lambda_+|^{\beta k_v}, \quad (6.10)$$

for a suitable constant  $\bar{C}$ . We must require that  $|\lambda_+|^\beta \lambda < 1$ , and in order for the sum  $\sum_{k_v \geq 1} (|\lambda_+|^\beta \lambda)^{k_v}$  to be still proportional to  $\mu^{-1}$ , we can take

$$\beta = c\mu, \quad 0 < c < |\Gamma| / \log |\lambda_+|, \quad (6.11)$$

which gives  $\beta = c\varepsilon$  for  $\mu = \varepsilon$ . This concludes the proof of the theorem.

## 7 Concluding remarks

Let us add a few comments on the role of the specific hypotheses in the theorem. The assumption that there is  $w_0$  such that  $\int_0^{2\pi} dt g(x, t, w_0 + t)$  is zero has been heavily used as well as the assumption that  $\int_0^{2\pi} dt \partial_w g(x, t, w_0 + t) = \Gamma$ , for  $\Gamma < 0$ . Both assumptions can be relaxed into

- (a)  $\int_0^{2\pi} dt g(x, w_0 + t, t) = \varepsilon \bar{g}(x)$ , for some  $\bar{g}(x)$ ,
- (b)  $\int_0^{2\pi} dt \partial_w g(x, w_0 + t, t) \leq \Gamma$ , for  $\Gamma < 0$ ,

and the statement of the theorem remains valid (and its proof is essentially unchanged).

Assumption (a) is strong and it does not allow for perturbations of the form  $\varepsilon g$  because the  $\varepsilon$ -dependence has to involve higher powers of  $\varepsilon$  to satisfy (a) unless  $\bar{g}(x) \equiv 0$  (as in the theorem proved here).

It is unclear whether the results discussed in this paper would hold under the weaker assumptions that

- (a')  $\int_0^{2\pi} dt g(x, w_0 + t, t) = \tilde{g}(x)$ , with  $\tilde{g}(x)$  with 0 average,
- (b')  $\int_0^{2\pi} dt \partial_w g(x, w_0 + t, t) = \tilde{g}_1(x)$ , with  $\tilde{g}_1(x)$  with negative average.

Preliminary computations and numerical evidence suggest that the existence of an attractor  $\mathcal{A} = \{(x, 0, W(x))\}$  for  $S_g^{2\pi}$  may still be true, possibly “just” for a.e.- $x$  and with a non-smooth function  $W(x)$ : in other words, a similar phase-locking phenomenon as the one proved in this paper may still take place, with  $\mathcal{A}$  a fractal rather than a smooth surface.

It would be interesting to check that the central Lyapunov exponent in the theorem is also expressed by an expansion in tree graphs: the procedure should be the same illustrated in the theory of the tangent map for perturbations of the simple Anosov map (e.g.,  $A_2$  in Section 1), see proposition 10.3.1 in [6].

Finally a natural conjecture is that the assumption  $f = 0$  in Eq.(2.1) can be eliminated as long as  $f$  is analytic. However, we do not have specific indications of what is the general structure of the attractor in this case.

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