Exact RG computation of the optical conductivity of graphene

Alessandro Giuliani[1] and Vieri Mastropietro[2]

[1] *Università di Roma Tre, L.go S. L. Murialdo 1, 00146 Roma, Italy*

[2] *Università di Roma Tor Vergata, Viale della Ricerca Scientifica 00133 Roma, Italy*

The optical conductivity of a system of electrons on the honeycomb lattice interacting through an electromagnetic field is computed by truncated exact Renormalization Group (RG) methods. We find that the conductivity has the universal value $\pi/2$ times the conductivity quantum up to negligible corrections vanishing as a power law in the limit of low frequencies.

PACS numbers: 71.10.Fd, 72.80.Vp, 05.10.Cc, 05.30.Fk

Among the remarkable properties of graphene[1], the optical conductivity is of special interest. Recent experiments[2] found that the conductivity in monolayer graphene is essentially constant in a wide range of frequencies between the temperature and the band-width. The observed value of the conductivity is equal, within experimental errors, to $\sigma_0 = \pi e^2/(2h)$, a universal value that only depends on fundamental constants and not on the material parameters like the Fermi velocity. This fact can be nicely explained in terms of the standard graphene's model of massless Dirac particles in 2+1 dimensions: in this case, by neglecting interactions and disorder, Ref.[3] predicted a value of the conductivity $\sigma_\beta(\omega)$ at inverse temperature β and frequency ω satisfying $\lim_{\omega\to 0}\lim_{\beta\to\infty}\sigma_\beta(\omega) = \sigma_0$. The inclusion of lattice effects does not change the value of this limit[4].

This remarkable agreement with a theoretical value computed by *neglecting* many body interaction is, however, surprising and needs an explanation[5]. Indeed, the strength of the interactions in graphene is measured by the ratio $\alpha = \frac{e^2}{\hbar v_0} \sim 2.2$ (e is the electric charge and v_0 the Fermi velocity), which is 300 times larger than the usual fine structure constant. The effects of the interactions are clearly seen in experiments on the Fermi velocity[6]. Therefore, why is not there an essential many body renormalization of the optical conductivity, too?

On the theoretical side, a theorem proved in Ref.[7] establishes that the conductivity of electrons hopping on the honeycomb lattice and interacting via a weak Hubbard interaction is equal to σ_0 in the limit $\omega \to 0$. Note that, even if dimensionally irrelevant, the interaction can produce finite many body renormalizations: for instance, the Fermi velocity is *renormalized* by the interaction. Therefore, the universality of the conductivity is a non trivial statement, following from an exact cancellation of all the many body corrections.

It is, however, believed that the interaction in clean suspended graphene is not at all short-ranged as the Hubbard interaction (no screening), so that a more realistic description of the clean system requires the inclusion of the long-ranged electromagnetic (e.m.) interactions. In the case of static Coulomb interactions, Ref.[8] predicted a logarithmic renormalization of the Fermi velocity, namely $v(q) = v_0\big(1 + \frac{\alpha}{4}\log\frac{\varepsilon}{q}\big)$ where q is the momentum measured from the Fermi points and ε is the bandwidth. First attempts to include the effects of a Coulomb potential on the conductivity[9] led to the conclusion that the interaction radically changes its behavior, that is $\lim_{\omega\to 0}\sigma(\omega) = 0$, where $\sigma(\omega)$ is the conductivity in the limit of zero temperature. Later, Ref.[10,11] obtained the qualitatively different result $\lim_{\omega\to 0}\sigma(\omega) = \sigma_0$, based on scaling arguments. In particular, Ref.[10] found the formula

$$\sigma(\omega) = \sigma_0\big[1 + O(\frac{1}{\log(\varepsilon/\omega)})\big] \ . \tag{1}$$

Note that the inverse logarithmic correction in Eq.(1) is a consequence of the logarithmic divergence of the Fermi velocity, and should be read as $O(\alpha v_0/v(\omega))$. As pointed out in Ref.[5], this correction is in general *larger* than the experimental error[2]. Ref.[12,13] proposed that the way out from this apparent contradiction should be found in the constant in front of the inverse log corrections, whose correct value should be much smaller than the one computed in Ref.[10]. However, Ref.[14] raised objections against the new value proposed in Ref.[12,13], because the regularizations used in these works can produce unphysical results. The disagreement between the big (inverse log) corrections to the conductivity and the experimental data suggested[5] to phenomenologically postulate a Fermi liquid description of the interacting system: this assumption implies that the universal conductivity is reached at low frequencies polynomially fast (i.e., as $\sim \omega^2$) but is in contrast with the experiments in Ref.[6].

Eq.(1) was derived by assuming that the electrons interact via a static Coulomb interaction: however, the logarithmic increase of the Fermi velocity shows that the assumption of instantaneous interactions becomes unphysical at very low energy scales[15]. Therefore, the use of Eq.(1) and of the divergence of the Fermi velocity to predict the universality of the conductivity as $\omega \to 0$ is questionable. The unbounded increase of the Fermi velocity is absent in the case that the interaction with the e.m. field is introduced via the Peierls substitution in order to preserve gauge invariance. It is well known[16,17] that in this case the Fermi velocity stops flowing at the speed of light c and Lorentz symmetry spontaneously emerges in the infrared. We compute the optical conductivity at imaginary frequency ω in a lattice gauge invariant model for graphene using *truncated exact RG methods*. We find an expression that is qualitatively different from Eq.(1), namely[18]

$$\sigma(\omega) = \sigma_0\big[1 + O(\frac{\omega}{\varepsilon}\log\frac{\varepsilon}{\omega})\big]\ , \tag{2}$$

which is very close to the universal one at low frequencies, up to a really negligible power law correction, compatible with the experimental results in Ref.[2]. The $\omega\log\omega$ dependence of the correction is not necessarily optimal, it may just be a byproduct of our estimates.

We derived Eq.(2) under the assumption that the values of the bare parameters are sufficiently close to the infrared fixed point (i.e., the bare Fermi velocity v_0 is sufficiently large). The extension of its validity to real frequencies and to a larger range of parameters, including those measured in actual graphene's samples, requires a microscopic justification that is quite difficult in view of the strength of interactions in graphene[19]; of course, this is a caveat that applies to all the approaches based on expansions, resummations and truncations. In any case, it is reassuring to see that encoding a fundamental physical principle like gauge invariance into the model is sufficient to obtain results that are in good qualitative agreement with the experimental data, in particular with the observed dramatic increase of the Fermi velocity and with the universality of the conductivity up to negligible power law corrections at low frequencies.

The model we consider was defined in detail in Ref.[17]. Let us just remind here the main definitions. The grand-canonical Hamiltonian at half-filling is $H = H_0 + H_C + H_A$, with

$$H_0 = -t\sum_{\substack{\vec x\in\Lambda_A\\ j=1,2,3}}\sum_{\sigma=\uparrow\downarrow} a^+_{\vec x,\sigma} b^-_{\vec x+\vec\delta_j,\sigma} e^{ie\int_0^1 \vec\delta_j\cdot\vec A(\vec x+s\vec\delta_j,0)\,ds} + c.c.$$

the gauge invariant nearest neighbor hopping term (here t is the hopping strength, $\vec\delta_j$ the nearest neighbor vectors and $a^\pm, b^\pm$ the creation/annihilation operators of electrons sitting at the sites of the A or B sublattice of the honeycomb lattice),

$$H_C = \frac{e^2}{2}\sum_{\vec x,\vec y\in\Lambda_A\cup\Lambda_B}(n_{\vec x}-1)\varphi(\vec x-\vec y)(n_{\vec y}-1)\ ,$$

where e is the electric charge, $\hat\varphi_{\vec p}$ is a regularized version of the static Coulomb potential and $n_{\vec x}$ the electron number at site $\vec x$. Finally, H_A is the energy (in the presence of an ultraviolet cutoff) of the three-dimensional photon field $\underline{A} = (\vec A, A^3)$ in the Coulomb gauge. Units are fixed in such a way that the speed of light $c = 1$. Note that the interaction with the quantum e.m. field is introduced via the Peierls substitution in order to preserve Gauge invariance.

Proceeding as in Ref.[7], where we computed the conductivity in the case of short range interactions, we define a "space-time" three-components vector $\hat J_{\vec p,\mu}$, $\mu = 0,1,2$, with

$$\hat J_{\vec p,0} = e\sum_{\substack{\vec x\in\Lambda_A\\ \sigma=\uparrow\downarrow}} e^{-i\vec p\vec x} a^+_{\vec x,\sigma}a^-_{\vec x,\sigma} + \sum_{\substack{\vec x\in\Lambda_B\\ \sigma=\uparrow\downarrow}} e^{-i\vec p\vec x} b^+_{\vec x,\sigma}b^-_{\vec x,\sigma}\ , \tag{3}$$

the density operator and $\hat J_{\vec p,1}, \hat J_{\vec p,2}$ the two components of the *paramagnetic current*

$$\vec J_{\vec p} = iet\sum_{\substack{\vec x\in\Lambda\\ \sigma,j}} e^{-i\vec p\vec x}\vec\delta_j\eta^j_{\vec p}\big(a^+_{\vec x,\sigma}b^-_{\vec x+\vec\delta_j,\sigma} - b^+_{\vec x+\vec\delta_j,\sigma}a^-_{\vec x,\sigma}\big)$$

where $\eta^j_{\vec p} = (1-e^{-i\vec p\vec\delta_j})/(i\vec p\vec\delta_j)$. Let also $\mathbf{p} = (\omega,\vec p)$, with ω the Matsubara frequency, and $\hat K_{\mu\nu}(\mathbf{p})$ is the current-current response function, i.e., the Fourier transform of $\lim_{\beta\to\infty}\langle J_{\mathbf{x},\mu}; J_{\mathbf{y},\nu}\rangle_\beta$.

We are interested in the *conductivity*, defined via Kubo formula as[4,7] (here $l,m = 1,2$):

$$\sigma_{lm}(\omega) = -\frac{2}{3\sqrt3}\frac{1}{\omega}\Big[\hat K_{lm}(\omega,\vec 0) + \hat\Delta_{lm}(\vec 0)\Big]\ ,$$

where $3\sqrt3/2$ is the area of the hexagonal cell of the honeycomb lattice and

$$\hat\Delta_{lm}(\vec p) = \lim_{\beta,L\to\infty}\frac{1}{L^2}\sum_{\substack{\vec x\in\Lambda\\ j=1,2,3}}(\vec\delta_j)_l(\vec\delta_j)_m|\eta^j_{\vec p}|^2\langle\Delta_{\vec x,j}\rangle_\beta\ ,$$

with $\Delta_{\vec x,j} = -e^2t\sum_\sigma(a^+_{\vec x,\sigma}b^-_{\vec x+\vec\delta_j,\sigma} + b^+_{\vec x+\vec\delta_j,\sigma}a^-_{\vec x,\sigma})$ the *diamagnetic tensor*.

The current-current response function can be computed via the generating functional that, in the Feynman gauge, reads

$$e^{\mathcal W_{h^*}(J,\lambda)} = \int P(d\psi)\int P_{h^*}(dA)e^{\mathcal V(A+J,\psi)+(\psi,\lambda)} \tag{4}$$

which has been studied in great detail in Ref.[17]. In Eq.(4): (i) $\psi^\pm_{\mathbf{k},\sigma}$ are Grassman spinors (of the form $\psi = (a,b)$, with a and b the electron fields associated to the two sublattices of the honeycomb net) and $P(d\psi)$ is the fermionic gaussian integration with propagator

$$g(\mathbf{k}) = -\frac{1}{Z_0}\begin{pmatrix} ik_0 & v_0\Omega^*(\vec k)\\ v_0\Omega(\vec k) & ik_0\end{pmatrix}^{-1} \tag{5}$$

where $Z_0 = 1$ is the free wave function renormalization, $v_0 = \frac32 t$ is the free Fermi velocity and $\Omega(\vec k) = \frac23\sum_{j=1,2,3}e^{i\vec k(\vec\delta_j-\vec\delta_1)}$ is the complex dispersion relation. Note that $g(\mathbf{k})$ is singular only at the Fermi points $\mathbf{p}^r_F = (0,\frac{2\pi}{3}, r\frac{2\pi}{3\sqrt3})$, where $r = \pm$ is the *valley index*. Moreover, $A_\mu(\mathbf{p})$, $\mu = 0,1,2$, are the Fourier transform of real gaussian variables and $P_{h^*}(dA)$ is the gaussian integration with propagator $w_{\mu\nu}(\mathbf{p}) = \delta_{\mu\nu}(2|\mathbf{p}|)^{-1}\chi_{[h^*,0]}(|\vec p|)$, where $\chi_{[h^*,0]}$ is a smooth compact support function that acts both as an ultraviolet cutoff on scale $|\mathbf{p}|\sim 1$ *and* as an infrared cutoff on scale $|\mathbf{p}| \sim 2^{h^*}$ (to be eventually removed). Finally $\mathcal{V}$ is the interaction whose explicit form can be easily inherited from H[17]. The current-current response function can be obtained by taking the limit $h^* \to -\infty$ and by deriving twice with respect to the external field J and then setting $J = \lambda = 0$. Field-field correlations or field-current correlations can be obtained similarly, by suitably deriving with respect to the external fields λ and/or J. Note that in writing the generating functional as in Eq.(4) we exploited gauge invariance and, more precisely, the equivalence between the Feynman and the Coulomb gauges. Another crucial consequence of gauge invariance is the following equation

$$0 = \frac{\partial}{\partial \hat{\alpha}_{\mathbf{p}}} \mathcal{W}(\Phi, J + \partial\alpha, \lambda e^{ie\alpha})\Big|_{\hat{\alpha}=0} . \qquad (6)$$

By performing derivatives with respect to the external fields, this equation also implies a sequence of exact lattice *Ward Identities*, valid for each finite choice of the cutoff scale h^*. In particular, proceeding as in Ref.[7] and defining $p^0 = -i\omega$, the current-current response function satisfies the Ward Identities $\sum_{\mu=0}^{2} p^\mu \hat{K}_{\mu 0}(\mathbf{p}) = 0$ and, for $j = 1, 2$,

$$\sum_{\mu=0}^{2} p^\mu \hat{K}_{\mu j}(\mathbf{p}) = -\sum_{i=1}^{2} p_i \hat{\Delta}_{ij}(\vec{p}) . \qquad (7)$$

An immediate consequence of Eq.(7) *and* of the continuity of $\hat{K}_{\mu,\nu}(\mathbf{p})$ in $\mathbf{p} = \mathbf{0}$ (proved at all orders of renormalized perturbation theory[17]) is that, for $i, j \in \{1, 2\}$,

$$\sigma_{ij}(\omega) = -\frac{2}{3\sqrt{3}} \frac{1}{\omega} [\hat{K}_{ij}(\omega, 0) - \hat{K}_{ij}(0, \vec{0})] , \qquad (8)$$

see[7] for the simple argument leading to Eq.(8).

The generating function (4) can be computed by exact RG methods[17], which allowed us to prove that the response functions can be written in terms of a renormalized perturbation theory that is finite at all orders in the effective coupling constants, with explicit bounds on the n-th order contributions. In particular, after the integration of the degrees of freedom corresponding to momenta larger than 2^h, $h < 0$, we rewrite (setting for simplicity $\lambda = 0$): $e^{\mathcal{W}_{h^*}(J,0)} =$

$$= \int \prod_{r=\pm} P(d\psi_r^{(\leq h)}) P(dA^{(\leq h)}) e^{\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)},\, A^{(\leq h)}+J)} ,$$

where $P(d\psi_r^{(\leq h)})$ and $P(dA^{(\leq h)})$ have propagators

$$\hat{g}_r^{(\leq h)}(\mathbf{k}') = -\frac{\chi_h(\vec{k}')}{Z_h} \begin{pmatrix} ik_0 & v_h \Omega^*(\vec{k}' + \vec{p}_F^{\,r}) \\ v_h \Omega(\vec{k}' + \vec{p}_F^{\,r}) & ik_0 \end{pmatrix}^{-1} \qquad (9)$$

and $w_{\mu\nu}^{(\leq h)}(\mathbf{p}) = \delta_{\mu\nu} (2|\mathbf{p}|)^{-1} \chi_{[h^*,h]}(\vec{p})$, where: (i) $\chi_h(\vec{k}')$ is a smooth cutoff function vanishing for momenta larger than $|\vec{k}'| \sim 2^h$; (ii) $\chi_{[h^*,h]}(\vec{p}) = \chi_h(\vec{p}) - \chi_{h^*}(\vec{p})$; (iii) Z_h, v_h are the effective wave function renormalization and Fermi velocity at scale h. Moreover $\mathcal{V}^{(h)}$ is the *effective potential*, expressed by a sum of monomials in $\psi^{(\leq h)}, A^{(\leq h)}$ of any degree:

$$\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}, A^{(\leq h)}) = \int \frac{d\mathbf{p}}{(2\pi)^3} \big[Z_h^{(\mu)} e \hat{\jmath}_{\mu,\mathbf{p}}^{(\leq h)} \hat{A}_{\mu,\mathbf{p}}^{(\leq h)} - 2^h \nu_{\mu,h} \hat{A}_{\mu,-\mathbf{p}}^{(\leq h)} \hat{A}_{\mu,\mathbf{p}}^{(\leq h)} \big] + \mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}, A^{(\leq h)}) \qquad (10)$$

where $\mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}, A^{(\leq h)})$ is the *irrelevant part* of the effective potential (sum of all the terms with more than three fields) and

$$\hat{\jmath}_{\mu,\mathbf{p}}^{(\leq h)} := \frac{i}{\beta L^2} \sum_{r,\sigma,\mathbf{k}'} \hat{\psi}_{\mathbf{k}'+\mathbf{p},\sigma,r}^{(\leq h)+} \Gamma_r^\mu(\vec{k}' + \vec{p}_F^{\,r}) \hat{\psi}_{\mathbf{k}',\sigma,r}^{(\leq h)-} , \qquad (11)$$

with $\Gamma^0(\vec{k}) = \mathbb{1}$ and

$$\Gamma^i(\vec{k}) = \frac{2}{3} \sum_{j=1}^{3} (\vec{\delta}_j)_i \begin{pmatrix} 0 & ie^{-i\vec{k}(\vec{\delta}_j - \vec{\delta}_1)} \\ -ie^{i\vec{k}(\vec{\delta}_j - \vec{\delta}_1)} & 0 \end{pmatrix} .$$

We can summarize the previous discussion by saying that after the integration of the degrees of freedom corresponding to momenta $\geq 2^h$, we get an effective theory that is qualitatively very similar to the original one, modulo the renormalization of a finite number of effective parameters, namely the Fermi velocity v_h, the wave function renormalization Z_h, the vertex function $Z_h^{(\mu)}$ and the photon mass $\nu_{\mu,h}$. The discrete rotational symmetries of the model imply that $Z_h^{(1)} = Z_h^{(2)}$ and $\nu_{1,h} = \nu_{2,h}$.

These parameters verify suitable flow equations well defined at all orders in the renormalized expansion: this is an instance of the renormalizability at all orders of the model. Note that the effective charges at scale h are: $e_{0,h} = eZ_h^{(0)}/Z_h$ and $e_{i,h} = eZ_h^{(i)}/(Z_h v_h)$, for $i \in \{1, 2\}$. Thanks to the WIs induced by Eq.(6), we proved in Ref.[17] that $\nu_{\mu,h} = O(e^2 2^h)$ (i.e., the effective photon mass vanishes in the infrared) and that the beta function for the effective charge is asymptotically vanishing, i.e.,

$$e_{0,h} = e\frac{Z_h^{(0)}}{Z_h} = e\big(1 + O(e_{\mu,h}^2)\big) ,$$
$$e_{i,h} = e\frac{Z_h^{(i)}}{Z_h v_h} = e\big(1 + O(e_{\mu,h}^2)\big) \qquad (12)$$

and $e_{0,-\infty} = e_{1,-\infty} = e_{2,-\infty}$. Moreover, see[17], the wave function renormalization diverges in the infrared, while the effective Fermi velocity increases up to the speed of light, both approaching their limits with an anomalous power law:

$$Z_h \sim 2^{-\eta h} \qquad 1 - v_h \sim 2^{\tilde{\eta} h} \qquad (13)$$

with $\eta = \frac{e_{-\infty}^2}{12\pi^2} + O(e_{-\infty}^4)$ and $\tilde{\eta} = \frac{2e_{-\infty}^2}{5\pi^2} + O(e_{-\infty}^4)$ the two critical exponents.

The above integration procedure leads to an expansion of the conductivity in terms of powers of $e_{\mu,h}$; such renormalized expansion is a resummation of the naive perturbative expansion in e. It must be stressed that there is a big difference between these two expansions: while the one in $e_{\mu,h}$ is order by order finite (with explicit bounds on the growth of the n-th order contributions[17]), the naive one in e is plagued by $O(\log^n\omega)$ divergences at order n. Therefore, the truncation of the renormalized expansion is expected to give much more accurate predictions than the naive one.

By truncating the exact RG expression for the conductivity at one loop, we get contributions from the bubble diagrams in Fig.1,

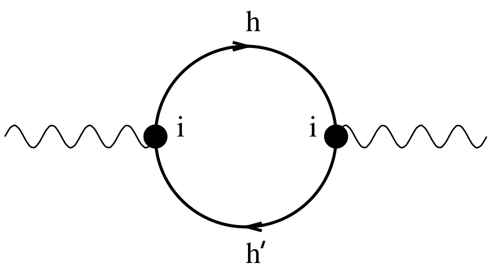

FIG. 1. The one-loop bubble diagram contributing to the longitudinal conductivity $\sigma_{ii}(\omega)$. The labels h, h' indicate the scale of the two loop propagators, which depend on the dressed Fermi velocities $v_h, v_{h'}$ and on the effective wave function renormalizations $Z_h, Z_{h'}$. A summation over h, h' is understood. The big dots correspond to dressed vertex functions $Z^{(i)}_{\bar h}$, with $\bar h = \max\{h, h'\}$. The external momentum flowing in the wavy lines is $\mathbf{p} = (\omega, \vec 0)$.

which give (defining $\sigma_0 = \frac{\pi}{2}\frac{e^2}{h}$):

$$\frac{\sigma_{ii}(\omega)}{\sigma_0} = \frac{16}{3\sqrt{3}}\frac{1}{\omega}\sum_{h,h'\le 0}^{r=\pm}\int\frac{dk_0}{2\pi}\int_{\mathcal{B}}\frac{d\vec k'}{|\mathcal{B}|}\frac{(Z^{(i)}_{\bar h})^2}{Z_h Z_{h'}}\cdot \tag{14}$$
$$\cdot\mathrm{Tr}\Big\{\Gamma^i_r(\vec k')g^{(h)}_r(\mathbf{k}')\Gamma^i_r(\vec k')\big[g^{(h)}_r(\mathbf{k}'+(\omega,\vec 0)) - g^{(h)}_r(\mathbf{k}')\big]\Big\}\,.$$

where: (i) $\mathcal{B}$ is the first Brillouin zone and $|\mathcal{B}| = 8\pi^2/(3\sqrt{3})$ its area; (ii) $\bar h = \max\{h, h'\}$; (iii) $\Gamma^i_r(\vec k') = \Gamma^i(\vec k' + \vec p^{\,r}_F)$; (iv) $g^{(h)}_r$ is the effective propagator on scale h, given by the same expression as Eq.(9) with $\chi_h(\vec k')$ replaced by the smooth compact support function $f_h(\vec k') := \chi_h(\vec k') - \chi_{h-1}(\vec k')$, which is non vanishing only if $\vec k'$ is on scale h, i. e., $2^{h-1} \le |\vec k'| \le 2^{h+1}$. Note that the effective parameters $Z^{(i)}_h, Z^h, v_h$ entering Eq.(14) are all functions of e: if we expanded them in e we would recover infinitely many graphs of the naive perturbation theory, all plagued by logarithmic divergences. Note also that Eq.(14) is not simply the "bubble graph" with the dressed propagator and vertices: e.g., if one thinks of the dressed propagator with momentum $\mathbf{k}$ as being obtained by resummations of the chain of self-energies, one has to take into account that the scales of the momenta flowing inside such self-energy sub-diagrams are higher than the scale of $\mathbf{k}$, according to the rules of exact RG (which avoid the problem of overlapping divergences and, correspondingly, the emergence of $n!$ factors at higher orders).

The computation of Eq.(14) can be explicitly performed, by making use of Eq.(12) and by carefully exploiting symmetry cancellations that make the apparent logarithmic divergence of Eq.(14) finite, see Appendix A for details. The result is Eq.(2).

Our analysis is based on a truncation of the exact RG equations, and the question of how to generalize it to the full RG expansion is a very interesting and important theoretical problem; so far, we succeeded in performing the full RG computation only in the case of short range interactions[7]. Another important open problem is to understand the analytic extension of the conductivity to real frequencies.

In conclusion, we considered a model of electrons on the honeycomb lattice interacting via a quantized photon field, previously investigated in Ref.[17]. The coupling with the e.m. field is introduced via the Peierls substitution in order to preserve gauge invariance. We showed that at low frequencies the conductivity is equal to the universal value σ_0 up to corrections $O(\omega\log\omega)$, which are much smaller than the $1/\log\omega$ corrections found for static Coulomb interactions. Our results are a priori valid close to the infrared fixed point and the extension of their validity to a larger range of bare parameters (including those measured in actual graphene's samples) is based on a phenomenological assumption. Still, it is reassuring to see that it is enough to encode gauge invariance in a microscopic model for clean graphene to recover good qualitative agreement of the predictions with the experimental data.

Acknowledgements We acknowledge financial support from the ERC Starting Grant CoMBoS-239694.

Appendix A: Computation of the conductivity

Using the definition of $g^{(h)}_r$ into Eq.(14), we can rewrite (shifting the momenta by $\vec p^{\,r}_F$ and neglecting for consistency terms of order $O(e^4)$, which should be combined with the two-loops contributions)

$$\frac{\sigma_{ii}(\omega)}{\sigma_0} = \frac{4}{\pi^2}\frac{1}{\omega}\sum_{h,h'\le 0}^{r=\pm}\int\frac{dk_0}{2\pi}\int_{\mathcal{B}}d\vec k'\frac{(Z^{(i)}_{\bar h})^2}{Z_h Z_{h'}}\frac{f_h(\vec k')f_{h'}(\vec k')}{k_0^2 + v_h^2|\Omega_r(\vec k')|^2}\cdot$$
$$\cdot\Bigg[\frac{-k_0(k_0+\omega)|a_{i,r}(\vec k')|^2 + v_h v_{h'}\mathrm{Re}\big(\Omega_r^2(\vec k')a^2_{i,r}(\vec k')\big)}{(k_0+\omega)^2 + v_{h'}^2|\Omega_r(\vec k')|^2} +$$
$$+\frac{k_0^2|a_{i,r}(\vec k')|^2 - v_h v_{h'}\mathrm{Re}\big(\Omega_r^2(\vec k')a^2_{i,r}(\vec k')\big)}{k_0^2 + v_{h'}^2|\Omega_r(\vec k')|^2}\Bigg]\,, \tag{A1}$$

where $\Omega_r(\vec k') = \Omega(\vec k' + \vec p^{\,r}_F)$ and

$$a_{i,r}(\vec k') = \frac{2}{3}\sum_{j=1}^{3}(\vec\delta_j)_i\, i e^{-i(\vec k' + \vec p^{\,r}_F)(\vec\delta_j - \vec\delta_1)}\,.$$

Note that $\Omega_r(\vec{k}') = ik'_1 + rk'_2 + O(|\vec{k}'|^2)$, $a_{1,r}(\vec{k}') = i + O(|\vec{k}'|)$ and $a_{2,r}(\vec{k}') = -r + O(|\vec{k}'|)$. The "relativistic approximation" consists in replacing $\Omega_r(\vec{k}')$, $a_{1,r}(\vec{k}')$ and $a_{2,r}(\vec{k}')$ in Eq.(A1) by $ik'_1 + rk'_2$, by i and by $-r$, respectively. By performing this replacement, it becomes apparent that the r.h.s. of Eq.(A1) behaves dimensionally as $\frac{1}{\omega}\int d^3k'[\frac{1}{(k'+\omega)^2} - \frac{1}{(k')^2}]$, which is logarithmically divergent as $\omega \to 0$. In order to prove the finiteness of $\sigma_{ii}(\omega)$ in the low frequency limit, it is necessary to exploit cancellations, which follow from the use of a Ward Identity combined with an essentially explicit computation (using the residues' theorem to integrate k_0 out) of the r.h.s. of Eq.(A1).

Let ε be a small but finite fraction of the bandwidth (say, $\varepsilon = t/10$) and let us distinguish the contributions to the integral coming from the region $v_0|\Omega_r(\vec{k}')| \geq \varepsilon$ from those $v_0|\Omega_r(\vec{k}')| \leq \varepsilon$. The former correspond to non-singular contributions, which can be estimated as follows: we expand in Taylor series the expression in square brackets up to $O(\omega^2)$; we note that the term linear in ω is vanishing by parity in k_0; we bound dimensionally the term quadratic in ω as:

$$(\text{const.})\frac{1}{\omega}\int dk_0 \int_{\varepsilon/v_0}^{1} dk'\, k' \frac{v_0^2\omega^2}{(k_0^2 + v_0^2(k')^2)^2} \leq (\text{const.})\frac{\omega}{\varepsilon}\ . \tag{A2}$$

Let us now look at the the terms coming from the region $v_0|\Omega_r(\vec{k}')| \leq \varepsilon$. Note that the contributions from $r = +$ or $r = -$ are equal among each other, thanks to the symmetry under valley exchange. We introduce some shorthands: we define $\Delta_h = v_h|\Omega_+(\vec{k}')|$ and $W_{h,h'} = v_h v_{h'}\mathrm{Re}\big(\Omega_+^2(\vec{k}')a_{i,+}^2(\vec{k}')\big)$, so that, by performing the integral over k_0 using the residues' theorem, we can write the contribution to the conductivity coming from the region $v_0|\Omega_r(\vec{k}')| \leq \varepsilon$ as

$$\begin{aligned}&\frac{8}{\pi^2}\frac{1}{\omega}\int_{|\vec{k}'|\leq\varepsilon/v_0} d\vec{k}' \sum_{h,h'\leq 0}\frac{(Z_{\bar{h}}^{(i)})^2}{Z_h Z_{h'}} f_h(\vec{k}')f_{h'}(\vec{k}')\,\cdot\\ &\cdot\Big\{\frac{\Delta_h(\Delta_h - i\omega)|a_{i,+}(\vec{k}')|^2 + W_{h,h'}}{2\Delta_h\big[\omega^2 + 2i\omega\Delta_h - (\Delta_h^2 - \Delta_{h'}^2)\big]} +\\ &+\frac{\Delta_{h'}(\Delta_{h'} + i\omega)|a_{i,+}(\vec{k}')|^2 + W_{h,h'}}{2\Delta_{h'}\big[\omega^2 - 2i\omega\Delta_{h'} + (\Delta_h^2 - \Delta_{h'}^2)\big]} +\\ &+\frac{\Delta_h^2|a_{i,+}(\vec{k}')|^2 + W_{h,h'}}{2\Delta_h(\Delta_h^2 - \Delta_{h'}^2)} - \frac{\Delta_{h'}^2|a_{i,+}(\vec{k}')|^2 + W_{h,h'}}{2\Delta_{h'}(\Delta_h^2 - \Delta_{h'}^2)}\Big\}\ .\end{aligned} \tag{A3}$$

Let us first consider in Eq.(A3) the terms with the integrand proportional to $W_{h,h'}$, which can be rewritten as (defining $V_{h,h'} = \Delta_h + \Delta_{h'}$)

$$\int_{|\vec{k}'|\leq\varepsilon/v_0} \frac{d\vec{k}'}{\pi^2}\sum_{h,h'\leq 0}\frac{(Z_{\bar{h}}^{(i)})^2}{Z_h Z_{h'}}\frac{W_{h,h'}}{\Delta_h\Delta_{h'}}\frac{-4\omega f_h(\vec{k}')f_{h'}(\vec{k}')}{V_{h,h'}(\omega^2 + V_{h,h'}^2)}\ . \tag{A4}$$

Now note that $W_{h,h'} = v_h v_{h'}\big[(k'_2)^2 - (k'_1)^2 + O(|\vec{k}'|^3)\big]$: therefore, the term proportional to $(k'_2)^2 - (k'_1)^2$ is zero by symmetry, and we are left with a contribution dimensionally bounded as (using that Δ_h, $\Delta_{h'}$ and $V_{h,h'}$ behave dimensionally as $\sim k'$ close to the singularity)

$$(\text{const.})\,\omega\int_0^{\varepsilon/v_0} dk'\, k' \frac{v_0(k')^3}{(k')^3(\omega^2 + v_0^2(k')^2)} \leq (\text{const.})\,\frac{\omega}{v_0}\log\frac{\varepsilon}{\omega}\ , \tag{A5}$$

for $\omega \ll \varepsilon$. We are left with the terms obtained by replacing $W_{h,h'}$ with 0 in Eq.(A3), which are given by

$$\begin{aligned}&\frac{4}{\pi^2}\omega\int_{|\vec{k}'|\leq\varepsilon/v_0} d\vec{k}' \sum_{h,h'\leq 0}\frac{(Z_{\bar{h}}^{(i)})^2}{Z_h Z_{h'}} f_h(\vec{k}')f_{h'}(\vec{k}')\,\cdot\\ &\cdot\frac{|a_{i,+}(\vec{k}')|^2}{|\Omega_+(\vec{k}')|(v_h + v_{h'})\big[\omega^2 + (v_h + v_{h'})^2|\Omega_+(\vec{k}')|^2\big]}\ .\end{aligned}$$

Using the Ward Identity Eq.(12) and the rewritings $\Omega_+(\vec{k}') = ik'_1 + k'_2 + O(|\vec{k}'|^2)$, $a_{1,+}(\vec{k}') = i + O(|\vec{k}'|)$ and $a_{2,+}(\vec{k}') = -1 + O(|\vec{k}'|)$ (valid close to the singularity $\vec{k}' = \vec{0}$), this is equal (up to terms coming from the "non-relativistic" parts of $\Omega_+(\vec{k}')$ and $a_{i,+}(\vec{k}')$, which are bounded as in Eq.(A5)) to

$$\frac{4}{\pi}\omega\sum_{h,h'\leq 0}\int_0^{\varepsilon v_{\bar{h}}/v_0} dk' \frac{Z_{\bar{h}}^2}{Z_h Z_{h'}}\frac{f_h(k')f_{h'}(k')}{(\frac{v_h + v_{h'}}{2v_{\bar{h}}})\big[\omega^2 + (\frac{v_h + v_{h'}}{2v_{\bar{h}}})^2 4(k')^2\big]}\ . \tag{A6}$$

By the compact support properties of f_h, the integrand is non vanishing only of $|h - h'| \leq 1$, in which case

$$\frac{v_h + v_{h'}}{v_{\bar{h}}} = 1 + O(e^2)\ ,\qquad \frac{Z_{\bar{h}}^2}{Z_h Z_{h'}} = 1 + O(e^2)\ ,$$

uniformly in h. Therefore, Eq.(A6) is equal to

$$\int_0^{\varepsilon}\frac{dk'}{\pi}\frac{4\omega}{\big[\omega^2 + 4(k')^2\big]} = \frac{2}{\pi}\arctan(\frac{2\varepsilon}{\omega}) = 1 - \frac{\omega}{\varepsilon\pi} + O(\omega^2)\ , \tag{A7}$$

up to terms bounded by $(1 - v_0)\omega/\varepsilon$ and terms bounded *uniformly* in ω by

$$(\text{const.})e^2\int_0^{\varepsilon} dk' \frac{\omega}{(k')^2 + \omega^2} \leq (\text{const.})e^2\ . \tag{A8}$$

Of course, these correction terms should be neglected, for consistency, because Eq.(14) is obtained by truncating the exact RG expansion at one loop. Putting all together we get Eq.(2).

[1] A. K. Geim and K. S. Novoselov, Nature Materials **6**, 183 (2007).
[2] R. R. Nair et al., *Science* **320**, 1308 (2008); Z. Q. Li et al., *Nature Phys.* **4**, 532 (2008).
[3] A. W. W. Ludwig et al., Phys. Rev. B **50**, 7526 (1994).
[4] T. Stauber, N. M. Peres and A. K. Geim, Phys. Rev. B **78**, 085432 (2008).
[5] M. I. Katsnelson, Europhys. Lett. **84**, 37001(2008).
[6] D. C. Elias et al., Nature Phys. **7**, 701 (2011).
[7] A. Giuliani, V. Mastropietro and M. Porta, Phys. Rev. B **83**, 195401 (2011); Comm. Math. Phys., in press.
[8] J. González, F. Guinea and M. A. H. Vozmediano, Phys. Rev. B **59**, R2474 (1999).
[9] E. G. Mishchenko, Phys. Rev. Lett. **98**, 216801 (2007).
[10] I. F. Herbut, V. Juricic and O. Vafek, Phys. Rev. Lett **100**, 046403 (2008).
[11] D. E. Sheehy and J. Schmalian Phys. Rev. Lett. **99**, 226803 (2007).
[12] E. G. Mishchenko, Europhys. Lett. **83**, 17005 (2008).
[13] D. E. Sheehy and J. Schmalian, Phys. Rev. B **80**, 193411 (2009).
[14] V. Juricic, O. Vafek and I. F. Herbut, Phys. Rev. B **82**, 235402 (2010); arXiv:0809.0725.
[15] M. A. H. Vozmediano, Phil. Trans. Roy. Soc. A **369**, 2625 (2011).
[16] J. Gonzalez, F. Guinea and M. A. H. Vozmediano, Nucl. Phys. B **424**, 595 (1994).
[17] A. Giuliani, V. Mastropietro and M. Porta, *Phys. Rev. B* **82**, 121418 (2010); arXiv:1107.4741.
[18] In Eq.(2), by writing $O(\frac{\omega}{\varepsilon}\log\frac{\varepsilon}{\omega})$, we mean that the correction is bounded from above by (const.)$\frac{\omega}{\varepsilon}\log\frac{\varepsilon}{\omega}$ for $\omega \ll \varepsilon$.
[19] J. P. Reed et al., Science **330**, 805 (2010).