

Kosterlitz-Thouless Transition Line for the Two Dimensional Coulomb Gas

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Abstract

With a renormalization group approach, we study the pressure of the two dimensional Coulomb Gas along a small piece of the Kosterlitz-Thouless transition line, i.e. the boundary of the dipole region in the activity-temperature phase-space.

1 Introduction

Occasionally in Statistical Mechanics the study of a special model inspired major advances of the general theory. This was the case, for example, of the exact solutions of the two dimensional Ising and Six-Vortex models in the ambit of the theory of universality; and was certainly the case of the Renormalization Group (RG) analysis of two dimensional Coulomb Gas as prototype of the *Kosterlitz-Thouless* (KT) transition for systems with long range interactions.

Two dimensional Coulomb Gas is the statistical system of point particles on a plane, carrying a charge ± 1 , and interacting through the two-dimensional electrostatic potential that, for large distances, is

$$V(x - y) \sim -\frac{1}{2\pi} \ln |x - y| .$$

An ideal realization of the model is a system of infinite, parallel, uniformly charged wires in thermal equilibrium; historically it was proposed as a model of strongly magnetized plasma (see introduction of [15] and references therein). Very soon, anyways, the Coulomb Gas acquired a great theoretical importance, for Berezinskii, [4], and Kosterlitz and Thouless, [26], found in it the solution of a puzzling dichotomy in the theory of the two dimensional *XY model*: the *spin-wave* approximation, quite reliable for low temperatures, predicted, in agreement with the general Mermin-Wagner argument, absence of order and power law fall-off of the spin correlation; on the other hand, high temperature expansion clearly demonstrated

exponential decay of the correlations. The two scenarios were merged together by the fundamental observation that the former picture did not take into account spin configurations with *vortex excitations*, which interacted through the logarithmic potential written above.

Using RG ideas, Kosterlitz and Thouless, [26], [25], were able to obtain the diagram of phases of the Coulomb Gas (and so of the XY model) depicted in Fig.1, where z is the activity and β the inverse temperature. At high temperatures, the gas

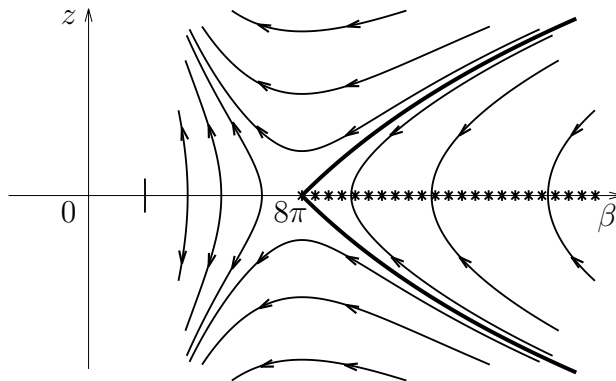


Figure 1: Diagram of phases: the separatrix is the KT line.

is in the *plasma phase* - or *Debye screening phase*: the configurations of significant probability are those in which the long-range electrostatic interactions generated by a uniform density of particles of opposite charges almost cancel each others, resulting in an effective, short range potential; as consequence, the correlation length is expected to be finite, and certain *screening sum rules* are conjectured. Low temperatures, on the contrary, favor the formation of pairs of opposite charges, the dipoles: the effective range of the interactions remains long, and the correlations length is infinite; this regime is the *dipole phase* - or *KT phase*. In between the two phases, there is (at least) one critical curve, that is called *KT transition line*, and was found to intersect the $z = 0$ axis at $\beta_c(0) = 8\pi$, [25]. This picture represented a *new kind* of phase transition, because all the temperatures below $\beta_c(z)$ are, in a sense, critical; in fact, approaching $\beta_c(z)$ from higher temperatures the correlations length is expected to diverge as $\xi \sim \exp[c(z)|\beta - \beta_c(z)|^{-\frac{1}{2}}]$, as opposed to the $\xi \sim |\beta - \beta_c|^{-\nu}$ (or $\xi \sim \ln|\beta - \beta_c|$) of the second-order phase transitions.

After the pioneering analysis of Kosterlitz and Thouless, efforts of many authors were addressed to the topic, in search of stronger evidence of the KT phase transition: the reader interested in theoretical physics works can find useful discussions and a good selection of references in [15], [1], [36], [27] and [23] (see also the conjecture in [20] of an infinite series of thresholds, intersecting the $z = 0$ axis at $\beta_{c,n}(0) = 8\pi(1 - 1/2n)$, $n = 1, 2, \dots \infty$, where more and more coefficients of the power series of the pressure in the activity are divergent; criticism is in [34]).

The first major rigorous result was the proof of Fröhlich and Park, [17], of the existence of the thermodynamic limit for pressure and correlations. Later on, Fröhlich and Spencer, [18], [19], proved, for β large enough, an upper and lower power-law bound for the correlations of *fractional* (i.e. non-integer) charges; then, refinements of the same technique allowed Marchetti, Klain and Peres, [29], [28], to cover increasing regions in the dipole phase that eventually included the point $(\beta, z) = (8\pi, 0)$, but not the rest of the KT transition line. Despite its fast improvement, Fröhlich-Spencer method seemed to have some unavoidable limitations: it could not provide the exact power of the correlations fall-off, nor could exclude logarithmic corrections to such decay (which actually were expected along the KT transition line); and it did not provide any useful bound for correlations of *integer* charges. For this reasons, different authors started developing an RG approach to the model - at the beginning in some approximate form: hierarchical metric, or order by order in perturbation theory; see [2], [30], [12], [33], [24], [3]. Later, Dimock and Hurd, [14], achieved a rigorous construction, under no approximation, of the pressure in a region of the dipole phase that included $(\beta, z) = (8\pi, 0)$ but not the rest of the KT transition line; they could not discuss charge correlations, though. Finally, the only rigorous result on the plasma phase is the work of Yang, [35], that extended to the two dimensional case the proof of dynamical mass generation for $\beta \ll \beta_{c,D}(z)$ obtained in higher dimension in [6], [8].

The objective of this paper is to study the Coulomb Gas *along* the KT transition line, for small activity. Using the general RG approach of [5] and some model-specific ideas in [14], we are able to give a constructive proof of the existence of the pressure. We shall not discuss here the critical exponents of the correlation functions; but in view of the bounds of this paper, the task should not be difficult. Besides, we shall not consider in these pages more complicated models studied by Fröhlich and Spencer, such as the XY, Villain, discrete Gaussian, Z_n -Clock and the solid-on-solid models, because they require (perhaps) a large-activity framework. Even more interesting - and more difficult to treat - is the surprising (formal) equivalence between the KT transition and the second-order phase transitions of certain two-dimensional, lattice models, such as Ashkin-Teller, Six-Vertex and Eight-Vertex, Q -states and anti-ferromagnetic Potts models, $O(n)$ models (including the $n = 0$ case, i.e. the Self Avoiding Walks) etc. (see [22], [32] and references therein). Future efforts should be addressed to these appealing applications of the Coulomb Gas.

2 Definitions and Results

A possible microscopic realization of the two-dimensional Coulomb Gas is the following. Consider a system of point particles labeled with numbers $1, 2, 3 \dots$ and a finite square lattice $\Lambda \subset \mathbb{Z}^2$ endowed with periodic boundary conditions: a configuration is the assignment to each of the particles, say the j -th, of a charge $\sigma_j = \pm 1$ and a position $x_j \in \Lambda$. Particles interact through a two-body electrostatic potential

$W_\Lambda(x_i, x_j) \equiv W_\Lambda(x_i - x_j)$, so that, if Ω_n is the set of all the possible configurations of n particles, the total energy (*self-energy* included) of the configuration $\omega \in \Omega_n$ is

$$H_\Lambda(\omega) = \frac{1}{2} \sum_{i,j=1}^n \sigma_i \sigma_j W_\Lambda(x_i - x_j) . \quad (2.1)$$

If z is the activity and $\beta > 0$ the inverse temperature, the Grand Canonical partition function is

$$Z_\Lambda(\beta, z) = \sum_{n \geq 0} \frac{z^n}{n!} \sum_{\omega \in \Omega_n} e^{-\beta H_\Lambda(\omega)} . \quad (2.2)$$

Before stating the main theorem, we have to give a precise definition of electrostatic potential W_Λ . It is to be the inverse of $-\Delta_\Lambda$, the Laplacian operator on Λ ; anyways, because of the periodic boundary conditions on Λ , $-\Delta_\Lambda$ has zero modes, and we have to assign a regularization procedure to make sense of $(-\Delta_\Lambda)^{-1}$. Define the Yukawa interaction on Λ with mass m

$$W_\Lambda(x; m) = \frac{1}{|\Lambda|} \sum_{k \in \Lambda^*} \frac{e^{ikx}}{m^2 - \widehat{\Delta}(k)} \quad (2.3)$$

where Λ^* is the reciprocal lattice of Λ and $-\widehat{\Delta}(k) = 2 \sum_{j=0,1} (1 - \cos k_j)$ is the Fourier transform of $-\Delta_\Lambda$; then, if $H_\Lambda(\omega; m)$ is the Yukawa energy of the configuration ω , re-define (2.1) such that

$$e^{-\beta H_\Lambda(\omega)} := \lim_{m \rightarrow 0} e^{-\frac{\beta}{2} \sum_{i,j=1}^n \sigma_i \sigma_j W_\Lambda(x_i - x_j; m)} . \quad (2.4)$$

Of course in the massless limit $m \rightarrow 0$ the Yukawa potential is, by itself, ill defined; though we shall see in Sec. A.1 that (2.4) makes sense and, in fact, assigns weight zero to configurations of non-neutral total charge - so it is apparent that the system is symmetric under $z \rightarrow -z$. Finally, for R and L positive integers, L odd and bigger than one, assume that Λ is a square with a side of L^R lattice sites; then the *thermodynamic limits* of the pressure is

$$p(\beta, z) = \lim_{R \rightarrow \infty} \frac{1}{\beta |\Lambda|} \ln Z_\Lambda(\beta, z) . \quad (2.5)$$

The main result of the paper is the following theorem.

Theorem 2.1 *For $\varepsilon > 0$ small enough, there exists a function $\Sigma(z) \geq \Sigma(0) = 8\pi$ such that, if $|z| \leq \varepsilon$ and $\beta = \Sigma(z)$, the limit (2.5) exists.*

As stated at this point, the Theorem is slightly imprecise, because we have not found a way to characterize the Kosterlitz-Thouless transition line in terms of the pressure only; though it will become apparent in the next section that the curve $\beta = \Sigma(z)$ is exactly such line. Lengthier computations, simple but not explicitly pursued in this paper, would prove that $\Sigma(z)$ is a smooth function.

3 Strategy of the Proof

The starting point of this analysis is the functional integral representation of the partition function that allows for a more standard RG approach. In Sec. A.1 we shall prove an equivalent formula for the partition function for finite lattice Λ :

$$Z_\Lambda(\beta, z) = \lim_{m \rightarrow 0} \int dP_{\geq R}(\zeta^{(R)}; m) \int dP_{R-1}(\zeta^{(R-1)}) \cdots \int dP_1(\zeta^{(1)}) \cdot \int dP_0(\zeta^{(0)}) e^{\mathcal{V}(\zeta^{(0)} + \zeta^{(1)} + \cdots + \zeta^{(R-1)} + \zeta^{(R)})} \quad (3.1)$$

where $dP_{\geq R}(\zeta; m)$ is a Gaussian with *massive* covariance, $\int dP_{\geq R}(\zeta; m) \zeta_x \zeta_y = \Gamma_{\geq R}(x - y; m)$, while dP_j , for $j = 0, \dots, R-1$, is a Gaussian measure with *massless* covariance $\int dP_j(\zeta) \zeta_x \zeta_y = \Gamma_j(x - y)$. Explicit definitions of $\Gamma_{\geq R}$ and Γ_j (and precise meaning of 'massive' and 'massless') are given in Sec. A.1. Here we stress that, besides being positive-definite functions, $\Gamma_0, \Gamma_1, \dots, \Gamma_{R-1}$ satisfy the following properties

$$\Gamma_j(x) = 0 \quad \text{for } |x| \geq L^{j+1}/2, \quad (3.2)$$

$$|\partial^a \Gamma_j(x)| \leq C_a L^{-j|a|} \quad \text{for } |x| \leq L^{j+1}/2, \quad \text{if } |a| > 0, \quad (3.3)$$

$$\Gamma_j(0) = \frac{1}{2\pi} \ln L + c_j(L) \quad \text{for } |c_j(L)| \leq cL^{-\frac{j}{4}}; \quad (3.4)$$

where C_a and c are independent of L . Namely, Γ_j has compact support $O(L^{j+1})$ and typical momentum $O(L^{-j})$. Finally, to complete the explanation of (3.1),

$$\mathcal{V}(\varphi) := |\Lambda| \frac{1}{2} \ln(1 - s) + \frac{s}{2} \sum_{\substack{x \in \Lambda \\ \mu \in \hat{e}}} (\partial^\mu \varphi_x)^2 + z \sum_{\substack{x \in \Lambda \\ \sigma = \pm}} e^{i\sigma \alpha \varphi_x}, \quad (3.5)$$

where $\hat{e} = \{(1, 0), (0, 1), (-1, 0), (0, -1)\}$ and $\partial^\mu \varphi_x = \varphi_{x+\mu} - \varphi_x$; the notation $\sum_{\mu \in \hat{e}}$ implies also a factor $1/2$ that we do not write explicitly (so that the Fourier transform of $\sum_{\mu \in \hat{e}} \partial^{-\mu} \partial^\mu$ coincides with $\hat{\Delta}(k)$ defined after (2.3)). The parameter s is in $[0, 1/2)$ and will be chosen as function of z : in the final limit $R \rightarrow \infty$, it will fix the relation between α and the inverse temperature β through the formula $\alpha^2 = (1-s)\beta$.

The RG approach consists in computing the integrals in (3.1) progressively from the random variable with highest momentum to the one with lowest. After each integration we define \mathcal{V}_j , the effective potential on scale j , such that $\mathcal{V}_0 = \mathcal{V}$,

$$e^{\mathcal{V}_{j+1}(\varphi)} = \int dP_j(\zeta) e^{\mathcal{V}_j(\varphi + \zeta)}, \quad j = 0, 1, \dots, R-1; \quad (3.6)$$

and, at last,

$$Z_\Lambda(\beta, z) = \lim_{m \rightarrow 0} \int dP_{\geq R}(\zeta; m) e^{\mathcal{V}_R(\zeta)}. \quad (3.7)$$

In this way the evaluation of the partition function is transformed into the flow of a dynamical system of effective interactions $\mathcal{V}_0, \mathcal{V}_1, \dots, \mathcal{V}_R$. To have control on it, we must distinguish the *irrelevant part* of \mathcal{V}_j , namely the terms that, along the flow, become smaller and smaller by simple 'power counting' arguments, from the *relevant part*, namely the terms that require a more careful study. In order to do that, it is important to introduce some special kind of lattice domains: *blocks* and *polymers* ([5], [10]). Define $|x| := \max\{|x_0|, |x_1|\}$. Recall that L was chosen odd; and, for $j = 0, 1, \dots, R$, pave the periodic lattice Λ with $L^{2(R-j)}$ disjoint squares of size L^j in a natural way: there is the central square,

$$\{x \in \Lambda : |x| \leq L^j/2\}$$

and all the other squares are translations of this one by vectors in $L^j\mathbb{Z}$. We call these squares j -blocks, and we denote the set of j -blocks by $\mathcal{B}_j \equiv \mathcal{B}_j(\Lambda)$. 0-blocks are made of single points, so, for example, $\mathcal{B}_0(\Lambda) = \Lambda$. A union of j -blocks is called j -polymer, and the set of all j -polymers in Λ is denoted $\mathcal{P}_j \equiv \mathcal{P}_j(\Lambda)$. Suppose X is a j -polymer: ∂X is the set of sites in X with a nearest neighbor outside X ; $\mathcal{B}_j(X)$ is the set of the j -blocks in X ; $|X|_j$ is the cardinality of $\mathcal{B}_j(X)$; and \bar{X} is the smallest polymer in $\mathcal{P}_{j+1}(\Lambda)$ that contains X . A polymer made of two blocks, $B, D \in \mathcal{B}_j(\Lambda)$, is connected if there exist $x \in B$ and $y \in D$ s.t. $|x - y| = 1$; the definition extends to connected polymers of more blocks in the usual way. We call $\mathcal{P}_j^c \equiv \mathcal{P}_j^c(\Lambda)$ and $\mathcal{S}_j \equiv \mathcal{S}_j(\Lambda)$ the set of the connected j -polymers and the set of the connected j -polymers that are made of no more than 4 j -blocks - the "small polymers" - respectively. $\mathcal{S}_j \equiv \mathcal{S}_j(\Lambda)$ is the set of the connected polymers that are not small; and S is the (j -independent) number of small j -polymers that contain a given j -block. Given a j -polymer X , the collection of its maximal connected parts (each of which is a j -polymer by construction) is called $\mathcal{C}_j(X)$; while its *small set neighborhood* is $X^* = \cup\{Y \in \mathcal{S}_j(\Lambda) : Y \cap X \neq \emptyset\}$. The empty set is considered as an element of $\mathcal{P}_j(\Lambda)$, but not of $\mathcal{P}_j^c(\Lambda)$.

Given a block $B \in \mathcal{B}_j$, we define the interaction

$$U_j(s, z, \varphi, B) = V_j(s, z, \varphi, B) + W_j(s, z, \varphi, B) \quad (3.8)$$

where $W_j(s, z, \varphi, B)$ is quadratic in s, z and will be defined precisely below; while V_j is basically given by the original interaction

$$V_j(s, z, \varphi, B) = \frac{S}{2} \sum_{\substack{x \in B \\ \mu \in \bar{e}}} (\partial^\mu \varphi_x)^2 + zL^{-2j} \sum_{\substack{x \in B \\ \sigma = \pm}} e^{i\sigma\alpha\varphi_x} . \quad (3.9)$$

Notice anyways the factor L^{-2j} that makes V_j explicitly j -dependent; besides, it actually depends upon $\{\varphi_x\}_x$ for x in a domain slightly bigger than B (because includes the outer boundary sites). We assume the $W_j(s, z, \varphi, B)$ depends upon

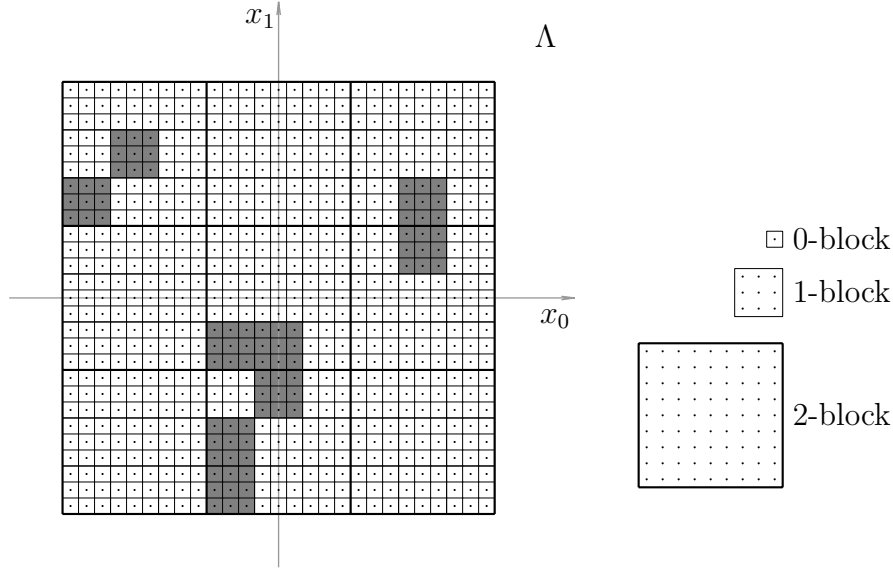


Figure 2: Lattice paving with blocks of different sizes in the case $L = 3$ and $R = 3$

$\{\varphi_x\}_{x \in B^*}$. We extend these definitions to polymers $X \in \mathcal{P}_j$ additively:

$$U_j(s, z, \varphi, X) := \sum_{B \in \mathcal{B}_j(X)} U_j(s, z, \varphi, B); \quad (3.10)$$

$V_j(s, z, \varphi, X)$ and $W_j(s, z, \varphi, X)$ are defined in the same way. If we drop the first two variable of U_j , V_j and W_j , it means they are s_j and z_j - i.e. their have the same label j of the potential. Finally, inductively assume the following formula for the effective potential

$$\mathcal{V}_j(\varphi) = E_j |\Lambda| + \ln \left[\sum_{X \in \mathcal{P}_j(\Lambda)} e^{U_j(\varphi, \Lambda \setminus X)} \prod_{Y \in \mathcal{C}_j(X)} K_j(\varphi, Y) \right] \quad (3.11)$$

where the *polymer activity*, $K_j(\varphi, Y)$, is a function of $\{\varphi_x\}_{x \in Y^*}$. Finally, from (3.7) and (3.11) with $j = R$

$$Z_\Lambda(\beta, z) = e^{E_R |\Lambda|} \lim_{m \rightarrow 0} \int dP_{\geq R}(\zeta; m) [e^{U_R(\zeta, \Lambda)} + K_R(\zeta, \Lambda)] := e^{E_{R+1} |\Lambda|}. \quad (3.12)$$

The details of these constructions are in Sec. 4; as opposed to the original RG method in [11], we will not need any 'cluster expansion' to handle the tails of the covariances, for in our setting they have compact support (this idea was devised in [31], [5]). We shall be more specific on the regularity of U_j and K_j later. At this stage, we just mention that U_j is going to contain the second order part of *marginal* terms of the iteration, whereas the *irrelevant* terms, as well higher order marginal terms, will be stored in K_j .

Assumption (3.11) holds at $j = 0$, for $(E_0, s_0, z_0) = (\frac{1}{2} \ln(1-s), s, z)$, $W_0 \equiv 0$ and $K_0 \equiv 0$. In Sec. 4 we shall prove that, iterating (3.6), assumption (3.11) holds at any scale $j = 1, \dots, R$, with (s_j, z_j, K_j) recursively given by: for $j = 0$,

$$\begin{aligned} s_1 &= s \\ z_1 &= L^2 e^{-\frac{\alpha^2}{2} \Gamma_0(0)} z \\ K_1 &= \mathcal{R}_0(z, s); \end{aligned} \quad (3.13)$$

for $j = 1, 2, \dots, R-1$,

$$\begin{aligned} s_{j+1} &= s_j - a_j z_j^2 + \mathcal{F}_j(K_j) \\ z_{j+1} &= L^2 e^{-\frac{\alpha^2}{2} \Gamma_j(0)} \left[z_j - b_j s_j z_j + \mathcal{M}_j(K_j) \right] \\ K_{j+1} &= \mathcal{L}_j(K_j) + \mathcal{R}_j(z_j, s_j, K_j). \end{aligned} \quad (3.14)$$

\mathcal{F}_j and \mathcal{M}_j are real functions of the polymer activity; \mathcal{L}_j and \mathcal{R}_j are a linear and high order maps of the polymer activities (and functions of z_j and s_j). Note that $a_j \equiv a_j(L)$, $b_j \equiv b_j(L)$.

Lemma 3.1 *If $\alpha^2 = 8\pi$, set $a := 8\pi^2 e^c \ln L$ and $b := 2 \ln L$, where c is a constant introduced in (A.29); then there exist C and $C(L)$ such that*

$$\left| L^2 e^{-\frac{\alpha^2}{2} \Gamma_j(0)} - 1 \right| \leq CL^{-\frac{j}{4}}, \quad |a_j - a|, |b_j - b| \leq C(L)L^{-\frac{j}{4}}. \quad (3.15)$$

The former inequality is a consequence of (3.4); the latter is proven in Sec. A.3.2. The energy parameters, E_0, E_1, \dots, E_R , are recursively defined by

$$E_{j+1} = E_j + L^{-2j} \left[s_j^2 \widehat{e}_{3,j} + z_j^2 \widehat{e}_{4,j} + s_j \widehat{e}_{2,j} + \widehat{e}_{1,j}(K_j) \right], \quad (3.16)$$

where $\widehat{e}_{1,j}(K_j)$ is linear in K_j , while the other $\widehat{e}_{m,j}$'s are independent of s_j, z_j and K_j . E_{R+1} is defined in (3.12). For $j = 0, 1, \dots, R$, let $\mathcal{E}_j := E_{j+1} - E_j$. In Sec. 5.1, we will set up a norm $\|\cdot\|_{h,T_j}$ for polymer activities on scale j depending on two parameters h and A : if properly chosen, the following lemmas hold.

Lemma 3.2 *There exists $C(\alpha, L)$ such that, for any given $j = 0, 1, \dots, R$, if $|s_j|, |z_j|, \|K_j\|_{h,T_j} \leq \varepsilon_0$,*

$$|\mathcal{E}_j| \leq C(\alpha, L)L^{-2j}\varepsilon_0. \quad (3.17)$$

Besides, $\mathcal{E}_0, \dots, \mathcal{E}_{R-1}$ (but not \mathcal{E}_R) are the same on Λ and on \mathbb{Z}^2 .

The proof is in Sec. 6.1. (3.14) is the *RG map*. The solution of (3.14), for initial data $(s_0, z_0, K_0) = (s, z, 0)$ will be called the *RG flow*; the sequence of energies given by (3.16) has no influence on the RG flow, hence can be seen as the history of an observable. Consider the case $\alpha^2 > 8\pi$: by (3.4), for L large enough, $L^2 e^{-\frac{\alpha^2}{2} \Gamma_j(0)} <$

1; consequently, if also $|z|$ is small enough, the flow goes to zero as was discussed in [14] (in the sub-case of $|z|$ small w.r.t the value of β). This paper is focused on the more complicated case $\alpha^2 = 8\pi$, when $L^2 e^{-\frac{\alpha^2}{2}\Gamma_j(0)}$ is basically 1 and then the flow is determined by the second order terms (see Lemma 3.1). In fact, neglecting higher orders and neglecting the RG map for $(K_j)_j$, (3.14) is the equation that Kosterlitz obtained with a (formal) RG technique in coordinate space, [25]. The corresponding approximate solution is, for $j \geq 1$,

$$s_j = \frac{1}{b}|q_j|, \quad z_j = \frac{1}{\sqrt{ab}}q_j, \quad \text{with } q_j := \frac{q_1}{1 + |q_1|(j-1)}. \quad (3.18)$$

(3.18) describes a line, *i.e.* a first approximation of the separatrix in Fig. E.1, which is the Kosterlitz-Thouless transition line. Our goal is a proof that the solution of the full RG map (3.13), (3.14), is, qualitatively, not too different from (3.18). In order to achieve that, we need to know the smoothness of the remainder terms.

Lemma 3.3 *There exist $C, C(\alpha) > 1$ such that, if L is large enough,*

$$|\mathcal{F}_j(K_j)| \leq CA^{-1}\|K_j\|_{h,T_j}, \quad |\mathcal{M}_j(K_j)| \leq C(\alpha)A^{-1}\|K_j\|_{h,T_j}. \quad (3.19)$$

Besides, $\mathcal{F}_1, \dots, \mathcal{F}_{R-1}$ and $\mathcal{M}_1, \dots, \mathcal{M}_{R-1}$ are the same on Λ and on \mathbb{Z}^2 .

Lemma 3.4 *There exist $C(\alpha) > 1$ and $\eta, \vartheta > 0$ such that, for L large enough,*

$$\|\mathcal{L}_j(K_j)\|_{h,T_{j+1}} \leq C(\alpha) (L^{-\vartheta} + A^{-\eta}) \|K_j\|_{h,T_j}. \quad (3.20)$$

Lemma 3.5 *If $\varepsilon_j > 0$ is small enough, there exists $C \equiv C(A, L, \alpha) > 1$ such that, for any (s_j, z_j, K_j) and $(\dot{s}_j, \dot{z}_j, \dot{K}_j)$ satisfying the conditions $|s_j|, |z_j|, |\dot{s}_j|, |\dot{z}_j| \leq \varepsilon_j$ and $\|K_j\|_{h,T_j}, \|\dot{K}_j\|_{h,T_j} \leq \varepsilon_j^2$ we have: for $j = 0$*

$$\|\mathcal{R}_0(z, s) - \mathcal{R}_0(\dot{z}, \dot{s})\|_{h,T_1} \leq C\varepsilon_0 [|s - \dot{s}| + |z - \dot{z}|]; \quad (3.21)$$

while, for $j = 1, 2, \dots, R-1$,

$$\begin{aligned} & \|\mathcal{R}_j(z_j, s_j, K_j) - \mathcal{R}_j(\dot{z}_j, \dot{s}_j, \dot{K}_j)\|_{h,T_{j+1}} \\ & \leq C \left[\varepsilon_j^2 |s_j - \dot{s}_j| + \varepsilon_j^2 |z_j - \dot{z}_j| + \varepsilon_j \|K_j - \dot{K}_j\|_{h,T_j} \right]. \end{aligned} \quad (3.22)$$

The proofs are in Sec. 6. In particular, Lemma 3.4 is crucial: to prove the contraction of \mathcal{L}_j we have to show that $\mathcal{L}_j(K_j)$ is made of irrelevant terms. The role of constants and parameters so far introduced is the following: $\alpha^2 = 8\pi$, although in many sub-results of the paper we will just assume $\alpha^2 \geq 8\pi$; h is a numerical constant related to the propagator, see (5.7); L will be taken large; A is to be large enough w.r.t. L ; finally, ε_0 , the size of z , is to be small enough w.r.t. A and L (and α). In a sense, the major improvement w.r.t. [14], crucial for studying the Kosterlitz-Thouless line, is that in this paper h is independent of L .

Thanks to Lemma 3.3, 3.4 and 3.5, we have the following result on the RG flow.

Theorem 3.6 *Given L, A large enough and an $\varepsilon \equiv \varepsilon(A, L, \alpha)$, in correspondence of any $z, |z| \leq \varepsilon$, there exists an $s = \Sigma(z)$ such that the solution of (3.14) with initial data (z, s) is*

$$s_j = \frac{|q_j|}{b} + O(j^{-\frac{3}{2}}), \quad z_j = \frac{q_j}{\sqrt{ab}} + O(j^{-\frac{3}{2}}), \quad \|K_j\|_{h, T_j} = O(j^{-3}), \quad (3.23)$$

for $j = 0, \dots, R$ and $q_1 = O(z)$. Besides, $\Sigma(z)$ can be chosen independent of Λ .

The proof is given in Sec.7, and uses the fixed point theorem for Banach spaces. Since the flow of (s_j, z_j, K_j) remains bounded, Lemma 3.2 applies, and

$$\beta p(\beta, z) = \lim_{R \rightarrow \infty} \frac{\ln Z_\Lambda(\beta, z)}{|\Lambda|} = \lim_{R \rightarrow \infty} E_{R+1} = E_0 + \sum_{j \geq 0} \mathcal{E}_j,$$

where the series is convergent because of (3.17) (valid also for \mathcal{E}_{R+1}). This completes the proof of the main result, Theorem 2.1; in the remaining sections, we shall take up the task of proving all the above sub-results. Note that $C, C(\alpha), C(L, \alpha)$ and $C(A, L, \alpha)$, will indicate (possibly) different values in different equations.

4 Renormalization Group Map

In this and the following section we adopt an abridged notation for the fields. In general, we remove the labels j because they will be clear from the context, and label the sum of the fields on higher scales with a prime, so that $\zeta_x := \zeta_x^{(j)}$ and $\varphi'_x := \zeta_x^{(R)} + \zeta_x^{(R-1)} + \dots + \zeta_x^{(j+1)}$; besides, $\varphi_x := \varphi'_x + \zeta_x$. We also use $\mathbb{E}_j[\cdot]$ for the expectations w.r.t. the measure $dP_j(\zeta^{(j)})$. Therefore the RG map for the effective potential is

$$e^{\mathcal{V}_{j+1}(\varphi')} = \mathbb{E}_j \left[e^{\mathcal{V}_j(\varphi' + \zeta)} \right], \quad (4.1)$$

for $\mathcal{V}_j(\varphi)$ given by (3.11). As function of the fields, \mathcal{V}_0 is made of a periodic term of period $2\pi/\alpha$, and a derivative term: then \mathcal{V}_0 is invariant under $\varphi_x \rightarrow \varphi_x + \frac{2\pi}{\alpha}t$ for the a constant, integer field t . The latter property remains true, by induction, for \mathcal{V}_j and then for K_j . As consequence, we shall prove in appendix B the following decomposition of $K_j(\varphi, Y)$ into *charged components*.

Lemma 4.1 *For any $j = 0, 1, \dots, R$, there exists a decomposition*

$$K_j(\varphi, X) = \sum_{q \in \mathbb{Z}} \widehat{K}_j(q, \varphi, X) \quad (4.2)$$

such that, if ϑ is a constant field,

$$\widehat{K}_j(q, \varphi + \vartheta, X) = e^{iq\alpha\vartheta} \widehat{K}_j(q, \varphi, X). \quad (4.3)$$

This simple result is borrowed from [14] and for completeness is reviewed in Sec. B. The 'power counting' argument - that we shall make rigorous in the rest of the paper - implies: a) terms with charge q contract by a factor $L^{-\frac{\alpha^2}{4\pi}q^2}$; b) terms proportional to $(\partial\varphi')^n$ contract by a factor L^{-n} ; c) all terms are increased by a volume factor L^2 (the ratio of volumes of $j+1$ - and j -blocks). Therefore, at $\alpha^2 = 8\pi$, RG reduces the size of the components with charge $|q| \geq 2$; of the components with charge $|q| = 1$, if the 0-th order Taylor expansion in $\partial\varphi'$ has been taken away; and also the of neutral charge component, if the 2-th order Taylor expansion in $\partial\varphi'$ has been taken away. The terms so removed are absorbed in E_j, s_j, z_j to give $E_{j+1}, s_{j+1}, z_{j+1}$. Guided by these ideas, we pass to a technical description of RG; because of technical reasons, it is more convenient to define the first RG steps in a different (in fact simpler) way.

4.1 First RG step

After integrating the field $\zeta^{(0)}$ in V_0 , we want to recast the effective potential \mathcal{V}_1 into the form (3.11). There are many ways to achieve this.

Lemma 4.2 *Given s and z , define s_1 and z_1 as in (3.13), and*

$$E_1 = -\frac{s}{2} \sum_{\mu \in \hat{e}} (\partial^\mu \partial^\mu \Gamma_0)^2(0). \quad (4.4)$$

There exists a choice of K_1 and U_1 such that (3.11) holds with $K_1 = O(V_0^2)$.

Proof of Lemma 4.2 - For $D \in \mathcal{B}_1$ and $Y \in \mathcal{P}_1^c$ define

$$\begin{aligned} P_0(\varphi', \zeta, D) &:= e^{V_0(\varphi, D)} - e^{V_1(\varphi', D) + E_1|D|}, \\ P_0^Y(\varphi', \zeta) &:= \prod_{D \in \mathcal{B}_1(Y)} P_0(\varphi', \zeta, D). \end{aligned} \quad (4.5)$$

(recall $V_0(\varphi, D) \equiv V_0(s, z, \varphi, D)$ and $V_1(\varphi', D) \equiv V_1(s_1, z_1, \varphi', D)$). Expand formula (3.6) at $j = 0$ w.r.t. P_0 , namely

$$\mathbb{E}_0 [e^{\mathcal{V}_0(\varphi, \Lambda)}] = \mathbb{E}_0 \left[\prod_{D \in \mathcal{B}_1(\Lambda)} \left(P_0(\varphi', \zeta, D) + e^{V_1(\varphi', D) + E_1|D|} \right) \right], \quad (4.6)$$

to obtain (3.11) for $U_1 \equiv V_1$ (i.e. $W_1 \equiv 0$) and

$$K_1(\varphi', Y) = e^{-E_1|Y|} \mathbb{E}_0 [P_0^Y(\varphi', \zeta)]. \quad (4.7)$$

The linear order in V_0 of (4.7) may not be zero only when Y is a 1-block D :

$$[\mathcal{L}_0 K_0](\varphi', D) = \mathbb{E}_0[V_0(\varphi, D)] - V_1(\varphi', D) - E_1|D|. \quad (4.8)$$

But this expression is zero by the choice of s_1, z_1 and E_1 . ■

4.2 j -th RG Step

Consider now $j = 1, 2, \dots, R - 1$. After integrating the field ζ , we want to recast the effective potential V_{j+1} into the form (3.11):

$$e^{\mathcal{V}_{j+1}(\varphi')} = e^{E_{j+1}|\Lambda|} \sum_{X \in \mathcal{P}_{j+1}(\Lambda)} e^{U_{j+1}(\varphi', \Lambda \setminus X)} \prod_{Y \in \mathcal{C}_{j+1}(X)} K_{j+1}(\varphi', Y). \quad (4.9)$$

Again, this formula does not determine K_{j+1} and U_{j+1} in a unique way; we want to take advantage of this freedom in order to make K_{j+1} either irrelevant or third order in V_j . To achieve that, we have to extract a $\overline{Q}_j(\varphi', X)$ from $\mathbb{E}_j[K_j]$ and a $Q_j(\varphi', Y)$ from $\mathbb{E}_j^T[V_j; V_j]$: the next lemma, purely combinatorial, furnishes such a K_{j+1} and the corresponding U_{j+1} .

Lemma 4.3 *Suppose to be given two activities: $\overline{Q}_j(\varphi', X) = O(K_j)$ with support on sets $X \subset \mathcal{S}_j$, and $Q_j(\varphi', Y) = O(V_j^2)$ with support on sets $Y \subset \mathcal{S}_{j+1}$; and suppose the difference between (s_j, z_j, E_j) and $(s_{j+1}, z_{j+1}, E_{j+1})$ is such that*

$$(E_{j+1} - E_j)|D| + V_{j+1}(\varphi', D) - \mathbb{E}_j[V_j(\varphi, D)] = O(K_j, V_j^2). \quad (4.10)$$

Then there exists a choice of K_{j+1} and U_{j+1} such that (4.9) holds with

$$K_{j+1}(\varphi', Y') := [\mathcal{L}_j K_j](\varphi', Y') + \mathcal{R}_j(\varphi', Y'), \quad (4.11)$$

where $\mathcal{R}_j(\varphi', Y') = O(K_j^2, K_j V_j, V_j^3)$ and,

$$\begin{aligned} [\mathcal{L}_j K_j](\varphi', Y') &= \sum_{X \in \mathcal{P}_j^c(Y')}^{\overline{X}=Y'} \left[\mathbb{E}_j[K_j(\varphi, X)] - \overline{Q}_j(\varphi', X) \right] \\ &+ \frac{1}{2} \sum_{B_0, B_1 \in \mathcal{B}_j(Y')}^{\overline{B_0 \cup B_1}=Y'} \mathbb{E}_j^T[V_j(\varphi, B_0); V_j(\varphi, B_1)] - Q_j(\varphi', Y') \\ &- \sum_{D \in \mathcal{B}_{j+1}}^{D=Y'} \left[W_{j+1}(\varphi', D) - \mathbb{E}_j[W_j(s_{j+1}, z_{j+1}, \varphi, D)] - \sum_{Y \in \mathcal{S}_{j+1}}^{Y \supset D} \frac{Q_j(\varphi', Y)}{|Y|_{j+1}} \right] \\ &- \sum_{B \in \mathcal{B}_j}^{\overline{B}=Y'} \left[\mathcal{E}_j|B| + V_{j+1}(\varphi', B) - \mathbb{E}_j[V_j(\varphi, B)] - \sum_{X \in \mathcal{S}_j}^{X \supset B} \frac{\overline{Q}_j(\varphi', X)}{|X|_j} \right] \end{aligned} \quad (4.12)$$

(recall the short notations for V_j , V_{j+1} and W_{j+1} defined after (3.10)).

In the third line there is $W_j(s_{j+1}, z_{j+1}, \varphi, D)$ as opposed to $W_j(s_j, z_j, \varphi, D)$: the former is more convenient for our analysis and the difference with the latter is $O(V_j^3, K_j V_j^2, K_j^2)$, which can be left inside \mathcal{R}_j . Besides, (4.11) gives the third line of (3.14), with $\mathcal{R}_j(\varphi', Y') \equiv [\mathcal{R}_j(s_j, z_j, K_j)](\varphi', Y')$.

Before giving the proof we elaborate on (4.12). As planned before, in the first and second line of the r.h.s. member we read the extraction of \bar{Q}_j and Q_j from $\mathbb{E}_j[K_j]$ and $\mathbb{E}_j^T[V_j; V_j]$; the same terms are re-absorbed into E_j, s_j, z_j in the third line, so obtaining $\bar{E}_{j+1}, s_{j+1}, z_{j+1}$. Therefore, to determine the latter parameters, we have to choose \bar{Q}_j and Q_j first.

The choice of \bar{Q}_j requires Taylor expansion in $\nabla\varphi'$, that we now define. Let $F(\xi, X)$ be a smooth function of the field $(\xi_x)_{x \in X^*}$; the n -order Taylor expansion of $F(\xi, X)$ at $\xi = 0$ is

$$(\text{Tay}_n F)(\xi, X) := \sum_{m=0}^n \frac{1}{m!} \sum_{x_1, \dots, x_m \in X^*} \xi_{x_1} \cdots \xi_{x_m} \frac{\partial^m F}{\partial \xi_{x_1} \cdots \partial \xi_{x_m}}(0, X); \quad (4.13)$$

the n -order remainder is $(\text{Rem}_n F)(\xi, X) := F(\xi, X) - (\text{Tay}_n F)(\xi, X)$. The Taylor expansion of K_j is to be in the field $\xi \sim \nabla\varphi'$, namely we have to single out the part of the activity that is *purely dipolar*. That is accomplished by (4.3): for any point $x_0 \in X$, if $(\delta\varphi')_x := \varphi'_x - \varphi'_{x_0}$ (which is a sum of $\nabla\varphi'$'s),

$$\widehat{K}_j(q, \varphi, X) = e^{i\alpha q \varphi'_{x_0}} \widehat{K}_j(q, \delta\varphi' + \zeta, X).$$

Therefore, our choice for \bar{Q}_j is

$$\begin{aligned} \bar{Q}_j(\varphi', X) &= \frac{1}{|X|} \sum_{x_0 \in X} \text{Tay}_2 \mathbb{E}_j \left[\widehat{K}_j(0, \delta\varphi' + \zeta, X) \right] \\ &+ \frac{1}{|X|} \sum_{x_0 \in X} \sum_{\sigma = \pm 1} e^{i\sigma \alpha \varphi'_{x_0}} \text{Tay}_0 \mathbb{E}_j \left[\widehat{K}_j(\sigma, \delta\varphi' + \zeta, X) \right] \end{aligned} \quad (4.14)$$

where Taylor expansions are in $\xi = \delta\varphi'$ (the special point x_0 is averaged over X); in (4.14) we also used the fact that K_j is even in ζ , therefore the expectation of one derivative of K_j is zero. We shall prove in Sec. 6.2 that (4.14) makes irrelevant the first line of (4.12). Then, define intermediate parameters $\bar{\mathcal{E}}_j = \bar{E}_j - E_j, \bar{s}_j$ and \bar{z}_j such that

$$\begin{aligned} \bar{s}_j &= s_j + \mathcal{F}_j(K_j), \quad \bar{z}_j = L^2 e^{-\frac{\alpha^2}{2} \Gamma_j(0)} [z + \mathcal{M}_j(K_j)] \\ \bar{\mathcal{E}}_j &= L^{-2j} [\widehat{e}_{1,j}(K_j) + s_j \widehat{e}_{2,j}], \end{aligned} \quad (4.15)$$

where

$$\begin{aligned} \widehat{e}_{1,j}(K_j) &= \sum_{X \ni 0} \frac{1}{|X|^j} \mathbb{E}_j \left[\widehat{K}_j(0, \zeta, X) \right], \quad \widehat{e}_{2,j} = -\frac{L^{2j}}{2} \sum_{\mu \in \widehat{e}} (\partial^\mu \partial^\mu \Gamma_j)(0). \\ \mathcal{F}_j(K_j) &= \sum_{X \ni 0} \frac{L^{-2j}}{|X|^j |X|} \sum_{\substack{x_0 \in X \\ x_1, x_2 \in X^*}} \mathbb{E}_j \left[\frac{\partial^2 \widehat{K}_j}{\partial \varphi_{x_1} \partial \varphi_{x_2}}(0, \zeta, X) \right] \sum_{\mu \in \widehat{e}} (x_1 - x_0)^\mu (x_2 - x_0)^\mu \end{aligned}$$

$$\mathcal{M}_j(K_j) = \frac{e^{\frac{\alpha^2}{2}\Gamma_j(0)}}{2} \sum_{X \in \mathcal{S}_j} \frac{X \ni 0}{|X|_j} \sum_{q=\pm 1} \mathbb{E}_j \left[\widehat{K}_j(q, \zeta, X) \right]. \quad (4.16)$$

In Sec. 6.2 we shall also prove that, with the above choices, the following quantity (compare it with the third line of (4.12)) is irrelevant:

$$\sum_{B \in \mathcal{B}_j} \left[\overline{\mathcal{E}}_j |B| + V_{j+1}(\overline{s}_j, \overline{z}_j, \varphi', B) - \mathbb{E}_j [V_j(\varphi, B)] - \sum_{X \in \mathcal{S}_j} \frac{X \supset B}{|X|_j} \overline{Q}_j(\varphi', X) \right] \quad (4.17)$$

Then set Q_j so that the second line of (4.12) is vanishing:

$$Q_j(\varphi', Y') := \frac{1}{2} \sum_{B_0, B_1 \in \mathcal{B}_j(Y')}^{\overline{B_0 \cup B_1} = Y'} \mathbb{E}_j^T [V_j(\varphi, B_0); V_j(\varphi, B_1)] ; \quad (4.18)$$

and choose $\mathcal{E}_j, s_{j+1}, z_{j+1}$ so that the remaining part of (4.12) vanish:

$$\begin{aligned} & \sum_{D \in \mathcal{B}_{j+1}}^{D=Y'} \left[(\mathcal{E}_j - \overline{\mathcal{E}}_j) |D| + V_{j+1}(s_{j+1} - \overline{s}_j, z_{j+1} - \overline{z}_j, \varphi', D) \right. \\ & \left. + W_{j+1}(\varphi', D) - \mathbb{E}_j [W_j(s_{j+1}, z_{j+1}, \varphi, D)] - \sum_{Y \in \mathcal{S}_{j+1}} \frac{Y \supset D}{|Y|_{j+1}} Q_j(\varphi', Y) \right] = 0. \quad (4.19) \end{aligned}$$

Because of the simple identity

$$\sum_{Y \in \mathcal{S}_{j+1}} \frac{Y \supset D}{|Y|_{j+1}} Q_j(\varphi', Y) = \frac{1}{2} E^T [V_j(s_j, z_j, \varphi, D); V_j(s_j, z_j, \varphi, D^*)] \quad (4.20)$$

and computations in Sec. C.2, cancellation (4.19) is obtained if

$$\begin{aligned} s_{j+1} &= \overline{s}_j - a_j z_j^2, & z_{j+1} &= \overline{z}_j - L^2 e^{-\frac{\alpha^2}{2}\Gamma_j(0)} b_j s_j z_j \\ \mathcal{E}_j &= \overline{\mathcal{E}}_j + L^{-2j} [s_j^2 \widehat{e}_{3,j} + z_j^2 \widehat{e}_{4,j}] \end{aligned} \quad (4.21)$$

and if W_j is given by

$$s^2 W_{a,j}(\varphi, B) + z^2 [W_{b,j}(\varphi, B) + W_{c,j}(\varphi, B)] + z s W_{d,j}(\varphi, B), \quad (4.22)$$

where, setting $\Gamma_{j,n}(x) := \Gamma_n(x) + \Gamma_{n+1}(x) + \dots + \Gamma_j(x)$ and $f(0|x) := f(0) - f(x)$, the definitions of the functions used in (4.21) and (4.22) are

$$a_j := \frac{\alpha^2}{2} \sum_{y \in \mathbb{Z}} |y|^2 \left[w_{b,j}(y) \left(e^{-\alpha^2 \Gamma_j(0|y)} - 1 \right) + e^{-\alpha^2 \Gamma_j(0)} \left(e^{\alpha^2 \Gamma_j(y)} - 1 \right) L^{-4j} \right]$$

$$\begin{aligned}
b_j &:= \frac{\alpha^2}{2} \sum_{\substack{y \in \mathbb{Z}^2 \\ \mu \in \widehat{e}}} \left[(\partial^\mu \Gamma_j)^2(y) + 2 \sum_{n=0}^{j-1} (\partial^\mu \Gamma_n)(y) (\partial^\mu \Gamma_j)(y) e^{-\frac{\alpha^2}{2} \Gamma_{j-1,n}(0)} L^{2(j-n)} \right] \\
\widehat{e}_{3,j} &:= \frac{L^{2j}}{4} \sum_{y \in \mathbb{Z}^2} \sum_{\substack{\mu \in \widehat{e} \\ \nu \in \widehat{e}}} \left[(\partial^\mu \partial^\nu \Gamma_{j,0})(y) + 2(\partial^\mu \partial^\nu \Gamma_{j-1,0})(y) \right] (\partial^\mu \partial^\nu \Gamma_j)(y|0) , \\
\widehat{e}_{4,j} &:= 2L^{2j} \sum_y w_{b,j}(y) \left[e^{-\alpha^2 \Gamma_j(0|y)} - 1 - \frac{\alpha^2}{2} \sum_{\mu, \nu} \partial^\mu \partial^\nu \Gamma_j(0) y^\mu y^\nu \right] \\
&\quad + L^{-2j} \sum_y e^{-\alpha^2 \Gamma_j(0)} \left(e^{\alpha^2 \Gamma_j(y)} - 1 \right) , \tag{4.23}
\end{aligned}$$

and $W_{m,0}(s, z, \varphi, B) = 0$ for $m = a, b, c, d, e$, while for $j \geq 1$,

$$\begin{aligned}
W_{a,j}(\varphi, B) &= - \sum_{\substack{\mu \in \widehat{e} \\ \nu \in \widehat{e}}} \sum_{y \in \mathbb{Z}^2} w_{a,j}^{\mu\nu}(y) \sum_{x \in B} (\partial^\mu \varphi_x) \left[(\partial^\nu \varphi_{x+y}) - (\partial^\nu \varphi_x) \right] \\
W_{b,j}(\varphi, B) &= \sum_{y \in \mathbb{Z}^2} w_{b,j}(y) \sum_{\substack{x \in B \\ \sigma = \pm}} \left[e^{i\sigma\alpha(\varphi_x - \varphi_{x+y})} - 1 + \frac{\alpha^2}{2} \sum_{\mu\nu} (\partial^\mu \varphi_x) (\partial^\nu \varphi_x) y^\mu y^\nu \right] \\
W_{c,j}(\varphi, B) &= \sum_{y \in \mathbb{Z}^2} w_{c,j}(y) \sum_{\substack{x \in B \\ \sigma = \pm}} e^{i\sigma\alpha(\varphi_x + \varphi_{x+y})} \\
W_{d,j}(\varphi, B) &= \sum_{\mu} \sum_{y \in \mathbb{Z}^2} w_{d,j}^\mu(y) \sum_{\substack{x \in B \\ \sigma = \pm}} i\sigma \left[e^{i\sigma\alpha\varphi_x} (\partial^\mu \varphi_{x+y}) - e^{i\sigma\alpha\varphi_{x+y}} (\partial^\mu \varphi_x) \right] \\
&\quad - \sum_{y \in \mathbb{Z}^2} w_{e,j}(y) \sum_{\substack{x \in B \\ \sigma = \pm}} \left(e^{i\sigma\alpha\varphi_{x+y}} - e^{i\sigma\alpha\varphi_x} \right) \tag{4.24}
\end{aligned}$$

for

$$\begin{aligned}
w_{a,j}^{\mu\nu}(y) &= \frac{1}{2} (\partial^\mu \partial^\nu \Gamma_{j-1,0})(y) ; \\
w_{b,j}(y) &= \sum_{n=0}^{j-1} e^{-\alpha^2 \Gamma_{j-1,n+1}(0|y)} e^{-\alpha^2 \Gamma_n(0)} \left(e^{\alpha^2 \Gamma_n(y)} - 1 \right) L^{-4n} ; \\
w_{c,j}(y) &= \frac{1}{2} \sum_{n=0}^{j-1} e^{-\alpha^2 [\Gamma_{j-1,n+1}(0) + \Gamma_{j-1,n+1}(y)]} e^{-\alpha^2 \Gamma_n(0)} \left(e^{-\alpha^2 \Gamma_n(y)} - 1 \right) L^{-4n} ; \\
w_{d,j}^\mu(y) &= \frac{\alpha}{2} \sum_{n=0}^{j-1} e^{-\frac{\alpha^2}{2} \Gamma_{j-1,n}(0)} (\partial^\mu \Gamma_n)(y) L^{-2n} \\
w_{e,j}(y) &= \frac{\alpha^2}{4} \sum_{n=0}^{j-1} e^{-\frac{\alpha^2}{2} \Gamma_{j-1,n}(0)} \sum_{\mu} \left[(\partial^\mu \Gamma_{j-1,n})^2(y) - (\partial^\mu \Gamma_{j-1,n+1})^2(y) \right] L^{-2n} .
\end{aligned}$$

$$(4.25)$$

By (3.2), $W_j(\varphi, B)$ depends on the field φ_x for x in a neighborhood of B of diameter $L^j/2$, which is a subset of B^* . Finally, joining (4.21) with (4.15) we obtain (3.14) and fulfill condition (4.10).

Proof of Lemma 4.3. - The *re-blocking* operation used here is different from the one in [5], partly because the extraction of Q_j (that for the dipoles of [5] is not required), partly because of a different *large field regulators* introduced below.

Starting from (3.11), reblock the polymers on scale $j + 1$ and obtain:

$$e^{V_j(\varphi)} = e^{E_j|\Lambda|} \sum_{X \in \mathcal{P}_{j+1}} e^{U_j(\varphi, \Lambda \setminus X)} \prod_{Y \in \mathcal{C}_{j+1}(X)} \tilde{K}_j(\varphi, Y) \quad (4.26)$$

for

$$\tilde{K}_j(\varphi, Y) := \sum_{X' \in \mathcal{P}_j(Y)}^{\bar{X}'=Y} e^{U_j(\varphi, Y \setminus X')} \prod_{Y' \in \mathcal{C}_j(X')} K_j(\varphi, Y'). \quad (4.27)$$

If D is a $j + 1$ -block, and Z a $j + 1$ -polymer, define

$$\begin{aligned} P_j(\varphi', \zeta, D) &:= e^{U_j(\varphi', D)} - e^{U_{j+1}(\varphi', D) + \mathcal{E}_j|D|}, \\ P_j^Z(\varphi', \zeta) &:= \prod_{D \in \mathcal{B}_{j+1}(Z)} P_j(\varphi', \zeta, D); \end{aligned} \quad (4.28)$$

besides, if Y and X are $j + 1$ -polymers, define

$$\begin{aligned} R_j(\varphi', \zeta, Y) &:= \tilde{K}_j(\varphi', Y) - \sum_{D \in \mathcal{B}_{j+1}(Y)} J_j(\varphi', D, Y), \\ R_j^X(\varphi', \zeta) &:= \prod_{Y \in \mathcal{C}_{j+1}(X)} R_j(\varphi', \zeta, Y) \end{aligned} \quad (4.29)$$

where $J_j(\varphi', D, Y)$ contains the extracted terms \bar{Q}_j and Q_j (which have support on $j + 1$ -small and j -small polymers, respectively): if $Y \notin \mathcal{S}_{j+1}$, or $D \not\subset Y$, $J_j(\varphi', D, Y) \equiv 0$; otherwise

$$\begin{aligned} J_j(\varphi', D, Y) &:= \frac{Q_j(\varphi', Y)}{|Y|_{j+1}} + \sum_{B \in \mathcal{B}_j(D)} \sum_{\substack{X \in \mathcal{S}_j \\ X \supset B}}^{\bar{X}=Y} \frac{\bar{Q}_j(\varphi', X)}{|X|_j} \\ &- \delta_{D,Y} \sum_{Y' \in \mathcal{S}_{j+1}}^{Y' \supset D} \left[\frac{Q_j(\varphi', Y')}{|Y'|_{j+1}} + \sum_{B \in \mathcal{B}_j(D)} \sum_{\substack{X \in \mathcal{S}_j \\ X \supset B}}^{\bar{X}=Y'} \frac{\bar{Q}_j(\varphi', X)}{|X|_j} \right]; \end{aligned} \quad (4.30)$$

hence $J_j(\varphi', D, Y)$ is a function of $\{\varphi'_x\}_{x \in Y^*}$; if $Y \notin \mathcal{S}_{j+1}$; and, by construction,

$$\sum_{Y \in \mathcal{P}_{j+1}^c} J_j(\varphi', D, Y) = 0. \quad (4.31)$$

Plugging decompositions (4.28) and (4.29) in (4.26) and expanding we find (4.9), for

$$\begin{aligned} K_{j+1}(\varphi', Y') &= \sum_{\substack{\rightarrow Y' \\ X_0, X_1 \\ Z, (D)}} e^{-\mathcal{E}_j |W| + U_{j+1}(\varphi', Y' \setminus W)}. \\ &\cdot \mathbb{E}_j [P_j^Z(\varphi', \zeta) R_j^{X_1}(\varphi', \zeta)] J_j^{X_0, (D)}(\varphi'). \end{aligned} \quad (4.32)$$

In this formula, we abridged $X_0 \cup X_1 \cup Z$ into W . And we labeled $\rightarrow Y'$ the sum over three $j+1$ -polymers, X_0, X_1, Z , and over one $j+1$ -block, D_Y , for each polymer $Y \in \mathcal{C}_{j+1}(X_0)$, such that: a) X_0 and X_1 are separated by at least by one $j+1$ -block, so that $\mathcal{C}_{j+1}(X_0 \cup X_1) = \mathcal{C}_{j+1}(X_0) + \mathcal{C}_{j+1}(X_1)$; b) Z is a subset of $Y' \setminus (X_0 \cup X_1)$; c) each connected component of X_0 is $j+1$ -small; d) $\cup_Y D_Y^* \cup Z \cup X_1 = Y'$. Finally, given X_0 and a D_Y for each $Y \in \mathcal{C}_{j+1}(X_0)$, we defined

$$J_j^{X_0, (D)}(\varphi') := \prod_{Y \in \mathcal{C}_{j+1}(X_0)} J_j(\varphi', D_Y, Y).$$

For (4.32) we also used the factorization of \mathbb{E}_j over sets that are in two different connected components of a $j+1$ -polymer (hence at distance not smaller than L^{j+1} , while the range of Γ_j is $L^{j+1}/2$). Besides, by construction, $W \subset Y'$ so that that $K_{j+1}(\varphi', Y)$ depends on the fields $(\varphi'_y)_{y \in Y^*}$ as required.

Finally, we have to prove that the linear part in K_j and second order part in V_j of this choice of K_{j+1} is (4.12): this follows from two simple identities,

$$\begin{aligned} \sum_{D \in \mathcal{B}_{j+1}(Y')} J_j(\varphi', D, Y') &= Q_j(\varphi', Y') + \sum_{\substack{\bar{X}=Y' \\ X \in \mathcal{S}_j}} \bar{Q}_j(\varphi', X) \\ &- \sum_{D \in \mathcal{B}_{j+1}} \sum_{Y \supset D} \frac{Q_j(\varphi', Y)}{|Y|_{j+1}} - \sum_{B \in \mathcal{B}_j} \sum_{X \supset B} \frac{\bar{Q}_j(\varphi', X)}{|X|_j}; \end{aligned} \quad (4.33)$$

and, by (4.31),

$$\sum_{Y \in \mathcal{S}_{j+1}} \sum_{D \in \mathcal{B}_{j+1}(Y)}^{D^*=Y'} J_j(\varphi', D, Y) = \sum_{D \in \mathcal{B}_{j+1}} \sum_{Y \supset D}^{D^*=Y'} J_j(\varphi', D, Y) = 0. \quad (4.34)$$

This completes the proof. ■

5 Estimates

5.1 Norms and Regulators

Unless otherwise stated, $j = 0, 1, \dots, R-1$. Given $X \in \mathcal{P}_j^c$, let $\mathcal{C}_j^2(X)$ be the vector space of the bounded functions $\varphi : X^* \rightarrow \mathbb{C}$. For $n = 0, 1, 2$, define (for ∂^μ the discrete derivative defined after (3.5))

$$\|\nabla_j^n \varphi\|_{L^\infty(X)} := \max_{\mu_1, \dots, \mu_n} \max_{x \in X} L^{nj} |\partial^{\mu_1} \dots \partial^{\mu_n} \varphi_x| \quad (5.1)$$

and the norm $\|\varphi\|_{\mathcal{C}_j^2(X)} := \max_{n=0,1,2} \|\nabla_j^n \varphi\|_{L^\infty(X^*)}$. Also, consider L^2 norms:

$$\begin{aligned} \|\nabla_j^n \varphi\|_{L_j^2(X)}^2 &= L^{-2j} \sum_{x \in X} \sum_{\mu_1, \dots, \mu_n} L^{2nj} |\partial^{\mu_1} \dots \partial^{\mu_n} \varphi_x|^2, \\ \|\nabla_j^n \varphi\|_{L_j^2(\partial X)}^2 &= L^{-j} \sum_{x \in \partial X} \sum_{\mu_1, \dots, \mu_n} L^{2nj} |\partial^{\mu_1} \dots \partial^{\mu_n} \varphi_x|^2. \end{aligned} \quad (5.2)$$

Bounds on expectations of \mathcal{C}_j^2 -norms can be obtained by these L_j^2 -norms.

Let $\mathcal{N}_j(X)$ be the space of the smooth complex activities of the polymer X^* , i.e. the set of C^∞ functions $F(\varphi, X) : \mathcal{C}_j^2(X) \rightarrow \mathbb{C}$. The n -order derivative of F along the directions $f_1, \dots, f_n \in \mathcal{C}_j^2(X)$ is

$$D_\varphi^n F(\varphi, X) \cdot (f_1, \dots, f_n) = \sum_{x_1, \dots, x_n \in X^*} (f_1)_{x_1} \dots (f_n)_{x_n} \frac{\partial^n F}{\partial \varphi_{x_1} \dots \partial \varphi_{x_n}}(\varphi, X).$$

The size of the differential of order n is given by

$$\|F(\varphi, X)\|_{T_j^n(\varphi, X)} = \sup_{\|f_i\|_{\mathcal{C}_j^2(X)} \leq 1} |D_\varphi^n F(\varphi, X) \cdot (f_1, \dots, f_n)|. \quad (5.3)$$

Then, given any $h > 1$, we define the norm

$$\|F(\varphi, X)\|_{h, T_j(\varphi, X)} = \sum_{n \geq 0} \frac{h^n}{n!} \|F(\varphi, X)\|_{T_j^n(\varphi, X)}. \quad (5.4)$$

In order to control the norm of the activities as function of the field φ , for any scale j and any $X \in \mathcal{P}_j^c$ introduce the *field regulators*, $G_j(\varphi, X) \geq 1$, that depends upon derivatives of φ only. Accordingly, define

$$\|F(X)\|_{h, T_j(X)} = \sup_{\varphi \in \mathcal{C}_j^2(X)} \frac{\|F(\varphi, X)\|_{h, T_j(\varphi, X)}}{G_j(\varphi, X)}. \quad (5.5)$$

Finally, we have to weight the polymer activity w.r.t. the size of the set. Given a parameter $A > 1$, define

$$\|F\|_{h, T_j} = \sup_{X \in \mathcal{P}_j^c} A^{|X|_j} \|F(X)\|_{h, T_j(X)}. \quad (5.6)$$

As already announced, in this paper, as opposed to [14], h is independent L and of the scale j (as well as independent of α , s and z). Consider the function $h_j(x) := \Gamma_j(x) - \Gamma_j(0)$: by (3.3), there exists a numerical constant $C_0 > 0$ such that, for any $X \in \mathcal{S}_j$ that contains 0, $\|h_j\|_{\mathcal{C}_j^2(X)} \leq C_0$; then any fixed $h > 1$ that satisfies

$$h \geq 2\alpha C_0 \tag{5.7}$$

will work for our purposes.

A convenient choice for the functions G_j , that guarantees the integrability of the polymer activities at any scale, is inspired by [5]: for X a j -polymer, define

$$W_j(\varphi, X)^2 := \sum_{B \in \mathcal{B}_j(X)} \|\varphi\|_{L^\infty(B^*)}^2 ; \tag{5.8}$$

then, given two positive constants c_1, c_3 , and a positive function of L, κ_L , if $X \in \mathcal{P}_j^c$

$$\ln G_j(\varphi, X) = c_1 \kappa_L \|\nabla_j \varphi\|_{L_j^2(X)}^2 + c_3 \kappa_L \|\nabla_j \varphi\|_{L_j^2(\partial X)}^2 + c_1 \kappa_L W_j(\nabla_j^2 \varphi, X)^2 . \tag{5.9}$$

$\mathcal{N}_j(X)$ with the norm $\|\cdot\|_{h, T_j(X)}$ is a Banach space. For the field dependence of U_j we shall use the *strong field regulator*, G_j^{str} : for $B \in \mathcal{B}_j$ and $X \in \mathcal{P}_j$,

$$\ln G_j^{\text{str}}(\varphi, B) = \kappa_L \max_{n=1,2} \|\nabla_j^n \varphi\|_{L^\infty(B^*)}^2 , \quad G_j^{\text{str}}(\varphi, X) = \prod_{B \in \mathcal{B}_j(X)} G_j^{\text{str}}(\varphi, B) . \tag{5.10}$$

These definitions suits the Coulomb Gas, more than those ones in [5] (Sec. 6.2.4-5.), designed for the Dipole Gas; the reason is the need of a *finer decomposition* of the covariance suggested in [7]; we will discuss this in Sec. D. Anyways the basic structure of the regulators is unchanged, therefore we still refer to [5] in most of the proofs below.

We now list some useful features of the field regulator. As apparent from the definition, if $X \in \mathcal{P}_{j+1}$,

$$G_j^{\text{str}}(\varphi', X) \leq G_{j+1}^{\text{str}}(\varphi', X) . \tag{5.11}$$

For any polymer $X \in \mathcal{P}_j$, the sets Y 's in $\mathcal{C}_j(X)$ have disjoint boundaries, then the norm $L_j^2(\partial Y)$ (besides the norm $L_j^2(Y)$) is additive in Y ; hence

$$\prod_{Y \in \mathcal{C}_j(X)} G_j(\varphi, Y) = G_j(\varphi, X) . \tag{5.12}$$

In the following results, borrowed or inspired by [5], c_3 and c_1 must be large enough, but independent of the scale j and of the parameters L, s and z .

Lemma 5.1 For $X \in \mathcal{P}_j$,

$$G_j^{\text{str}}(\varphi, X) \leq G_j(\varphi, X) . \tag{5.13}$$

For $X \in \mathcal{P}_j$ and B a j -block that is not inside X ,

$$G_j^{\text{str}}(\varphi, B) G_j(\varphi, X) \leq G_j(\varphi, B \cup X) . \tag{5.14}$$

Lemma 5.2 *If $\kappa_L = c(\log L)^{-1}$ with $c > 0$ and small enough:*

a) for $X \in \mathcal{P}_j^c$,

$$\mathbb{E}_j [G_j(\varphi, X)] \leq 2^{|X|_j} G_{j+1}(\varphi', \bar{X}) ; \quad (5.15)$$

b) if $X \in \mathcal{S}_j$, for a $C > 1$,

$$\left(1 + \max_{n=1,2} \|\nabla_{j+1}^n \varphi'\|_{L^\infty(X^*)} \right)^3 \mathbb{E}_j [G_j(\varphi, X)] \leq C \frac{2^{|X|_j}}{\kappa_L^{3/2}} G_{j+1}(\varphi', \bar{X}) . \quad (5.16)$$

Besides (5.16) holds also if $G_j(\varphi, X)$ is replaced by $\sup_{t \in [0,1]} G_j(t\varphi' + \zeta, X)$.

For the proof of Lemma 5.1 see Lemma 6.21 and formula (6.52) in [5]; for the former we need $c_1 > 4$, in the latter we have to assume $c_3 < cc_1$, for a certain geometrical constant c . Lemma 5.2 has analogies with formulas (6.53) and (6.58) in [5], but the proof is a substantial re-shuffling of the one in that paper, and so is given in Sec. D. It means that $\kappa_L = O(1/\ln L)$, as opposed to $\kappa_L = O(1/L^2)$ of [14]: this makes an important difference, because the r.h.s. of (5.16) is then proportional to $O(\ln L)^{3/2}$, and is beaten, for large L , by any power of L .

Lemma 5.2 holds for $j = 0, 1, \dots, R-1$. Let \mathbb{E}_R be the expectation with covariance $\Gamma_{\geq R}(x; m)$, followed by the limit $m \rightarrow 0$.

Lemma 5.3 *If $\kappa_L = c(\log L)^{-1}$ with $c > 0$ and small enough,*

$$\mathbb{E}_R [G_R(\varphi, \Lambda)] \leq 2 . \quad (5.17)$$

We conclude this section with two bounds. For $\lambda \in (0, 1)$ and $\rho = s, l$, set

$$k_\rho(A, \lambda) = \sup_{V \in \mathcal{P}_{j+1}^c} A^{|V|_{j+1}} \sum_{Y \in O_\rho}^{\bar{Y}=V} (\lambda A)^{-|Y|_j} \quad (5.18)$$

where $O_s = \mathcal{S}_j$ and $O_l = \mathcal{S}_j$.

Lemma 5.4 *There exist $c > 0$ and $\eta > 0$ such that, for A large enough*

$$k_s(A, \lambda) \leq cL^2 \quad k_l(A, \lambda) \leq A^{-\eta} \quad (5.19)$$

In brief, when the sum is over small sets the bound is proportional to a volume factor L^2 ; when the sum is over large sets, the bound is finite in L and vanishing for large A . For the proof see Lemma 6.19 and Lemma 6.18 in [5].

5.2 Power Counting Analysis

For the definition set up so far, we can easily derive some fundamental bounds (mostly introduced in [5]), that will be repeatedly used in the rest of the paper. From the definitions one can easily verify the two following facts: first, for any $\varphi \in \mathcal{C}_{j+1}^2(X)$, we have $\|\varphi\|_{\mathcal{C}_j^2(X)} \leq \|\varphi\|_{\mathcal{C}_{j+1}^2(X)}$, so that, for any $F \in \mathcal{N}_j(X)$

$$\|F(\varphi, X)\|_{h, T_{j+1}(\varphi, X)} \leq \|F(\varphi, X)\|_{h, T_j(\varphi, X)} ; \quad (5.20)$$

second, if $Y \subset X$, for any $\varphi \in \mathcal{C}_j^2(X)$ we have $\|\varphi\|_{\mathcal{C}_j^2(Y)} \leq \|\varphi\|_{\mathcal{C}_j^2(X)}$, so that $\mathcal{C}_j^2(X) \subset \mathcal{C}_j^2(Y)$ and

$$\|F(\varphi, X)\|_{h, T_j(\varphi, X)} \leq \|F(\varphi, X)\|_{h, T_j(\varphi, Y)} . \quad (5.21)$$

For any two polymers Y_1, Y_2 not necessarily disjoint and such that $Y_1 \cup Y_2 \subset X$, and any two polymer activities, $F_1 \in \mathcal{N}_j(Y_1)$ and $F_2 \in \mathcal{N}_j(Y_2)$, we have the *triangular inequality*

$$\|F_1(\varphi, Y_1) + F_2(\varphi, Y_2)\|_{h, T_j(\varphi, X)} \leq \|F_1(\varphi, Y_1)\|_{h, T_j(\varphi, Y_1)} + \|F_2(\varphi, Y_2)\|_{h, T_j(\varphi, Y_2)} , \quad (5.22)$$

and the *factorization property*

$$\|F_1(\varphi, Y_1)F_2(\varphi, Y_2)\|_{h, T_j(\varphi, X)} \leq \|F_1(\varphi, Y_1)\|_{h, T_j(\varphi, Y_1)} \|F_2(\varphi, Y_2)\|_{h, T_j(\varphi, Y_2)} . \quad (5.23)$$

(The former is a consequence of (5.21) and of the triangular inequality for norms; the latter follows from differentiation rules.) By (5.20) and (5.15), for any $X \in \mathcal{P}_j^c$

$$\|\mathbb{E}_j [K_j(\varphi, X)]\|_{h, T_{j+1}(\varphi', X)} \leq \|K_j\|_{h, T_j} \left(\frac{A}{2}\right)^{-|X|_j} G_{j+1}(\varphi', \bar{X}) ; \quad (5.24)$$

and similarly, for each charged component

$$\|\mathbb{E}_j [\widehat{K}_j(q, \varphi, X)]\|_{h, T_{j+1}(\varphi', X)} \leq \|K_j\|_{h, T_j} \left(\frac{A}{2}\right)^{-|X|_j} G_{j+1}(\varphi', \bar{X}) . \quad (5.25)$$

We passed from scale $j+1$ to scale j by the bound (5.20) which is of general validity. But, under special circumstances, this step can be done in a more efficient way and the above bounds can be considerably improved.

Theorem 5.5 *If L is large enough, there exists $\vartheta > 0$ such that, for any $X \in \mathcal{S}_j$,*

$$\|\text{Rem}_2 \mathbb{E}_j [\widehat{K}_j(0, \varphi, X)]\|_{h, T_{j+1}(\varphi', X)} \leq L^{-(2+\vartheta)} \|K_j\|_{h, T_j} \left(\frac{A}{2}\right)^{-|X|_j} G_{j+1}(\varphi', \bar{X}) . \quad (5.26)$$

Theorem 5.6 *If L is large enough, $\alpha^2 \geq 8\pi$ and $|q| \geq 1$, then, for any $X \in \mathcal{S}_j$,*

$$\|\mathbb{E}_j [\widehat{K}_j(q, \varphi, X)]\|_{h, T_{j+1}(\varphi', X)} \leq C(\alpha) L^{-2|q|} \|K_j\|_{h, T_j} \left(\frac{A}{2}\right)^{-|X|_j} G_{j+1}(\varphi', \bar{X}) . \quad (5.27)$$

Theorem 5.7 *If L is large enough, $\alpha^2 \geq 8\pi$ and $q = \pm 1$, there exists $\vartheta > 0$ such that, for any $X \in \mathcal{S}_j$,*

$$\|\text{Rem}_0 \mathbb{E}_j \left[\widehat{K}_j(q, \varphi, X) \right] \|_{h, T_{j+1}(\varphi', X)} \leq L^{-(2+\vartheta)} \|K_j\|_{h, T_j} \left(\frac{A}{2} \right)^{-|X|_j} G_{j+1}(\varphi', \overline{X}) \quad (5.28)$$

These results make rigorous power counting arguments mentioned below Lemma 4.1. Theorem 5.6 is borrowed from [14] and means that the charged terms - on small sets - are *irrelevant* as soon as $L^{-2|q|}$ beats the volume factor L^2 , i.e. for $|q| \geq 2$. Theorem 5.5 is taken from [5] and means that the second order Taylor remainder - again, on small sets only - is an *irrelevant term* because the factor $L^{-3}(\ln L)^{3/2}$ on the r.h.s. member beats the volume factor. Theorem 5.7 is a fusion of the first two: it was missing in [14] and prevented that paper to cover the case of the Kosterlitz-Thouless line. It means that a charged component with $|q| = 1$ is irrelevant if the zeroth order Taylor term has been taken away. Although we do not need any essential change respect to the formulations in [14] (even in the case of Theorem 5.7, once the L -independence of h , found in this paper, is taken into account) we reviewed the proofs of these fundamental results in Sec. E.

6 Smoothness of RG Map

6.1 Proof of Lemma 3.3 and 3.2

Given $X \in \mathcal{S}_j$ and $x_0 \in X$, set $f_x^\mu := (x - x_0)^\mu$.

$$|\mathcal{F}_j(K_j)| \leq C \sum_{X \in \mathcal{S}_j} \frac{L^{-2j}}{|X|_j} \|\mathbb{E}_j \left[\widehat{K}_j(0, \zeta, X) \right] \|_{h, T_{j+1}(\varphi', X)} \sum_{\mu \in \widehat{e}} \|f^\mu\|_{\mathcal{C}_j^2(X)}^2; \quad (6.1)$$

as $\|f^\mu\|_{\mathcal{C}_j^2(X)} \leq CL^j$ and by (5.25), (6.1) is bounded by $CA^{-1}\|K_j\|_{h, T_j}$.

$$|\mathcal{M}_j(K_j)| \leq \sum_{X \in \mathcal{S}_j} \frac{e^{\frac{\alpha^2}{2}\Gamma_j(0)}}{|X|_j} \sum_{\sigma=\pm 1} \|\mathbb{E}_j \left[\widehat{K}_j(\varepsilon, \zeta, X) \right] \|_{h, T_{j+1}(\varphi', X)}, \quad (6.2)$$

which, by (3.4) and (5.27), is bounded by $C(\alpha)A^{-1}\|K_j\|_{h, T_j}$. Lemma 3.3 is proven.

By (3.2), (3.3), (3.4) and as $\partial^\mu \Gamma_j(0) \equiv 0$, for $|y| \leq C(L)L^j$,

$$\begin{aligned} |e^{-\alpha^2 \Gamma_j(0|y)} - 1 - \frac{\alpha^2}{2} \sum_{\mu, \nu \in \widehat{e}} (\partial^\mu \partial^\nu \Gamma_j)(0) y^\mu y^\nu| &\leq C(L)L^{-3j}|y|^3, \\ e^{-\alpha^2 \Gamma_j(0)} |e^{\pm \alpha^2 \Gamma_j(y)} - 1| &\leq C(L, \alpha) |\Gamma_j(y)|, \quad |e^{-\alpha^2 \Gamma_j(0|y)} - 1| \leq C(L, \alpha) |y| L^{-j}, \\ |(\partial^\mu \partial^\nu \Gamma_j)(0|y)| &\leq C(L) |y| L^{-3j}, \end{aligned} \quad (6.3)$$

so that (using also the bounds (C.9), (C.10))

$$|\widehat{e}_{3,j}| \leq C(L), \quad |\widehat{e}_{2,j}| \leq C, \quad |\widehat{e}_{4,j}| \leq C(L, \alpha), \quad (6.4)$$

$$|a_j| \leq C(L, \alpha), \quad |b_j| \leq C(L, \alpha). \quad (6.5)$$

Then

$$|\widehat{e}_{1,j}(K_j)| \leq \sum_{X \in \mathcal{S}_j} \frac{1}{|X|_j} \|\mathbb{E}_j [\widehat{K}_j(0, \zeta, X)]\|_{h, T_{j+1}(\varphi', X)}, \quad (6.6)$$

which, by (5.25), is bounded by $CA^{-1}\|K_j\|_{h, T_j}$. Finally, a bound for \mathcal{E}_R is

$$|\mathcal{E}_R| \leq \frac{1}{|\Lambda|} \ln \mathbb{E}_R [1 + \|e^{U_R(\zeta, \Lambda)} - 1\|_{h, T_R(\zeta, \Lambda)} + \|K_R(\zeta, \Lambda)\|_{h, T_R(\zeta, \Lambda)}] ;$$

therefore (3.17) with $j = R$ follows from (5.17). Proof of Lemma 3.2 is complete.

6.2 Proof of Theorem 3.4

Taking into account cancellation (4.19),

$$(\mathcal{L}_j K_j)(\varphi', V) = \sum_{p=1}^5 (\mathcal{L}_j^{(p)} K_j)(\varphi', V)$$

for (Taylor expansions are in $\delta\varphi'$)

$$(\mathcal{L}_j^{(1)} K_j)(\varphi', V) := \sum_{Y \in \mathcal{S}_j(V)}^{\bar{Y}=V} \mathbb{E}_j [K_j(\varphi, Y)] \quad (6.7)$$

$$(\mathcal{L}_j^{(2)} K_j)(\varphi', V) := \sum_{q:|q|\geq 2} \sum_{Y \in \mathcal{S}_j(V)}^{\bar{Y}=V} \mathbb{E}_j [\widehat{K}_j(q, \varphi, Y)] \quad (6.8)$$

$$(\mathcal{L}_j^{(3)} K_j)(\varphi', V) := \sum_{Y \in \mathcal{S}_j(V)}^{\bar{Y}=V} \text{Rem}_2 \mathbb{E}_j [\widehat{K}_j(0, \varphi, Y)] \quad (6.9)$$

$$(\mathcal{L}_j^{(4)} K_j)(\varphi', V) := \sum_{q=\pm 1} \sum_{Y \in \mathcal{S}_j(V)}^{\bar{Y}=V} \text{Rem}_0 \mathbb{E}_j [\widehat{K}_j(q, \varphi, Y)] \quad (6.10)$$

$$\begin{aligned} (\mathcal{L}_j^{(5)} K_j)(\varphi', V) := & - \sum_{B \in \mathcal{B}_j(V)}^{\bar{B}=V} \left[V_{j+1}(\bar{s}_j, \bar{z}_j, \varphi', B) - \mathbb{E}_j [V_j(\varphi, B)] \right. \\ & \left. + \bar{\mathcal{E}}_j |B| - \sum_{X \in \mathcal{S}_j}^{X \supset B} \frac{\bar{Q}(\varphi', X)}{|X|_j} \right] \end{aligned} \quad (6.11)$$

Consider each term separately: we want to prove that

$$\|\mathcal{L}_j^{(p)} K_j\|_{h, T_{j+1}} \leq \rho(A, L) \|K_j\|_{h, T_j} \quad (6.12)$$

for $p = 1, \dots, 5$; the constant $\rho(A, L)$ is to be small enough if L and A are large enough, uniformly in the scale j .

6.2.1 Norm of $\mathcal{L}^{(1)}$

Use (5.22), (5.24) and definition (5.18) to find

$$\begin{aligned} \|(\mathcal{L}_j^{(1)} K_j)(\varphi', V)\|_{h, T_{j+1}(\varphi', V)} &\leq \sum_{Y \in \mathcal{S}_j}^{\bar{Y}=V} \|\mathbb{E}_j [K_j(\varphi, Y)]\|_{h, T_{j+1}(\varphi', Y)} \\ &\leq G_{j+1}(\varphi', V) \|K_j\|_{h, T_j} \sum_{Y \in \mathcal{S}_j}^{\bar{Y}=V} A^{-|Y|_j} 2^{|Y|_j} \\ &\leq G_{j+1}(\varphi', V) A^{-|V|_{j+1}} \|K_j\|_{h, T_j} k_l(A, 1/2); \end{aligned} \quad (6.13)$$

by (5.19), we find (6.12) for $\rho(A, L) \geq A^{-\eta}$.

6.2.2 Norm of $\mathcal{L}^{(2)}$

Use (5.27) and definition (5.18) to find

$$\begin{aligned} \|(\mathcal{L}_j^{(2)} K_j)(\varphi', V)\|_{h, T_{j+1}(\varphi', V)} &\leq \sum_{q: |q| \geq 2} \sum_{Y \in \mathcal{S}_j(V)}^{\bar{Y}=V} \|\mathbb{E}_j [\widehat{K}_j(q, \varphi, Y)]\|_{h, T_{j+1}(\varphi', Y)} \\ &\leq C(\alpha) G_{j+1}(\varphi', V) A^{-|V|_{j+1}} \|K_j\|_{h, T_j} k_s(A, 1/2) \sum_{q: |q| \geq 2} L^{-2|q|}; \end{aligned} \quad (6.14)$$

by (5.19) we obtain (6.12) for $\rho(A, L) \geq C(\alpha)L^{-2}$.

6.2.3 Norm of $\mathcal{L}^{(3)}$

By (5.26)

$$\begin{aligned} \|(\mathcal{L}_j^{(3)} K_j)(\varphi', V)\|_{h, T_{j+1}(\varphi', V)} &\leq \sum_{Y \in \mathcal{S}_j(V)}^{\bar{Y}=V} \|\text{Rem}_2 \mathbb{E}_j [\widehat{K}_j(0, \varphi, Y)]\|_{h, T_{j+1}(\varphi', Y)} \\ &\leq G_{j+1}(\varphi', V) A^{-|V|_{j+1}} \|K_j\|_{h, T_j} L^{-(2+\vartheta)} k_s(A, 1/2); \end{aligned} \quad (6.15)$$

by (5.19) we obtain (6.12) for $\rho(A, L) \geq CL^{-\vartheta}$.

6.2.4 Norm of $\mathcal{L}^{(4)}$

By (5.28),

$$\begin{aligned} \|(\mathcal{L}_j^{(4)} K_j)(\varphi', V)\|_{h, T_{j+1}(\varphi', V)} &\leq \sum_{q=\pm 1} \sum_{Y \in \mathcal{S}_j(V)}^{\bar{Y}=V} \|\text{Rem}_0 \mathbb{E}_j \left[\widehat{K}_j(q, \varphi, Y) \right]\|_{h, T_{j+1}(\varphi', Y)} \\ &\leq G_{j+1}(\varphi', V) A^{-|V|_{j+1}} \|K_j\|_{h, T_j} L^{-(2+\vartheta)} k_s(A, 1/2). \end{aligned} \quad (6.16)$$

by (5.19) we obtain (6.12) for $\rho(A, L) \geq CL^{-\vartheta}$.

6.2.5 Norm of $\mathcal{L}^{(5)}$

Here we prove that (4.17) is irrelevant as claimed before (4.17). Further decompose this term as

$$(\mathcal{L}_j^{(5)} K_j)(\varphi', V) := (\mathcal{L}_j^{(5,a)} K_j)(\varphi', V) + (\mathcal{L}_j^{(5,b)} K_j)(\varphi', V),$$

where

$$\begin{aligned} (\mathcal{L}_j^{(5,a)} K_j)(\varphi', V) &= \sum_{B \in \mathcal{B}_j(V)}^{\bar{B}=V} \left[\sum_{X \in \mathcal{S}_j}^{X \supset B} \sum_{x_0 \in X} \frac{\text{Tay}_2 \mathbb{E}_j \left[\widehat{K}_j(0, \zeta, X) \right]}{|X| |X|_j} \right. \\ &\quad \left. - V_{j+1}(\mathcal{F}_j(K_j), 0, \varphi', B) - \widehat{e}_{1,j}(K_j) \right] \\ (\mathcal{L}_j^{(5,b)} K_j)(\varphi', V) &= \sum_{B \in \mathcal{B}_j(V)}^{\bar{B}=V} \left[\sum_{X \in \mathcal{S}_j}^{X \supset B} \sum_{q=\pm} \sum_{x_0 \in X} \frac{e^{iq\alpha\varphi'_{x_0}} \mathbb{E}_j \left[\widehat{K}_j(q, \zeta, X) \right]}{|X| |X|_j} \right. \\ &\quad \left. - V_{j+1}(0, L^2 e^{-\frac{\alpha^2}{2} \Gamma_j(0)} \mathcal{M}_j(K_j), \varphi', B) \right] \end{aligned} \quad (6.17)$$

Consider $\mathcal{L}_j^{(5,a)}$. The term proportional to $\widehat{e}_{1,j}$ exactly cancels the one proportional to the zero order of $\text{Tay}_2 \mathbb{E}_j \left[\widehat{K}_j(0, \varphi, X) \right]$. We are left with the difference between $V_{j+1}(\mathcal{F}_j(K_j), 0, \varphi', B)$ and the 2-nd order of $\text{Tay}_2 \mathbb{E}_j \left[\widehat{K}_j(0, \varphi, X) \right]$ (the first being zero by symmetry), that, by invariance under space translations of the problem, yields

$$(\mathcal{L}_j^{(5,a)} K_j)(\varphi', V) = \sum_{B \in \mathcal{B}_j(V)}^{\bar{B}=V} \sum_{X \in \mathcal{S}_j}^{X \supset B} \sum_{\substack{x_0 \in X, \\ x_1, x_2 \in X^*}} \frac{R_{x_1, x_2}^{x_0}(\varphi')}{2|X| |X|_j} \mathbb{E}_j \left[\frac{\partial^2 \widehat{K}_j}{\partial \varphi_{x_1} \partial \varphi_{x_2}}(0, \zeta, X) \right]$$

where $R_{x_1, x_2}^{x_0}(\varphi')$ is given by

$$(\varphi'_{x_2} - \varphi'_{x_0})(\varphi'_{x_1} - \varphi'_{x_0}) - L^{-2j} \sum_{x \in B} \sum_{\mu, \nu \in \widehat{e}} (\partial^\mu \varphi')_x (\partial^\nu \varphi')_x (x_2 - x_0)^\mu (x_1 - x_0)^\nu$$

$$= L^{-2j} \sum_{x \in B} \left[u_{x_1, x_0}(x, \varphi') u_{x_2, x_0}(x, \varphi') + v_{x_1, x_0}(x, \varphi') u_{x_2, x_0}(x, \varphi') \right. \\ \left. + u_{x_1, x_0}(x, \varphi') v_{x_2, x_0}(x, \varphi') \right] \quad (6.18)$$

and

$$u_{x_1, x_0}(x, \varphi) := \varphi_{x_1} - \varphi_{x_0} - \sum_{\mu \in \hat{e}} (\partial^\mu \varphi)_x (x_1 - x_0)^\mu \\ v_{x_1, x_0}(x, \varphi) := \sum_{\mu \in \hat{e}} (\partial^\mu \varphi)_x (x_1 - x_0)^\mu. \quad (6.19)$$

Notice that $|x_0 - x_1|, |x_1 - x_2| \leq CL^j$; it implies

$$\|u_{x_1, x_0}(x, \varphi')\|_{h, T_{j+1}(\varphi, X)} \leq CL^{-2} (1 + \max_{n=1,2} \|\nabla_{j+1}^n \varphi'\|_{L^\infty(X^*)}) \\ \|v_{x_1, x_0}(x, \varphi')\|_{h, T_{j+1}(\varphi, X)} \leq CL^{-1} (1 + \max_{n=1,2} \|\nabla_{j+1}^n \varphi'\|_{L^\infty(X^*)}) \quad (6.20)$$

and then a bound for $\|(\mathcal{L}_j^{(5,a)} K_j)(\varphi', V)\|_{h, T_{j+1}(\varphi', V)}$ is

$$C \|K_j\|_{h, T_j} \sum_{B \in \mathcal{B}_j(V)} \sum_{X \supset B} \left(\frac{A}{2}\right)^{-|X|_j} \sup_{x_0, x_1, x_2} \|R_{x_0, x_1, x_2}^{x_0}(\varphi')\|_{h, T_{j+1}(\varphi', V)} \\ \leq C' \|K_j\|_{h, T_j} A^{-|V|_{j+1}} L^{-1} \left(1 + \max_{n=1,2} \|\nabla_{j+1}^n \varphi'\|_{L^\infty(V^*)}^2\right) \\ \leq C'' \kappa_L^{-1} L^{-1} \|K_j\|_{h, T_j} G_{j+1}^{\text{str}}(\varphi', V) A^{-|V|_{j+1}} \quad (6.21)$$

(the sums over B and X have no more than CL^2 terms, in all of which $|V|_{j+1} = 1$). $G_{j+1}^{\text{str}}(\varphi', V)$ can be replaced by $G_{j+1}(\varphi', V)$ by (5.13). Next consider $\mathcal{L}_j^{(5,b)}$. Simple computations and symmetry $\widehat{K}_j(-1, \zeta, X) = \widehat{K}_j(1, \zeta, X)$, give

$$(\mathcal{L}_j^{(5,b)} K_j)(\varphi', V) = \sum_{B \in \mathcal{B}_j(V)} \sum_{\substack{x_0 \in B \\ q = \pm 1}} L^{-2j} e^{iq\alpha\varphi'_{x_0}} \\ \cdot \sum_{X \supset B} \sum_{y_0 \in X} \frac{(e^{iq\alpha(\varphi'_{y_0} - \varphi'_{x_0})} - 1)}{2|X||X|_j} \sum_{\sigma = \pm 1} \mathbb{E}_j \left[\widehat{K}_j(\varepsilon, \zeta, X) \right]. \quad (6.22)$$

Notice that by construction $|x_0 - y_0| \leq CL^j$ so that, by (5.13),

$$\|e^{iq\alpha(\varphi'_{y_0} - \varphi'_{x_0})} - 1\|_{h, T_{j+1}(\varphi', X)} \leq C(\alpha) L^{-1} (1 + \max_{n=1,2} \|\nabla_{j+1}^n \varphi'\|_{L^\infty(X^*)}) \\ \leq C'(\alpha) \kappa_L^{-1} L^{-1} G_{j+1}(\varphi', V) \quad (6.23)$$

and then, using (5.27), a bound for $\|(\mathcal{L}_j^{(5,b)} K_j)(\varphi', V)\|_{h, T_{j+1}(\varphi', V)}$ is

$$\begin{aligned} C(\alpha)\kappa_L^{-1}L^{-1}G_{j+1}(\varphi', V) \sum_{\sigma=\pm} \sum_{B \in \mathcal{B}_j(V)} \sum_{X \supset B} \frac{1}{|X|_j} \|\mathbb{E}_j \left[\widehat{K}_j(\varepsilon, \zeta, X) \right]\|_{h, T_{j+1}(\varphi', V)} \\ \leq C'(\alpha)\kappa_L^{-1}L^{-1}G_{j+1}(\varphi', V)A^{-|V|_{j+1}}\|K_j\|_{h, T_j} . \end{aligned} \quad (6.24)$$

Again we obtain (6.12) for $\rho(A, L) \geq C(\alpha)\kappa_L^{-1}L^{-1}$.

6.3 Preparation Bounds

We begin with bound (6.74) of [5].

Lemma 6.1 *There exists $\eta > 0$ such that, for any $X \in P_j$,*

$$(1 + 2\eta)|\overline{X}|_{j+1} \leq |X|_j + 8(1 + 2\eta)|\mathcal{C}_j(X)| . \quad (6.25)$$

In the next section we shall need bounds on the ‘‘building blocks’’ of the RG map. Let us introduce a vector notation and a norm with weight $\mu \geq 0$,

$$\mathcal{K}_j := (s_j, z_j, K_j) , \quad |\mathcal{K}_j|_{\mu, j} := \mu^{-1}|s_j| + \mu^{-1}|z_j| + \mu^{-2}\|K_j\|_{h, T_j} . \quad (6.26)$$

Lemma 6.2 *There exist $\eta > 0$, $C \equiv C(A, L, \alpha) > 1$ such that, for ε_0 small enough, any \mathcal{K}_j and $\dot{\mathcal{K}}_j$ with $|\mathcal{K}_j|_{1, j}, |\dot{\mathcal{K}}_j|_{1, j} \leq \varepsilon_0$ and $D \in \mathcal{B}_{j+1}$ or $Y \in \mathcal{P}_{j+1}^c$*

$$\|e^{U_j(\varphi, D)} - e^{\dot{U}_j(\varphi, D)}\|_{h, T_j(\varphi', D)} \leq C\varepsilon_0|\mathcal{K}_j - \dot{\mathcal{K}}_j|_{\varepsilon_0, j}G_j^{\text{str}}(\varphi, D) , \quad (6.27)$$

$$\|e^{U_{j+1}(\varphi', D)} - e^{\dot{U}_{j+1}(\varphi', D)}\|_{h, T_j(\varphi', D)} \leq C\varepsilon_0|\mathcal{K}_j - \dot{\mathcal{K}}_j|_{\varepsilon_0, j}G_{j+1}^{\text{str}}(\varphi', D) , \quad (6.28)$$

$$\begin{aligned} \|P_j(\varphi', \zeta, D) - \dot{P}_j(\varphi', \zeta, D)\|_{h, T_j(\varphi', D)} \\ \leq C\varepsilon_0|\mathcal{K}_j - \dot{\mathcal{K}}_j|_{\varepsilon_0, j}A^{-(1+\eta)} [G_{j+1}^{\text{str}}(\varphi', D) + G_j^{\text{str}}(\varphi, D)] , \end{aligned} \quad (6.29)$$

$$\begin{aligned} \|J_j(\varphi', D, Y) - \dot{J}_j(\varphi', D, Y)\|_{h, T_j(\varphi', Y)} \\ \leq C\varepsilon_0^2|\mathcal{K}_j - \dot{\mathcal{K}}_j|_{\varepsilon_0, j}A^{-(1+\eta)|D^*|_{j+1}}G_{j+1}^{\text{str}}(\varphi', D) , \end{aligned} \quad (6.30)$$

$$\begin{aligned} \|R_j(\varphi', \zeta, Y) - \dot{R}_j(\varphi', \zeta, Y)\|_{h, T_j(\varphi', Y)} \\ \leq C\varepsilon_0^2|\mathcal{K}_j - \dot{\mathcal{K}}_j|_{\varepsilon_0, j}A^{-(1+\eta)|Y|_{j+1}} [G_{j+1}^{\text{str}}(\varphi', Y) + G_j(\varphi, Y)] , \end{aligned} \quad (6.31)$$

where $\dot{U}_j, \dot{U}_{j+1}, \dot{P}_j, \dot{R}_j$ and \dot{J}_j contains dotted parameters $\dot{\mathcal{K}}_j = (\dot{s}_j, \dot{z}_j, \dot{K}_j)$.

Remark. Although the $T_{j+1}(\varphi', Y)$ norm requires a decay in the size of Y as $A^{-|Y|_{j+1}}$, in the r.h.s. of (6.29), (6.30) and (6.31) the decay is (basically) $A^{-(1+\eta)|Y|_{j+1}}$; this is because the extra factor $A^{-\eta|Y|_{j+1}}$ is needed to control the sum in (4.32) (cfr. Sec. 6.4). In (6.29) and (6.30), $|Y|_{j+1}$ is bounded and a $A^{-\eta|Y|_{j+1}}$ factor can be extracted from $C(A, L, \alpha)$; in (6.31), instead, we shall need Lemma 6.1 to have such an improvement.

Proof. Let $B \in \mathcal{B}_j(D)$ and use (5.22) and (5.23) to obtain

$$\begin{aligned} \|V_j(\varphi, B) - \dot{V}_j(\varphi, B)\|_{h, T_j(\varphi', B)} &\leq |s_j - \dot{s}_j| (h + \|\nabla_j \varphi\|_{L^\infty(B)})^2 + 2|z_j - \dot{z}_j| e^{\alpha h} \\ &\leq C(\alpha) \left[|s_j - \dot{s}_j| + |z_j - \dot{z}_j| \right] \left(1 + \max_{n=1,2} \|\nabla_j^n \varphi\|_{L^\infty(B^*)}^2 \right). \end{aligned} \quad (6.32)$$

A similar bound holds for W_j that, by support property of the propagator, depends on fields at B^* (more details are in Appendix C.3):

$$\begin{aligned} \|W_j(\varphi, B) - \dot{W}_j(\varphi, B)\|_{h, T_j(\varphi', B)} \\ \leq C(\alpha) \left[|s_j - \dot{s}_j| + |z_j - \dot{z}_j| \right] A_*^{-1} \left(1 + \max_{n=1,2} \|\nabla_j^n \varphi\|_{L^\infty(B^*)}^2 \right). \end{aligned} \quad (6.33)$$

From the definition of G_j^{str}

$$\sum_{B \in \mathcal{B}_j(D)} \left(1 + \max_{n=1,2} \|\nabla_j^n \varphi\|_{L^\infty(B^*)}^2 \right) \leq C(L) G_j^{\text{str}}(\varphi, D)^{\frac{1}{3}}; \quad (6.34)$$

and for ε_0 smaller than a $\varepsilon_0(L, \alpha)$,

$$\prod_{B \in \mathcal{B}_j(D)} \exp \left\{ C(\alpha) \varepsilon_0 \left(1 + \max_{n=1,2} \|\nabla_j^n \varphi\|_{L^\infty(B^*)}^2 \right) \right\} \leq 2G_j^{\text{str}}(\varphi, D)^{\frac{1}{3}}. \quad (6.35)$$

Then (6.27) follows by factorization property (5.14) and these inequalities. (6.28) is obtained by repeating (6.32) and (6.33) on scale $j+1$; and by taking into account that, by (3.16), (3.14) and the bounds in Sec. 6.1, if A is large enough,

$$\max \left\{ |\mathcal{E}_j - \dot{\mathcal{E}}_j| L^{2j+2}, |s_{j+1} - \dot{s}_{j+1}|, |z_{j+1} - \dot{z}_{j+1}| \right\} \leq C(L, \alpha) |\mathcal{K}_j - \dot{\mathcal{K}}_j|_{1,j}. \quad (6.36)$$

Accordingly, for ε_0 smaller than a $\varepsilon_0(L, \alpha)$,

$$\begin{aligned} \|e^{U_{j+1}(\varphi', D) + \mathcal{E}_j |D|} - e^{\dot{U}_{j+1}(\varphi', D) + \dot{\mathcal{E}}_j |D|}\|_{h, T_j(\varphi', D)} \\ \leq C(L, \alpha) |\mathcal{K}_j - \dot{\mathcal{K}}_j|_{1,j} G_j^{\text{str}}(\varphi', D), \end{aligned} \quad (6.37)$$

which, together to (6.27), proves (6.29). Next, consider (6.30). By (4.14), for any $X \in \mathcal{S}_j$ that contains a given $B \in \mathcal{B}_j(D)$ (if $L > 8$ then $X^* \subset D^*$)

$$\|\bar{Q}_j(\varphi', X) - \dot{\bar{Q}}_j(\varphi', X)\|_{h, T_j(\varphi', X)} \leq C(\alpha) A^{-|X|_j} \|K - \dot{K}\|_{h, T_j} G_{j+1}^{\text{str}}(\varphi', D); \quad (6.38)$$

then, as $|D^*| \leq 4S$, replace $C(\alpha)A^{-|X|_j}$ with $C(A, \alpha)A^{-(1+\eta)|D^*|_{j+1}}$. Besides, for any $Y \in \mathcal{S}_{j+1}$ that contains a given block $D \in \mathcal{B}_{j+1}$, by the second order computation in in Appendix C.2,

$$\begin{aligned} & \|Q_j(\varphi', Y) - \dot{Q}_j(\varphi', Y)\|_{h, T_j(\varphi', Y)} \\ & \leq C(L, \alpha)\varepsilon_0 \left[|s_j - \dot{s}_j| + |z_j - \dot{z}_j| \right] G_{j+1}^{\text{str}}(\varphi', D); \end{aligned} \quad (6.39)$$

again we replace $C(L, \alpha)$ with $C(A, L, \alpha)A^{-(1+\eta)|D^*|_{j+1}}$ as $|D^*|_{j+1} \leq 4S$. Together, (6.38) and (6.39) prove (6.30). Bounds so far have been straightforward consequences of the properties of the norms. As announced, bound (6.31) is slightly more sophisticated because requires Lemma 6.1.

$$\begin{aligned} \tilde{K}_j(\varphi, Y) - \check{K}_j(\varphi, Y) &= \sum_{X \in \mathcal{P}_j^c}^{\bar{X}=Y} \left[U_j(\varphi, Y \setminus X) - \dot{U}_j(\varphi, Y \setminus X) \right] \prod_{Z \in \mathcal{C}_j(X)} K_j(\varphi, Z) \\ & \quad \cdot \int_0^1 dt \left[e^{tU_j(\varphi, Y \setminus X) + (1-t)\dot{U}_j(\varphi, Y \setminus X)} \right] \\ & + \sum_{X \in \mathcal{P}_j^c}^{\bar{X}=Y} e^{\dot{U}_j(\varphi, Y \setminus X)} \sum_{X_0 \in ((X))_j} \prod_{Z \in \mathcal{C}_j(X_0)} \left[K_j(\varphi, Z) - \dot{K}_j(\varphi, Z) \right] \prod_{Z \in \mathcal{C}_j(X \setminus X_0)} \dot{K}_j(\varphi, Z) \end{aligned}$$

where $((X))_j$ is the family of sets made of unions of connected parts of X , empty set excluded. Therefore, using the factorization property (5.14) and previous bounds

$$\begin{aligned} & \|\tilde{K}_j(\varphi, Y) - \check{K}_j(\varphi, Y)\|_{h, T_j(\varphi, Y)} \\ & \leq \left[|s_j - \dot{s}_j| + |z_j - \dot{z}_j| \right] G_j(\varphi, Y) \sum_{X \in \mathcal{P}_j^c}^{\bar{X}=Y} C(\alpha)^{|Y \setminus X|_j} A^{-|X|_j} \varepsilon_0^{|\mathcal{C}_j(X)|} \\ & \quad + \varepsilon_0^{-1} \|K_j - \dot{K}_j\|_{h, T_j} G_j(\varphi, Y) \sum_{X \in \mathcal{P}_j^c}^{\bar{X}=Y} 2^{|Y \setminus X|_j} 4^{|\mathcal{C}_j(X)|} A^{-|X|_j} \varepsilon_0^{|\mathcal{C}_j(X)|} \\ & \leq \varepsilon_0 |\mathcal{K}_j - \dot{\mathcal{K}}_{j, \varepsilon_0, j}| G_j(\varphi, Y) C'(\alpha)^{|Y|_j} \sum_{X \in \mathcal{P}_j^c}^{\bar{X}=Y} A^{-|X|_j} (4\varepsilon_0)^{|\mathcal{C}_j(X)|} \end{aligned} \quad (6.40)$$

In order to extract the factor $A^{-(1+\eta)|Y|_{j+1}}$ and, at the same time, to control the sum over X , which is made of no more than $2^{|Y|_j} = 2^{L^2|Y|_{j+1}}$ terms, use (6.25) and obtain

$$A^{-|X|_j} A^{-8(1+2\eta)|\mathcal{C}_j(X)|} \leq A^{-(1+2\eta)|Y|_{j+1}}. \quad (6.41)$$

Therefore, for $4A^{8(1+2\eta)}\varepsilon_0 < 1$, bound (6.40) is smaller than

$$A^{-(1+\eta)|Y|_{j+1}} \varepsilon_0 |\mathcal{K}_j - \dot{\mathcal{K}}_{j, \varepsilon_0, j}| G_j(\varphi, Y) \left(C(\alpha)^{L^2} A^{-\eta} \right)^{|Y|_{j+1}} 4A^{8(1+2\eta)} \varepsilon_0;$$

finally, for A large enough, $C(\alpha)^{L^2} A^{-\eta} \leq 1$ and choosing $C(A, \alpha, L) \geq 4A^{8(1+2\eta)}$,

$$\begin{aligned} & \|\tilde{K}_j(\varphi, Y) - \check{K}_j(\varphi, Y)\|_{h, T_j(\varphi', Y)} \\ & \leq C(A, L, \alpha) A^{-(1+\eta)|Y|_{j+1}} \varepsilon_0^2 |\mathcal{K}_j - \dot{\mathcal{K}}_j|_{\varepsilon_0, j} G_j(\varphi, Y). \end{aligned} \quad (6.42)$$

To conclude, (6.31) follows from (6.30) and (6.42). This completes the proof. ■

6.4 Proof of Theorem 3.5

We need an explicit formula for $\mathcal{R}_j(s_j, z_j, K_j)$. Expanding (4.32) we obtain:

$$\begin{aligned} K_{j+1}(\varphi', Y') &= \mathbb{E}_j \left[P_j^{Y'}(\varphi', \zeta) \right] + \left(e^{-\varepsilon_j |Y'|} - 1 \right) \mathbb{E}_j \left[P_j^{Y'}(\varphi', \zeta) \right] \\ &+ \sum_{\substack{\rightarrow Y' \\ c_{j+1}(X_0 \cup X_1) \geq 1}} \mathbb{E}_j \left[P_j^Z(\varphi', \zeta) R_j^{X_1}(\varphi', \zeta) \right] J_j^{X_0, (D)}(\varphi') \\ &+ \sum_{\substack{\rightarrow Y' \\ c_{j+1}(X_0 \cup X_1) \geq 1}} \left(e^{-\varepsilon_j |W| + U_{j+1}(\varphi', W \setminus Y')} - 1 \right) \\ &\cdot \mathbb{E}_j \left[P_j^Z(\varphi', \zeta) R_j^{X_1}(\varphi', \zeta) \right] J_j^{X_0, (D)}(\varphi'). \end{aligned} \quad (6.43)$$

Next, we have to remove the part $\mathcal{L}_j K_j$: to this purpose, further expand the first term in the r.h.s. of (6.43)

$$\begin{aligned} & \sum_{D \in \mathcal{B}_{j+1}}^{D=Y'} \mathbb{E}_j [P_j(\varphi', \zeta, D)] + \frac{1}{2} \sum_{\substack{D_0 \cup D_1 = Y' \\ D_0, D_1 \in \mathcal{B}_{j+1} \\ D_0 \neq D_1}} \mathbb{E}_j [P_j(\varphi', \zeta, D_0) P_j(\varphi', \zeta, D_1)] \\ &+ \sum_{\substack{Z=Y' \\ Z \in \mathcal{P}_{j+1} \\ |Z|_{j+1} \geq 3}} \mathbb{E}_j [P_j^Z(\varphi', \zeta)] ; \end{aligned} \quad (6.44)$$

and also further expand the second line of the r.h.s. member of (6.43)

$$\begin{aligned} & \mathbb{E}_j [R_j(\varphi', \zeta, Y')] + \sum_{Y \in \mathcal{S}_{j+1}} \sum_{D \in \mathcal{B}_{j+1}(Y)}^{D=Y'} J_j(\varphi', D, Y) \\ &+ \sum_{\substack{\rightarrow Y' \\ c_{j+1}(X_0 \cup X_1) \geq 1 \\ |Z|_{j+1} + c_{j+1}(X_0 \cup X_1) \geq 2}} \mathbb{E}_j [P_j^Z(\varphi', \zeta) R_j^{X_1}(\varphi', \zeta)] J_j^{X_0, (D)}(\varphi'). \end{aligned} \quad (6.45)$$

Grouping together the above decompositions and definition (4.12), we obtain

$$\mathcal{R}_j(\varphi', Y') := K_{j+1}(\varphi', Y') - (\mathcal{L}_j K_j)(\varphi', Y') = \sum_{s=1}^6 \mathcal{R}_j^{(s)}(\varphi', Y') \quad (6.46)$$

where the eight summands are

$$\begin{aligned}
\mathcal{R}_j^{(1)}(\varphi', Y') &:= \sum_{D \in \mathcal{B}_{j+1}}^{D=Y'} \left[\mathbb{E}_j [P_j(\varphi', \zeta, D)] + \mathcal{E}_j |D| + U_{j+1}(\varphi', D) \right. \\
&\quad \left. - \mathbb{E}_j [U_j(\varphi, D)] - \frac{1}{2} \mathbb{E}_j^T [V_j(\varphi, D); V_j(\varphi, D)] \right] \\
\mathcal{R}_j^{(2)}(\varphi', Y') &:= \sum_{D \in \mathcal{B}_{j+1}}^{D=Y'} \left[\mathbb{E}_j [W_j(s_{j+1}, z_{j+1}, \varphi, D)] - \mathbb{E}_j [W_j(s_j, z_j, \varphi, D)] \right] \\
\mathcal{R}_j^{(3)}(\varphi', Y') &:= \frac{1}{2} \sum_{\substack{D_0 \cup D_1 = Y' \\ D_0, D_1 \in \mathcal{B}_{j+1} \\ D_0 \neq D_1}} \left[\mathbb{E}_j [P_j(\varphi', \zeta, D_0) P_j(\varphi', \zeta, D_1)] \right. \\
&\quad \left. - \mathbb{E}_j^T [V_j(\varphi, D_0); V_j(\varphi, D_1)] \right] \\
\mathcal{R}_j^{(4)}(\varphi', Y') &:= \sum_{\substack{\rightarrow Y' \\ c_{j+1}(X_0 \cup X_1) \geq 1 \\ |Z|_{j+1} + c_{j+1}(X_0 \cup X_1) \geq 2}} \mathbb{E}_j [P_j^Z(\varphi', \zeta) R_j^{X_1}(\varphi', \zeta)] J_j^{X_0, (D)}(\varphi') \\
\mathcal{R}_j^{(5)}(\varphi', Y') &:= \sum_{c_{j+1}(X_0 \cup X_1) \geq 1}^{\rightarrow Y'} \left(e^{-\mathcal{E}_j |W| + U_{j+1}(\varphi', Y' \setminus W)} - 1 \right) \cdot \\
&\quad \cdot \mathbb{E}_j [P_j^Z(\varphi', \zeta) R_j^{X_1}(\varphi', \zeta)] J_j^{X_0, (D)}(\varphi') \\
\mathcal{R}_j^{(6)}(\varphi', Y') &:= \left(e^{-\mathcal{E}_j |Y'|} - 1 \right) \mathbb{E}_j [P_j^{Y'}(\varphi', \zeta)] + \sum_{\substack{Z=Y' \\ Z \in \mathcal{P}_{j+1} \\ |Z|_{j+1} \geq 3}} \mathbb{E}_j [P_j^Z(\varphi', \zeta)] \\
\mathcal{R}_j^{(7)}(\varphi', Y') &:= \sum_{\substack{\bar{X}'=Y' \\ X' \in \mathcal{P}_j \\ c_j(X') \geq 2}} \mathbb{E}_j \left[e^{U_j(\varphi, Y' \setminus X')} \prod_{Y \in \mathcal{C}_j(X')} K_j(\varphi, Y) \right] \\
\mathcal{R}_j^{(8)}(\varphi', Y') &:= \sum_{X' \in \mathcal{P}_j}^{\bar{X}'=Y'} \mathbb{E}_j \left[\left(e^{U_j(\varphi, Y' \setminus X')} - 1 \right) \prod_{Y \in \mathcal{C}_j(X')} K_j(\varphi, Y) \right]. \tag{6.47}
\end{aligned}$$

Notice that, for $j = 0$, we instead have

$$\begin{aligned} \mathcal{R}_0(\varphi', Y') := & \sum_{D \in \mathcal{B}_1}^{D=Y'} \left[\mathbb{E}_0 [P_0(\varphi', \zeta, D)] + \mathcal{E}_1 |D| + V_1(\varphi', D) - \mathbb{E}_1 [V_1(\varphi, D)] \right] \\ & + \left(e^{-\mathcal{E}_1 |Y'|} - 1 \right) \mathbb{E}_j \left[P_0^{Y'}(\varphi', \zeta) \right] + \sum_{\substack{Z \in \mathcal{P}_1 \\ |Z|_1 \geq 2}}^{Z=Y'} \mathbb{E}_j \left[P_j^Z(\varphi', \zeta) \right] \end{aligned} \quad (6.48)$$

Then, Lemma 3.5 is a direct consequence of the following result.

Lemma 6.3 *There exists $C(A, L, \alpha) > 1$ such that, for ε_0 small enough, any \mathcal{K}_j and $\dot{\mathcal{K}}_j$ with $|\mathcal{K}_j|_{\varepsilon_0, j}, |\dot{\mathcal{K}}_j|_{\varepsilon_0, j} \leq 1$ and $Y' \in \mathcal{P}_{j+1}^e$: if $j=0$,*

$$\|\mathcal{R}_0 - \dot{\mathcal{R}}_0\|_{h, T_0} \leq C(A, L, \alpha) \varepsilon_0 \left[|s - \dot{s}| + |z - \dot{z}| \right]; \quad (6.49)$$

while, if $j = 1, \dots, R-1$, for any $m = 1, 2, \dots, 8$,

$$\|\mathcal{R}_j^{(m)} - \dot{\mathcal{R}}_j^{(m)}\|_{h, T_j} \leq C(A, L, \alpha) \varepsilon_0^3 |\mathcal{K}_j - \dot{\mathcal{K}}_j|_{\varepsilon_0, j}. \quad (6.50)$$

Remark. Many details of the construction of K_{j+1} out of K_j and U_j , (4.32), are designed with the goal of having (6.49), (6.50). In particular, the preliminary re-blocking of K_j , with support on j -polymers, into \tilde{K}_j , with support on $j+1$ -polymers - a step not done in [5] - seems necessary here to obtain (6.52) from (6.51) through the new integration property (5.15).

Proof. First consider $\mathcal{R}_j^{(4)}$ that is the most instructive term. By Lemma 6.2, its bound is made of three kind of factors: a product of field regulators; a product of A^{-1} 's; a product of $\|\mathcal{K}_j\|_{h, T_j}$'s and $C(A, L, \alpha)$'s. The field regulators stemming from the P_j and R_j factors can be merged together by Lemma 5.1: given disjoint $Z, X_1 \in \mathcal{P}_{j+1}$,

$$\begin{aligned} & \prod_{D \in \mathcal{B}_{j+1}(Z)} [G_j^{\text{str}}(\varphi, D) + G_{j+1}^{\text{str}}(\varphi', D)] \prod_{Y \in \mathcal{C}_{j+1}(X_1)} [G_j(\varphi, Y) + G_{j+1}^{\text{str}}(\varphi', Y)] \\ & \leq \sum_{\substack{W_1 \in \mathcal{P}_{j+1}(Z) \\ W_2 \in ((X_1))_{j+1}}} G_{j+1}^{\text{str}}(\varphi', Z \setminus W_1) G_{j+1}^{\text{str}}(\varphi', X_1 \setminus W_2) G_j(\varphi, W_1 \cup W_2) \end{aligned} \quad (6.51)$$

where $((X_1))_{j+1}$ is the collections of subsets of X_1 made of unions of connected parts of X_1 . There are no more than $2^{|\mathcal{C}_{j+1}(X_1)| + |Z|_{j+1}}$ terms in the sum above. Then take the expectation \mathbb{E}_j in (6.51) using (5.15), and merge also the (strong) field regulators stemming from the bounds for the J_j 's factors, to obtain the upper bound for (6.51)

$$2^{|\mathcal{C}_{j+1}(X_1)| + |Z|_{j+1}} 2^{L^2 |X_1 \cup Z|_{j+1}} G_{j+1}(\varphi', \cup_Y D_Y \cup X_1 \cup Z). \quad (6.52)$$

Next, collect all the A^{-1} factors from the bounds of P_j 's, R_j 's and J_j 's to obtain $A^{-(1+\eta)|Y'|_{j+1}}$. Finally, since the variation in the vector \mathcal{K}_j can occur in each of the factors, that are no more than $|Y'|_{j+1}$, we obtain:

$$\begin{aligned} & \|\mathcal{R}_j^{(4)}(\varphi', Y') - \dot{\mathcal{R}}_j^{(4)}(\varphi', Y')\|_{h, T_j(\varphi', Y')} \\ & \leq G_{j+1}(\varphi', Y') A^{-(1+\eta)|Y'|_{j+1}} |\mathcal{K}_j - \dot{\mathcal{K}}_j|_{\varepsilon_0, j} |Y'|_{j+1} 2^{L^2|Y'|_{j+1}} \cdot \\ & \quad \cdot \sum_{\substack{\rightarrow Y' \\ c_{j+1}(X_0 \cup X_1) \geq 1 \\ |Z|_{j+1} + c_{j+1}(X_0 \cup X_1) \geq 2}} [2C(A, L, \alpha) \varepsilon_0]^{|Z|_{j+1} + 2|c_{j+1}(X_0 \cup X_1)|} . \end{aligned} \quad (6.53)$$

Notice the following facts: the number of term in the sum is not bigger than $4^{|Y'|_{j+1}} 4^{|c_{j+1}(X_0)|}$; if ε_0 is small enough, $8C(A, L, \alpha) \varepsilon_0 < 1$; by the constraint on the sum there is a least factor $[8C(A, L, \alpha) \varepsilon_0]^3$ in each summand; finally, for A large enough, $(A^{-\eta} 2^{(L^2+2)})^{|Y'|_{j+1}} |Y'|_{j+1} < 1$. Therefore, for $C'(A, L, A) > [8C(A, L, \alpha)]^3$, an upper bound for (6.53) is

$$C'(A, L, \alpha) \varepsilon_0^3 |\mathcal{K}_j - \dot{\mathcal{K}}_j|_{\varepsilon_0, j} G_{j+1}(\varphi', Y') A^{-|Y'|_{j+1}} . \quad (6.54)$$

This proves (6.50) for $m = 4$. Now consider $\mathcal{R}_j^{(5)}$. It can be recasted as

$$\begin{aligned} & \sum_{c_{j+1}(X_0 \cup X_1) \geq 1}^{\rightarrow Y'} \sum_{\substack{|W_0 \cup W_1|_{j+1} \geq 1 \\ W_0 \in \mathcal{P}_j(W) \\ W_1 \in \mathcal{P}_j(Y' \setminus W)}} \left(e^{-\varepsilon_j L^{2j+2}} - 1 \right)^{|W_0|_{j+1}} J_j^{X_0, (D)}(\varphi') \cdot \\ & \cdot \mathbb{E}_j \left[P_j^Z(\varphi', \zeta) R_j^{X_1}(\varphi', \zeta) \right] \prod_{D_1 \in \mathcal{B}_{j+1}(W_1)} \left(e^{U_{j+1}(\varphi', D_1)} - 1 \right) . \end{aligned} \quad (6.55)$$

Bound (6.50) for $m = 5$ follows, mostly, as for $m = 4$; so we just stress one detail: after field regulators are multiplied and integrated, we have to merge strong field regulators originated by the bound on (possibly, the variation of) J_j 's, but also $e^{-U_{j+1}(\varphi', D_1)} - 1$'s; still, by construction, the final bound for the field regulators is $G_{j+1}(\varphi', Y')$, as wanted.

Bounds (6.50) for $m = 7, 8$ can be obtained in the very same fashion as (6.31); while cases $m = 1, 2, 3, 6$ are even simpler than the previous ones. This completes the proof of Lemma 3.5. ■

7 Stable Manifold Theorem

In this discussion, $\alpha^2 = 8\pi$. Then, a natural way to recast (3.14) is

$$s_{j+1} = s_j - az_j^2 + \mathcal{F}_j(s_j, z_j, K_j)$$

$$\begin{aligned} z_{j+1} &= z_j - bs_j z_j + \mathcal{M}_j(s_j, z_j, K_j) \\ K_{j+1} &= \mathcal{L}_j(K_j) + \mathcal{R}_j(z_j, s_j, K_j) \end{aligned} \quad (7.1)$$

for $a \equiv a(L)$ and $b \equiv b(L)$ given in Lemma 3.1, and

$$\begin{aligned} \mathcal{F}_j(s_j, z_j, K_j) &:= \mathcal{F}_j(K_j) - (a_j - a)z_j^2, \\ \mathcal{M}_j(s_j, z_j, K_j) &:= L^2 e^{-\frac{\alpha^2}{2}\Gamma_j(0)} \mathcal{M}_j(K_j) + \left[L^2 e^{-\frac{\alpha^2}{2}\Gamma_j(0)} - 1 \right] z_j \\ &\quad - \left[L^2 e^{-\frac{\alpha^2}{2}\Gamma_j(0)} b_j - b \right] s_j z_j. \end{aligned} \quad (7.2)$$

The flow equation for $j = 0$ is simpler and given in (3.13). We want to redefine the dynamical system so that it begins at step 1 instead of step 0 (and so to absorb the constants a and b). To this purpose, define $(x_j, y_j, W_j)_{j \geq 1}$ such that: $(x_1, y_1, W_1) := (bs_1, \sqrt{ab}z_1, 0)$, while, for $j \geq 2$, $(x_j, y_j, W_j) := (bs_j, \sqrt{ab}z_j, K_j)$. Accordingly the flow equation becomes

$$\begin{aligned} x_{j+1} - x_j &= -y_j^2 + \tilde{\mathcal{F}}_j(x_j, y_j, W_j) \\ y_{j+1} - y_j &= -x_j y_j + \tilde{\mathcal{M}}_j(x_j, y_j, W_j) \\ W_{j+1} &= \mathcal{L}_j(W_j) + \tilde{\mathcal{R}}_j(x_j, y_j, W_j). \end{aligned} \quad (7.3)$$

with, for $j = 1$ and $\lambda = \left[L^2 e^{-\frac{\alpha^2}{2}\Gamma_0(0)} \right]^{-1}$,

$$\begin{aligned} \tilde{\mathcal{F}}_1(x_1, y_1, W_1) &:= b\mathcal{F}_1(s_1, z_1, \mathcal{R}_0(s_1, \lambda z_1)), \\ \tilde{\mathcal{M}}_1(x_1, y_1, W_1) &:= \sqrt{ab}\mathcal{M}_1(s_1, z_1, \mathcal{R}_0(s_1, \lambda z_1)), \\ \tilde{\mathcal{R}}_1(x_1, y_1, W_1) &:= \mathcal{L}_1(\mathcal{R}_0(s_1, \lambda z_1)) + \mathcal{R}_1(s_1, z_1, \mathcal{R}_0(s_1, \lambda z_1)); \end{aligned} \quad (7.4)$$

while, for $j \geq 2$,

$$\begin{aligned} \tilde{\mathcal{R}}_j(x_j, y_j, W_j) &:= \mathcal{R}_j(s_j, z_j, K_j), \\ \tilde{\mathcal{F}}_j(x_j, y_j, W_j) &:= b\mathcal{F}_j(s_j, z_j, K_j), \\ \tilde{\mathcal{M}}_j(x_j, y_j, W_j) &:= \sqrt{ab}\mathcal{M}_j(s_j, z_j, K_j). \end{aligned} \quad (7.5)$$

Then, equations (3.13) and (3.14) for the RG flow are equivalent to (7.3). Note the following consequences of Lemmas 3.1, 3.3, 3.4 and 3.5: if $|x_j|, |y_j|, |\dot{x}_j|, |\dot{y}_j| \leq \varepsilon_j$ and $\|W_j\|_{h,j}, \|\dot{W}_j\|_{h,j} \leq \varepsilon_j^2$,

$$\begin{aligned} |\tilde{\mathcal{F}}_j(x_j, y_j, W_j) - \tilde{\mathcal{F}}_j(\dot{x}_j, \dot{y}_j, \dot{W}_j)| &\leq C(L)L^{-\frac{1}{4}}|y_j^2 - \dot{y}_j^2| \\ &\quad + C(L)A^{-1} \left[\|W_j - \dot{W}_j\|_{h,T_j} + \delta_{j,1}\varepsilon_1(|x_1 - \dot{x}_1| + |y_1 - \dot{y}_1|) \right] \end{aligned} \quad (7.6)$$

$$\begin{aligned}
|\widetilde{\mathcal{M}}_j(x_j, y_j, W_j) - \widetilde{\mathcal{M}}_j(\dot{x}_j, \dot{y}_j, \dot{W}_j)| &\leq cL^{-\frac{j}{4}}|y_j - \dot{y}_j| + C(L)L^{-\frac{j}{4}}|x_j y_j - \dot{x}_j \dot{y}_j| \\
&+ C(L)A^{-1} \left[\|W_j - \dot{W}_j\|_{h, T_j} + \delta_{j,1} \varepsilon_1 (|x_1 - \dot{x}_1| + |y_1 - \dot{y}_1|) \right] \quad (7.7)
\end{aligned}$$

$$\begin{aligned}
|\widetilde{\mathcal{R}}_j(x_j, y_j, W_j) - \widetilde{\mathcal{R}}_j(\dot{x}_j, \dot{y}_j, \dot{W}_j)| &\leq c(L^{-\vartheta} + A^{-\eta})\delta_{j,1}\varepsilon_1(|x_1 - \dot{x}_1| + |y_1 - \dot{y}_1|) \\
&+ C(A, L) \left[\varepsilon_j^2 |x_j - \dot{x}_j| + \varepsilon_j^2 |y_j - \dot{y}_j| + \varepsilon_j \|W_j - \dot{W}_j\|_{h, T_j} \right] \quad (7.8)
\end{aligned}$$

The RG equations that we are describing here are well defined, and so considered, in “infinite volume”, *i.e.* for any $j \geq 1$ and with polymer activities defined on \mathbb{Z}^2 . Theorem 3.6 (which is about the RG on finite lattice Λ and with $j = 1, \dots, R$) will be a corollary of the following result - see the remark after the proof. Hypotheses slightly more general than what we need, since W_1 is not assumed to be 0.

Theorem 7.1 *Consider the solution of (7.3) with initial data (x_1, y_1, W_1) . If L and A are large enough, if $|y_1| \leq \varepsilon_1$ for a small $\varepsilon_1 \equiv \varepsilon_1(A, L)$ and if $\|W_1\|_{h, T_1}$ is small enough w.r.t $|y_1|$, there exist $x_1 = \Sigma(y_1)$ such that $|x_1| \leq 2\varepsilon_1$ and*

$$x_j = |q_j| + O(j^{-\frac{3}{2}}), \quad y_j = q_j + O(j^{-\frac{3}{2}}), \quad \|W_j\|_{h, T_j} = O(j^{-3}), \quad (7.9)$$

where q_j is the same defined in (3.18) with $q_1 := y_1$.

Proof. Without loss of generality, assume y_1 positive. The strategy - partially inspired to [10] - is to mimic the treatment of the continuous version of (7.3). We look for a solution (x_j, y_j, W_j) of (7.3) into the form $x_j = q_j + u_j$, $y_j = q_j + v_j$, where u_j, v_j, W_j are unknowns with initial data $u_1 = x_1 - y_1$, $v_1 = 0$ and W_1 . By the identity $q_{j+1} - q_j = -q_j q_{j+1}$, the equations for u_j and v_j are

$$\begin{aligned}
u_{j+1} - u_j &= -2v_j q_j + \mathcal{U}_j(u_j, v_j, W_j) \\
v_{j+1} - v_j &= -u_j q_j - v_j q_j + \mathcal{V}_j(u_j, v_j, W_j) \quad (7.10)
\end{aligned}$$

for $\mathcal{U}_j(u_j, v_j, W_j) := -v_j^2 - q_j^2 q_{j+1} + \widetilde{\mathcal{F}}_j(x_j, y_j, W_j)$ and $\mathcal{V}_j(u_j, v_j, W_j) := -u_j v_j - q_j^2 q_{j+1} + \widetilde{\mathcal{M}}_j(x_j, y_j, W_j)$. In order to diagonalize the linear part, introduce the *stable direction*, $w_j^+ := u_j + 2v_j$, and the *unstable direction*, $w_j^- := u_j - v_j$, that have initial data $w_1^+ = w_1^- = x_1 - y_1$ and satisfy

$$\begin{aligned}
w_{j+1}^+ - w_j^+ &= (-2q_{j+1} + q_{j+1}^2)w_j^+ + \mathcal{W}_j^+(u_j, v_j, W_j) \\
w_{j+1}^- - w_j^- &= q_j w_j^- + \mathcal{W}_j^-(u_j, v_j, W_j) \quad (7.11)
\end{aligned}$$

for $\mathcal{W}_j^+ := \mathcal{U}_j + 2\mathcal{V}_j + [2q_j - q_{j+1}]q_{j+1}w_j^+$ and $\mathcal{W}_j^- := \mathcal{U}_j - \mathcal{V}_j$ (changing the linear order of the former equation into $(-2q_{j+1} + q_{j+1}^2)w_j^+$, up to a correction $O(q_j^2)w_j^+$ that is absorbed into \mathcal{W}_j^+ , will make easier the next step). Since q_j is strictly positive, w_j^+ is driven to zero by the linear term in any case; whereas w_j^- converges to zero only for a special initial $w_1^+ = w_1^- = x_1 - y_1$, to be found. With some simple

algebra, turn (7.11), together to the condition $w_\infty^- = 0$, into a form that resembles the integral equation for the continuous flow:

$$\begin{aligned} w_j^+ &= \left(\frac{q_j}{q_1}\right)^2 w_1^+ + \sum_{s=1}^{j-1} \left(\frac{q_j}{q_{s+1}}\right)^2 \mathcal{W}_s^+(u_j, v_j, W_j) \\ w_j^- &= - \sum_{s \geq j} \frac{q_{s+1}}{q_j} \mathcal{W}_s^-(u_j, v_j, W_j) \\ W_j &= \mathcal{L}_{j-1,1} W_1 + \sum_{s=1}^{j-1} \mathcal{L}_{j-1,s+1} \mathcal{W}_s^0(u_j, v_j, W_j) \end{aligned} \quad (7.12)$$

for $\mathcal{L}_{n,m} := \mathcal{L}_n \circ \mathcal{L}_{n-1} \circ \dots \circ \mathcal{L}_m$ and $\mathcal{W}_j^0(u_j, v_j, W_j) := \tilde{\mathcal{R}}_j(x_j, y_j, K_j)$. (As in standard conventions, when a counter ranges over an empty set, the corresponding summation is zero, the corresponding chain of convolutions is the identity.)

Introduce a vector notation, $\underline{w} = (w_j^+, w_j^-, W_j)_{j \geq 1}$, and the norm

$$\|\underline{w}\| = \sup_{j \geq 1} \max\{(\tau h_j)^{-1} |w_j^+|, 2(\tau h_j)^{-1} |w_j^-|, (\tau h_j)^{-2} \|W_j\|_{h, T_j}\},$$

where h_j is the sequence $h_j = y_1[1 + y_1(j-1)]^{-\frac{3}{2}}$ and $\tau > 0$. It is easy to see that $\mathbb{W} = \{\underline{w} : \|\underline{w}\| \leq 1\}$ is a Banach space. Then define the operator $\mathcal{T} = (\mathcal{T}_j^+, \mathcal{T}_j^-, \mathcal{T}_j^0)_j$ such that (7.12) reads $\underline{w} = \mathcal{T}\underline{w}$, *i.e.*

$$w_j^+ = \mathcal{T}_j^+(\underline{w}), \quad w_j^- = \mathcal{T}_j^-(\underline{w}), \quad W_j = \mathcal{T}_j^0(\underline{w}). \quad (7.13)$$

We shall prove that, if τ is small enough (independently of A and L), \mathcal{T} is a contraction of \mathbb{W} ; then $\underline{w} = \mathcal{T}\underline{w}$ has a unique solution in \mathbb{W} .

Define $p_j := 2^{-(j-1)}$ and the operators

$$T_j^+(\mathcal{W}) := \sum_{s=1}^{j-1} \left(\frac{q_j}{q_{s+1}}\right)^2 \mathcal{W}_s, \quad T_j^-(\mathcal{W}) := - \sum_{s \geq j} \frac{q_{s+1}}{q_j} \mathcal{W}_s.$$

For $\alpha = \pm$,

$$|T_j^\alpha(h^2)| \leq Ch_j, \quad |T_j^\alpha(pq)| \leq Ch_j. \quad (7.14)$$

By definition of \mathcal{W}_j^α , for $\alpha = \pm$, if L is large enough,

$$|\mathcal{W}_j^\alpha(u_j, v_j, W_j)| \leq Ch_j^2[\tau^2 + q_1] + C(L) [p_j q_j^2 + A^{-1} h_j^2] + CL^{-\frac{1}{4}} p_j q_j; \quad (7.15)$$

therefore, using (7.14),

$$|T_j^\alpha(\mathcal{W}^\alpha)| \leq Ch_j \left[\tau^2 + q_1 + L^{-\frac{1}{4}} + C(L)(A^{-1} + q_1) \right]. \quad (7.16)$$

Now consider two vectors in \mathbb{W} , $\underline{w} = (u_j, v_j, W_j)_j$ and $\underline{w} = (\dot{u}_j, \dot{v}_j, \dot{W}_j)_j$; then

$$\begin{aligned} & |\mathcal{W}_j^\alpha(u_j, v_j, W_j) - \mathcal{W}_j^\alpha(\dot{u}_j, \dot{v}_j, \dot{W}_j)| \\ & \leq C \|\underline{w} - \underline{\dot{w}}\| \tau \left[h_j^2 [\tau + q_1] + C(L) A^{-1} h_j^2 + L^{-\frac{1}{4}} p_j q_j \right]; \end{aligned} \quad (7.17)$$

hence, by (7.14),

$$|T_j^\alpha(\mathcal{W}^\alpha) - T_j^\alpha(\dot{\mathcal{W}}^\alpha)| \leq C \|\underline{w} - \underline{\dot{w}}\| \tau h_j \left[\tau + q_1 + L^{-\frac{1}{4}} + C(L) A^{-1} \right]. \quad (7.18)$$

Finally, from (7.16) and (7.18), assuming τ small enough so that $16C\tau \leq 1$, L large enough so that $16CL^{-\frac{1}{4}} \leq \tau$, A large enough so that $16C(L)A^{-1}C \leq \tau$ and finally $q_1 = |y_1|$ small enough so that $16C(L)q_1C \leq \tau$,

$$|T_j^\alpha(\mathcal{W}^\alpha)| \leq \frac{\tau h_j}{2} \quad |T_j^\alpha(\mathcal{W}^\alpha) - T_j^\alpha(\dot{\mathcal{W}}^\alpha)| \leq \|\underline{w} - \underline{\dot{w}}\| \frac{\tau h_j}{4}. \quad (7.19)$$

Now consider the third of (7.12). For a polymer activity \mathcal{W}_j , define

$$T_j^0(\mathcal{W}) = \sum_{s=1}^{j-1} \mathcal{L}_{j-1, s+1} \mathcal{W}_s.$$

Suppose that $\|\mathcal{W}_j\|_{h, T_j} \leq h_j^2$; if $(L^{-\vartheta} + A^{-\eta})$ is small enough (*cfr.* Lemma 3.4),

$$\|(T_0 Q)_j\|_j \leq C h_j^2. \quad (7.20)$$

By (7.8)

$$\|\mathcal{W}_j^0(u_j, v_j, W_j)\|_{h, T_{j+1}} \leq C \left[L^{-\vartheta} + A^{-\eta} + C(A, L) q_1 \right] h_j^2, \quad (7.21)$$

and then

$$\|T_j^0(\mathcal{W}^0)\|_{h, T_j} \leq C \left[L^{-\vartheta} + A^{-\eta} + C(A, L) q_1 \right] h_j^2, \quad (7.22)$$

Besides, given two vectors in \mathbb{W} , $\underline{w} = (u_j, v_j, W_j)_j$ and $\underline{w} = (\dot{u}_j, \dot{v}_j, \dot{W}_j)_j$,

$$\begin{aligned} & \|\mathcal{W}_j^0(u_j, v_j, W_j) - \mathcal{W}_j^0(\dot{u}_j, \dot{v}_j, \dot{W}_j)\|_{h, T_{j+1}} \\ & \leq \tau \|\underline{w} - \underline{\dot{w}}\| C \left[L^{-\vartheta} + A^{-\eta} + C(A, L) q_1 \right] h_j^2. \end{aligned} \quad (7.23)$$

and using (7.20) again,

$$\|T_j^0(\mathcal{W}^0) - T_j^0(\dot{\mathcal{W}}^0)\|_{h, T_j} \leq \tau \|\underline{w} - \underline{\dot{w}}\| C \left[L^{-\vartheta} + A^{-\eta} + C(A, L) q_1 \right] h_j^2. \quad (7.24)$$

For L and A large enough, $4C[L^{-\vartheta} + A^{-\eta}] \leq \tau^2$; then, for q_1 small enough, $4C(A, L, \alpha) q_1 C < \tau^2$; then

$$\|T_j^0(\mathcal{W}^0)\|_{h, T_j} \leq (\tau h_j)^2, \quad \|T_j^0(\mathcal{W}^0) - T_j^0(\dot{\mathcal{W}}^0)\|_{h, T_j} \leq \frac{(\tau h_j)^2}{2} \|\underline{w} - \underline{\dot{w}}\|. \quad (7.25)$$

Finally, consider the operator \mathcal{T} . By (7.19) and (7.25), if $\underline{w} \in \mathbb{W}$,

$$\begin{aligned} |\mathcal{T}_j^+(\underline{w})| &\leq \left(\frac{q_j}{q_1}\right)^2 |w_1^-| + |T_j^+(\mathcal{W}^+)| \leq \tau h_j, & |\mathcal{T}_j^-(\underline{w})| &\leq |T_j^-(\mathcal{W}^-)| \leq \frac{\tau h_j}{2}, \\ \|\mathcal{T}_j^0(\underline{w})\|_{h, \mathcal{T}_j} &\leq \|T_j^0(\mathcal{W}^0)\|_{h, \mathcal{T}_j} \leq (\tau h_j)^2; \end{aligned} \quad (7.26)$$

and, if $\underline{w}, \underline{\dot{w}} \in \mathbb{W}$, with similar derivation

$$\begin{aligned} |\mathcal{T}_j^+(\underline{w}) - \mathcal{T}_j^+(\underline{\dot{w}})| &\leq \frac{\tau h_j}{2} \|\underline{w} - \underline{\dot{w}}\|, & |\mathcal{T}_j^-(\underline{w}) - \mathcal{T}_j^-(\underline{\dot{w}})| &\leq \frac{\tau h_j}{4} \|\underline{w} - \underline{\dot{w}}\|, \\ \|\mathcal{T}_j^0(\underline{w}) - \mathcal{T}_j^0(\underline{\dot{w}})\|_{h, \mathcal{T}_j} &\leq \frac{(\tau h_j)^2}{2}. \end{aligned} \quad (7.27)$$

Hence \mathcal{T} is a contraction on \mathbb{W} with norm 1/2. The Theorem is proven. ■

Remark. Using the same choice $\Sigma(y_1)$ for x_1 found above in the infinite-volume case, the very same bounds for the decay of x_j, y_j, W_j are valid also for the finite-volume RG flow. In fact, the flows of x_j and y_j are unchanged, for \tilde{F}_j and $\tilde{\mathcal{M}}_j$ are built out of W_j 's living on small sets only. The flow of W_j , instead, is changed, but only when the support of W_j is a polymer that wraps around the torus Λ - a non-small at any scale $j = 1, 2, \dots, R-1$: since W_j is anyways a stable (*i.e.* contracting) direction of the flow, this change does not require a different initial datum (x_1, y_1) . Therefore, Theorem 7.1 indirectly implies Theorem 3.6.

A Functional Integral Formula

A.1 Siegert-Kac transformation

We begin with stressing two properties of the lattice Yukawa potential in two dimensions, (2.3). First, the *self-energy* of a particle,

$$W_\Lambda(0; m) = \frac{1}{|\Lambda|} \sum_{p \in \Lambda^*} \frac{1}{m^2 - \widehat{\Delta}(p)} \quad (A.1)$$

is positive divergent. Next, if we define the *normalized interaction* $W_\Lambda(x|0; m) := W_\Lambda(x; m) - W_\Lambda(0; m)$, we find

$$W_\Lambda(x|0) := \lim_{m \rightarrow 0} W_\Lambda(x|0; m) = \frac{1}{|\Lambda|} \sum_{p \in \Lambda^* \setminus \{0\}} \frac{e^{ipx} - 1}{-\widehat{\Delta}(p)}; \quad (A.2)$$

then, in the infinite volume, $W(x|0) := \lim_{\Lambda \rightarrow \infty} W_\Lambda(x|0)$ has large $|x|$ behavior

$$W(x|0) = -\frac{1}{2\pi} \ln |x| + O(1), \quad (A.3)$$

which is what is called the Coulomb potential in two-dimensions. Now consider (2.4) written in terms of normalized interactions and self-energies,

$$H_\Lambda(\omega; m) = \frac{1}{2} \sum_{i,j=1}^n \sigma_i \sigma_j W_\Lambda(x_i - x_j | 0; m) + \frac{1}{2} \left(\sum_{i=1}^n \sigma_i \right)^2 W_\Lambda(0; m). \quad (\text{A.4})$$

As consequence of (A.2) and (A.3), in the limit $m \rightarrow 0$, the latter term of (A.4) assign an infinite energy, *i.e.* a zero probabilistic weight, to configurations of Ω_n that are not globally neutral. For neutral configurations, instead, the latter term is zero and the particle interacts through the normalized potential. This justifies the use of the Yukawa potential as regularization of the Coulomb one. Notice that $W_\Lambda(x; m)$ is positive definite and thus $e^{-H_\Lambda(\omega; m)} \leq 1$; then, in the formula for the partition function, the sum on n can be taken before the limit $m \rightarrow 0$, for the resulting series is convergent uniformly in m :

$$Z_\Lambda(\beta) = \lim_{m \rightarrow 0} \sum_{n \geq 0} \frac{z^n}{n!} \sum_{\omega \in \Omega_n} e^{-\beta H_\Lambda(\omega; m)}. \quad (\text{A.5})$$

Now we introduce the Siegert-Kac construction. Since $W_\Lambda(x; m)$ is strictly-positive definite, introduce at any site $x \in \Lambda$ a random variable, or 'field', φ_x , with Gaussian measure $dP_{R,0}(\varphi; m)$ determined by $\int dP_{R,0}(\varphi; m) \varphi_x = 0$ and $\int dP_{R,0}(\varphi; m) \varphi_x \varphi_y = W_\Lambda(x - y; m)$. Accordingly, Boltzmann weights of a given configuration in Ω_n can be re-written as characteristic functions of the Gaussian measure

$$\int dP_{R,0} \left(\frac{\varphi}{\sqrt{\beta}}; m \right) \exp \left(i \sum_{j=1}^n \sigma_j \varphi_{x_j} \right) = e^{-\beta H_\Lambda(\omega; m)}.$$

Notice that at this point of the analysis the Gaussian measure is finite-dimensional and non-degenerate; then its density has an explicit formula. For RG analysis purposes, we want to consider a part of the Gaussian density as an integrand: for any $s \in [0, 1/2]$, if $\alpha^2 := \beta(1 - s)$ and replacing m with $\frac{m}{\sqrt{1-s}}$,

$$dP_{R,0} \left(\frac{\varphi}{\sqrt{\beta}}; \frac{m}{\sqrt{1-s}} \right) = dP_{R,0} \left(\frac{\varphi}{\alpha}; m \right) \exp \left\{ \frac{s}{2\alpha^2} \sum_{\substack{x \in \Lambda \\ \mu \in \hat{e}}} (\partial^\mu \varphi_x)^2 \right\} \mathcal{N}_\Lambda^{1/2}(s; m) \quad (\text{A.6})$$

where \mathcal{N}_Λ takes into account the different normalization of the two measures

$$\mathcal{N}_\Lambda(s; m) = \prod_{k \in \Lambda^*} \frac{m^2 - (1-s)\hat{\Delta}(k)}{m^2 - \hat{\Delta}(k)}. \quad (\text{A.7})$$

With some algebra, the functional integral representation of the partition function of the Coulomb Gas is

$$Z_\Lambda(\beta) = \lim_{m \rightarrow 0} \int dP_{R,0}(\varphi; m) e^{\mathcal{V}(\varphi)} \quad (\text{A.8})$$

for \mathcal{V} given by (3.5). To obtain (3.1) we need a decomposition of the fields.

A.2 Multiscale decomposition.

Suppose there exist positive-definite functions $\Gamma_0(x; m), \Gamma_1(x; m), \dots, \Gamma_{R-1}(x; m)$ and $\Gamma_{\geq R}(x; m)$ such that

$$W_\Lambda(x; m) = \sum_{j=0}^{R-1} \Gamma_j(x; m) + \Gamma_{\geq R}(x; m) . \quad (\text{A.9})$$

By standard theory of (finite dimensional) Gaussian processes, φ can be decomposed into $R + 1$ random variables,

$$\varphi_x = \zeta_x^{(R)} + \zeta_x^{(R-1)} + \dots + \zeta_x^{(0)}$$

that are independent and Gaussian:

$$dP_{R,0}(\varphi; m) = dP_{\geq R}(\zeta^{(R)}; m) dP_{R-1}(\zeta^{(R-1)}; m) \dots dP_0(\zeta^{(0)}; m) \quad (\text{A.10})$$

for dP_j and $dP_{\geq R}$ Gaussian measures with covariances Γ_j and $\Gamma_{\geq R}$, respectively.

Decomposition (A.9) can be obtained in many ways. Here we take advantage of a similar decomposition for $\overline{W}(x; m)$, the massive covariance for the *infinite* lattice, \mathbb{Z}^2 , derived in [9], but implemented on a finer set of scales (*cfr.* Sec. D). Suppose $L = \gamma^M$, for γ and M positive integers; γ odd. The decomposition of the covariance that we shall consider is done on scales γ^h , for integer $h > 0$. Define, for $j = 0, \dots, R-1$, $I_{j,j+1} := \{j + m \log_L \gamma \mid m = 0, 1, \dots, M-1\}$; also, define $Q_h := [-\pi\gamma^h, \pi\gamma^h]$ and $Q := Q_0$.

Theorem A.1 *There exists a positive-definite $\overline{\Gamma}_{\geq R}(x; m)$ and, for any $j = 0, 1, \dots, R-1$ and any $s \in I_{j,j+1}$, a positive-definite $C_s(x; m)$ such that*

$$\overline{W}(x; m) = \sum_{j=0}^{R-1} \overline{\Gamma}_j(x; m) + \overline{\Gamma}_{\geq R}(x; m) , \quad \overline{\Gamma}_j(x; m) = \sum_{s \in I_{j,j+1}} C_s(x; m) , \quad (\text{A.11})$$

with $C_s(x; m) = 0$ if $|x| \equiv \max\{x_0, x_1\} > \gamma^{sM+1}/2$. Besides, for any $h/M \in I_{j,j+1}$, there exists $F_h(p; m)$, defined for $p \in Q_h$, such that, if $h = sM$,

$$\begin{aligned} C_s(x; m) &= \int_Q \frac{d^2p}{(2\pi)^2} \frac{e^{ipx}}{m^2 - \widehat{\Delta}(p)} [F_h(\gamma^h p; m) - F_{h+1}(\gamma^{h+1} p; m)] , \\ \overline{\Gamma}_{\geq R}(x; m) &= \int_Q \frac{d^2p}{(2\pi)^2} \frac{e^{ipx}}{m^2 - \widehat{\Delta}(p)} F_{MR}(\gamma^{MR} p; m) . \end{aligned} \quad (\text{A.12})$$

The proof (without the finer scales, *i.e.* for γ replaced by $12L$) is in [9]. Notice that, as opposed to the full interactions $W_\Lambda(x; m)$ or $\overline{W}(x; m)$ that are *strictly* positive-definite, the C_s 's might be positive-definite only, *i.e.* the corresponding Gaussian measure might be degenerate. Anyways notice that, for example, if $C_s = g_s \circ g_s$ for

positive-definite g_s , and if \mathbb{E}_C is the expectation w.r.t. the Gaussian measure with covariance $C(x - y)$, then, for any integrable function F , $\mathbb{E}_{C_s}[F(z)] = \mathbb{E}_I[F(g_s \circ z)]$ where $I(x - y) := \delta_{x,y}$ and hence *strictly* positive-definite (see App. of [13]).

There is a standard way to construct the periodic $W_\Lambda(x; m)$ out of its infinite volume version $\overline{W}(x; m)$,

$$W_\Lambda(x; m) = \sum_{y \in \mathbb{Z}^2} \overline{W}(x + L^R y; m) . \quad (\text{A.13})$$

By finite support of C_s , $\overline{\Gamma}_j(x; m) = 0$ if $|x| > L^{j+1}/2$; therefore by simple arguments (A.11), through (A.13), implies (A.9) for

$$\Gamma_j(x; m) = \overline{\Gamma}_j(x; m) , \quad \Gamma_{\geq R}(x; m) = \frac{1}{|\Lambda|} \sum_{p \in \Lambda^*} \frac{e^{ipx}}{m^2 - \widehat{\Delta}(p)} F_{MR}(L^R p; m) . \quad (\text{A.14})$$

[9] provides also some properties of $F_h(p; m)$'s that we shall need below.

Lemma A.2 *Given an integer $h \geq 0$, for $n = 1, 2, \dots, h$, there are complex functions $\widehat{A}_{h,n}(p, m)$, defined and differentiable for $p \in Q_h$ and m in a neighborhood of 0, such that*

$$F_0(p; m) = 1 , \quad F_h(p; m) = \prod_{n=0}^{h-1} |\widehat{A}_{h,n}(p, m)|^2 .$$

Besides, there exists a constant $c \equiv c(\gamma)$ such that

$$|\widehat{A}_{h,n}(p; m)| \leq 1 , \quad \widehat{A}_{h,n}(0; 0) = 1 , \quad |1 - \widehat{A}_{h,n}(p; 0)| \leq c|p|\gamma^{-(h-n)} , \quad (\text{A.15})$$

$$|\widehat{A}_{h,n}(p; m)| \leq \frac{c}{1 + (\gamma^{-(h-n-1)} p)^4} , \quad (\text{A.16})$$

$$|\widehat{A}_{h,n}(p; 0) - \widehat{A}_{h+1,n+1}(p; 0)| \leq 2c\gamma^{-n} . \quad (\text{A.17})$$

W.r.t. [9], here we abridged the notation $\widehat{A}_{\varepsilon_h, h-n}^{m^2}(R_{h-n})(p)$ into $\widehat{A}_{h,n}(p; \gamma^{-h} m)$. Therefore the above estimates (A.15), (A.16) (A.17) correspond to formula (3.12), formula (6.17), Lemma 5.4 and Lemma 6.7 of [9], respectively; to optimize the constants, read off that paper the case $n = 1$ only; the other values of n can be obtained by scaling (3.26) of [9].

A direct consequence of (A.15) and (A.16) is that

$$0 \leq 1 - F_h(p; 0) = c|p| , \quad |F_h(p; m)| \leq \frac{c}{1 + p^4} . \quad (\text{A.18})$$

Now we can complete the proof of (3.1). Define $dP_{R-1,0}(\varphi; m)$ as the Gaussian measure corresponding to the covariance $\Gamma_{R-1,0}(x; m) = \Gamma_{R-1}(x; m) + \Gamma_{R-2}(x; m) + \dots + \Gamma_0(x; m)$, and neglect the m dependence in measures when $m = 0$, i.e. $dP_{R-1,0}(\varphi) := dP_{R-1,0}(\varphi; 0)$, $dP_j(\varphi) := dP_j(\varphi; 0)$. (3.1) is a consequence of the following result.

Lemma A.3

$$\lim_{m \rightarrow 0} \int dP_{R,0}(\varphi; m) e^{\mathcal{V}(\varphi)} = \lim_{m \rightarrow 0} \int dP_{\geq R}(\zeta; m) \int dP_{R-1,0}(\varphi) e^{\mathcal{V}(\zeta+\varphi)}. \quad (\text{A.19})$$

Proof. It is an explicit check. For masses m and m' , define the Yukawa potential

$$T_{\Lambda}(x; m, m') := \frac{1}{|\Lambda|} \sum_{p \in \Lambda^*} e^{ipx} \frac{(1-s) [\widehat{\Gamma}_{R-1,0}(p; m') + \widehat{\Gamma}_{\geq R}(p; m)]}{1+s [\widehat{\Gamma}_{R-1,0}(p; m') + \widehat{\Gamma}_{\geq R}(p; m)] \widehat{\Delta}(p)}. \quad (\text{A.20})$$

In particular, notice that $W_{\Lambda}(x; \frac{m}{\sqrt{1-s}}) = T_{\Lambda}(x; m, m)$. Consider $T_{\Lambda}(x; m, 0)$ instead: it is well-defined and positive-definite; $T_{\Lambda}(0; m, 0)$ is positive divergent for $m \rightarrow 0$; and $T_{\Lambda}(x|0) := \lim_{m \rightarrow 0} [T_{\Lambda}(x; m, 0) - T_{\Lambda}(0; m, 0)] = W_{\Lambda}(x|0)$. This means that there is no change in (A.5) if we replace $W_{\Lambda}(x; \frac{m}{\sqrt{1-s}})$ with $T_{\Lambda}(x; m, 0)$. The r.h.s. member of (A.19) stems from repeating all the derivations of Section A.1 for the latter potential. ■

A.3 Explicit Computations

In order to prove (3.4) and (3.15) we need explicit computations that involve the covariances. For $p \in \mathbb{R}^2$, consider the differentiable function $u(p) = \lim_{h \rightarrow \infty} F_h(p; 0)$; and define the covariance

$$\widetilde{C}(x) := \int \frac{d^2 p}{(2\pi)^2} e^{ixp} \frac{u(p) - u(\gamma p)}{p^2}. \quad (\text{A.21})$$

By construction, $u(0) = 1$ and, uniformly in ϑ , $\lim_{\rho \rightarrow \infty} u(\rho \cos \vartheta, \rho \sin \vartheta) = 0$

Lemma A.4 *For any $j = 0, 1, \dots, R-1$ and $s \in I_{j,j+1}$, if $h = sM$,*

$$|C_s(x; 0) - \widetilde{C}(\gamma^{-h}x)| \leq c\gamma^{-\frac{h}{4}}, \quad |\partial^{\mu} C_s(x; 0) - \gamma^{-h} \widetilde{C}^{\mu}(\gamma^{-h}x)| \leq c\gamma^{-\frac{5h}{4}}, \quad (\text{A.22})$$

where the upper label μ indicates the continuous derivative (as opposed to ∂^{μ} that is the lattice one).

As consequence, $\widetilde{C}(x) = \lim_{h \rightarrow \infty} C_s(\gamma^h x; 0)$, and $\widetilde{C}(x) = 0$ if $|x| \geq \gamma/2$.

Proof. We discuss the details for the former inequality only; for the latter the argument is similar. Consider the formula for the Fourier transform of the covariance C_h given in (A.12) for $m = 0$: the idea is to replace the h -dependent cutoff function $F_h(p; 0)$ with $u(p)$; and to replace $-\widehat{\Delta}(p)$ with p^2 and the interval of integration Q with \mathbb{R}^2 . We have to prove that the errors so generated are $O(\gamma^{-\vartheta h})$ for some $\vartheta > 0$. If $p \in Q$, $-\widehat{\Delta}(p) \geq (4p^2/\pi^2)$ and $|p^2 + \widehat{\Delta}(p)| \leq (\sqrt{2}/3)|p|^3$; then, by the second of (A.18),

$$\left| \frac{1}{-\widehat{\Delta}(p)} - \frac{1}{p^2} \right| F_h(\gamma^h p; 0) \leq \frac{C}{|p|} \frac{1}{1 + |\gamma^h p|^4}. \quad (\text{A.23})$$

The replacement of $\widehat{\Delta}(p)$ with p^2 in C_s gives an error not larger than the integral of (A.23) over \mathbb{R}^2 , which is $O(\gamma^{-h})$. Add and subtract $u(\gamma^h p)$ and $u(\gamma^{h+1} p)$ in place of $F_h(\gamma^h p; 0)$ and $F_{h+1}(\gamma^{h+1} p; 0)$, respectively, to get

$$\begin{aligned} C_s(x) &= \int_Q \frac{d^2 p}{(2\pi)^2} e^{ixp} \frac{u(\gamma^h p) - u(\gamma^{h+1} p)}{p^2} + \int_Q \frac{d^2 p}{(2\pi)^2} e^{ixp} \frac{F_h(\gamma^h p; 0) - u(\gamma^h p)}{p^2} \\ &\quad - \int_Q \frac{d^2 p}{(2\pi)^2} e^{ixp} \frac{F_{h+1}(\gamma^{h+1} p; 0) - u(\gamma^{h+1} p)}{p^2} + O(\gamma^{-h}). \end{aligned} \quad (\text{A.24})$$

Now assume that (proof is below)

$$\int_{Q_h} \frac{d^2 p}{(2\pi)^2} \frac{|F_h(p; 0) - u(p)|}{p^2} \leq c(\gamma) \gamma^{-\frac{h}{4}}; \quad (\text{A.25})$$

(A.24) and (A.25) directly give (A.22). We are only left with proving assumption (A.25), that is a consequence of

$$\int_{Q_{h'}} \frac{d^2 p}{(2\pi)^2} \frac{|F_h(p; 0) - F_{h+1}(p; 0)|}{p^2} \leq c(\gamma) \gamma^{-\frac{h}{4}}, \quad (\text{A.26})$$

for any $h > h' > 0$. Define

$$\begin{aligned} O_h(p) &:= \prod_{n=1}^h |\widehat{A}_{h,n}(p; 0)|^2 - \prod_{n=0}^h |\widehat{A}_{h+1,n+1}(p; 0)|^2 \\ &= \sum_{m=1}^h \left[|\widehat{A}_{h,m}(p; 0)|^2 - |\widehat{A}_{h+1,m+1}(p; 0)|^2 \right] \prod_{n=1}^{m-1} |\widehat{A}_{h,n}(p; 0)|^2 \prod_{n=m+1}^h |\widehat{A}_{h+1,n+1}(p; 0)|^2 \\ &\quad + \left[1 - |\widehat{A}_{h+1,1}(p; 0)|^2 \right] \prod_{n=1}^h |\widehat{A}_{h+1,n+1}(p; 0)|^2 := \sum_{m=0}^h O_{h,m}(p); \end{aligned} \quad (\text{A.27})$$

then, using (A.15), (A.16) and (A.17), for $0 \leq m \leq h$,

$$|O_{h,m}(p)| \leq C(\gamma) \frac{\min\{\gamma^{-m}, |p|\gamma^{-(h-m)}\}}{1 + p^4};$$

therefore

$$\sum_{m=1}^{h+1} \int \frac{d^2 p}{(2\pi)^2} \frac{|O_{h,m}(p)|}{p^2} \leq C(\gamma) \gamma^{-\frac{h}{2}} h$$

(when $m \leq h/2$, use the bound $|p|\gamma^{-(h-m)}$; when $m > h/2$, use the bound $|p|\gamma^{-(h-m)}$ for the integration over $|p| \leq \gamma^{-h/2}$ and the bound γ^{-m} for the integration over $\gamma^{-h/2} \leq |p| \leq 1$ and $|p| \geq 1$). This proves (A.26). ■

A.3.1 Proof of (3.4)

Integrating in polar coordinates, only using that $u(0) = 1$ and $\lim_{\rho \rightarrow \infty} u(\rho \cos \vartheta, \rho \sin \vartheta) = 0$,

$$\tilde{C}(0) = \int \frac{d^2 p}{(2\pi)^2} \frac{u(p) - u(\gamma p)}{p^2} = \frac{1}{2\pi} \ln \gamma ; \quad (\text{A.28})$$

(3.4) follows.

A.3.2 Proof of Lemma 3.1

A preliminary result is that, for large $|x|$,

$$\tilde{\Gamma}_{\infty,0}(x|0) := \int \frac{d^2 p}{(2\pi)^2} (e^{ipx} - 1) \frac{u(p)}{p^2} = -\frac{1}{2\pi} \ln |x| + c + o(1) . \quad (\text{A.29})$$

This formula is an easy consequence of the decomposition of the integral into three parts: for $I_1 = \{|p| \leq |x|^{-1}\}$ and $I_2 = \{|p| > |x|^{-1}\}$, the above integral is equal to

$$\int_{I_1} \frac{d^2 p}{(2\pi)^2} (e^{ipx} - 1) \frac{u(p)}{p^2} + \int_{I_2} \frac{d^2 p}{(2\pi)^2} e^{ipx} \frac{u(p)}{p^2} - \int_{I_2} \frac{d^2 p}{(2\pi)^2} \frac{u(p)}{p^2} ;$$

the only term that is divergent for large $|x|$ is the third, and its leading term can be explicitly computed. Now consider the coefficient a_j . Let $\tilde{w}_{b,j}$ and $\tilde{\Gamma}_j(0|y)$ be the same function defined in Sec.4.2, but with $\tilde{C}(\gamma^{-h}x)$ in place of $C_s(x)$. It is not difficult to see, also using (A.22), that an equivalent formula for the coefficient a_j is, up to $O(L^{-\frac{j}{4}})$ errors,

$$\begin{aligned} & \frac{\alpha^2}{2} \int d^2 y y^2 \left[\tilde{w}_{b,j}(y) \left(e^{-\alpha^2 \tilde{\Gamma}_j(0|y)} - 1 \right) + e^{-\alpha^2 \tilde{\Gamma}_j(0)} \left(e^{\alpha^2 \tilde{\Gamma}_j(y)} - 1 \right) L^{-4j} \right] \\ & = \frac{\alpha^2}{2} \int d^2 y y^2 \left[\sum_{n=0}^j R_n^{(j)}(y) - \sum_{n=0}^{j-1} R_n^{(j-1)}(y) \right] \end{aligned} \quad (\text{A.30})$$

for

$$R_n^{(j-1)}(y) := e^{-\alpha^2 \tilde{\Gamma}_{j-1,n+1}(0|y)} e^{-\alpha^2 \tilde{\Gamma}_n(0)} \left(e^{\alpha^2 \tilde{\Gamma}_n(y)} - 1 \right) L^{-4n} .$$

From now on we only consider the case $\alpha^2 = 8\pi$. Since $R_n^{(j-1)}(y) = L^4 R_{n+1}^{(j)}(yL)$, (A.30) becomes

$$\begin{aligned} \frac{\alpha^2}{2} \int d^2 y y^2 R_0^{(j)}(y) & = \frac{\alpha^2}{2} \int d^2 y y^2 e^{-\alpha^2 \tilde{\Gamma}_{\infty,1}(0|y)} e^{-\alpha^2 \tilde{\Gamma}_0(0)} \left(e^{-\alpha^2 \tilde{\Gamma}_0(y)} - 1 \right) + O(L^{-j}) \\ & = \frac{\alpha^2}{2} \int \frac{d^2 y}{y^2} \left[w(y) - w(yL^{-1}) \right] + O(L^{-j}) \end{aligned} \quad (\text{A.31})$$

for $w(y) = y^4 e^{-\alpha^2 \tilde{\Gamma}_{\infty,0}(0|y)}$ and the new error $O(L^{-j})$ in the first line is due to the replacement of $\tilde{\Gamma}_{j,1}(0|y)$ with $\tilde{\Gamma}_{\infty,1}(0|y)$. The last integral can be computed in polar

coordinates only using the fact that $w(0) = 0$ and, by (A.29), $\lim_{y \rightarrow \infty} w(y) = e^c$. This proves the first of (3.15).

Now consider the coefficient b_j (still $\alpha^2 = 8\pi$). With the same argument used for a_j , an equivalent formula for b_j is, up to an error term $O(L^{-\frac{j}{4}})$,

$$\frac{\alpha^2}{2} \sum_{\mu \in \widehat{e}} \int d^2 y \left[(\widetilde{\Gamma}_{j,0}^\mu)^2(y) - (\widetilde{\Gamma}_{j-1,0}^\mu)^2(y) \right]. \quad (\text{A.32})$$

As $\widetilde{\Gamma}_{j-1,0}^\mu(y) = L\widetilde{\Gamma}_{j,1}^\mu(yL)$, (A.32) becomes, up another error term $O(L^{-j})$,

$$\begin{aligned} & \frac{\alpha^2}{2} \sum_{\mu \in \widehat{e}} \int d^2 y \left[2\widetilde{\Gamma}_{\infty,1}^\mu(y)\widetilde{\Gamma}_0^\mu(y) + (\widetilde{\Gamma}_0^\mu)^2(y) \right] \\ &= \frac{\alpha^2}{2} \sum_{\mu \in \widehat{e}} \int d^2 y \left[(\widetilde{\Gamma}_{\infty,0}^\mu)^2(y) - (\widetilde{\Gamma}_{\infty,1}^\mu)^2(y) \right] = \frac{\alpha^2}{2} \int \frac{d^2 p}{(2\pi)^2} \frac{[u(\rho)]^2 - [u(L\rho)]^2}{p^2}. \end{aligned} \quad (\text{A.33})$$

This proves the second of (3.15).

B Charged Components

The following constructions was introduced in [14]. By iteration on scale j , the polymer activities are showed to satisfy, for any $m \in \mathbb{Z}$,

$$K_j(\varphi + \frac{2m\pi}{\alpha}, X) = K_j(\varphi, X). \quad (\text{B.1})$$

Accordingly, given $X \in \mathcal{P}_j^c$ and a field $\{\varphi_x\}_{x \in X}$, the function of real variable $F(t) := K_j(\varphi + t, X)$ is smooth and periodic of period $2\pi/a$: expand $F(t)$ in (absolutely convergent) Fourier series and, at $t = 0$, obtain (4.2) with charged components $\widehat{K}_j(q, \varphi, X) := \frac{\alpha}{2\pi} \int_0^{2\pi} ds K_j(\varphi + t, X) e^{-iqas}$. Besides, since $G_j(\varphi, X)$ only depends upon the derivatives of φ ,

$$\|\widehat{K}_j(q, \varphi, X)\|_{h, T_j(\varphi, X)} \leq \|K_j(X)\|_{h, T_j(X)} G_j(\varphi, X). \quad (\text{B.2})$$

C Lowest Orders Computation

C.1 First Order

Let $B \in \mathcal{B}_j$. An explicit computation of $\mathbb{E}_j [V_j(\varphi, B)]$ yields

$$\begin{aligned} & \frac{s_j}{2} \sum_{\substack{x \in B \\ \mu \in \widehat{e}}} (\partial^\mu \varphi'_x)^2 - |B| \frac{s_j}{2} \sum_{\mu \in \widehat{e}} (\partial^\mu \partial^\mu \Gamma_j)(0) + z_j L^{-2j} e^{-\frac{\alpha^2}{2} \Gamma_j(0)} \sum_{\substack{x \in B \\ \sigma = \pm}} e^{i\sigma \alpha \varphi'_x} \\ &= -\frac{s_j}{2} |B| \sum_{\mu \in \widehat{e}} (\partial^\mu \partial^\mu \Gamma_j)(0) + V_{j+1}(s_j, L^2 e^{-\frac{\alpha^2}{2} \Gamma_j(0)} z_j, \varphi', B). \end{aligned} \quad (\text{C.1})$$

C.2 Second Order

Notice the following ‘‘cancellations’’:

$$\begin{aligned} \sum_y e^{-\alpha^2 \Gamma_j(0)} \left(e^{\alpha^2 \Gamma_j(y)} - 1 \right) y^\mu y^\nu &= \delta^{\mu\nu} \sum_y e^{-\alpha^2 \Gamma_j(0)} \left(e^{\alpha^2 \Gamma_j(y)} - 1 \right) \frac{|y|^2}{2}; \\ \sum_y (\partial^\mu \partial^\nu \Gamma_j(y)) &= -k^\mu k^\nu \widehat{\Gamma}_j(k) \Big|_{k=0} = 0. \end{aligned} \quad (\text{C.2})$$

Let $D = \overline{B}$, and $B \in \mathcal{B}_j$. An explicit computation yields:

$$\begin{aligned} \frac{1}{2} E_j^T [V_j(\varphi, B); V_j(\varphi, D^*)] &= s_j^2 F_{a,j+1}(\varphi', B) + z_j^2 F_{b,j+1}(\varphi', B) \\ &\quad + z_j^2 F_{c,j+1}(\varphi', B) + z_j s_j F_{d,j+1}(\varphi', B) \end{aligned} \quad (\text{C.3})$$

where we have defined:

$$\begin{aligned} F_{a,j+1}(\varphi', B) &:= -\frac{1}{2} \sum_{y \in \mathbb{Z}^2} \sum_{\substack{\mu \in \widehat{e} \\ \nu \in \widehat{e}}} (\partial^\mu \partial^\nu \Gamma_j(y)) \sum_{x \in B} (\partial^\mu \varphi'_x) [(\partial^\nu \varphi'_{x+y}) - (\partial^\nu \varphi'_x)] \\ &\quad + |B| \frac{1}{4} \sum_{y \in \mathbb{Z}^2} \sum_{\substack{\mu \in \widehat{e} \\ \nu \in \widehat{e}}} (\partial^\mu \partial^\nu \Gamma_j(y)) (\partial^\mu \partial^\nu \Gamma_j(y)|0) \end{aligned} \quad (\text{C.4})$$

$$\begin{aligned} F_{b,j+1}(\varphi', B) &:= \sum_{y \in \mathbb{Z}^2} \frac{1}{2} e^{-\alpha^2 \Gamma_j(0)} \left(e^{\alpha^2 \Gamma_j(y)} - 1 \right) L^{-4j} \\ &\quad \cdot \sum_{\substack{x \in B \\ \sigma = \pm}} \left[e^{i\sigma \alpha (\varphi'_x - \varphi'_{x+y})} - 1 + \frac{\alpha^2}{2} \sum_{\substack{\mu \in \widehat{e} \\ \nu \in \widehat{e}}} (\partial^\mu \varphi'_x) (\partial^\nu \varphi'_x) y^\mu y^\nu \right] \\ &\quad + \sum_y e^{-\alpha^2 \Gamma_j(0)} \left(e^{\alpha^2 \Gamma_j(y)} - 1 \right) L^{-4j} |B| \\ &\quad - \sum_{y \in \mathbb{Z}^2} \frac{\alpha^2}{2} e^{-\alpha^2 \Gamma_j(0)} |y|^2 \left(e^{\alpha^2 \Gamma_j(y)} - 1 \right) L^{-4j} \frac{1}{2} \sum_{x \in B} \sum_{\mu \in \widehat{e}} (\partial^\mu \varphi'_x)^2 \end{aligned} \quad (\text{C.5})$$

$$F_{c,j+1}(\varphi', B) := \frac{L^{-4j}}{2} \sum_{y \in \mathbb{Z}^2} e^{-\alpha^2 \Gamma_j(0)} \left(e^{-\alpha^2 \Gamma_j(y)} - 1 \right) \sum_{x \in B} \sum_{\varepsilon} e^{i\sigma \alpha (\varphi'_x + \varphi'_{x+y})} \quad (\text{C.6})$$

$$F_{d,j+1}(\varphi', B) := \frac{L^{-2j}}{2} \sum_{\substack{y \in \mathbb{Z}^2 \\ \nu \in \widehat{e}}} e^{-\frac{\alpha^2}{2} \Gamma_j(0)} (\partial^\nu \Gamma_j(y)).$$

$$\begin{aligned}
& \cdot \sum_{\substack{x \in B \\ \sigma = \pm}} i\sigma\alpha \left[e^{i\sigma\alpha\varphi'_x} (\partial^\nu \varphi'_{x+y}) - e^{i\sigma\alpha\varphi'_{x+y}} (\partial^\nu \varphi'_x) \right] \\
& - \frac{L^{-2j}}{4} \sum_{\substack{y \in \mathbb{Z}^2 \\ \nu \in \widehat{e}}} e^{-\frac{\alpha^2}{2}\Gamma_j(0)} (\partial^\nu \Gamma_j)^2(y) \alpha^2 \sum_{\substack{x \in B \\ \sigma = \pm}} \left(e^{i\sigma\alpha\varphi'_{x+y}} - e^{i\sigma\alpha\varphi'_x} \right) \\
& - \frac{L^{-2j}}{2} \sum_{\substack{y \in \mathbb{Z}^2 \\ \nu \in \widehat{e}}} e^{-\frac{\alpha^2}{2}\Gamma_j(0)} (\partial^\nu \Gamma_j)^2(y) \alpha^2 \sum_{\substack{x \in B \\ \sigma = \pm}} e^{i\sigma\alpha\varphi'_x} .
\end{aligned} \tag{C.7}$$

C.3 Estimates for W_j

Let $B \in \mathcal{B}_j$. If $|y| \leq L^{n+1}/2$, uniformly in j

$$\begin{aligned}
& |\Gamma_{j-1,n+1}(0) - \Gamma_{j-1,n+1}(y)| \leq C , \\
& |\Gamma_{j-1,n+1}(0) + \Gamma_{j-1,n+1}(y) - (j-n-1)\frac{1}{\pi} \ln L| \leq C .
\end{aligned} \tag{C.8}$$

Accordingly,

$$L^{-j} \sum_{y \in \mathbb{Z}^2} |w_{a,j}^{\mu\nu}(y)| |y| \leq C(L) , \quad L^{-j} \sum_{y \in \mathbb{Z}^2} |w_{b,j}^{\mu\nu}(y)| |y|^3 \leq C(L) , \tag{C.9}$$

$$L^{2j} \sum_{y \in \mathbb{Z}^2} |w_{c,j}(y)| \leq C(L) , \quad L^j \sum_{y \in \mathbb{Z}^2} |w_{d,j}^\mu(y)| \leq C(L) ,$$

$$L^j \sum_{y \in \mathbb{Z}^2} |w_{e,j}(y)| |y| \leq C(L) . \tag{C.10}$$

The above inequalities yield

$$\begin{aligned}
& \|W_{a,j}(s, z, \varphi, B)\|_{h, T_j(\varphi, B)} \\
& \leq s^2 \sum_{\mu, \nu, \rho \in \widehat{e}} \sum_{y \in \mathbb{Z}^2} |w_{a,j}^{\mu\nu}(y)| |y^\rho| |B| \sup_{x, z \in B} \|(\partial^\mu \varphi_x)(\partial^\rho \varphi_z)\|_{h, T_j(\varphi, B)} \\
& \leq s^2 C(L) \left(1 + \max_{n=1,2} \|\nabla_j^n \varphi\|_{L^\infty(B^*)}^2 \right)
\end{aligned} \tag{C.11}$$

$$\begin{aligned}
& \|W_{b,j}(s, z, \varphi, B)\|_{h, T_j(\varphi, B)} \leq z^2 C(\alpha, L) \sum_{y \in \mathbb{Z}^2} |w_{b,j}(y)| |y|^3 \cdot \\
& \quad \cdot |B| \sum_{\mu, \nu, \rho \in \widehat{e}} \sup_{x, z \in B} \left[\|(\partial^\mu \varphi_z)^3 + (\partial^\rho \partial^\mu \varphi_x)(\partial^\nu \varphi_z)\|_{h, T_j(\varphi, B)} \right] \\
& \leq z^2 C(\alpha, L) \left(1 + \max_{n=1,2} \|\nabla_j^n \varphi\|_{L^\infty(B^*)}^2 \right)
\end{aligned} \tag{C.12}$$

$$\|W_{c,j}(s, z, \varphi, B)\|_{h, T_j(\varphi, B)} \leq z^2 C(\alpha) \sum_{y \in \mathbb{Z}^2} |w_{c,j}(y)| |B| \leq z^2 C(L, \alpha) \quad (\text{C.13})$$

$$\begin{aligned} \|W_{d,j}(s, z, \varphi, B)\|_{h, T_j(\varphi, B)} &\leq |zs| C(\alpha) \sum_{\mu} \sum_{y \in \mathbb{Z}^2} [|w_{d,j}^{\mu}(y)| + |w_{e,j}(y)| |y|] \cdot \\ &\cdot |B| \sup_{z \in B^*} \|(\partial^{\mu} \varphi_z)\|_{h, T_j(\varphi, B)} \\ &\leq |zs| C(\alpha, L) \left(1 + \max_{n=1,2} \|\nabla_j^n \varphi\|_{L^{\infty}(B^*)}^2 \right) \end{aligned} \quad (\text{C.14})$$

All together these bounds give (6.33).

D Proof of Lemma 5.2.

The goal is to implement the idea of finer decomposition of the covariances, [7], in order to have control over the different field regulators needed for the Coulomb Gas. By (A.11), decompose the covariances and the field

$$\Gamma_j(x) = \sum_{s \in I_{j,j+1}} C_s(x), \quad \zeta_x^{(j)} = \sum_{s \in I_{j,j+1}} \xi_x^{(s)}, \quad (\text{D.1})$$

so that $\xi_x^{(s)}$ is a Gaussian random field with covariance $E_s[\xi_x^{(s)} \xi_y^{(s)}] = C_s(x - y)$, with range $L^{s + \log_L \gamma} / 2 = L^j \gamma^{m+1} / 2$ and typical momentum $L^{-s} = L^{-j} \gamma^{-m}$. The notation for the fields is now $\varphi_x^{(s)} = \varphi_x^{(s')} + \xi_x^{(s)}$, for $s' = s + \log_L \gamma$. It is clear how to extend previous definitions to have s -blocks, s -polymers, L_s^2 norms and G_s field regulators. We use \overline{X}_s to indicate the closure of the set X on scale s (hence, for any j -polymer, $X \overline{X}_j \equiv X$ and $\overline{X}_{j+1} \equiv \overline{X}$). Given $c_4 > 0$, define (cfr. (5.9))

$$\ln g_s(\xi, \overline{X}_s) = c_4 \kappa_L \sum_{n=0,1,2} W_s(\nabla_s^n \xi, \overline{X}_s)^2. \quad (\text{D.2})$$

Lemma D.1 *There exists a choice of the constants c_1, c_3, c_4 and γ such that:*

1) for any $X \in \mathcal{P}_s^c$,

$$G_s(\varphi^{(s)}, \overline{X}_s) \leq G_{s'}(\varphi^{(s')}, \overline{X}_{s'}) g_s(\xi^{(s)}, \overline{X}_s); \quad (\text{D.3})$$

2) for any $X \in \mathcal{S}_s$,

$$G_s(\varphi^{(s)}, \overline{X}) \leq G_{s'}(\varphi^{(s')}, \overline{X}_{s'})^{2/3} g_s(\xi^{(s)}, \overline{X}_s). \quad (\text{D.4})$$

Proof. Only in this proof, let us shorten the notations $\varphi^{(s)}$, $\varphi^{(s')}$, $\xi^{(s)}$, \overline{X}_s and $\overline{X}_{s'}$ into φ , φ' , ξ , X and \overline{X} , respectively. Since $\varphi = \varphi' + \xi$, for any $0 < \alpha < 1$, by discrete partial integration (see (6.117) of [5])

$$\begin{aligned} \|\nabla_s \varphi\|_{L_s^2(X)}^2 &\leq \|\nabla_s \varphi'\|_{L_s^2(X)}^2 + \alpha \|\nabla_s \varphi'\|_{L_s^2(\partial X)}^2 + \alpha \|\nabla_s^2 \varphi'\|_{L_s^2(X)}^2 \\ &\quad + \|\nabla_s \xi\|_{L_s^2(X)}^2 + \alpha^{-1} \|\xi\|_{L_s^2(\partial X)}^2 + \alpha^{-1} \|\xi\|_{L_s^2(X)}^2 ; \end{aligned} \quad (\text{D.5})$$

whereas, by squaring triangular inequalities

$$\|\nabla_s \varphi\|_{L_s^2(\partial X)}^2 \leq 2\|\nabla_s \varphi'\|_{L_s^2(\partial X)}^2 + 2\|\nabla_s \xi\|_{L_s^2(\partial X)}^2, \quad (\text{D.6})$$

$$W_s(\nabla_s^2 \varphi, X)^2 \leq 2W_s(\nabla_s^2 \varphi', X)^2 + 2W_s(\nabla_s^2 \xi, X)^2. \quad (\text{D.7})$$

Collecting the last three formulas, by (6.110) of [5],

$$\begin{aligned} \ln G_s(\varphi, X) &\leq c_1 \kappa_L \|\nabla_s \varphi'\|_{L_s^2(X)}^2 + f_3 \kappa_L \|\nabla_s \varphi'\|_{L_s^2(\partial X)}^2 + f_1 \kappa_L W_s(\nabla_s^2 \varphi', X)^2 \\ &\quad + f_4 \kappa_L \sum_{n=0,1,2} W_s(\nabla_s^n \xi, X)^2 \end{aligned} \quad (\text{D.8})$$

for $f_3 = \alpha c_1 + 2c_3$, $f_1 = (4\alpha + 2)c_1 < 6c_1$ and $f_4 = 4c_1 \alpha^{-1} + 8c_3 + 4c_1$. In order to pass to scale s' , use the inequality

$$\|\nabla_s \varphi'\|_{L_s^2(\partial X)}^2 \leq \|\nabla_s \varphi'\|_{L_s^2(\partial \overline{X})}^2 + c \|\nabla_s \varphi'\|_{L_s^2(\overline{X} \setminus X)}^2 + c W_s(\nabla_s^2 \varphi', \overline{X} \setminus X)^2 \quad (\text{D.9})$$

for a certain constant $c > 0$ (for the proof, *cfr.* the proof of (6.105) of [5]). If α and c_3 are so small that $c f_3 \leq c_1$,

$$\begin{aligned} \ln G_s(\varphi, X) &\leq c_1 \kappa_L \|\nabla_s \varphi'\|_{L_s^2(\overline{X})}^2 + f_3 \kappa_L \|\nabla_s \varphi'\|_{L_s^2(\partial \overline{X})}^2 + 6c_1 \kappa_L W_s(\nabla_s^2 \varphi', \overline{X})^2 \\ &\quad + f_4 \kappa_L \sum_{n=0,1,2} W_s(\nabla_s^n \xi, X)^2 \end{aligned} \quad (\text{D.10})$$

Finally, to restore the old constant c_3 and c_1 in the 2nd and 3rd terms, observe that these two terms are irrelevant, i.e. passing to scale s' gives γ^{-1} 's factors:

$$\begin{aligned} \|\nabla_s \varphi'\|_{L_s^2(\partial \overline{X})}^2 &\leq \gamma^{-1} \|\nabla_{s'} \varphi'\|_{L_{s'}^2(\partial \overline{X})}^2 \\ W_s(\nabla_s^2 \varphi', \overline{X})^2 &\leq \gamma^{-2} W_{s'}(\nabla_{s'}^2 \varphi', \overline{X})^2 ; \end{aligned} \quad (\text{D.11})$$

the first term is, instead, scale invariant (and the fourth is relevant but there is no need to scale it). Therefore, for $c_4 \geq f_4$, if γ is so large that $f_3 \gamma^{-1} < c_3$ and $6\gamma^{-2} < 1$ (D.3) is proven.

Then consider the case when $X \in \mathcal{S}_s$. If $\gamma > 8$, it is possible to find the polymer TX , a translation of X , such that TX is disjoint from X , but $(TX)^* \cap X^* \neq \emptyset$ and $TX \subset \bar{X}$; by (6.129) of [5], setting $X_T = TX \cup X$ and for a $c > 0$,

$$\frac{3}{2} \|\nabla_s \varphi'\|_{L_s^2(X)}^2 \leq \|\nabla_s \varphi'\|_{L_s^2(X_T)}^2 + cW_s(\nabla^2 \varphi'; X_T)^2; \quad (\text{D.12})$$

besides, as X and TX have disjoint boundaries,

$$\|\nabla_s \varphi'\|_{L_s^2(\partial X)}^2 \leq \|\nabla_s \varphi'\|_{L_s^2(\partial X_T)}^2, \quad W_s(\nabla_s^2 \varphi', X)^2 \leq W_s(\nabla_s^2 \varphi', X_T)^2. \quad (\text{D.13})$$

Multiply both sides of (D.8) times $3/2$; then use (D.12) and (D.13) to obtain

$$\begin{aligned} \frac{3}{2} \ln G_s(\varphi, X) &\leq c_1 \kappa_L \|\nabla_s \varphi'\|_{L_s^2(X_T)}^2 + \frac{3f_3}{2} \kappa_L \|\nabla_s \varphi'\|_{L_s^2(\partial X_T)}^2 \\ &\quad + t_1 \kappa_L W_s(\nabla_s^2 \varphi', X_T)^2 + c_4 \sum_{n=0,1,2} W_s(\nabla_s^n \xi, X)^2 \end{aligned} \quad (\text{D.14})$$

for $t_1 = 3f_1/2 + cc_1$ and $c_4 = 3f_4/2$. This inequality is of the same form as (D.8), except for the fact that X has been replaced by X_T in the first three terms of the r.h.s, and the constants are different. Therefore one can proceed as done from (D.8) to obtain (D.3): since by construction $\bar{X}_T = \bar{X}$, (D.4) follows for α and c_3 small enough, and γ large enough. ■

To conclude the proof of Lemma 5.2, we need two results of [5].

Lemma D.2 *If κ_L is small enough (w.r.t γ^{-2}), for any $X \in \mathcal{P}_s$*

$$\mathbb{E}_s [g_s(\xi^{(s)}, X)] \leq e^{c\kappa_L |X|_s}, \quad (\text{D.15})$$

where $|X|_s$ is the number of s -blocks in X .

Lemma D.3 *For $X \in \mathcal{P}_j$*

$$\left(1 + \max_{n=1,2} \|\nabla_{j+1}^n \varphi^{(j+1)}\|_{L^\infty(X^*)}\right)^3 \leq \frac{c}{\kappa_L^{3/2}} [G_{j+1}(\varphi^{(j+1)}, \bar{X})]^{1/3}. \quad (\text{D.16})$$

The former is Lemma 6.31 of [5] and its hypothesis is fulfilled if M is large enough; the latter corresponds to formula (6.127) of [5]. Use (D.3) iteratively, then (D.15), to find, for any $X \in \mathcal{P}_j$,

$$\begin{aligned} \mathbb{E}_j [G_j(\varphi^{(j)}, X)] &\leq G_{j+1}(\varphi^{(j+1)}, \bar{X}) \prod_{s \in I_{j,j+1}} \mathbb{E}_s [g_s(\xi^{(s)}, \bar{X}_s)] \\ &\leq G_{j+1}(\varphi^{(j+1)}, \bar{X}) e^{c\kappa_L \sum_{s \in I_{j,j+1}} |\bar{X}_s|_s}. \end{aligned} \quad (\text{D.17})$$

This, for γ large enough proves (5.15) for $|\overline{X}_s|_s \leq |X|_j$ and $c\kappa_L = C/\ln L = C(\log_\gamma e)/M$. In the same fashion, using by (D.4) iteratively, and then (D.15), (D.16), for any $X \in \mathcal{S}_j$ (so that $\overline{X}_s \in \mathcal{S}_s$ for any s),

$$\begin{aligned} & \left(1 + \max_{n=1,2} \|\nabla_{j+1}^n \varphi^{(j+1)}\|_{L^\infty(X^*)}\right)^3 \mathbb{E}_j [G_j(\varphi^{(j)}, X)] \\ & \leq \left(1 + \max_{n=1,2} \|\nabla_{j+1}^n \varphi^{(j+1)}\|_{L^\infty(X^*)}\right)^3 G_{j+1}^{2/3}(\varphi^{(j+1)}, \overline{X}) \prod_{s \in I_{j,j+1}} \mathbb{E}_s [g_s(\xi^{(s)}, \overline{X}_s)] \\ & \leq \frac{c}{\kappa_L^{3/2}} G_{j+1}(\varphi^{(j+1)}, \overline{X}) e^{c\kappa_L \sum_{s \in I_{j,j+1}} |\overline{X}_s|_s}. \end{aligned} \quad (\text{D.18})$$

This proves (5.16). (When $G_j(\varphi^{(j)}, X)$ is replaced by $\sup_{t \in [0,1]} G_j(t\varphi^{(j+1)} + \zeta, X)$, only little changes are required.)

E Proof of Theorems 5.5, 5.6 and 5.7

In this section we shall use $\|\cdot\|_{h, T_j^{\geq m}(\varphi, X)}$ for $\sum_{n \geq m} \frac{h^n}{n!} \|\cdot\|_{T_j^n(\varphi, X)}$.

Lemma E.1 *Let $F \in \mathcal{N}_j(X)$ and $X \in \mathcal{P}_j$:*

a) *if $X \in \mathcal{S}_j$ and $x_0 \in X$, for $(\delta\varphi)_x := \varphi_x - \varphi_{x_0}$ and $\rho = 5L^{-1}$,*

$$\|F(\delta\varphi)\|_{h, T_{j+1}(\varphi, X)} \leq \|F(\xi)\|_{\rho h, T_j(\xi, X)} \Big|_{\xi_x = \delta\varphi_x}; \quad (\text{E.1})$$

b) *given $\psi \in \mathcal{C}_j^2(X)$, if $\Delta := \|\psi\|_{\mathcal{C}_j^2(X)}$ is finite,*

$$\|F(\varphi + \psi)\|_{h, T_j(\varphi, X)} \leq \|F(\varphi)\|_{h+\Delta, T_j(\varphi, X)}; \quad (\text{E.2})$$

c) *for $m = 0, 2$ and any $h > 0$, (for Taylor remainder is ξ)*

$$\|\text{Rem}_m F(\xi)\|_{h, T_j(\xi, X)} \leq (1 + h^{-1} \|\xi\|_{\mathcal{C}_j^2(X)})^m \sup_{t \in [0,1]} \|F(t\xi)\|_{h, T_j^{\geq m}(t\xi, X)}. \quad (\text{E.3})$$

Proof. (E.1) follows from the identity $\sum_x f_x \frac{\partial}{\partial \varphi_x} = \sum_x (\delta f)_x \frac{\partial}{\partial \delta \varphi_x}$ and the fact that, for X small, $\|\delta f\|_{\mathcal{C}_j^2(X)} \leq 5L^{-1} \|f\|_{\mathcal{C}_{j+1}^2(X)}$. (E.2) is a direct consequence of Taylor series. Consider (E.3) for $m = 2$ - the case $m = 0$ can be proved with similar arguments - and set $O(\xi) := \text{Rem}_2 F(\xi)$; then using test functions f_1, f_2, \dots such that $\|f_j\|_{\mathcal{C}_j^2(X)} \leq 1$, for $n = 0, 1, 2$

$$\frac{h^n}{n!} |D_\xi^n O(\xi) \cdot (f_1, \dots, f_n)| \leq \frac{h^n}{n!} \int_0^1 dt \frac{(1-t)^{2-n}}{(2-n)!} |D_\xi^3 F(t\xi) \cdot (f_1, \dots, f_n, \xi, \dots, \xi)|$$

$$\leq \frac{3!}{(3-n)! n!} [h^{-1} \|\xi\|_{\mathcal{C}_j^2(X)}]^{3-n} \frac{h^3}{3!} \sup_{t \in [0,1]} \|F(t\xi)\|_{h, T_j^3(t\xi, X)}, \quad (\text{E.4})$$

whereas, for $n \geq 3$, $\frac{h^n}{n!} |D_\xi^n O(\xi) \cdot (f_1, \dots, f_n)| \leq \frac{h^n}{n!} \|F(\xi)\|_{h, T_j^n(\xi, X)}$. The result follows by first summing the terms with $n = 0, 1, 2, 3$ with the binomial theorem, and then summing the terms with $n > 3$. ■

E.1 Proof of Theorem 5.6

Consider the measure \mathbb{E}_j and an integrable function $F(\varphi)$. Factorize the covariance into $\Gamma_j = g_j \circ g_j$ and call \mathbb{E}_I the Gaussian expectation with covariance $I = (\delta_{i,j})$; under the imaginary translation $\zeta_x \rightarrow \zeta_x + i(g_j f)_x$ where f is any test function with finite support,

$$\mathbb{E}_j[F(\varphi)] = \mathbb{E}_I[F(\varphi' + (g_j \zeta))] = e^{\frac{1}{2}(f, \Gamma_j f)} \mathbb{E}_j [e^{-i(\zeta, f)} F(\varphi + i(\Gamma_j f))] .$$

Now apply this identity with $F(\varphi) = \widehat{K}_j(q, \varphi, X)$: calling $\psi_x := (\Gamma_j f)_x$ and $(\delta\psi)_x := (\Gamma_j f)_x - (\Gamma_j f)_{x_0}$, by (4.3),

$$\begin{aligned} \mathbb{E}_j \left[\widehat{K}_j(q, \varphi, X) \right] &= e^{\frac{1}{2}(f, \Gamma_j f)} \mathbb{E}_j \left[e^{-i(\zeta, f)} \widehat{K}_j(q, \varphi + i\psi, X) \right] \\ &= e^{\frac{1}{2}(f, \Gamma_j f) - \alpha q (\delta_{x_0, \Gamma_j f})} \mathbb{E}_j \left[\widehat{K}_j(q, \varphi + i\delta\psi, X) \right] \end{aligned} \quad (\text{E.5})$$

where $(\delta_{x_0})_x := \delta_{x, x_0}$. Then, choosing $f_x = \alpha \text{sgn}(q) \delta_{x, x_0}$ as in [14],

$$\begin{aligned} &\| \mathbb{E}_j \left[\widehat{K}_j(q, \varphi, X) \right] \|_{h, T_{j+1}(\varphi', X)} \\ &\leq e^{-(|q| - \frac{1}{2}) \alpha^2 \Gamma_j(0)} \mathbb{E}_j \left[\| \widehat{K}_j(q, \varphi + i\delta\psi, X) \|_{h, T_{j+1}(\varphi', X)} \right] . \end{aligned} \quad (\text{E.6})$$

This inequality is an important result of [14]. Now consider the expectation on the r.h.s. of (E.6): for $\rho := 5L^{-1}$, $\eta := \zeta + i\delta\psi$, $\Delta := \|\delta\psi\|_{\mathcal{C}_j^2(X)}$, since $\|e^{iq\varphi_{x_0}}\|_{h, T_{j+1}(\varphi', X)}$ is less than $e^{h|q|\alpha}$, by (4.3), (E.1), (E.2), (B.2), and for L so large that $\rho h + \Delta \leq h$,

$$\begin{aligned} \| \widehat{K}_j(q, \varphi + i\delta\psi, X) \|_{h, T_{j+1}(\varphi', X)} &\leq e^{h|q|\alpha} \| \widehat{K}_j(q, \delta\varphi' + \eta, X) \|_{h, T_{j+1}(\varphi', X)} \\ &\leq e^{h|q|\alpha} \| \widehat{K}_j(q, \xi + \eta, X) \|_{\rho h, T_j(\xi, X)} \Big|_{\xi := \delta\varphi'} \\ &\leq e^{h|q|\alpha} \| \widehat{K}_j(q, \xi + \zeta, X) \|_{\rho h + \Delta, T_j(\xi, X)} \Big|_{\xi := \delta\varphi'} \\ &\leq e^{h|q|\alpha} \| K_j(X) \|_{h, T_j(X)} G_j(\varphi, X) . \end{aligned} \quad (\text{E.7})$$

(The final inequality is due to the fact that $G_j(\varphi, X)$ depends on the derivatives of φ , and then $G_j(\xi + \zeta, X) = G_j(\varphi, X)$.) Finally, plugging (E.7) into (E.6), for any $\vartheta > 0$ and L large enough,

$$\| \mathbb{E}_j \left[\widehat{K}_j(q, \varphi, X) \right] \|_{h, T_{j+1}(\varphi', X)}$$

$$\leq e^{c(\alpha)|q|} L^{-(2|q|-1)\frac{\alpha^2}{4\pi}} \|K_j\|_{h,T_j} \left(\frac{A}{2}\right)^{-|X|_j} G_{j+1}(\varphi', \bar{X}). \quad (\text{E.8})$$

This inequality proves Theorem 5.6.

Remark. In order to minimize the prefactor in the r.h.s. of (E.5), one should rather choose $f_x = \alpha q \delta_{x,x_0}$; then, the prefactor in (E.6) would be $e^{-\frac{q^2}{2}\alpha^2 \Gamma_j(0)}$, as in heuristic arguments. Though this choice conflicts with the condition $\rho h + \Delta \leq h$, for the corresponding Δ grows with q , whereas the radius of analyticity, h , is assumed independent of q .

E.2 Proof of Theorem 5.5

With Taylor remainder in $\delta\varphi'$, by (E.1), (E.3),

$$\begin{aligned} \|\text{Rem}_2 \widehat{K}_j(0, \delta\varphi' + \zeta, X)\|_{h,T_{j+1}(\varphi', X)} &\leq \|\text{Rem}_2 \widehat{K}_j(0, \xi + \zeta, X)\|_{\rho h, T_j(\xi, X)} \Big|_{\xi=\delta\varphi'} \\ &\leq \left(1 + (\rho h)^{-1} \|\xi\|_{C_j^2(X)}\right)^3 \sup_{t \in [0,1]} \|\widehat{K}_j(0, t\xi + \zeta, X)\|_{\rho h, T_j^{\geq 3}(t\xi, X)} \Big|_{\xi=\delta\varphi'} \\ &\leq C \left(1 + L \|\xi\|_{C_j^2(X)}\right)^3 \rho^3 \sup_{t \in [0,1]} \|\widehat{K}_j(0; t\xi + \zeta, X)\|_{h, T_j(t\xi, X)} \Big|_{\xi=\delta\varphi'} \\ &\leq C' \left(1 + L \|\xi\|_{C_j^2(X)}\right)^3 L^{-3} \|K_j(X)\|_{h, T_j(X)} \sup_{t \in [0,1]} G_j(t\varphi' + \zeta, X). \end{aligned} \quad (\text{E.9})$$

(We used that, as $\rho = 5L^{-1} < 1$, then $\|\cdot\|_{\rho h, T_j^{\geq 3}} \leq \rho^3 \|\cdot\|_{h, T_j}$.) As $X \in \mathcal{S}_j$,

$$L \|\delta\varphi'\|_{C_j^2(X)} \leq C \max_{p=1,2} \|\nabla_{j+1}^p \varphi'\|_{L^\infty(X^*)}. \quad (\text{E.10})$$

Therefore, by (4.3), (5.16) a bound for $\|\text{Rem}_2 \mathbb{E}_j [\widehat{K}_j(0, \varphi, X)]\|_{h, T_{j+1}(\varphi', X)}$ is

$$C \frac{L^{-3}}{\kappa_L^{3/2}} \|K_j(X)\|_{h, T_j(X)} \left(\frac{A}{2}\right)^{-|X|_j} G_{j+1}(\varphi', \bar{X}), \quad (\text{E.11})$$

which proves Theorem 5.5.

E.3 Proof of Theorem 5.7

Set $\eta := \zeta + i\delta\psi$. By (E.1), (E.3), (E.2), and since, by definition of h , $\frac{h}{2} + \Delta \leq h$,

$$\begin{aligned} \|\text{Rem}_0 \widehat{K}_j(q, \delta\varphi' + \eta, X)\|_{h, T_{j+1}(\varphi', X)} &\leq \|\text{Rem}_0 \widehat{K}_j(q, \xi + \eta, X)\|_{\rho h, T_j(\xi, X)} \Big|_{\xi=\delta\varphi'} \\ &\leq C \left(1 + L \|\xi\|_{C_j^2(X)}\right) \sup_{0 \leq t \leq 1} \|\widehat{K}_j(q, t\xi + \eta, X)\|_{\rho h, T_j^{\geq 1}(t\xi, X)} \Big|_{\xi=\delta\varphi'} \end{aligned}$$

$$\begin{aligned}
&\leq C' \left(1 + L\|\xi\|_{\mathcal{C}_j^2(X)}\right) L^{-1} \sup_{0 \leq t \leq 1} \|\widehat{K}_j(q, t\xi + \eta, X)\|_{\frac{h}{2}, T_j(\xi, X)} \\
&\leq C' \left(1 + L\|\xi\|_{\mathcal{C}_j^2(X)}\right) L^{-1} \sup_{0 \leq t \leq 1} \|\widehat{K}_j(q, t\xi + \zeta, X)\|_{h, T_j(\xi, X)} \\
&\leq C' \left(1 + L\|\xi\|_{\mathcal{C}_j^2(X)}\right) L^{-1} \|K_j(X)\|_{h, T_j} \sup_{0 \leq t \leq 1} G_j(t\varphi' + \zeta, X). \quad (\text{E.12})
\end{aligned}$$

As for (E.6), a bound for $\|\mathbb{E}_j \left[\text{Rem}_0 \widehat{K}_j(q, \varphi, X) \right]\|_{h, T_{j+1}(\varphi', X)}$ is

$$e^{-(|q|-\frac{1}{2})\alpha^2\Gamma_j(0)} \mathbb{E}_j \left[\|\text{Rem}_0 \widehat{K}_j(q, \varphi + i\delta\psi, X)\|_{h, T_{j+1}(\varphi', X)} \right]; \quad (\text{E.13})$$

so that, continuing the chain of inequalities by (E.10) and (E.12), (5.16),

$$\begin{aligned}
&\leq e^{-(|q|-\frac{1}{2})\alpha^2\Gamma_j(0)} e^{h|q|\alpha} \mathbb{E}_j \left[\|\text{Rem}_0 \widehat{K}_j(q, \delta\varphi' + \eta, X)\|_{h, T_{j+1}(\varphi', X)} \right] \\
&\leq C e^{-(|q|-\frac{1}{2})\alpha^2\Gamma_j(0)} e^{h|q|\alpha} \frac{L^{-1}}{\kappa_L^{3/2}} \|K_j\|_{h, T_j} \left(\frac{A}{2}\right)^{-|X|_j} G_{j+1}(\varphi', \bar{X}), \quad (\text{E.14})
\end{aligned}$$

which proves Theorem 5.7.

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