The ground state construction of the two-dimensional Hubbard model on the honeycomb lattice

Alessandro Giuliani

Dipartimento di Matematica, Università di Roma Tre

L.go S. L. Murialdo 1, 00146 Roma, Italy

In these lectures I consider the half-filled two-dimensional (2D) Hubbard model on the honeycomb lattice and I review the rigorous construction of its ground state properties by making use of constructive fermionic Renormalization Group methods.

1. INTRODUCTION

There are very few quantum interacting systems whose ground state properties (thermodynamic functions, reduced density matrices, etc.) can be computed without approximations. Among these, the Luttinger and the Thirring model [31, 37, 40, 45, 46] (one-dimensional spinless relativistic fermions), the one-dimensional Hubbard model [16, 36] (non-relativistic lattice fermions with spin), the Lieb-Liniger model [34, 35] (one-dimensional bosons with repulsive delta interactions) and the BCS model [1, 15, 42] (d-dimensional spinning fermions with mean field interactions, $d \geq 1$); the construction of the ground states of these systems is based on some remarkable exact solutions, which make use of bosonization techniques and Bethe ansatz. Unfortunately, in most cases these exact solutions do not allow to compute the long-distance decay of the n-particles correlation functions in a closed form (with some notable exceptions, namely the Luttinger and Thirring models, see [32, 38]). Moreover, even the computation of the thermodynamic functions crucially relies on a very special choice of the particle-particle interaction; as soon as the interparticle potential is slightly modified the exact solvability of these models is destroyed and no conclusion on the new "perturbed" system can be drawn from their solution. This is, of course, very annoying and unsatisfactory from a physical point of view.

There is another rigorous powerful method that allows in a few cases to fully construct the ground state of a system of interacting particles, known as Renormalization Group (RG). This method, whenever it works, has the advantage that it provides full information on the zero or low temperatures state of the system, including correlations and that, typically, it is robust under small modifications of the interparticle potential; even better, it usually gives a very precise meaning to what "small modifications" means: it allows one to classify perturbations into "relevant" and "irrelevant" and to show that the addition of a small irrelevant perturbation does not change the asymptotic behavior of correlations.

Unfortunately, it only works in the weak coupling regime and it has only been successfully applied to a limited number of interacting quantum systems. Most of the available results on the ground (or thermal) states of interacting quantum systems obtained via RG concern one-dimensional (1D) weakly interacting fermions, e.g., ultraviolet $O(N)$ models with $N \geq 2$ [17, 22], ultraviolet QED-like models [33] and non-relativistic lattice systems [6, 9, 23]. In more than one dimension, most of the result derived by rigorous RG techniques concern the finite temperature properties of two-dimensional (2D) fermionic systems above the BCS critical temperature [4, 7, 8, 13, 14, 18]. Two remarkable exceptions are the Fermi liquid construction by Feldman, Knörrer and Trubowitz [19], which concerns zero temperature properties of a system of 2D interacting fermions with highly asymmetric Fermi surface, and the ground state construction of the short range half-filled 2D Hubbard model on the honeycomb lattice by Giuliani and Mastropietro [24, 25], which will be reviewed here.

The 2D Hubbard model on the honeycomb lattice is a basic model for describing *graphene*, a newly discovered material consisting of a one-atom thick layer of graphite [41], see [12] for a comprehensive and up-to-date review of its low temperature properties. One of the most remarkable features of graphene is that at half-filling the Fermi surface is highly degenerate and it consists of just two isolated points. This makes the infrared properties of the system very peculiar: in the absence of interactions, it behaves in the same way as a system of non-interacting $(2 + 1)$-dimensional Dirac particles [47]. Therefore, the interacting system is a sort of $(2+1)$-dimensional QED, with some peculiar differences that make its study new and non-trivial [29, 44].

The goal of these lectures is to give a self-contained proof of the analyticity of the ground state energy of the Hubbard model on the 2D honeycomb lattice at half filling and weak coupling via constructive RG methods. A simple extension of the proof of convergence of the series for the specific ground state energy presented below allows one to construct the whole set of reduced density matrices at weak coupling (see [24]): it turns out that the off-diagonal elements of these matrices decay to zero at infinity, with the same decay exponents as the non-interacting system; in this sense, the construction presented below rigorously exclude the presence of long range order in the ground state, and the absence of anomalous critical exponents (in other words, the interacting system is in the same universality class as the non-interacting one). Let me also mention that more sophisticated extensions of the methods exposed here also allowed us to: (i) give an order by order construction of the ground state correlations of the same model in the presence of electromagnetic interactions [26, 27]; (ii) predict a possible mechanism for the spontaneous generation of the Peierls-Kekulé instability [27]; (iii) prove the universality of the optical conductivity [28].

The plan of these lectures is the following:

- In Section 2, I introduce the model and state the main result.
- In Section 3, I review the non-interacting case.
- In Section 4, I describe the formal series expansion for the ground state energy, I explain how to conveniently re-express it in terms of Grassmann functional integrals and estimate by naive power-counting the generic N-th order in perturbation theory, so identifying two main issues in the convergence of the series: a combinatorial problem, related to the large number of Feynman graphs contributing at a generic perturbative order, and a divergence problem, related to the (very mild) ultraviolet singularity of the propagator and to its (more serious) infrared singularity.
- In Section 5, I describe a way to reorganize and estimate the perturbation theory, via the so-called *determinant expansion*, that allows one to prove convergence of the series, for any fixed choice of the ultraviolet and infrared cut-offs.
- In Section 6, I describe how to resum the determinant expansion in order to cure the mild ultraviolet divergences appearing in perturbation theory, via a multiscale expansion, whose result is conveniently expressed in terms of *Gallavotti-Nicolò* trees [20, 21].
- In Section 7, I describe how to resum and cure the infrared divergences and conclude the proof of the main result.
- Finally, in Section 8, I draw the conclusions. A few technical aspects of the proof are described in the Appendixes

The material presented in this lectures is mostly taken from [24]. Some technical proofs concerning the determinant and the tree expansions are taken from the review [23]. Other reviews of the constructive RG methods discussed here, which the reader may find useful to consult, are [5, 20, 39, 43].

2. THE MODEL AND THE MAIN RESULTS

The grandcanonical Hamiltonian of the 2D Hubbard model on the honeycomb lattice at half filling in second quantized form is given by:

$$H_\Lambda = -t \sum_{\substack{\vec{x}\in\Lambda_A \\ i=1,2,3}} \sum_{\sigma=\uparrow\downarrow} \left(a^+_{\vec{x},\sigma} b^-_{\vec{x}+\vec{\delta}_i,\sigma} + b^+_{\vec{x}+\vec{\delta}_i,\sigma} a^-_{\vec{x},\sigma} \right) + \tag{2.1}$$

$$+U \sum_{\vec{x}\in\Lambda_A} \big(a^+_{\vec{x},\uparrow} a^-_{\vec{x},\uparrow} - \frac{1}{2}\big)\big(a^+_{\vec{x},\downarrow} a^-_{\vec{x},\downarrow} - \frac{1}{2}\big) + U \sum_{\vec{x}\in\Lambda_B} \big(b^+_{\vec{x},\uparrow} b^-_{\vec{x},\uparrow} - \frac{1}{2}\big)\big(b^+_{\vec{x},\downarrow} b^-_{\vec{x},\downarrow} - \frac{1}{2}\big)$$

where:

1. $\Lambda_A = \Lambda$ is a periodic triangular lattice, defined as $\Lambda = \mathbb{B}/L\mathbb{B}$, where $L \in \mathbb{N}$ and $\mathbb{B}$ is the infinite triangular lattice with basis $\vec{l}_1 = \frac{1}{2}(3, \sqrt{3})$, $\vec{l}_2 = \frac{1}{2}(3, -\sqrt{3})$. $\Lambda_B = \Lambda_A + \vec{\delta}_i$ is obtained by translating Λ_A by a nearest neighbor vector $\vec{\delta}_i$, $i = 1, 2, 3$, where

$$\vec{\delta}_1 = (1, 0)\ , \quad \vec{\delta}_2 = \frac{1}{2}(-1, \sqrt{3})\ , \quad \vec{\delta}_3 = \frac{1}{2}(-1, -\sqrt{3})\ . \tag{2.2}$$

   The honeycomb lattice we are interested in is the union of the two triangular sublattices Λ_A and Λ_B, see Fig.1

2. $a^\pm_{\vec{x},\sigma}$ are creation or annihilation fermionic operators with spin index $\sigma =\uparrow\downarrow$ and site index $\vec{x} \in \Lambda_A$, satisfying periodic boundary conditions in $\vec{x}$. Similarly, $b^\pm_{\vec{x},\sigma}$ are creation or annihilation fermionic operators with spin index $\sigma =\uparrow\downarrow$ and site index $\vec{x} \in \Lambda_B$, satisfying periodic boundary conditions in $\vec{x}$.

3. U is the strength of the on–site density–density interaction; it can be either positive or negative.

Note that the terms in the second line of Eq.(2.1) can be rewritten as the sum of a truly quartic term in the creation/annihilation operators (the density-density interaction), plus a quadratic term (a chemical potential term, of the form $-\mu N$, with $\mu = U/2$ the chemical potential and N the total particles number operator), plus a constant (which plays no role in the thermodynamic properties of the system). The Hamiltonian (2.1) is hole-particle symmetric, i.e., it is invariant under the exchange $a^\pm_{\vec{x},\sigma} \longleftrightarrow a^\mp_{\vec{x},\sigma}$, $b^\pm_{\vec{x}+\vec{\delta}_1,\sigma} \longleftrightarrow -\, b^\mp_{\vec{x}+\vec{\delta}_1,\sigma}$. This invariance implies in particular that, if we define the average density of the system to be $\rho = (2|\Lambda|)^{-1}\langle N\rangle_{\beta,\Lambda}$, with $N = \sum_{\vec{x},\sigma}(a^+_{\vec{x},\sigma}a^-_{\vec{x},\sigma} + b^+_{\vec{x}+\vec{\delta}_1,\sigma} b^-_{\vec{x}+\vec{\delta}_1,\sigma})$ the total particle number operator and $\langle\cdot\rangle_{\beta,\Lambda} = \mathrm{Tr}\{e^{-\beta H_\Lambda}\cdot\}/\mathrm{Tr}\{e^{-\beta H_\Lambda}\}$ the average with respect to the (grandcanonical) Gibbs measure at inverse temperature β, one has $\rho \equiv 1$, for

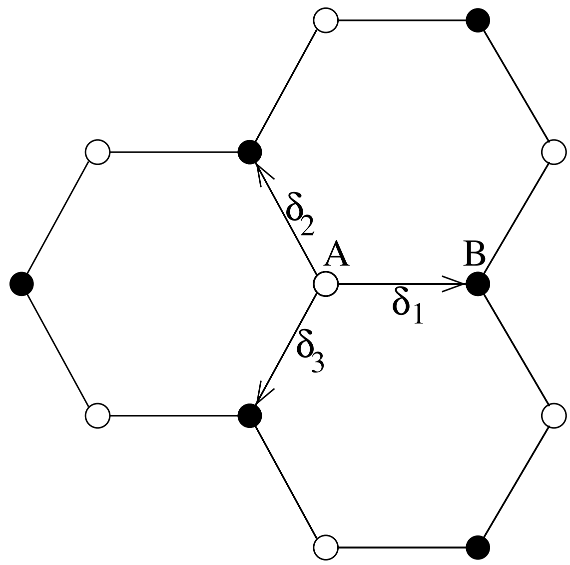

FIG. 1. A portion of the honeycomb lattice Λ. The white and black dots correspond to the sites of the Λ_A and Λ_B triangular sublattices, respectively. These two sublattices are one the translate of the other. They are connected by nearest neighbor vectors $\vec{\delta}_1, \vec{\delta}_2, \vec{\delta}_3$ that, in our units, are of unit length.

any $|\Lambda|$ and any β; in other words, the grand-canonical Hamiltonian Eq.(2.1) describes the system at half-filling, for all U, β, Λ. Let

$$f_\beta(U) = -\frac{1}{\beta} \lim_{|\Lambda|\to\infty} |\Lambda|^{-1} \log \mathrm{Tr}\{e^{-\beta H_\Lambda}\} \ . \tag{2.3}$$

be the specific free energy of the system and $e(U) = \lim_{\beta\to\infty} f_\beta(U)$ the specific ground state energy. We will prove the following Theorem.

Theorem 2.1 *There exists a constant $U_0 > 0$ such that, if $|U| \leq U_0$, the specific free energy $f_\beta(U)$ of the 2D Hubbard model on the honeycomb lattice at half filling is an analytic function of U, uniformly in β as $\beta \to \infty$, and so is the specific ground state energy $e(U)$.*

The proof is based on RG methods, which will be reviewed below. A straightforward extension of the proof of Theorem 2.1 allows one to prove that the correlation functions (i.e., the off-diagonal elements of the reduced density matrices of the system) are analytic functions of U and they decay to zero at infinity with the same decay exponents as in the non-interacting ($U = 0$) case, see [24]. This rigorously excludes the presence of LRO in the ground state and proves that the interacting system is in the same universality class as the non-interacting system.

3. THE NON-INTERACTING SYSTEM

Let us begin by reviewing the construction of the finite and zero temperature states for the non-interacting ($U = 0$) case. In this case the Hamiltonian of interest reduces to

$$H^0_\Lambda = -t \sum_{\substack{\vec{x}\in\Lambda \\ i=1,2,3}} \sum_{\sigma=\uparrow\downarrow} \left(a^+_{\vec{x},\sigma} b^-_{\vec{x}+\vec{\delta}_i,\sigma} + b^+_{\vec{x}+\vec{\delta}_i,\sigma} a^-_{\vec{x},\sigma} \right) , \tag{3.1}$$

with Λ, $a^\pm_{\vec{x},\sigma}$, $b^\pm_{\vec{x}+\vec{\delta}_i,\sigma}$ defined as in items (1)–(4) after (2.1). We aim at computing the spectrum of H^0_Λ by diagonalizing the right hand side (r.h.s.) of (3.1). To this purpose, we pass to Fourier space. We identify the periodic triangular lattice Λ with the set of vectors of the infinite triangular lattice within a "box" of size L, i.e.,

$$\Lambda = \{n_1 \vec{l}_1 + n_2 \vec{l}_2 \ : \ 0 \le n_1, n_2 \le L-1\} , \tag{3.2}$$

with $\vec{l}_1 = \frac{1}{2}(3, \sqrt{3})$ and $\vec{l}_2 = \frac{1}{2}(3, -\sqrt{3})$. The reciprocal lattice Λ^* is the set of vectors $\vec{K}$ such that $e^{i\vec{K}\vec{x}} = 1$, if $\vec{x} \in \Lambda$. A basis $\vec{G}_1, \vec{G}_2$ for Λ^* can be obtained by the inversion formula:

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = 2\pi \begin{pmatrix} l_{11} & l_{21} \\ l_{12} & l_{22} \end{pmatrix}^{-1} , \tag{3.3}$$

which gives

$$\vec{G}_1 = \frac{2\pi}{3}(1, \sqrt{3}) , \qquad \vec{G}_2 = \frac{2\pi}{3}(1, -\sqrt{3}) . \tag{3.4}$$

We call $\mathcal{B}_L$ the set of quasi-momenta $\vec{k}$ of the form

$$\vec{k} = \frac{m_1}{L}\vec{G}_1 + \frac{m_2}{L}\vec{G}_2 , \qquad m_1, m_2 \in \mathbb{Z} , \tag{3.5}$$

identified modulo Λ^*; this means that $\mathcal{B}_L$ can be identified with the vectors $\vec{k}$ of the form (2.2) and restricted to the *first Brillouin zone*:

$$\mathcal{B}_L = \{\vec{k} = \frac{m_1}{L}\vec{G}_1 + \frac{m_2}{L}\vec{G}_2 \ : \ 0 \le m_1, m_2 \le L-1\} . \tag{3.6}$$

Given a periodic function $f : \Lambda \to \mathbb{R}$, its Fourier transform is defined as

$$f(\vec{x}) = \frac{1}{|\Lambda|} \sum_{\vec{k}\in\mathcal{B}_L} e^{i\vec{k}\vec{x}} \hat{f}(\vec{k}) , \tag{3.7}$$

which can be inverted into

$$\hat{f}(\vec{k}) = \sum_{\vec{x}\in\Lambda} e^{-i\vec{k}\vec{x}} f(\vec{x}) , \qquad \vec{k} \in \mathcal{B}_L , \tag{3.8}$$

where we used the identity

$$\sum_{\vec{x}\in\Lambda} e^{i\vec{k}\vec{x}} = |\Lambda|\delta_{\vec{k},\vec{0}} \tag{3.9}$$

and δ is the periodic Kronecker delta function over Λ^*.

We now associate to the set of creation/annihilation operators $a^\pm_{\vec{x},\sigma}$, $b^\pm_{\vec{x}+\vec{\delta}_i,\sigma}$ the corresponding set of operators in momentum space:

$$a^\pm_{\vec{x},\sigma} = \frac{1}{|\Lambda|}\sum_{\vec{k}\in\mathcal{B}_L} e^{\pm i\vec{k}\vec{x}}\hat{a}^\pm_{\vec{k},\sigma}\,, \qquad b^\pm_{\vec{x}+\vec{\delta}_1,\sigma} = \frac{1}{|\Lambda|}\sum_{\vec{k}\in\mathcal{B}_L} e^{\pm i\vec{k}\vec{x}}\hat{b}^\pm_{\vec{k},\sigma}\,. \tag{3.10}$$

Using (3.7)–(3.9), we find that

$$\hat{a}^\pm_{\vec{k},\sigma} = \sum_{\vec{x}\in\Lambda} e^{\mp i\vec{k}\vec{x}} a^\pm_{\vec{x},\sigma}\,, \qquad \hat{b}^\pm_{\vec{k},\sigma} = \sum_{\vec{x}\in\Lambda} e^{\mp i\vec{k}\vec{x}} b^\pm_{\vec{x}+\vec{\delta}_1,\sigma} \tag{3.11}$$

are fermionic creation/annihilation operators, *periodic over* Λ^*, satisfying

$$\{a^\varepsilon_{\vec{k},\sigma}, a^{\varepsilon'}_{\vec{k}',\sigma'}\} = |\Lambda|\delta_{\vec{k},\vec{k}'}\delta_{\varepsilon,-\varepsilon'}\delta_{\sigma,\sigma'}\,, \qquad \{b^\varepsilon_{\vec{k},\sigma}, b^{\varepsilon'}_{\vec{k}',\sigma'}\} = |\Lambda|\delta_{\vec{k},\vec{k}'}\delta_{\varepsilon,-\varepsilon'}\delta_{\sigma,\sigma'} \tag{3.12}$$

and $\{a^\varepsilon_{\vec{k},\sigma}, b^{\varepsilon'}_{\vec{k}',\sigma'}\} = 0$ [48]. With the previous definitions, we can rewrite

$$\begin{aligned} H^0_\Lambda &= -t \sum_{\substack{\vec{x}\in\Lambda\\ i=1,2,3}} \sum_{\sigma=\uparrow\downarrow} (a^+_{\vec{x},\sigma} b^-_{\vec{x}+\vec{\delta}_i,\sigma} + b^+_{\vec{x}+\vec{\delta}_i,\sigma} a^-_{\vec{x},\sigma}) = \\ &= -\frac{v_0}{|\Lambda|} \sum_{\substack{\vec{k}\in\mathcal{B}_L\\ \sigma=\uparrow\downarrow}} \left(\hat{a}^+_{\vec{k},\sigma}, \hat{b}^+_{\vec{k},\sigma}\right) \begin{pmatrix} 0 & \Omega^*(\vec{k}) \\ \Omega(\vec{k}) & 0 \end{pmatrix} \begin{pmatrix} \hat{a}^-_{\vec{k},\sigma} \\ \hat{b}^-_{\vec{k},\sigma} \end{pmatrix} \end{aligned} \tag{3.13}$$

with $v_0 = \frac{3}{2}t$ the unperturbed Fermi velocity (if the hopping strength t is chosen to be the one measured in real graphene, v_0 turns out to be approximately 300 times smaller than the speed of light) and

$$\Omega(\vec{k}) = \frac{2}{3}\sum_{i=1}^{3} e^{i(\vec{\delta}_i-\vec{\delta}_1)\vec{k}} = \frac{2}{3}\Big[1 + 2e^{-i\frac{3}{2}k_1}\cos(\frac{\sqrt{3}}{2}k_2)\Big] \tag{3.14}$$

the complex *dispersion relation.* By looking at the second line of Eq.(3.13), one realizes that the natural creation/annihilation operator is a 2D spinor of components a and b:

$$\hat{\Psi}^+_{\vec{k},\sigma} = (\hat{a}^+_{\vec{k},\sigma}, \hat{b}^+_{\vec{k},\sigma})\,, \qquad \hat{\Psi}^-_{\vec{k},\sigma} = \begin{pmatrix} \hat{a}^-_{\vec{k},\sigma} \\ \hat{b}^-_{\vec{k},\sigma} \end{pmatrix}\,, \tag{3.15}$$

whose real space counterparts read

$$\Psi^+_{\vec{x},\sigma} = (a^+_{\vec{x},\sigma}, b^+_{\vec{x}+\vec{\delta}_1,\sigma})\,, \qquad \Psi^-_{\vec{x},\sigma} = \begin{pmatrix} \hat{a}^-_{\vec{x},\sigma} \\ \hat{b}^-_{\vec{x}+\vec{\delta}_1,\sigma} \end{pmatrix}\,. \tag{3.16}$$

In order to fully diagonalize the theory, one needs to perform the diagonalization of the 2×2 quadratic form in the second line of Eq.(3.13), which can be realized by the $\vec{k}$-dependent unitary transformation

$$U_{\vec{k}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \frac{\Omega^*(\vec{k})}{|\Omega(\vec{k})|} \\ -\frac{1}{\sqrt{2}} \frac{\Omega(\vec{k})}{|\Omega(\vec{k})|} & \frac{1}{\sqrt{2}} \end{pmatrix} , \tag{3.17}$$

in terms of which, defining

$$\hat{\Phi}^-_{\vec{k},\sigma} = \begin{pmatrix} \hat{\alpha}^-_{\vec{k},\sigma} \\ \hat{\beta}^-_{\vec{k},\sigma} \end{pmatrix} := U \hat{\Psi}^-_{\vec{k},\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{a}^-_{\vec{k},\sigma} + \frac{\Omega^*(\vec{k})}{|\Omega(\vec{k})|} \hat{b}^-_{\vec{k},\sigma} \\ -\frac{\Omega(\vec{k})}{|\Omega(\vec{k})|} \hat{a}^-_{\vec{k},\sigma} + \hat{b}^-_{\vec{k},\sigma} \end{pmatrix} , \tag{3.18}$$

we can rewrite

$$\begin{aligned} H^0_\Lambda &= -\frac{v_0}{|\Lambda|} \sum_{\substack{\vec{k} \in \mathcal{B}_L \\ \sigma=\uparrow\downarrow}} \hat{\Psi}^+_{\vec{k},\sigma} U^\dagger_{\vec{k}} U_{\vec{k}} \begin{pmatrix} 0 & \Omega^*(\vec{k}) \\ \Omega(\vec{k}) & 0 \end{pmatrix} U^\dagger_{\vec{k}} U_{\vec{k}} \hat{\Psi}^-_{\vec{k},\sigma} = \\ &= -\frac{v_0}{|\Lambda|} \sum_{\substack{\vec{k} \in \mathcal{B}_L \\ \sigma=\uparrow\downarrow}} \hat{\Phi}^+_{\vec{k},\sigma} \begin{pmatrix} |\Omega(\vec{k})| & 0 \\ 0 & -|\Omega(\vec{k})| \end{pmatrix} \hat{\Phi}^-_{\vec{k},\sigma} = \\ &= -\frac{v_0}{|\Lambda|} \sum_{\substack{\vec{k} \in \mathcal{B}_L \\ \sigma=\uparrow\downarrow}} \left(|\Omega(\vec{k})| \hat{\alpha}^+_{\vec{k},\sigma} \hat{\alpha}^-_{\vec{k},\sigma} - |\Omega(\vec{k})| \hat{\beta}^+_{\vec{k},\sigma} \hat{\beta}^-_{\vec{k},\sigma} \right) . \end{aligned} \tag{3.19}$$

The two energy bands $\pm v_0 |\Omega(\vec{k})|$ are plotted in Fig.2. They cross the Fermi energy $E_F = 0$ at the *Fermi points* $\vec{k} = \vec{p}^{\,\omega}_F$, $\omega = \pm$, with

$$\vec{p}^{\,\omega}_F = (\frac{2\pi}{3}, \omega \frac{2\pi}{3\sqrt{3}}) , \tag{3.20}$$

close to which the complex dispersion relation vanishes linearly:

$$\Omega(\vec{p}^{\,\omega}_F + \vec{k}') = ik'_1 + \omega k'_2 + O(|\vec{k}'|^2) , \tag{3.21}$$

resembling in this sense the relativistic dispersion relation of $(2+1)$-dimensional Dirac fermions. From Eq.(3.19) it is apparent that the ground state of the system consists of a Fermi sea such that all the negative energy states (the "α-states") are filled and all the positive energy states (the "β-states") are empty. The specific ground state energy $e_{0,\Lambda}$ is

$$e_{0,\Lambda} = -\frac{2v_0}{|\Lambda|} \sum_{\vec{k} \in \mathcal{B}_L} |\Omega(\vec{k})| , \tag{3.22}$$

from which we find that the specific ground state energy in the thermodynamic limit is

$$e(0) = -2v_0 \int_{\mathcal{B}} \frac{d\vec{k}}{|\mathcal{B}|} |\Omega(\vec{k})| , \tag{3.23}$$

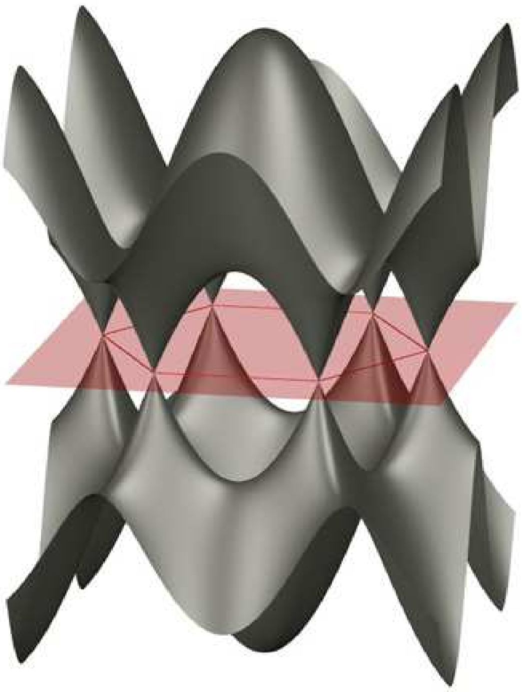

FIG. 2. A sketch of the energy bands of the free electron gas with nearest neighbor hopping on the honeycomb lattice. The red plane corresponds to the Fermi energy at half-filling. It cuts the bands at a discrete set of points, known as the Fermi points or Dirac points. From the picture, it seems that there are six distinct Fermi points. However, after identification of the points modulo vectors of the reciprocal lattice, it turns out that only two of them are independent.

where $\mathcal{B} := \{\vec{k} = t_1\vec{G}_1 + t_2\vec{G}_2 \ : \ t_i \in [0,1)\}$ and $|\mathcal{B}| = 8\pi^2/(3\sqrt{3})$. Similarly, the finite volume specific free energy $f^\beta_{0,\Lambda} := -(\beta|\Lambda|)^{-1}\log \mathrm{Tr}\{e^{-\beta H^0_\Lambda}\}$ is

$$f^\beta_{0,\Lambda} = -\frac{2}{\beta|\Lambda|}\sum_{\vec{k}\in\mathcal{B}_L}\log\Big[\big(1+e^{\beta v_0|\Omega(\vec{k})|}\big)\big(1+e^{-\beta v_0|\Omega(\vec{k})|}\big)\Big]\ , \qquad (3.24)$$

from which

$$f_\beta(0) = -\frac{2}{\beta}\int_{\mathcal{B}}\frac{d\vec{k}}{|\mathcal{B}|}\log\Big[\big(1+e^{\beta v_0|\Omega(\vec{k})|}\big)\big(1+e^{-\beta v_0|\Omega(\vec{k})|}\big)\Big]\ . \qquad (3.25)$$

Besides these thermodynamic functions, it is also useful to compute the *Schwinger functions* of the free gas, in terms of which, in the next section, we will write down the perturbation theory for the interacting system. These are defined as follows. Let $x_0 \in [0,\beta)$ be an *imaginary time*, let $\mathbf{x} := (x_0,\vec{x}) \in [0,\beta)\times\Lambda$ and let us consider the time-evolved operator $\Psi^\pm_{\mathbf{x},\sigma} = e^{Hx_0}\Psi^\pm_{\vec{x},\sigma}e^{-Hx_0}$, where $\Psi^\pm_{\vec{x},\sigma}$ is the two-components spinor defined in Eq.(3.16); in the following we shall denote its components by $\Psi^{\pm}_{\mathbf{x},\sigma,\rho}$, with $\rho \in \{1,2\}$, $\Psi^{\pm}_{\mathbf{x},\sigma,1} = a^{\pm}_{\mathbf{x},\sigma}$ and $\Psi^{\pm}_{\mathbf{x},\sigma,2} = b^{\pm}_{\mathbf{x}+\delta_1,\sigma}$ (here $\delta_1 = (0, \vec{\delta}_1)$). We define the n-points Schwinger functions at finite volme and finite temperature as:

$$S^{\beta,\Lambda}_n(\mathbf{x}_1, \varepsilon_1, \sigma_1, \rho_1; \ldots; \mathbf{x}_n, \varepsilon_n, \sigma_n, \rho_n) = \langle \mathbf{T}\{\Psi^{\varepsilon_1}_{\mathbf{x}_1,\sigma_1,\rho_1} \cdots \Psi^{\varepsilon_n}_{\mathbf{x}_n,\sigma_n,\rho_n}\}\rangle_{\beta,\Lambda} \qquad (3.26)$$

where: $\mathbf{x}_i \in [0,\beta] \times \Lambda$, $\sigma_i = \uparrow\downarrow$, $\varepsilon_i = \pm$, $\rho_i = 1, 2$ and $\mathbf{T}$ is the operator of fermionic time ordering, acting on a product of fermionic fields as:

$$\mathbf{T}(\Psi^{\varepsilon_1}_{\mathbf{x}_1,\sigma_1,\rho_1} \cdots \Psi^{\varepsilon_n}_{\mathbf{x}_n,\sigma_n,\rho_n}) = (-1)^\pi \Psi^{\varepsilon_{\pi(1)}}_{\mathbf{x}_{\pi(1)},\sigma_{\pi(1)},\rho_{\pi(1)}} \cdots \Psi^{\varepsilon_{\pi(n)}}_{\mathbf{x}_{\pi(n)},\sigma_{\pi(n)},\rho_{\pi(n)}} \qquad (3.27)$$

where π is a permutation of $\{1, \ldots, n\}$, chosen in such a way that $x_{\pi(1)0} \geq \cdots \geq x_{\pi(n)0}$, and $(-1)^\pi$ is its sign. [If some of the time coordinates are equal each other, the arbitrariness of the definition is solved by ordering each set of operators with the same time coordinate so that creation operators precede the annihilation operators.] Taking the limit $\Lambda \to \infty$ in (3.26) we get the finite temperature n-point Schwinger functions, denoted by $S^\beta_n(\mathbf{x}_1, \varepsilon_1, \sigma_1, \rho_1; \ldots; \mathbf{x}_n, \varepsilon_n, \sigma_n, \rho_n)$, which describe the properties of the infinite volume system at finite temperature. Taking the $\beta \to \infty$ limit of the finite temperature Schwinger functions, we get the zero temperature Schwinger functions, simply denoted by $S_n(\mathbf{x}_1, \varepsilon_1, \sigma_1, \rho_1; \ldots; \mathbf{x}_n, \varepsilon_n, \sigma_n, \rho_n)$, which by definition characterize the properties of the *thermal ground state* of (2.1) in the thermodynamic limit.

In the non-interacting case, i.e., if $H_\Lambda = H^0_\Lambda$, the Hamiltonian is quadratic in the creation/annihilation operators. Therefore, the $2n$-point Schwinger functions satisfy the Wick rule, i.e.,

$$\begin{aligned} &\langle \mathbf{T}\{\Psi^-_{\mathbf{x}_1,\sigma_1,\rho_1}\Psi^+_{\mathbf{y}_1,\sigma'_1,\rho'_1} \cdots \Psi^-_{\mathbf{x}_n,\sigma_n,\rho_n}\Psi^+_{\mathbf{y}_n,\sigma'_n,\rho'_n}\rangle_{\beta,\Lambda} = \det G \,, \\ &G_{ij} = \delta_{\sigma_i\sigma'_j}\langle \mathbf{T}\{\Psi^-_{\mathbf{x}_i,\sigma_i,\rho_i}\Psi^+_{\mathbf{y}_j,\sigma'_j,\rho'_j}\}\rangle_{\beta,\Lambda} \,. \end{aligned} \qquad (3.28)$$

Moreover, every n–point Schwinger function $S^{\beta,\Lambda}_n(\mathbf{x}_1, \varepsilon_1, \sigma_1, \rho_1; \ldots; \mathbf{x}_n, \varepsilon_n, \sigma_n, \rho_n)$ with $\sum_{i=1}^n \varepsilon_i \neq 0$ is identically zero. Therefore, in order to construct the whole set of Schwinger functions of H^0_Λ, it is enough to compute the 2–point function $S^{\beta,\Lambda}_0(\mathbf{x} - \mathbf{y}) = \langle \mathbf{T}\{\Psi^-_{\mathbf{x},\sigma,\rho}\Psi^+_{\mathbf{y},\sigma,\rho'}\}\rangle_{\beta,\Lambda}$. This can be easily reconstructed from the 2-point function of the α-fields and β-fields, see Eq.(3.18).

Let $\vec{x} \in \Lambda$, $\alpha^\pm_{\vec{x},\sigma} = |\Lambda|^{-1}\sum_{\vec{k}\in\mathcal{B}_L} e^{\pm i\vec{k}\vec{x}}\hat{\alpha}_{\vec{k},\sigma}$ and $\beta^\pm_{\vec{x},\sigma} = |\Lambda|^{-1}\sum_{\vec{k}\in\mathcal{B}_L} e^{\pm i\vec{k}\vec{x}}\hat{\alpha}_{\vec{k},\sigma}$; if $\mathbf{x} = (x_0, \vec{x})$, let $\alpha^\pm_{\mathbf{x},\sigma} = e^{H^0_\Lambda x_0}\alpha^\pm_{\vec{x},\sigma}e^{-H^0_\Lambda x_0}$ and $\beta^\pm_{\mathbf{x},\sigma} = e^{H^0_\Lambda x_0}\beta^\pm_{\vec{x},\sigma}e^{-H^0_\Lambda x_0}$. A straightforward computation shows that, if $-\beta < x_0 - y_0 \leq \beta$,

$$\begin{aligned} \langle \mathbf{T}\{\alpha^-_{\mathbf{x},\sigma}\alpha^+_{\mathbf{y},\sigma'}\}\rangle_{\beta,\Lambda} &= \frac{\delta_{\sigma,\sigma'}}{|\Lambda|}\sum_{\vec{k}\in\mathcal{B}_L} e^{-i\vec{k}(\vec{x}-\vec{y})} \,. \\ &\cdot\Big[\mathbb{1}\big(x_0 - y_0 > 0\big)\frac{e^{v_0(x_0-y_0)|\Omega(\vec{k})|}}{1 + e^{v_0\beta|\Omega(\vec{k})|}} - \mathbb{1}\big(x_0 - y_0 \leq 0\big)\frac{e^{v_0(x_0-y_0+\beta)|\Omega(\vec{k})|}}{1 + e^{v_0\beta|\Omega(\vec{k})|}}\Big] \,, \end{aligned} \qquad (3.29)$$

$$\langle \mathbf{T}\{\beta^-_{\mathbf{x},\sigma}\beta^+_{\mathbf{y},\sigma'}\}\rangle_{\beta,\Lambda} = \frac{\delta_{\sigma,\sigma'}}{|\Lambda|}\sum_{\vec k\in\mathcal{B}_L} e^{-i\vec k(\vec x-\vec y)}\ . \tag{3.30}$$
$$\cdot\Big[\mathbb{1}\big(x_0-y_0>0\big)\frac{e^{-v_0(x_0-y_0)|\Omega(\vec k)|}}{1+e^{-v_0\beta|\Omega(\vec k)|}} - \mathbb{1}\big(x_0-y_0\le 0\big)\frac{e^{-v_0(x_0-y_0+\beta)|\Omega(\vec k)|}}{1+e^{-v_0\beta|\Omega(\vec k)|}}\Big]$$

and $\langle \mathbf{T}\{\alpha^-_{\mathbf{x},\sigma}\beta^+_{\mathbf{y},\sigma'}\}\rangle_{\beta,\Lambda} = \langle \mathbf{T}\{\beta^-_{\mathbf{x},\sigma}\alpha^+_{\mathbf{y},\sigma'}\}\rangle_{\beta,\Lambda} = 0$. A priori, Eqs.(3.30) and (3.31) are defined only for $-\beta < x_0 - y_0 \le \beta$, but we can extend them periodically over the whole real axis; the periodic extension of the propagator is continuous in the time variable for $x_0 - y_0 \notin \beta\mathbb{Z}$, and it has jump discontinuities at the points $x_0 - y_0 \in \beta\mathbb{Z}$. Note that at $x_0 - y_0 = \beta n$, the difference between the right and left limits is equal to $(-1)^n\delta_{\vec x,\vec y}$, so that the propagator is discontinuous only at $\mathbf{x}-\mathbf{y} = \beta\mathbb{Z}\times\vec 0$. If we define $\mathcal{B}_{\beta,L} := \mathcal{B}_\beta\times\mathcal{B}_L$, with $\mathcal{B}_\beta = \{k_0 = \frac{2\pi}{\beta}(n_0+\frac12) : n_0\in\mathbb{Z}\}$, then for $\mathbf{x}-\mathbf{y}\notin\beta\mathbb{Z}\times\vec 0$ we can write

$$\langle \mathbf{T}\{\alpha^-_{\mathbf{x},\sigma}\alpha^+_{\mathbf{y},\sigma'}\}\rangle_{\beta,\Lambda} = \frac{\delta_{\sigma,\sigma'}}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{1}{-ik_0 - v_0|\Omega(\vec k)|}\ , \tag{3.31}$$
$$\langle \mathbf{T}\{\beta^-_{\mathbf{x},\sigma}\beta^+_{\mathbf{y},\sigma'}\}\rangle_{\beta,\Lambda} = \frac{\delta_{\sigma,\sigma'}}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{1}{-ik_0 + v_0|\Omega(\vec k)|}\ . \tag{3.32}$$

If we now re-express $\alpha^\pm_{\mathbf{x},\sigma}$ and $\beta^\pm_{\mathbf{x},\sigma}$ in terms of $a^\pm_{\mathbf{x},\sigma}$ and $b^\pm_{\mathbf{x}+\delta_1,\sigma}$, using Eq.(3.18), we find that, for $\mathbf{x}-\mathbf{y}\notin\beta\mathbb{Z}\times\vec 0$:

$$S_0^{\beta,\Lambda}(\mathbf{x}-\mathbf{y})_{\rho,\rho'} := S_2^{\beta,\Lambda}(\mathbf{x},\sigma,-,\rho;\mathbf{y},\sigma,+,\rho')\Big|_{U=0} =$$
$$= \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}}\frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k_0^2+v_0^2|\Omega(\vec k)|^2}\begin{pmatrix} ik_0 & -v_0\Omega^*(\vec k)\\ -v_0\Omega(\vec k) & ik_0\end{pmatrix}_{\rho,\rho'} \tag{3.33}$$

Finally, if $\mathbf{x}-\mathbf{y} = (0^-,\vec 0)$:

$$S_0^{\beta,\Lambda}(0^-,\vec 0) = -\frac12 + \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}}\frac{1}{k_0^2+v_0^2|\Omega(\vec k)|^2}\begin{pmatrix} 0 & -v_0\Omega^*(\vec k)\\ -v_0\Omega(\vec k) & 0\end{pmatrix}\ . \tag{3.34}$$

4. PERTURBATION THEORY AND GRASSMANN INTEGRATION

Let us now turn to the interacting case. The first step is to derive a formal perturbation theory for the specific free energy and ground state energy. In other words, we want to find rules to compute the generic perturbative order in U of $f_{\beta,\Lambda} := -(\beta|\Lambda|)^{-1}\log\mathrm{Tr}\{e^{-\beta H_\Lambda}\}$. We write $H_\Lambda = H^0_\Lambda + V_\Lambda$, with V_Λ the operator in the second line of Eq.(2.1) and we use Trotter's product formula

$$e^{-\beta H_\Lambda} = \lim_{n\to\infty}\Big[e^{-\beta H^0_\Lambda/n}(1-\frac{\beta}{n}V_\Lambda)\Big]^n \tag{4.1}$$

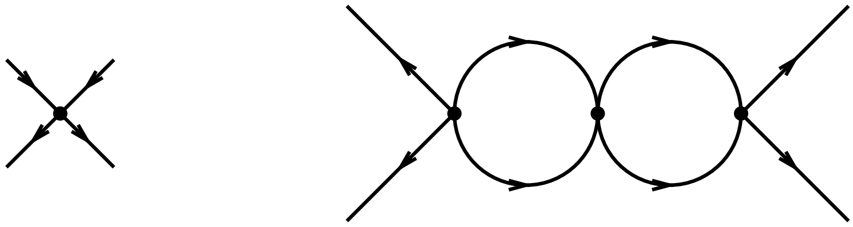

FIG. 3. The four-legged graph element (left); an example of a Feynman diagram of order 3 (right).

so that, defining $V_\Lambda(t) := e^{tH^0_\Lambda} V_\Lambda e^{-tH^0_\Lambda}$,

$$\frac{\mathrm{Tr}\{e^{-\beta H_\Lambda}\}}{\mathrm{Tr}\{e^{-\beta H^0_\Lambda}\}} = \tag{4.2}$$
$$= 1 + \sum_{N\geq 1} (-1)^N \int_0^\beta dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{N-1}} dt_N \frac{\mathrm{Tr}\{e^{-\beta H^0_\Lambda} V_\Lambda(t_1) \cdots V_\Lambda(t_N)\}}{\mathrm{Tr}\{e^{-\beta H^0_\Lambda}\}} \ .$$

Using the fermionic time-ordering operator defined in Eq.(3.27), we can rewrite Eq.(4.2) as

$$\frac{\mathrm{Tr}\{e^{-\beta H_\Lambda}\}}{\mathrm{Tr}\{e^{-\beta H^0_\Lambda}\}} = 1 + \sum_{N\geq 1} \frac{(-1)^N}{N!} \langle \mathbf{T}\{(V_{\beta,\Lambda}(\Psi))^N\}\rangle^0_{\beta,\Lambda} \ , \tag{4.3}$$

where $\langle\cdot\rangle^0_{\beta,\Lambda} = \mathrm{Tr}\{e^{-\beta H^0_\Lambda}\cdot\}/\mathrm{Tr}\{e^{-\beta H^0_\Lambda}\}$,

$$V_{\beta,\Lambda}(\Psi) := U \sum_{\rho=1,2} \int_{(\beta,\Lambda)} d\mathbf{x} \big(\Psi^+_{\mathbf{x},\uparrow,\rho}\Psi^-_{\mathbf{x},\uparrow,\rho} - \frac{1}{2}\big)\big(\Psi^+_{\mathbf{x},\downarrow,\rho}\Psi^-_{\mathbf{x},\downarrow,\rho} - \frac{1}{2}\big) \ , \tag{4.4}$$

and $\int_{(\beta,\Lambda)} d\mathbf{x}$ must be interpreted as $\int_{(\beta,\Lambda)} d\mathbf{x} = \int_{-\beta/2}^{\beta/2} dx_0 \sum_{\vec{x}\in\Lambda}$. Note that the N-th term in the sum in the r.h.s. of Eq.(4.3) can be computed by using the Wick rule (3.28) and the explicit expression for the 2-point function Eqs.(3.33)-(3.34). It is straightforward to check that the "Feynman rules" needed to compute $\langle \mathbf{T}\{(V_{\beta,\Lambda}(\Psi))^N\}\rangle^0_{\beta,\Lambda}$ are the following: (i) draw N graph elements consisting of 4-legged vertices, with the vertex associated to two labels $\mathbf{x}_i$ and ρ_i, $i = 1, \ldots, N$, and the four legs associated to two exiting fields (with labels $(\mathbf{x}_i, \uparrow, \rho_1)$ and $(\mathbf{x}_i, \downarrow, \rho_i)$) and two entering fields (with labels $(\mathbf{x}_i, \uparrow, \rho_i)$ and $(\mathbf{x}_i, \downarrow, \rho_i)$), respectively; (ii) pair the fields in all possible ways, in such a way that every pair consists of one entering and one exiting field, with the same spin index, see Fig.3; (iii) associate to every pairing a sign, corresponding to the sign of the permutation needed to bring every pair of contracted fields next to each other; (iv) associate to every paired pair of fields $[\Psi^-_{\mathbf{x}_i,\sigma_i,\rho_i},\Psi^+_{\mathbf{x}_j,\sigma_j,\rho_j}]$ an oriented line connecting the i-th with the j-th vertex, with orientation from j to i; (v) associate to every oriented line $[j\to i]$ a value equal to

$$g_{\rho_i,\rho_j}(\mathbf{x}_i-\mathbf{x}_j):=\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}}e^{-i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}\frac{\chi_0(2^{-M}|k_0|)}{k_0^2+v_0^2|\Omega(\vec{k})|^2}\begin{pmatrix}ik_0 & -v_0\Omega^*(\vec{k})\\ -v_0\Omega(\vec{k}) & ik_0\end{pmatrix}_{\rho_i,\rho_j} \tag{4.5}$$

where $\mathcal{B}^{(M)}_{\beta,L}=\mathcal{B}^{(M)}_{\beta}\times\mathcal{B}_L$, $\mathcal{B}^{(M)}_{\beta}=\mathcal{B}_\beta\cap\{k_0\ :\ \chi_0(2^{-M}|k_0|)>0\}$ and $\chi_0(t)$ is a smooth compact support function that is equal to 1 for (say) $|t|\le 1/3$ and equal to 0 for $|t|\ge 2/3$; (vi) associate to every pairing (i.e., to every Feynman graph) a value, equal to the product of the sign of the pairing times U^N times the product of the values of all the oriented lines; (vii) integrate over $\mathbf{x}_i$ and sum over ρ_i the value of each pairing, then sum over all pairings; (viii) finally, take the $M\to\infty$ limit: the result is equal to $\langle\mathbf{T}\{(V_{\beta,\Lambda}(\Psi))^N\}\rangle^0_{\beta,\Lambda}$. Note that the $M\to\infty$ limit of the *propagator* $g(\mathbf{x})$ is equal to $S_0^{\beta,\Lambda}(\mathbf{x})$ if $\mathbf{x}\neq\mathbf{0}$, while $\lim_{M\to\infty}g(\mathbf{0})=S_0^{\beta,\Lambda}(0^-,\vec{0})+\frac{1}{2}$, see Eq.(3.34): the difference between $\lim_{M\to\infty}g(\mathbf{x})$ and $S_0^{\beta,\Lambda}(\mathbf{x})$ takes into account the $-\frac{1}{2}$ terms in the definition of $V_{\beta,\Lambda}(\Psi)$.

An algebraically convenient way to re-express Eq.(4.3) is in terms of *Grassmann integrals.* Consider the set $\mathcal{A}_{M,\beta,L}=\{\hat\psi^\pm_{\mathbf{k},\sigma,\rho}\}^{\sigma=\uparrow\downarrow,\ \rho=1,2}_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}}$, where the *Grassmann variables* $\hat\psi^\pm_{\mathbf{k},\sigma,\rho}$ satisfy by the definition the anticommutation rules $\{\hat\psi^\varepsilon_{\mathbf{k},\sigma,\rho},\hat\psi^{\varepsilon'}_{\mathbf{k}',\sigma',\rho'}\}=0$. In particular, the square of a Grassmann variable is zero and the only non-trivial Grassmann monomials are at most linear in each variable. Let the Grassmann algebra generated by $\mathcal{A}_{M,\beta,L}$ be the set of all polynomials obtained by linear combinations of such non-trivial monomials. Let us also define the Grassmann integration $\int\big[\prod_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}}\prod^{\rho=1,2}_{\sigma=\uparrow\downarrow}d\hat\psi^+_{\mathbf{k},\sigma,\rho}d\hat\psi^-_{\mathbf{k},\sigma,\rho}\big]$ as the linear operator on the Grassmann algebra such that, given a monomial $Q(\hat\psi^-,\hat\psi^+)$ in the variables $\hat\psi^\pm_{\mathbf{k},\sigma,\rho}$, its action on $Q(\hat\psi^-,\hat\psi^+)$ is 0 except in the case $Q(\hat\psi^-,\hat\psi^+)=\prod_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}}\prod^{\rho=1,2}_{\sigma=\uparrow\downarrow}\hat\psi^-_{\mathbf{k},\sigma,\rho}\hat\psi^+_{\mathbf{k},\sigma,\rho}$, up to a permutation of the variables. In this case the value of the integral is determined, by using the anticommutation properties of the variables, by the condition

$$\int\Big[\prod_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}}\prod^{\rho=1,2}_{\sigma=\uparrow\downarrow}d\hat\psi^+_{\mathbf{k},\sigma,\rho}d\hat\psi^-_{\mathbf{k},\sigma,\rho}\Big]\prod_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}}\prod^{\rho=1,2}_{\sigma=\uparrow\downarrow}\hat\psi^-_{\mathbf{k},\sigma,\rho}\hat\psi^+_{\mathbf{k},\sigma,\rho}=1 \tag{4.6}$$

Defining the free propagator matrix $\hat g_{\mathbf{k}}$ as

$$\hat g_{\mathbf{k}}=\chi_0(2^{-M}|k_0|)\begin{pmatrix}-ik_0 & -v_0\Omega^*(\vec{k})\\ -v_0\Omega(\vec{k}) & -ik_0\end{pmatrix}^{-1} \tag{4.7}$$

and the "Gaussian integration" $P_M(d\psi)$ as

$$P_M(d\psi) = \Big[\prod_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}}^{\sigma=\uparrow\downarrow} \frac{-\beta^2|\Lambda|^2\,[\chi_0(2^{-M}|k_0|)]^2}{k_0^2+v_0^2|\Omega(\vec{k})|^2} d\hat\psi^+_{\mathbf{k},\sigma,1}d\hat\psi^-_{\mathbf{k},\sigma,1}d\hat\psi^+_{\mathbf{k},\sigma,2}d\hat\psi^-_{\mathbf{k},\sigma,2}\Big]\cdot$$

$$\cdot\ \exp\Big\{-(\beta|\Lambda|)^{-1}\sum_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}}^{\sigma=\uparrow\downarrow} \hat\psi^+_{\mathbf{k},\sigma}\,\hat g_{\mathbf{k}}^{-1}\hat\psi^-_{\mathbf{k},\sigma}\Big\}\,, \tag{4.8}$$

it turns out that

$$\int P(d\psi)\hat\psi^-_{\mathbf{k}_1,\sigma_1}\hat\psi^+_{\mathbf{k}_2,\sigma_2} = \beta|\Lambda|\delta_{\sigma_1,\sigma_2}\delta_{\mathbf{k}_1,\mathbf{k}_2}\hat g_{\mathbf{k}_1}\,, \tag{4.9}$$

while the average of an arbitrary monomial in the Grassmann variables with respect to $P_M(d\psi)$ is given by the fermionic Wick rule with propagator equal to the r.h.s. of Eq.(4.9). Using these definitions and the Feynman rules described above, we can rewrite Eq.(4.3) as

$$\frac{\mathrm{Tr}\{e^{-\beta H_\Lambda}\}}{\mathrm{Tr}\{e^{-\beta H^0_\Lambda}\}} = \lim_{M\to\infty}\int P_M(d\psi)e^{-\mathcal{V}(\psi)}\,, \tag{4.10}$$

where

$$\mathcal{V}(\psi) = U\sum_{\rho=1,2}\int_{(\beta,\Lambda)} d\mathbf{x}\,\psi^+_{\mathbf{x},\uparrow,\rho}\psi^-_{\mathbf{x},\uparrow,\rho}\psi^+_{\mathbf{x},\downarrow,\rho}\psi^-_{\mathbf{x},\downarrow,\rho}\,, \tag{4.11}$$

$$\psi^\pm_{\mathbf{x},\sigma,\rho} = \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}} e^{\pm i\mathbf{k}\mathbf{x}}\hat\psi^\pm_{\mathbf{k},\sigma,\rho}\,, \qquad \mathbf{x}\in(-\beta/2,\beta/2]\times\Lambda \tag{4.12}$$

and the exponential $e^{-\mathcal{V}(\psi)}$ in the r.h.s. of Eq.(4.10) must be identified with its Taylor series in U (which is finite for every finite M, due to the anticommutation rules of the Grassmann variables and the fact that the Grassmann algebra is finite for every finite M). Apriori, Eq.(4.10) must be understood as an equality between formal power series in U. However, it can be given a non-perturbative meaning, provided that we can prove the convergence of the Grassmann functional integral in the r.h.s., as shown by the following Proposition.

Proposition 4.1 *Let*

$$F^{(M)}_{\beta,\Lambda} := -\frac{1}{\beta|\Lambda|}\log\int P_M(d\psi)\big(e^{-\mathcal{V}(\psi)}\big) \tag{4.13}$$

and let β and $|\Lambda|$ be sufficiently large. Assume that there exists $U_0>0$ such that $F^{(M)}_{\beta,\Lambda}$ is analytic in the complex domain $|U|\le U_0$ and is uniformly convergent as $M\to\infty$. Then, if $|U|\le U_0$,

$$f_{\beta,\Lambda} = -\frac{2}{\beta|\Lambda|}\sum_{\vec{k}\in\mathcal{B}_L}\log\big(2+2\cosh(\beta v_0|\Omega(\vec{k})|)\big) + \lim_{M\to\infty}F^{(M)}_{\beta,\Lambda}\,. \tag{4.14}$$

Proof. We need to prove that

$$\frac{\mathrm{Tr}\{e^{-\beta H_\Lambda}\}}{\mathrm{Tr}\{e^{-\beta H_{0,\Lambda}}\}} = \exp\big\{ -\beta|\Lambda| \lim_{M\to\infty} F^{(M)}_{\beta,\Lambda}\big\} \tag{4.15}$$

under the given analyticity assumptions on $F^{(M)}_{\beta,\Lambda}$. The first key remark is that, if β,Λ are finite, the left hand side of Eq.(4.15) is a priori well defined and analytic on the whole complex plane. In fact, by the Pauli principle, the Fock space generated by the fermion operators $a^\pm_{\vec x,\sigma}, b^\pm_{\vec x+\vec\delta_1,\sigma}$, with $\vec x\in\Lambda, \sigma=\uparrow\downarrow$, is finite dimensional. Therefore, writing $H_\Lambda = H^0_\Lambda + V_\Lambda$, with H^0_Λ and V_Λ two bounded operators, we see that $\mathrm{Tr}\{e^{-\beta H_\Lambda}\}$ is an entire function of U, simply because $e^{-\beta H_\Lambda}$ converges in norm over the whole complex plane:

$$\begin{aligned}||e^{-\beta H_\Lambda}|| \le \sum_{n=0}^\infty \frac{\beta^n}{n!}\big(||H^0_\Lambda|| + ||V_\Lambda||\big)^n &= \sum_{k=0}^\infty \frac{\beta^k ||V_\Lambda||^k}{k!}\sum_{n\ge k}\frac{\beta^{n-k}||H^0_\Lambda||^{n-k}}{(n-k)!} = \\ &= e^{\beta ||H^0_\Lambda|| + \beta\, ||V_\Lambda||}\ ,\end{aligned} \tag{4.16}$$

where the norm $||\cdot||$ is, e.g., the Hilbert-Schmidt norm $||A|| = \sqrt{\mathrm{Tr}(A^\dagger A)}$.

On the other hand, by assumption, $F^{(M)}_{\beta,\Lambda}$ is analytic in $|U|\le U_0$, with U_0 independent of β,Λ,M, and uniformly convergent as $M\to\infty$. Hence, by Weierstrass theorem, the limit $F_{\beta,\Lambda} = \lim_{M\to\infty} F^{(M)}_{\beta,\Lambda}$ is analytic in $|U|\le U_0$ and its Taylor coefficients coincide with the limits as $M\to\infty$ of the Taylor coefficients of $F^{(M)}_{\beta,\Lambda}$. Moreover, $\lim_{M\to\infty} e^{-\beta|\Lambda| F^{(M)}_{\beta,\Lambda}} = e^{-\beta|\Lambda| F_{\beta,\Lambda}}$, again by Weierstrass theorem.

As discussed above, the Taylor coefficients of $e^{-\beta|\Lambda|F_{\beta,\Lambda}}$ coincide with the Taylor coefficients of $\mathrm{Tr}\{e^{-\beta H_\Lambda}\}/\mathrm{Tr}\{e^{-\beta H_{0,\Lambda}}\}$: therefore, $\mathrm{Tr}\{e^{-\beta H_\Lambda}\}/\mathrm{Tr}\{e^{-\beta H_{0,\Lambda}}\} = e^{-\beta|\Lambda|F_{\beta,\Lambda}}$ in the complex region $|U|\le U_0$, simply because the l.h.s. is entire in U, the r.h.s. is analytic in $|U|\le U_0$ and the Taylor coefficients at the origin of the two sides are the same. Taking logarithms at both sides proves Eq.(4.14). ∎

By Proposition 4.1, the Grassmann integral Eq.(4.13) can be used to compute the free energy of the original Hubbard model, provided that the r.h.s. of Eq.(4.13) is analytic in a domain that is uniform in M,β,Λ and that it converges to a well defined analytic function uniformly as $M\to\infty$. The rest of these notes are devoted to the proof of this fact. We start from Eq.(4.13), which can be rewritten as

$$F^{(M)}_{\beta,\Lambda} := -\frac{1}{\beta|\Lambda|}\sum_{N\ge1}\frac{(-1)^N}{N!}\mathcal{E}^T(\mathcal{V};N)\ , \tag{4.17}$$

where the *truncated expectation* $\mathcal{E}^T$ is defined as

$$\mathcal{E}^T(\mathcal{V};N) := \frac{\partial^N}{\partial\lambda^N}\log\int P_M(d\psi)e^{\lambda\mathcal{V}(\psi)}\big|_{\lambda=0}\ . \tag{4.18}$$

More in general,

$$\mathcal{E}^T(\mathcal{V}_1,\ldots,\mathcal{V}_N) := \frac{\partial^N}{\partial\lambda_1\cdots\partial\lambda_N}\log\int P_M(d\psi)e^{\lambda_1\mathcal{V}_1(\psi)+\cdots+\lambda_N\mathcal{V}_N(\psi)}\big|_{\lambda_i=0} \tag{4.19}$$

and $\mathcal{E}^T(\mathcal{V}_1,\ldots,\mathcal{V}_N)\big|_{\mathcal{V}_i=\mathcal{V}} = \mathcal{E}^T(\mathcal{V};N)$. It can be checked by induction that the truncated expectation is related to the simple expectation $\mathcal{E}(X(\psi)) = \int P_M(d\psi)X(\psi)$ by

$$\mathcal{E}(\mathcal{V}_1\cdots\mathcal{V}_N) = \sum_{m=1}^{N}\sum_{(Y^1,\ldots,Y^m)}\mathcal{E}^T(\mathcal{V}_{j_1^1},\ldots,\mathcal{V}_{j^1_{|Y^1|}})\cdots\mathcal{E}^T(\mathcal{V}_{j_1^m},\ldots,\mathcal{V}j^m_{|Y^m|})\ , \tag{4.20}$$

where the second sum in the r.h.s. runs over partitions of $\{1,\ldots,N\}$ of multiplicity m, i.e., over m-ples of disjoint sets such that $\cup_{i=1}^m Y^i = \{1,\ldots,N\}$, with $Y^i = \{j_1^i,\ldots,j^i_{|Y_i|}\}$. Note that $\mathcal{E}(\mathcal{V}^N) = \mathcal{E}(\mathcal{V}_1,\ldots,\mathcal{V}_N)\big|_{\mathcal{V}_i=\mathcal{V}}$ can be computed as a sum of Feynman diagrams whose values are determined by the same Feynman rules described after Eq.(4.4) (with the exception of rule (viii): of course, since $\mathcal{E}(X) = \int P_M(d\psi)X$, M should be temporarily kept fixed in the computation); we shall write

$$\mathcal{E}(\mathcal{V}^N) = \sum_{\mathcal{G}\in\Gamma_N}\widehat{\mathrm{Val}}(\mathcal{G})\ , \tag{4.21}$$

where Γ_N is the set of all Feynman diagrams with N vertices, constructed with the rules described above; $\widehat{\mathrm{Val}}(\mathcal{G})$ includes the integration over the space-time labels $\mathbf{x}_i$ and the sum over the component labels ρ_i: if $\mathcal{G}\in\Gamma_N^T$, we shall write symbolically

$$\widehat{\mathrm{Val}}(\mathcal{G}) = \sigma_{\mathcal{G}}U^N\sum_{\rho_1,\ldots,\rho_N}\int d\mathbf{x}_1\cdots d\mathbf{x}_N\prod_{\ell\in\mathcal{G}}\delta_{\sigma(\ell),\sigma'(\ell)}g_{\rho(\ell),\rho'(\ell)}(\mathbf{x}(\ell)-\mathbf{x}'(\ell))\ , \tag{4.22}$$

where $\sigma_{\mathcal{G}}$ is the sign of the permutation associated to the graph $\mathcal{G}$ and we denoted by $(\mathbf{x}(\ell),\sigma(\ell),\rho(\ell))$ and $(\mathbf{x}'(\ell),\sigma'(\ell),\rho'(\ell))$ the labels of the two vertices, which the line ℓ exits from and enters in, respectively. Using Eqs.(4.20)-(4.21), it can be proved by induction that

$$\mathcal{E}^T(\mathcal{V};N) = \sum_{\mathcal{G}\in\Gamma_N^T}\widehat{\mathrm{Val}}(\mathcal{G})\ , \tag{4.23}$$

where $\Gamma_N^T\subset\Gamma_N$ is the set of *connected* Feynman diagrams with N vertices. Combining Eq.(4.17) with Eq.(4.23) we finally have a formal power series expansion for the specific free energy of our model (more precisely, of its ultraviolet regularization associated to the imaginary-time ultraviolet cutoff $\chi_0(2^{-M}|k_0|)$). The Feynman rules for computing $\widehat{\mathrm{Val}}(\mathcal{G})$ allow us to derive a first *very naive* upper bound on the N-th order contribution to $F_{\beta,\Lambda}^{(M)}$, that is to

$$F_{\beta,\Lambda}^{(M;N)} := -\frac{1}{\beta|\Lambda|}\frac{(-1)^N}{N!}\mathcal{E}^T(\mathcal{V};N)\ . \tag{4.24}$$

We have:

$$|F_{\beta,\Lambda}^{(M;N)}| \le \frac{1}{\beta|\Lambda|}\frac{1}{N!}\sum_{\mathcal{G}\in\Gamma_N^T}|\widehat{\rm Val}(\mathcal{G})| \le$$
$$\le \frac{|\Gamma_N^T|}{N!}2^N|U|^N||g||_\infty^{N+1}\,||g||_1^{N-1}\,, \tag{4.25}$$

where $|\Gamma_N^T|$ is the number of connected Feynman diagrams of order N and $\beta|\Lambda|(2|U|)^N||g||_\infty^{N+1}\,||g||_1^{N-1}$ is a uniform bound on the value of a generic connected Feynman diagram of order N. The bound is obtained as follows: given $\mathcal{G}\in\Gamma_N^T$, select an arbitrary "spanning tree" in $\mathcal{G}$, i.e. a loopless subset of $\mathcal{G}$ that connects all the N vertices; now: the integrals over the space-time coordinates of the product of the propagators on the spanning tree can be bounded by $\beta|\Lambda|\,||g||_1^{N-1}$; the product of the remaining propagators can be bounded by $||g||_\infty^{N+1}$; finally, the sum over the ρ_i labels is bounded by 2^N. Using Eq.(4.25) and the facts that, for a suitable constant $C>0$: (i) $|\Gamma_N^T|\le C^N(N!)^2$ (see, e.g., [23, Appendix A.1.3] for a proof of this fact), (ii) $||g||_\infty\le CM$, (iii) $||g||_1\le C\beta$, we find:

$$|F_{\beta,\Lambda}^{(M;N)}| \le (2C^3|U|)^N N!\, M^{N+1}\beta^{N-1}\,. \tag{4.26}$$

Remark. While the bound $||g||_1\le C\beta$ (see Appendix A for a proof) is dimensionally optimal, the estimate $||g||_\infty\le CM$ could be improved to $||g||_\infty\le$ (const.), at the cost of a more detailed analysis of the definition of $g(\mathbf{x})$, which shows that the apparent ultraviolet logarithmic divergence associated to the sum over k_0 in Eq.(4.5) is in fact related to a jump singularity of $g(x_0,\vec{0})$ at $x_0=\beta\mathbb{Z}$. This can be proved along the lines of [3, Section 2]. However, to the purpose of the present discussion, the rough (and easier) bound $||g||_\infty\le CM$ is enough; see Appendix A for a proof.

The pessimistic bound Eq.(4.26) has two main problems: (i) a combinatorial problem, associated to the $N!$, which makes the r.h.s. of Eq.(4.26) not summable over N, not even for finite M and β; (ii) a divergence problem, associated to the factor $M^{N+1}\beta^{N-1}$, which diverges exponentially as $M\to\infty$ (i.e., as the ultraviolet regularization is removed) and as $\beta\to\infty$ (i.e., as the temperature is sent to 0). The combinatorial problem is solved by a smart reorganization of the perturbation theory, in the form of a determinant expansion, together with a systematic use of the Gram-Hadamard bound. The divergence problem is solved by systematic resummations of the series: we will first identify the class of contributions that produce ultraviolet or infrared divergences and then we show how to inductively resum them into a redefinition of the coupling constants of the theory; the inductive resummations are based on a multiscale integration of the theory: at the end of the construction, they will allow us to express the specific free energy in terms of modified Feynman diagrams, whose values are not affected anymore by ultraviolet or infrared divergences.

5. THE DETERMINANT EXPANSION

Let us now show how to attack the first of two problems that arose at the end of previous section. In other words, let us show how to solve the combinatorial problem by reorganizing the perturbative expansion discussed above into a more compact and more convenient form. In the previous section we discussed a Feynman diagram representation of the truncated expectation, see Eq.(4.23). A slightly more general version of Eq.(4.23) is the following. For a given set of indices $P = \{f_1, \ldots, f_{|P|}\}$, with $f_i = (\mathbf{x}_i, \sigma_i, \rho_i, \varepsilon_i)$, $\varepsilon_i \in \{+, -\}$, let

$$\psi_P := \prod_{f \in P} \psi^{\varepsilon(f)}_{\mathbf{x}(f), \sigma(f), \rho(f)} \ . \tag{5.1}$$

Each field $\psi^{\varepsilon(f)}_{\mathbf{x}(f), \sigma(f), \rho(f)}$ can be represented as an oriented half-line, emerging from the point $\mathbf{x}(f)$ and carrying an arrow, pointing in the direction entering or exiting the point, depending on whether $\varepsilon(f)$ is equal to $-$ or $+$, respectively; moreover, the half-line carries two labels, $\sigma(f) \in \{\uparrow, \downarrow\}$ and $\rho(f) \in \{1, 2\}$. Now, given s set of indices $P_1, \ldots, P_s$, we can enclose the points $\mathbf{x}(f)$ belonging to the set P_j, for some $j = 1, \ldots, s$, in a box: in this way, assuming that all the points $\mathbf{x}(f)$, $f \in \cup_i P_i$, are distinct, we obtain s disjoint boxes. Given $\mathcal{P} := \{P_1, \ldots, P_s\}$, we can associate to it the set $\Gamma^T(\mathcal{P})$ of connected Feynman diagrams, obtained by pairing the half-lines with consistent orientations, in such a way that the two half-lines of any connected pairs carry the same spin index, and in such a way that all the boxes are connected. Using a notation similar to Eq.(4.22), we have:

$$\begin{aligned} \mathcal{E}^T(\psi_{P_1}, \ldots, \psi_{P_s}) &= \sum_{\mathcal{G} \in \Gamma^T(\mathcal{P})} \mathrm{Val}(\mathcal{G}) \ , \\ \mathrm{Val}(\mathcal{G}) &= \sigma_{\mathcal{G}} \prod_{\ell \in \mathcal{G}} \delta_{\sigma(\ell), \sigma'(\ell)} g_{\rho(\ell), \rho'(\ell)}(\mathbf{x}(\ell) - \mathbf{x}'(\ell)) \ , \end{aligned} \tag{5.2}$$

A different a more compact representation for the truncated expectation, alternative to Eq.(5.2), is the following:

$$\mathcal{E}^T(\psi_{P_1}, \ldots, \psi_{P_s}) = \sum_{T \in \mathbf{T}(\mathcal{P})} \alpha_T \prod_{\ell \in T} g_\ell \int dP_T(\mathbf{t}) \det G^T(\mathbf{t}) \ , \tag{5.3}$$

where:

- any element T of the set $\mathbf{T}(\mathcal{P})$ is a set of lines forming an *anchored tree* between the boxes $P_1, \ldots, P_s$, i.e., T is a set of lines that becomes a tree if one identifies all the points in the same clusters;

- α_T is a sign (irrelevant for the subsequent bounds);
- g_ℓ is a shorthand for $\delta_{\sigma(\ell),\sigma'(\ell)}\, g_{\rho(\ell),\rho'(\ell)}(\mathbf{x}(\ell)-\mathbf{x}'(\ell))$;
- if $\mathbf{t}=\{t_{i,i'}\in[0,1], 1\le i,i'\le s\}$, then $dP_T(\mathbf{t})$ is a probability measure with support on a set of $\mathbf{t}$ such that $t_{i,i'}=\mathbf{u}_i\cdot\mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i\in\mathbb{R}^n$ of unit norm;
- if $2n=\sum_{i=1}^s|P_i|$, then $G^T(\mathbf{t})$ is a $(n-s+1)\times(n-s+1)$ matrix, whose elements are given by $G^T_{f,f'}=t_{i(f),i(f')}g_{\ell(f,f')}$, where: $f,f'\notin\cup_{\ell\in T}\{f^1_\ell,f^2_\ell\}$ and f^1_ℓ,f^2_ℓ are the two field labels associated to the two (entering and exiting) half-lines contracted into ℓ; $i(f)\in\{1,\dots,s\}$ is s.t. $f\in P_{i(f)}$; $g_{\ell(f,f')}$ is the propagator associated to the line obtained by contracting the two half-lines with indices f and f'.

If $s=1$ the sum over T is empty, but we can still use the Eq.(5.3) by interpreting the r.h.s. as equal to 1 if P_1 is empty and equal to $\det G^T(\mathbf{1})$ otherwise.

The proof of the determinant representation is described in Appendix B; this representation is due to a fermionic reinterpretation of the interpolation formulas by Battle, Brydges and Federbush [2, 10, 11], used originally by Gawedski-Kupianen [22] and by Lesniewski [33], among others, to study certain $(1+1)$-dimensional fermionic Quantum Field Theories. Using Eq.(5.3) we get an alternative representation for the N-th order contribution to the specific free energy:

$$F^{(M;N)}_{\beta,\Lambda}=-\frac{1}{\beta|\Lambda|}\frac{(-1)^N}{N!}\mathcal{E}^T(\mathcal{V};N)=-\frac{1}{\beta|\Lambda|}\frac{(-1)^N}{N!}U^N\sum_{\rho_1,\dots,\rho_N}\sum_{T\in\mathbf{T}_N}\alpha_T\int d\mathbf{x}_1\cdots d\mathbf{x}_N \cdot\prod_{\ell\in T}\delta_{\sigma(\ell),\sigma'(\ell)}\, g_{\rho(\ell),\rho'(\ell)}(\mathbf{x}(\ell)-\mathbf{x}'(\ell))\int dP_T(\mathbf{t})\det G^T(\mathbf{t})\;. \tag{5.4}$$

Using the fact that the number of anchored trees in $\mathbf{T}_N$ is bounded by $C^N N!$ for a suitable constant C (see, e.g., [23, Appendix A.3.3] for a proof of this fact), from Eq.(5.4) we get:

$$|F^{(M;N)}_{\beta,\Lambda}|\le(\text{const.})^N|U|^N\,||g||_1^{N-1}||\det G^T(\cdot)||_\infty\;. \tag{5.5}$$

In order to bound $\det G^T$, we use the *Gram-Hadamard inequality*, stating that, if M is a square matrix with elements M_{ij} of the form $M_{ij}=\langle A_i,B_j\rangle$, where A_i, B_j are vectors in a Hilbert space with scalar product $\langle\cdot,\cdot\rangle$, then

$$|\det M|\le\prod_i||A_i||\cdot||B_i||\;. \tag{5.6}$$

where $||\cdot||$ is the norm induced by the scalar product. See [23, Theorem A.1] for a proof of Eq.(5.6).

Let $\mathcal{H} = \mathbb{R}^n \otimes \mathcal{H}_0$, where $\mathcal{H}_0$ is the Hilbert space of the functions $\mathbf{F}$: $[-\beta/2, \beta/2) \times \Lambda \to \mathbb{C}^2$, with scalar product $\langle \mathbf{F}, \mathbf{G} \rangle = \sum_{\rho=1,2} \int d\mathbf{z}\, F^*_\rho(\mathbf{z}) G_\rho(\mathbf{z})$, where $F_\rho = [\mathbf{F}]_\rho$, $G_\rho = [\mathbf{G}]_\rho$, $\rho = 1, 2$, are the components of the vectors $\mathbf{F}$ and $\mathbf{G}$. It is easy to verify that

$$\begin{aligned} G^T_{f,f'} &= t_{i(f),i(f')}\, \delta_{\sigma(\ell),\sigma'(\ell)}\, g_{\rho(f),\rho(f')}(\mathbf{x}(f) - \mathbf{x}(f')) = \\ &= \langle \mathbf{u}_{i(f)} \otimes \mathbf{e}_{\sigma(f)} \otimes \mathbf{A}_{\mathbf{x}(f),\rho(f)}, \mathbf{u}_{i(f')} \otimes \mathbf{e}_{\sigma(f')} \otimes \mathbf{B}_{\mathbf{x}(f'),\rho(f')} \rangle \;, \end{aligned} \tag{5.7}$$

where: $\mathbf{u}_i \in \mathbb{R}^n$, $i = 1, \ldots, n$, are vectors such that $t_{i,i'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$; $\mathbf{e}_\uparrow = (1, 0)$, $\mathbf{e}_\downarrow = (0, 1)$; $\mathbf{A}_{\mathbf{x},\rho}$ and $\mathbf{B}_{\mathbf{x},\rho}$ have components:

$$[\mathbf{A}_{\mathbf{x},\rho}(\mathbf{z})]_i = \frac{1}{\beta|\Lambda|} \sum_{\mathbf{k} \in \mathcal{B}^{(M)}_{\beta,L}} \frac{\sqrt{\chi_0(2^{-M}|k_0|)}\, e^{-i\mathbf{k}(\mathbf{z}-\mathbf{x})}}{\left[k_0^2 + v_0^2 |\Omega(\vec{k})|^2\right]^{1/4}} \delta_{\rho,i} \;, \tag{5.8}$$

$$[\mathbf{B}_{\mathbf{x},\rho}(\mathbf{z})]_i = \frac{1}{\beta|\Lambda|} \sum_{\mathbf{k} \in \mathcal{B}^{(M)}_{\beta,L}} \frac{\sqrt{\chi_0(2^{-M}|k_0|)}\, e^{-i\mathbf{k}(\mathbf{z}-\mathbf{x})}}{\left[k_0^2 + v_0^2 |\Omega(\vec{k})|^2\right]^{3/4}} \begin{pmatrix} ik_0 & -v_0 \Omega^*(\vec{k}) \\ -v_0 \Omega(\vec{k}) & ik_0 \end{pmatrix}_{i,\rho} \;,$$

so that

$$||\mathbf{A}_{\mathbf{x},\rho}||^2 = ||\mathbf{B}_{\mathbf{x},\rho}||^2 = \frac{1}{\beta|\Lambda|} \sum_{\mathbf{k} \in \mathcal{B}^{(M)}_{\beta,L}} \frac{\chi_0(2^{-M}|k_0|)}{\left[k_0^2 + v_0^2 |\Omega(\vec{k})|^2\right]^{1/2}} \leq CM \;, \tag{5.9}$$

for a suitable constant C. Using the Gram-Hadamard inequality, we find $||\det G^T||_\infty \leq (\text{const.})^N M^{N+1}$; substituting this result into Eq.(5.5), we finally get:

$$|F^{(M;N)}_{\beta,\Lambda}| \leq (\text{const.})^N |U|^N M^{N+1} \beta^{N-1} \;, \tag{5.10}$$

which is similar to Eq.(4.26), but for the fact that there is no $N!$ in the r.h.s.! In other words, using the determinant expansion, we recovered the same dimensional estimate as the one obtained by the Feynman diagram expansion and we combinatorially gained a $1/N!$. The r.h.s. of Eq.(5.10) is now summable over N for $|U|$ sufficiently small, even though non uniformly in M and β. In the next section we will discuss how to systematically improve the dimensional bound by an iterative resummation method.

6. THE MULTISCALE INTEGRATION: THE ULTRAVIOLET REGIME

In this section we begin to illustrate the multiscale integration of the fermionic functional integral of interest. This method will later allow us to perform iterative resummations and to re-express the specific free energy in terms of a modified expansion, whose N-th order term is summable in N and uniformly convergent as $M \to \infty$ and $\beta \to -\infty$, as desired.

The first step in the computation of the partition function

$$\Xi_{M,\beta,\Lambda} := \int P_M(d\psi) e^{-\mathcal{V}(\psi)} \tag{6.1}$$

and of its logarithm is the integration of the ultraviolet degrees of freedom corresponding to the large values of k_0. We proceed in the following way. We decompose the free propagator $\hat{g}_{\mathbf{k}}$ into a sum of two propagators supported in the regions of k_0 "large" and "small", respectively. The regions of k_0 large and small are defined in terms of the smooth support function $\chi_0(t)$ introduced after Eq.(4.5); note that, by the very definition of χ_0, the supports of $\chi_0\Big(\sqrt{k_0^2 + |\vec{k} - \vec{p}_F^{\,+}|^2}\,\Big)$ and $\chi_0\Big(\sqrt{k_0^2 + |\vec{k} - \vec{p}_F^{\,-}|^2}\,\Big)$ are disjoint (here $|\cdot|$ is the euclidean norm over $\mathbb{R}^2/\Lambda^*$). We define

$$f_{u.v.}(\mathbf{k}) = 1 - \chi_0\Big(\sqrt{k_0^2 + |\vec{k} - \vec{p}_F^{\,+}|^2}\,\Big) - \chi_0\Big(\sqrt{k_0^2 + |\vec{k} - \vec{p}_F^{\,-}|^2}\,\Big) \tag{6.2}$$

and $f_{i.r.}(\mathbf{k}) = 1 - f_{u.v.}(\mathbf{k})$, so that we can rewrite $\hat{g}_{\mathbf{k}}$ as:

$$\hat{g}_{\mathbf{k}} = f_{u.v.}(\mathbf{k})\hat{g}_{\mathbf{k}} + f_{i.r.}(\mathbf{k})\hat{g}_{\mathbf{k}} \stackrel{def}{=} \hat{g}^{(u.v.)}(\mathbf{k}) + \hat{g}^{(i.r.)}(\mathbf{k}) \; . \tag{6.3}$$

We now introduce two independent set of Grassmann fields $\{\psi^{(u.v.)\pm}_{\mathbf{k},\sigma,\rho}\}$ and $\{\psi^{(i.r.)\pm}_{\mathbf{k},\sigma,\rho}\}$, with $\mathbf{k} \in \mathcal{B}^{(M)}_{\beta,L}$, $\sigma = \uparrow\downarrow$, $\rho = 1,2$, and the Gaussian integrations $P(d\psi^{(u.v.)})$ and $P(d\psi^{(i.r.)})$ defined by

$$\begin{aligned} &\int P(d\psi^{(u.v.)}) \hat{\psi}^{(u.v.)-}_{\mathbf{k}_1,\sigma_1} \hat{\psi}^{(u.v.)+}_{\mathbf{k}_2,\sigma_2} = \beta|\Lambda|\delta_{\sigma_1,\sigma_2}\delta_{\mathbf{k}_1,\mathbf{k}_2}\hat{g}^{(u.v.)}(\mathbf{k}_1) \; , \\ &\int P(d\psi^{(i.r.)}) \hat{\psi}^{(u.v.)-}_{\mathbf{k}_1,\sigma_1} \hat{\psi}^{(i.r.)+}_{\mathbf{k}_2,\sigma_2} = \beta|\Lambda|\delta_{\sigma_1,\sigma_2}\delta_{\mathbf{k}_1,\mathbf{k}_2}\hat{g}^{(i.r.)}(\mathbf{k}_1) \; . \end{aligned} \tag{6.4}$$

Similarly to $P_M(d\psi)$, the Gaussian integrations $P(d\psi^{(u.v.)})$, $P(d\psi^{(i.r.)})$ also admit an explicit representation analogous to (4.8), with $\hat{g}_{\mathbf{k}}$ replaced by $\hat{g}^{(u.v.)}(\mathbf{k})$ or $\hat{g}^{(i.r.)}(\mathbf{k})$ and the sum over $\mathbf{k}$ restricted to the values in the support of $f_{u.v.}(\mathbf{k})$ or $f_{i.r.}(\mathbf{k})$, respectively. The definition of Grassmann integration implies the following identity ("addition principle"):

$$\int P(d\psi) e^{-\mathcal{V}(\psi)} = \int P(d\psi^{(i.r.)}) \int P(d\psi^{(u.v.)}) e^{-\mathcal{V}(\psi^{(i.r.)} + \psi^{(u.v.)})} \tag{6.5}$$

so that we can rewrite the partition function as

$$\begin{aligned} \Xi_{M,\beta,\Lambda} &= e^{-\beta|\Lambda| F^{(M)}_{\beta,\Lambda}} = \int P(d\psi^{(i.r.)}) \exp\Big\{ \sum_{n\geq 1} \frac{1}{n!} \mathcal{E}^T_{u.v.}(-\mathcal{V}(\psi^{(i.r.)} + \cdot); n) \Big\} := \\ &:= e^{-\beta|\Lambda| F_{0,M}} \int P(d\psi^{(i.r.)}) e^{-\mathcal{V}_0(\psi^{(i.r.)})} \; , \end{aligned} \tag{6.6}$$

where the *truncated expectation* $\mathcal{E}^T_{u.v.}$ is defined, given any polynomial $V_1(\psi^{(u.v.)})$ with coefficients depending on $\psi^{(i.r.)}$, as

$$\mathcal{E}^T_{u.v.}(V_1(\cdot);n) = \frac{\partial^n}{\partial\lambda^n}\log\int P(d\psi^{(u.v.)})e^{\lambda V_1(\psi^{(u.v.)})}\Big|_{\lambda=0} \tag{6.7}$$

and $\mathcal{V}_0$ is fixed by the condition $\mathcal{V}_0(0)=0$. We will prove below that $\mathcal{V}_0$ can be written as

$$\mathcal{V}_0(\psi) = \sum_{n=1}^{\infty}(\beta|\Lambda|)^{-2n}\sum_{\sigma_1,\ldots,\sigma_n=\uparrow\downarrow}\sum_{\rho_1,\ldots,\rho_{2n}=1,2}\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_{2n}}\Big[\prod_{j=1}^{n}\hat\psi^{(i.r.)+}_{\mathbf{k}_{2j-1},\sigma_j,\rho_{2j-1}}\hat\psi^{(i.r.)-}_{\mathbf{k}_{2j},\sigma_j,\rho_{2j}}\Big]\cdot$$
$$\cdot\hat W_{M,2n,\underline{\rho}}(\mathbf{k}_1,\ldots,\mathbf{k}_{2n-1})\,\delta\Big(\sum_{j=1}^{n}(\mathbf{k}_{2j-1}-\mathbf{k}_{2j})\Big)\,, \tag{6.8}$$

where $\underline{\rho}=(\rho_1,\ldots,\rho_{2n})$ and we used the notation

$$\delta(\mathbf{k})=\delta(\vec k)\delta(k_0)\,,\qquad \delta(\vec k)=|\Lambda|\sum_{n_1,n_2\in\mathbb{Z}}\delta_{\vec k,n_1\vec G_1+n_2\vec G_2}\,,\qquad \delta(k_0)=\beta\delta_{k_0,0}\,, \tag{6.9}$$

with $\vec G_1,\vec G_2$ a basis of Λ^*. The possibility of representing $\mathcal{V}_M$ in the form (6.8), with the *kernels* $\hat W_{M,2n,\underline{\rho}}$ independent of the spin indices σ_i, follows from a number of remarkable symmetries, discussed in Appendix C, see in particular symmetries (1)–(3) in Lemma C.1. The regularity properties of the kernels are summarized in the following Lemma, which will be proved below.

Lemma 6.1 *The constant $F_{0,M}$ in (6.6) and the kernels $\hat W_{M,2n,\underline{\rho}}$ in (6.8) are given by power series in U, convergent in the complex disc $|U|\le U_0$, for U_0 small enough and independent of M,β,Λ; after Fourier transform, the $\mathbf{x}$-space counterparts of the kernels $\hat W_{M,2n,\underline{\rho}}$ satisfy the following bounds:*

$$\int d\mathbf{x}_1\cdots d\mathbf{x}_{2n}\big|W_{M,2n,\underline{\rho}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2n})\big|\le\beta|\Lambda|C^n|U|^{\max\{1,n-1\}} \tag{6.10}$$

for some constant $C>0$. Moreover, the limits $F_0=\lim_{M\to\infty}F_{0,M}$ and $W_{2n,\underline{\rho}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2n})=\lim_{M\to\infty}W_{M,2n,\underline{\rho}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2n})$ exist and are reached uniformly in M, so that, in particular, the limiting functions are analytic in the same domain $|U|\le U_0$ and so are their $\beta,|\Lambda|\to\infty$ limits (that, with some abuse of notation, we shall denote by the same symbols).

Remark. Once that the ultraviolet degrees of freedom have been integrated out, the remaining infrared problem (i.e., the computation of the Grassmann integral in the second line of Eq.(6.6)) is essentially independent of M, given the fact that the limit $W_{2n,\underline{\rho}}$ of the kernles $W_{M,2n,\underline{\rho}}$ is reached uniformly and that the limiting kernels are analytic and satisfy the same bounds as Eq.(6.10). For this reason, in the infrared integration described in the next two sections, M will not play any essential role and, whenever possible, we will simplify the notation by dropping the label M.

Before we present the proof of Lemma 6.1, let us note that the kernels $W_{M,2n,\underline{\rho}}$ satisfy a number of non-trivial invariance properties. We will be particularly interested in the invariance properties of the quadratic part $\hat W_{M,2,(\rho_1,\rho_2)}(\mathbf{k}_1,\mathbf{k}_2)$, which will be used below to show that the structure of the quadratic part of the new effective interaction has the same symmetries as the free integration. The crucial properties that we will need are summarized in the following Lemma, which is proved in Appendix C.

Lemma 6.2. *The kernel $\hat W_2(\mathbf{k}) := \hat W_{M,2}(\mathbf{k},\mathbf{k})$, thought as a 2×2 matrix with components $\hat W_{M,2,(i,j)}(\mathbf{k},\mathbf{k})$, satisfies the following symmetry properties:*

$$\hat W_2(\mathbf{k}) = e^{i\vec k(\vec\delta_1-\vec\delta_2)\frac{\sigma_3}{2}}\hat W_2\big((k_0, e^{i\frac{2\pi}{3}\sigma_2}\vec k)\big)e^{-i\vec k(\vec\delta_1-\vec\delta_2)\frac{\sigma_3}{2}} = \hat W_2^*(-\mathbf{k}) = \hat W_2\big((k_0,k_1,-k_2)\big)$$
$$= \sigma_1\hat W_2\big((k_0,-k_1,k_2)\big)\sigma_1 = \hat W_2^T\big((k_0,-\vec k)\big) = -\sigma_3\hat W_2\big((-k_0,\vec k)\big)\sigma_3\ , \qquad (6.11)$$

where $\sigma_1,\sigma_2,\sigma_3$ are the standard Pauli matrices. In particular, if $\hat W(\mathbf{k}) = \lim_{\beta,|\Lambda|\to\infty}\hat W_2(\mathbf{k})$, in the vicinity of the Fermi point $\mathbf{p}_F^\omega = (0,\vec p_F^{\,\omega})$, with $\omega=\pm$,

$$\hat W(\mathbf{k}) = -\begin{pmatrix} iz_0k_0 & \delta_0\Omega^*(\vec k)\\ \delta_0\Omega(\vec k) & iz_0k_0\end{pmatrix} + O(|\mathbf{k}-\mathbf{p}_F^\omega|^2)\ , \qquad (6.12)$$

for some real constants z_0,δ_0.

Note that Eq.(6.12) can be read by saying that, in the zero temperature and thermodynamic limits, the two-legged kernel has the same structure as the inverse of the free covariance, $\hat S_0^{-1}(\mathbf{k})$, modulo higher order terms in $\mathbf{k}-\mathbf{p}_F^\omega$. This fact will be used in the next section to define a dressed infrared propagator $[\hat S_0^{-1}(\mathbf{k})+\hat W(\mathbf{k})]^{-1}$, with the same infrared singularity structure as the free one. We will come back to this point in more detail. For the moment, let us turn to the proof of Lemma 6.1, which illustrates the main RG strategy that will be also used below, in the more difficult infrared integration.

Proof of Lemma 6.1. Let us rewrite the Fourier transform of $\hat g^{(u.v.)}(\mathbf{k})$ as

$$g^{(u.v.)}(\mathbf{x}) = \sum_{h=1}^{M} g^{(h)}(\mathbf{x})\ , \qquad (6.13)$$

where

$$g^{(h)}(\mathbf{x}) = \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}^{(M)}} e^{-i\mathbf{k}\mathbf{x}}\frac{f_{u.v.}(\mathbf{k})H_h(k_0)}{k_0^2+v_0^2|\Omega(\vec k)|^2}\begin{pmatrix} ik_0 & -v_0\Omega^*(\vec k)\\ -v_0\Omega(\vec k) & ik_0\end{pmatrix}\ , \qquad (6.14)$$

with $H_1(k_0)=\chi_0(2^{-1}|k_0|)$ and, if $h\ge 2$, $H_h(k_0)=\chi_0(2^{-h}|k_0|)-\chi_0(2^{-h+1}|k_0|)$. Note that $[g^{(h)}(\mathbf{0})]_{\rho\rho}=0$, $\rho=1,2$, and, for any integer $K\ge 0$, $g^{(h)}(\mathbf{x})$ satisfies the bound

$$||g^{(h)}(\mathbf{x})||\le \frac{C_K}{1+(2^h|x_0|_\beta+|\vec x|_\Lambda)^K}\ , \tag{6.15}$$

where $|x_0|_\beta=\min_{n\in\mathbb{Z}}|x_0+n\beta|$ is the distance over the one-dimensional torus of length β and $|\vec x|_\Lambda=\min_{\vec\ell\in\mathbb{B}}|\vec x+L\vec\ell|$ is the distance over the periodic lattice Λ (here $\mathbb{B}$ is the triangular lattice defined after Eq.(2.1)); see Appendix A for a proof. Moreover, $g^{(h)}(\mathbf{x})$ admits a Gram representation that, in notation analogous to Eq.(5.7), reads

$$g^{(h)}_{\rho,\rho'}(\mathbf{x}-\mathbf{y})=\langle \mathbf{A}^{(h)}_{\mathbf{x},\rho},\mathbf{B}^{(h)}_{\mathbf{y},\rho'}\rangle\ , \tag{6.16}$$

with

$$[\mathbf{A}^{(h)}_{\mathbf{x},\rho}(\mathbf{z})]_i=\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}}\sqrt{f_{u.v.}(\mathbf{k})H_h(k_0)}\frac{e^{-i\mathbf{k}(\mathbf{z}-\mathbf{x})}}{[k_0^2+v_0^2|\Omega(\vec k)|^2]^{1/4}}\delta_{\rho,i}\ , \tag{6.17}$$

$$[\mathbf{B}^{(h)}_{\mathbf{x},\rho}(\mathbf{z})]_i=\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}}\frac{\sqrt{f_{u.v.}(\mathbf{k})H_h(k_0)}\,e^{-i\mathbf{k}(\mathbf{z}-\mathbf{x})}}{[k_0^2+v_0^2|\Omega(\vec k)|^2]^{3/4}}\begin{pmatrix} ik_0 & -v_0\Omega^*(\vec k)\\ -v_0\Omega(\vec k) & ik_0\end{pmatrix}_{i,\rho}\ ,$$

and

$$||\mathbf{A}^{(h)}_{\mathbf{x},\rho}||^2=||\mathbf{B}^{(h)}_{\mathbf{x},\rho}||^2=\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}}\frac{f_{u.v.}(\mathbf{k})H_h(k_0)}{[k_0^2+v_0^2|\Omega(\vec k)|^2]^{1/2}}\le C\ , \tag{6.18}$$

for a suitable constant C. Our goal is to compute

$$\begin{aligned}e^{-\beta|\Lambda|F_{0,M}-\mathcal{V}_0(\psi^{(i.r)})}&=\int P(d\psi^{[1,M]})e^{-\mathcal{V}(\psi^{(i.r.)}+\psi^{[1,M]})}\\ &=\exp\Big\{\log\int P(d\psi^{[1,M]})e^{-\mathcal{V}(\psi^{(i.r.)}+\psi^{[1,M]})}\Big\}\ ,\end{aligned} \tag{6.19}$$

where $P(d\psi^{[1,M]})$ is the fermionic "Gaussian integration" associated with the propagator $\sum_{h=1}^M\hat g^{(h)}(\mathbf{k})$ (i.e., it is the same as $P(d\psi^{(u.v.)})$); moreover, we want to prove that $F_{0,M}$ and $\mathcal{V}_0(\psi^{(i.r)})$ are uniformly convergent as $M\to\infty$. We perform the integration of (6.19) in an iterative fashion: in fact, we will inductively prove that for $1\le h\le M$,

$$e^{-\beta|\Lambda|F_{0,M}-\mathcal{V}_0(\psi^{(i.r)})}=e^{-\beta|\Lambda|F_h}\int P(d\psi^{[1,h]})e^{-\mathcal{V}^{(h)}(\psi^{(i.r.)}+\psi^{[1,h]})} \tag{6.20}$$

where $P(d\psi^{[1,h]})$ is the fermionic "Gaussian integration" associated with the propagator $\sum_{k=1}^h\hat g^{(k)}(\mathbf{k})$ and $\mathcal{V}^{(M)}=\mathcal{V}$; for $1\le h<M$,

$$\begin{aligned}&\mathcal{V}^{(h)}(\psi)= \\ &=\sum_{n=1}^\infty\sum_{\underline\rho,\underline\sigma}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2n}\Big[\prod_{j=1}^n\psi^+_{\mathbf{x}_{2j-1},\sigma_j,\rho_{2j-1}}\psi^-_{\mathbf{x}_{2j},\sigma_j,\rho_{2j}}\Big]W^{(h)}_{M,2n,\underline\rho}(\mathbf{x}_1,\dots,\mathbf{x}_{2n})\ .\end{aligned} \tag{6.21}$$

In order to inductively prove (6.20)-(6.21) we use the addition principle to rewrite

$$\int P(d\psi^{[1,h]})e^{-\mathcal{V}^{(h)}(\psi^{(i.r.)}+\psi^{[1,h]})} = \int P(d\psi^{[1,h-1]})\int P(d\psi^{(h)})e^{-\mathcal{V}^{(h)}(\psi^{(i.r.)}+\psi^{[1,h-1]}+\psi^{(h)})} \tag{6.22}$$

where $P(d\psi^{(h)})$ is the fermionic Gaussian integration with propagator $\hat{g}^{(h)}(\mathbf{k})$. After the integration of $\psi^{(h)}$ we define

$$e^{-\mathcal{V}^{(h-1)}(\psi^{(i.r.)}+\psi^{[1,h-1]})-\beta|\Lambda|\bar{e}_h} = \int P(d\psi^{(h)})e^{-\mathcal{V}^{(h)}(\psi^{(i.r.)}+\psi^{[1,h-1]}+\psi^{(h)})}\ , \tag{6.23}$$

which proves (6.20) with

$$F_h = \sum_{k=h+1}^{M} \bar{e}_k\ . \tag{6.24}$$

Let $\mathcal{E}_h^T$ be the truncated expectation associated to $P(d\psi^{(h)})$: then we have

$$\bar{e}_h + \mathcal{V}^{(h-1)}(\psi) = \sum_{n\geq 1} \frac{1}{n!}(-1)^{n+1}\mathcal{E}_h^T(\mathcal{V}^{(h)}\big(\psi+\psi^{(h)}\big);n)\ . \tag{6.25}$$

Eq.(6.25) can be graphically represented as in Fig.4. The tree in the l.h.s., con-

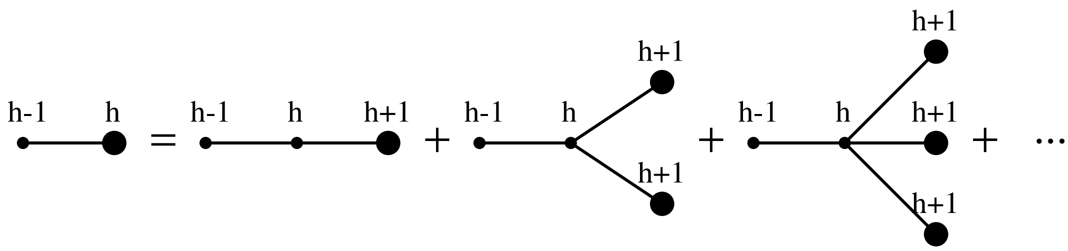

FIG. 4. The graphical representation of $\mathcal{V}^{(h-1)}$.

sisting of a single horizontal branch, connecting the left node (called the *root* and associated to the *scale label* $h-1$) with a big black dot on scale h, represents $\mathcal{V}^{(h-1)}(\psi)$. In the r.h.s., the term with n final points represents the corresponding term in the r.h.s. of Eq.(6.25): a scale label $h-1$ is attached to the leftmost node (the root); a scale label h is attached to the central node (corresponding to the action of $\mathcal{E}_h^T$); a scale label $h+1$ is attached to the n rightmost nodes with the big black dots (representing $\mathcal{V}^{(h)}$). Iterating the graphical equation in Fig.4 up to scale M, and representing the endpoints on scale $M+1$ as simple dots (rather than big black dots), we end up with a graphical representation of $\mathcal{V}^{(h)}$ in terms of *Gallavotti-Nicolò* trees, see Fig.5, defined in terms of the following features.

1. Let us consider the family of all trees which can be constructed by joining a point r, the *root*, with an ordered set of $n \geq 1$ points, the *endpoints* of

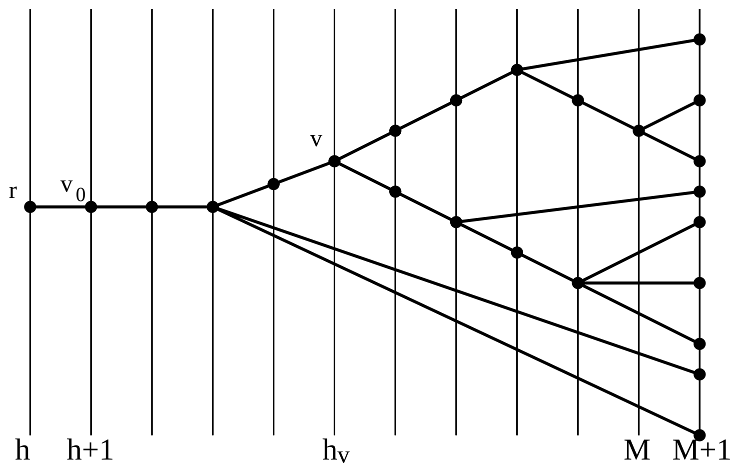

FIG. 5. A tree $\tau \in \widetilde{\mathcal{T}}_{M;h,n}$ with $n = 9$: the root is on scale h and the endpoints are on scale $M + 1$.

the *unlabeled tree*, so that r is not a branching point. n will be called the *order* of the unlabeled tree and the branching points will be called the *non trivial vertices*. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order. Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. It is then easy to see that the number of unlabeled trees with n end-points is bounded by 4^n (see, e.g., [23, Appendix A.1.2] for a proof of this fact). We shall also consider the *labelled trees* (to be called simply trees in the following); they are defined by associating some labels with the unlabelled trees, as explained in the following items.

2. We associate a label $0 \leq h \leq M - 1$ with the root and we denote by $\widetilde{\mathcal{T}}_{M;h,n}$ the corresponding set of labeled trees with n endpoints. Moreover, we introduce a family of vertical lines, labeled by an integer taking values in $[h, M+1]$, and we represent any tree $\tau \in \widetilde{\mathcal{T}}_{M;h,n}$ so that, if v is an endpoint, it is contained in the vertical line with index $h_v = M + 1$, while if it is a non trivial vertex, it is contained in a vertical line with index $h < h_v \leq M$, to be called the *scale* of v; the root r is on the line with index h. In general, the tree will intersect the vertical lines in set of points different from the root, the endpoints and the branching points; these points will be called *trivial vertices*. The set of the *vertices* will be the union of the endpoints, of the trivial vertices and of the non trivial vertices; note that the root is not a vertex. Every vertex v of a tree will be associated to its scale label

h_v, defined, as above, as the label of the vertical line whom v belongs to. Note that, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$.

3. There is only one vertex immediately following the root, called v_0 and with scale label equal to $h+1$.

4. Given a vertex v of $\tau \in \widetilde{\mathcal{T}}_{M;h,n}$ that is not an endpoint, we can consider the subtrees of τ with root v, which correspond to the connected components of the restriction of τ to the vertices $w \geq v$. If a subtree with root v contains only v and one endpoint on scale $h_v + 1$, it will be called a *trivial subtree.*

5. With each endpoint v we associate a factor $\mathcal{V}(\psi^{(i.r.)} + \psi^{[1,M]})$ and a set $\mathbf{x}_v$ of space-time points (the corresponding integration variables in the $\mathbf{x}$-space representation).

6. We introduce a *field label* f to distinguish the field variables appearing in the factors $\mathcal{V}(\psi^{(i.r.)} + \psi^{[1,M]})$ associated with the endpoints; the set of field labels associated with the endpoint v will be called I_v; note that if v is an endpoint $|I_v| = 4$. Analogously, if v is not an endpoint, we shall call I_v the set of field labels associated with the endpoints following the vertex v; $\mathbf{x}(f)$, $\varepsilon(f)$, $\sigma(f)$ and $\rho(f)$ will denote the space-time point, the ε index, the σ index and the ρ index, respectively, of the Grassmann field variable with label f.

In terms of trees, the effective potential $\mathcal{V}^{(h)}$, $0 \leq h \leq M$ (with $\mathcal{V}^{(0)}(\psi^{(i.r.)})$ identified with $\mathcal{V}_0(\psi^{(i.r.)})$), can be written as

$$\mathcal{V}^{(h)}(\psi^{(i.r.)} + \psi^{[1,h]}) + \beta|\Lambda|\bar{e}_{h+1} = \sum_{n=1}^{\infty} \sum_{\tau \in \widetilde{\mathcal{T}}_{M;h,n}} \widetilde{\mathcal{V}}^{(h)}(\tau, \psi^{(i.r.)} + \psi^{[1,h]}) , \qquad (6.26)$$

where, if v_0 is the first vertex of τ and $\tau_1, \ldots, \tau_s$ ($s = s_{v_0}$) are the subtrees of τ with root v_0, $\widetilde{\mathcal{V}}^{(h)}(\tau, \psi^{(i.r.)} + \psi^{[1,h]})$ is defined inductively as:

$$\widetilde{\mathcal{V}}^{(h)}(\tau, \psi^{[0,h]}) = \frac{(-1)^{s+1}}{s!} \mathcal{E}^T_{h+1}\big[\widetilde{\mathcal{V}}^{(h+1)}(\tau_1, \psi^{[0,h+1]}); \ldots ; \widetilde{\mathcal{V}}^{(h+1)}(\tau_s, \psi^{[0,h+1]})\big] . \qquad (6.27)$$

where $\psi^{[0,h]} := \psi^{(i.r.)} + \psi^{[1,h]}$ and, if τ is a trivial subtree with root on scale M, then $\widetilde{\mathcal{V}}^{(M)}(\tau, \psi^{[0,M]}) = \mathcal{V}(\psi^{[0,M]})$.

For what follows, it is important to specify the action of the truncated expectations on the branches connecting any endpoint v to the closest *non-trivial* vertex v' preceding it. In fact, if τ has only one end-point, it is convenient to rewrite $\widetilde{\mathcal{V}}^{(h)}(\tau, \psi^{[0,h]}) = \mathcal{E}^T_{h+1}\mathcal{E}^T_{h+2} \cdots \mathcal{E}^T_M(\mathcal{V}(\psi^{[0,M]}))$ as:

$$\widetilde{\mathcal{V}}^{(h)}(\tau, \psi^{[0,h]}) = \mathcal{V}(\psi^{[0,h]}) + \mathcal{E}^T_{h+1} \cdots \mathcal{E}^T_M\big(\mathcal{V}(\psi^{[0,M]}) - \mathcal{V}(\psi^{[0,h]})\big) . \qquad (6.28)$$

Now, the key observation is that, since $\mathcal{V}(\psi)$ is defined as in Eq.(4.11) and since our explicit choice of the ultraviolet cutoff makes the tadpoles equal to zero (i.e., $[g^{(h)}(\mathbf{0})]_{\rho\rho} = 0$), then the second term in the r.h.s. of Eq.(6.28) is identically zero:

$$\mathcal{E}^T_{h+1} \cdots \mathcal{E}^T_M \big(\mathcal{V}(\psi^{[0,M]}) - \mathcal{V}(\psi^{[0,h]})\big) = 0 \;, \tag{6.29}$$

for all $0 \le h < M$. Therefore, it is natural to shrink all the branches of $\tau \in \widetilde{\mathcal{T}}_{M;h,n}$ consisting of a subtree $\tau' \subseteq \tau$, having root r' on scale $h' \in [h, M]$ and only one endpoint on scale $M+1$, into a trivial subtree, rooted in r' and associated to the factor $\mathcal{V}(\psi^{[0,h']})$. By doing so, we end up with an alternative representation of the effective potentials, which is based on a slightly modified tree expansion. The set of modified trees with n endpoints contributing to $\mathcal{V}^{(h)}$ will be denoted by $\mathcal{T}_{M;h,n}$; every $\tau \in \mathcal{T}_{M;h,n}$ is characterized in the same way as the elements of $\widetilde{\mathcal{T}}_{M;h,n}$, but for two features: (i) the endpoints of $\tau \in \mathcal{T}_{M;h,n}$ are not necessarily on scale $M+1$; (ii) every endpoint v of τ is attached to a non-trivial vertex on scale $h_v - 1$ and is associated to the factor $\mathcal{V}(\psi^{[0,h_v-1]})$. See Fig.6. In terms of

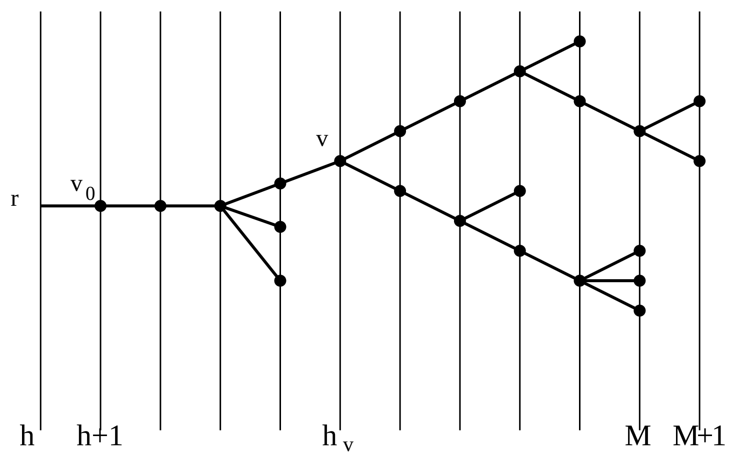

FIG. 6. A tree $\tau \in \mathcal{T}_{M;h,n}$ with $n = 9$: the root is on scale h and the endpoints are on scales $\le M+1$.

these modified trees, we have:

$$\mathcal{V}^{(h)}(\psi^{[0,h]}) + \beta|\Lambda|\bar{e}_{h+1} = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{M;h,n}} \mathcal{V}^{(h)}(\tau, \psi^{[0,h]}) \;, \tag{6.30}$$

where

$$\mathcal{V}^{(h)}(\tau, \psi^{[0,h]}) = \frac{(-1)^{s+1}}{s!} \mathcal{E}^T_{h+1}\big[\mathcal{V}^{(h+1)}(\tau_1, \psi^{[0,h+1]}); \ldots ; \mathcal{V}^{(h+1)}(\tau_s, \psi^{[0,h+1]})\big] \tag{6.31}$$

and, if τ is a trivial subtree with root on scale $k \in [h, M]$, then $\mathcal{V}^{(k)}(\tau, \psi^{[0,k]}) = \mathcal{V}(\psi^{[0,k]})$.

Using its inductive definition Eq.(6.31), the right hand side of Eq.(6.30) can be further expanded (it is a sum of several contributions, differing for the choices of the field labels contracted under the action of the truncated expectations $\mathcal{E}^T_{h_v}$ associated with the vertices v that are not endpoints), and in order to describe the resulting expansion we need some more definitions (allowing us to distinguish the fields that are contracted or not "inside the vertex v").

We associate with any vertex v of the tree a subset P_v of I_v, the *external fields* of v. These subsets must satisfy various constraints. First of all, if v is not an endpoint and $v_1, \ldots, v_{s_v}$ are the $s_v \geq 1$ vertices immediately following it, then $P_v \subseteq \cup_i P_{v_i}$; if v is an endpoint, $P_v = I_v$. If v is not an endpoint, we shall denote by Q_{v_i} the intersection of P_v and P_{v_i}; this definition implies that $P_v = \cup_i Q_{v_i}$. The union $\mathcal{I}_v$ of the subsets $P_{v_i} \setminus Q_{v_i}$ is, by definition, the set of the *internal fields* of v, and is non empty if $s_v > 1$. Given $\tau \in \mathcal{T}_{M;h,n}$, there are many possible choices of the subsets P_v, $v \in \tau$, compatible with all the constraints. We shall denote by $\mathcal{P}_\tau$ the family of all these choices and by $\mathbf{P}$ the elements of $\mathcal{P}_\tau$. With these definitions, we can rewrite $\mathcal{V}^{(h)}(\tau, \psi^{[0,h]})$ as:

$$\begin{aligned} \mathcal{V}^{(h)}(\tau, \psi^{[0,h]}) &= \sum_{\mathbf{P} \in \mathcal{P}_\tau} \mathcal{V}^{(h)}(\tau, \mathbf{P}) \;, \\ \mathcal{V}^{(h)}(\tau, \mathbf{P}) &= \int d\mathbf{x}_{v_0} \psi^{[0,h]}_{P_{v_0}} K^{(h+1)}_{\tau,\mathbf{P}}(\mathbf{x}_{v_0}) \;, \end{aligned} \tag{6.32}$$

where $\mathbf{x}_v = \cup_{f \in I_v} \{\mathbf{x}_v\}$,

$$\psi^{[0,h]}_{P_v} = \prod_{f \in P_v} \psi^{[0,h]\,\varepsilon(f)}_{\mathbf{x}(f),\sigma(f),\rho(f)} \tag{6.33}$$

and $K^{(h+1)}_{\tau,\mathbf{P}}(\mathbf{x}_{v_0})$ is defined inductively by the equation, valid for any $v \in \tau$ that is not an endpoint,

$$K^{(h_v)}_{\tau,\mathbf{P}}(\mathbf{x}_v) = \frac{1}{s_v!} \prod_{i=1}^{s_v} [K^{(h_v+1)}_{v_i}(\mathbf{x}_{v_i})] \;\; \mathcal{E}^T_{h_v}[\psi^{(h_v)}_{P_{v_1} \setminus Q_{v_1}}, \ldots, \psi^{(h_v)}_{P_{v_{s_v}} \setminus Q_{v_{s_v}}}] \;, \tag{6.34}$$

where $\psi^{(h_v)}_{P_{v_i} \setminus Q_{v_i}}$ has a definition similar to Eq.(6.33). Moreover, if v_i is an endpoint $K^{(h_v+1)}_{v_i}(\mathbf{x}_{v_i}) = U$; if v_i is not an endpoint, $K^{(h_v+1)}_{v_i} = K^{(h_v+1)}_{\tau_i,\mathbf{P}_i}$, where $\mathbf{P}_i = \{P_w, w \in \tau_i\}$. Using in the r.h.s. of Eq.(6.34) the determinant representation of the truncated expectation discussed in the previous section, we finally get:

$$\mathcal{V}^{(h)}(\tau, \mathbf{P}) = \sum_{T \in \mathbf{T}} \int d\mathbf{x}_{v_0} \psi^{[0,h]}_{P_{v_0}} W^{(h)}_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0}) := \sum_{T \in \mathbf{T}} \mathcal{V}^{(h)}(\tau, \mathbf{P}, T) \;, \tag{6.35}$$

where

$$W_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0}) = \tag{6.36}$$
$$= U^n \Big\{ \prod_{\substack{v \text{ not} \\ \text{e.p.}}} \frac{1}{s_v!} \int dP_{T_v}(\mathbf{t}_v)\, \det G^{h_v,T_v}(\mathbf{t}_v) \Big[\prod_{\ell\in T_v} \delta_{\sigma(\ell),\sigma'(\ell)}\, g^{(h_v)}_{\rho(\ell),\rho'(\ell)}(\mathbf{x}(\ell)-\mathbf{x}'(\ell)) \Big] \Big\}$$

and $G^{h_v,T_v}(\mathbf{t}_v)$ is a matrix analogous to the one defined in previous section, with g replaced by $g^{(h)}$. Note that $W_{\tau,\mathbf{P},T}$ and, therefore, $\mathcal{V}^{(h)}(\tau,\mathbf{P})$ do not depend on M: therefore, the effective potential $\mathcal{V}^{(h)}(\psi)$ depends on M only through the choice of the scale labels (i.e., the dependence on M is all encoded in $\mathcal{T}_{M;h,n}$). Using Eqs.(6.35)-(6.36), we finally get the bound:

$$\frac{1}{\beta|\Lambda|}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2l}|W^{(h)}_{M,2l,\underline{\rho}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2l})| \le \sum_{n\ge\max\{1,l-1\}} |U|^n \sum_{\tau\in\mathcal{T}_{M;h,n}} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau \\ |P_{v_0}|=2l}} \sum_{T\in\mathbf{T}} \cdot$$
$$\cdot \int \prod_{\ell\in T} d(\mathbf{x}(\ell)-\mathbf{x}'(\ell)) \Big[\prod_{\substack{v \text{ not} \\ \text{e.p.}}} \frac{1}{s_v!} \max_{\mathbf{t}_v} \big|\det G^{h_v,T_v}(\mathbf{t}_v)\big| \prod_{\ell\in T_v} \big|\big|g^{(h_v)}(\mathbf{x}(\ell)-\mathbf{x}'(\ell))\big|\big| \Big] . \tag{6.37}$$

Now, an application of the Gram–Hadamard inequality Eq.(5.6), combined with the representation Eq.(6.16) and the dimensional bounds Eq.(6.18), implies that

$$|\det G^{h_v,T_v}(\mathbf{t}_v)| \le (\text{const.})^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-2(s_v-1)} . \tag{6.38}$$

By the decay properties of $g^{(h)}(\mathbf{x})$ given by Eq.(6.15), it also follows that

$$\prod_{\substack{v \text{ not} \\ \text{e.p.}}} \frac{1}{s_v!} \int \prod_{\ell\in T_v} d(\mathbf{x}(\ell)-\mathbf{x}'(\ell))\, ||g^{(h_v)}(\mathbf{x}(\ell)-\mathbf{x}'(\ell))|| \le (\text{const.})^n \prod_{\substack{v \text{ not} \\ \text{e.p.}}} \frac{1}{s_v!} 2^{-h_v(s_v-1)} . \tag{6.39}$$

Plugging Eqs.(6.38)-(6.39) into Eq.(6.37), we find that the l.h.s. of Eq.(6.37) can be bounded from above by

$$\sum_{n\ge\max\{1,l-1\}} \sum_{\tau\in\mathcal{T}_{M;h,n}} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau \\ |P_{v_0}|=2l}} \sum_{T\in\mathbf{T}} (\text{const.})^n |U|^n \Big[\prod_{\substack{v \text{ not} \\ \text{e.p.}}} \frac{1}{s_v!} 2^{-h_v(s_v-1)} \Big] . \tag{6.40}$$

Using the following relation, which can be easily proved by induction,

$$\sum_{\substack{v \text{ not} \\ \text{e.p.}}} h_v(s_v-1) = h(n-1) + \sum_{\substack{v \text{ not} \\ \text{e.p.}}} (h_v-h_{v'})(n(v)-1) , \tag{6.41}$$

where v' is the vertex immediately preceding v on τ and $n(v)$ the number of endpoints following v on τ, we find that Eq.(6.40) can be rewritten as

$$\sum_{n\ge\max\{1,l-1\}} \sum_{\tau\in\mathcal{T}_{M;h,n}} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau \\ |P_{v_0}|=2l}} \sum_{T\in\mathbf{T}} (\text{const.})^n |U|^n 2^{-h(n-1)} \Big[\prod_{\substack{v \text{ not} \\ \text{e.p.}}} \frac{1}{s_v!} 2^{-(h_v-h_{v'})(n(v)-1)} \Big] , \tag{6.42}$$

where, by construction, we have that $n(v) > 1$ for any vertex v of $\tau \in \mathcal{T}_{M;h,n}$ that is not an endpoint (simply because every endpoint v of τ is attached to a non-trivial vertex on scale $h_v - 1$, see the discussion after Eq.(6.29)). Now, the number of terms in $\sum_{T\in\mathbf{T}}$ can be bounded by $(\text{const.})^n \prod_{v \text{ not e.p.}} s_v!$ (see [23, Appendix A.3.3]); moreover, $|P_v| \le 4n(v)$ and $n(v) - 1 \ge \max\{1, \frac{n(v)}{2}\}$, so that $n(v) - 1 \ge \frac{1}{2} + \frac{|P_v|}{16}$. Therefore,

$$\frac{1}{\beta|\Lambda|}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2l}|W^{(h)}_{M,2l,\underline{\rho}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2l})| \le \sum_{n\ge\max\{1,l-1\}}(\text{const.})^n|U|^n 2^{-h(n-1)}\cdot$$
$$\cdot\sum_{\tau\in\mathcal{T}_{M;h,n}}\Big(\prod_{v\text{ not e.p.}}2^{-\frac{1}{2}(h_v-h_{v'})}\Big)\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\|P_{v_0}|=2l}}\Big(\prod_{\substack{v\text{ not}\\\text{e.p.}}}2^{-|P_v|/16}\Big)\,. \tag{6.43}$$

Now, the sum over $\mathbf{P}$ can be bounded as follows: defining $p_v := |P_v|$ (so that $p_v \le p_{v_1} + \cdots + p_{v_{s_v}}$), then

$$\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\|P_{v_0}|=2l}}\Big(\prod_{\substack{v\text{ not}\\\text{e.p.}}}2^{-|P_v|/16}\Big) \le \prod_{\substack{v\text{ not}\\\text{e.p.}}}\sum_{p_v}2^{-p_v/16}\begin{pmatrix}p_{v_1}+\cdots+p_{v_{s_v}}\\p_v\end{pmatrix}\,. \tag{6.44}$$

The sum in the r.h.s. can be computed inductively, starting from the root and moving towards the endpoints; in fact, at the first step

$$\sum_{p_{v_0}}2^{-p_{v_0}/16}\begin{pmatrix}p_{v_1}+\cdots+p_{v_{s_{v_0}}}\\p_{v_0}\end{pmatrix} = \prod_{j=1}^{s_{v_0}}(1+2^{-1/16})^{p_{v_j}}\;; \tag{6.45}$$

at the second step,

$$\prod_{j=1}^{s_{v_0}}(1+2^{-1/16})^{p_{v_j}}\sum_{p_{v_j}}2^{-p_{v_j}/16}\begin{pmatrix}p_{v_{j,1}}+\cdots+p_{v_{j,s_{v_j}}}\\p_{v_j}\end{pmatrix} = \prod_{j=1}^{s_{v_0}}\prod_{j'=1}^{s_{v_j}}(1+2^{-\frac{1}{16}}+2^{-\frac{2}{16}})^{p_{v_{j,j'}}} \tag{6.46}$$

so that, iterating,

$$\prod_{\substack{v\text{ not}\\\text{e.p.}}}\sum_{p_v}2^{-p_v/16}\begin{pmatrix}p_{v_1}+\cdots+p_{v_{s_v}}\\p_v\end{pmatrix} = \prod_{v\text{ e.p.}}\Big[\sum_{k=1}^{L_v}2^{-\frac{k}{16}}\Big]^4 \tag{6.47}$$

where L_v is the distance from v to the root and we used the fact that $p_v = 4$ for all endpoints v. Plugging Eq.(6.47) into Eq.(6.44), we get

$$\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\|P_{v_0}|=2l}}\Big(\prod_{\substack{v\text{ not}\\\text{e.p.}}}2^{-|P_v|/16}\Big) \le \Big(\frac{1}{2^{1/16}-1}\Big)^n\,. \tag{6.48}$$

Similarly, one can prove that

$$\sum_{\tau\in\mathcal{T}_{M;h,n}}\Big(\prod_{\substack{v\text{ not}\\\text{e.p.}}}2^{-\frac{1}{2}(h_v-h_{v'})}\Big) \le \Big(\frac{4}{2^{1/2}-1}\Big)^n\,, \tag{6.49}$$

uniformly in M as $M \to \infty$, see [23, Lemma A.2]. Collecting all the previous bounds, we obtain

$$\frac{1}{\beta|\Lambda|}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2l}|W^{(h)}_{M,2l,\underline{\rho}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2l})| \leq \sum_{n\geq\max\{1,l-1\}}(\text{const.})^n|U|^n 2^{-h(n-1)}\;, \tag{6.50}$$

which implies Eq.(6.10). The constant $\overline{e}_h$ can be bounded by the r.h.s. of Eq.(6.37) with $l=0$ and $n\geq 2$ (because the contributions to $\overline{e}_h$ with $l=1$ are zero, by the condition that the tadpoles vanish), which implies

$$\overline{e}_h \leq \sum_{n\geq 2}(\text{const.})^n|U|^n 2^{-h(n-1)} \leq (\text{const.})|U|^2 2^{-h}\;. \tag{6.51}$$

Therefore, $F_{0,M}=\sum_{k=1}^M \overline{e}_k$ is given by an absolutely convergent power series in U, as desired. A critical analysis of the proof shows that all the bounds are uniform in M,β,Λ and all the expressions involved admit well-defined limits as $M,\beta,|\Lambda|\to\infty$. In particular, this implies that $F_0=\lim_{M\to\infty}F_{0,M}$ is analytic in U (and so is its $\beta,|\Lambda|\to\infty$ limit, which is reached uniformly) in the complex domain $|U|\leq U_0$, for U_0 small enough. See [24, Appendices C and D] for details on these technical aspects. This concludes the proof of Lemma 6.1. ∎

7. THE MULTISCALE INTEGRATION: THE INFRARED REGIME

We are now left with computing

$$\Xi_{M,\beta,\Lambda}=e^{-\beta|\Lambda|F_0}\int P(d\psi^{(i.r.)})e^{-\mathcal{V}_0(\psi^{(i.r.)})}\;. \tag{7.1}$$

We proceed in an iterative fashion, similar to the one described in the previous section for the integration of the large values of k_0. As a starting point, it is convenient to decompose the infrared propagator as:

$$g^{(i.r.)}(\mathbf{x}-\mathbf{y})=\sum_{\omega=\pm}e^{-i\vec{p}_F^{\,\omega}(\vec{x}-\vec{y})}g^{(\leq 0)}_{\omega}(\mathbf{x}-\mathbf{y})\;, \tag{7.2}$$

where, if $\mathbf{k}'=(k_0,\vec{k}')$,

$$g^{(\leq 0)}_{\omega}(\mathbf{x}-\mathbf{y})=\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}'\in\mathcal{B}^{\omega}_{\beta,L}}\chi_0(|\mathbf{k}'|)e^{-i\mathbf{k}'(\mathbf{x}-\mathbf{y})}\begin{pmatrix}-ik_0 & -v\Omega^*(\vec{k}'+\vec{p}_F^{\,\omega})\\ -v\Omega(\vec{k}'+\vec{p}_F^{\,\omega}) & -ik_0\end{pmatrix}^{-1} \tag{7.3}$$

and $\mathcal{B}^{\omega}_{\beta,L}=\mathcal{B}^{(M)}_{\beta}\times\mathcal{B}^{\omega}_L$, with $\mathcal{B}^{\omega}_L=\{\frac{n_1}{L}\vec{G}_1+\frac{n_2}{L}\vec{G}_2-\vec{p}_F^{\,\omega}\;,\;0\leq n_1,n_2\leq L-1\}$. Correspondingly, we rewrite $\psi^{(i.r.)}$ as a sum of two independent Grassmann fields:

$$\psi^{(i.r.)\pm}_{\mathbf{x},\sigma}=\sum_{\omega=\pm}e^{i\vec{p}_F^{\,\omega}\vec{x}}\psi^{(\leq 0)\pm}_{\mathbf{x},\sigma,\omega} \tag{7.4}$$

and we rewrite Eq.(6.6) in the form:

$$\Xi_{M,\beta,\Lambda} = e^{-\beta|\Lambda|F_0} \int P_{\chi_0,A_0}(d\psi^{(\leq 0)}) e^{-\mathcal{V}^{(0)}(\psi^{(\leq 0)})} \;, \tag{7.5}$$

where $\mathcal{V}^{(0)}(\psi^{(\leq 0)})$ is equal to $\mathcal{V}_0(\psi^{(i.r.)})$, once $\psi^{(i.r.)}$ is rewritten as in (7.4), i.e.,

$$\begin{aligned}
&\mathcal{V}^{(0)}(\psi^{(\leq 0)}) = \qquad\qquad (7.6)\\
&= \sum_{n=1}^{\infty} (\beta|\Lambda|)^{-2n} \sum_{\sigma_1,\ldots,\sigma_n=\uparrow\downarrow} \sum_{\rho_1,\ldots,\rho_{2n}=1,2}^{\omega_1,\ldots,\omega_{2n}=\pm} \sum_{\mathbf{k}'_1,\ldots,\mathbf{k}'_{2n}} \Big[\prod_{j=1}^{n} \hat\psi^{(\leq 0)+}_{\mathbf{k}'_{2j-1},\sigma_j,\rho_{2j-1},\omega_{2j-1}} \hat\psi^{(\leq 0)-}_{\mathbf{k}'_{2j},\sigma_j,\rho_{2j},\omega_{2j}}\Big] \cdot\\
&\qquad \cdot \hat W^{(0)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{k}'_1,\ldots,\mathbf{k}'_{2n-1})\, \delta\Big(\sum_{j=1}^{2n}(-1)^j(\mathbf{p}_F^{\omega_j}+\mathbf{k}'_j)\Big) =\\
&= \sum_{n=1}^{\infty}\sum_{\underline{\sigma},\underline{\rho},\underline{\omega}} \int d\mathbf{x}_1\cdots d\mathbf{x}_{2n} \Big[\prod_{j=1}^{n} \psi^{(\leq 0)+}_{\mathbf{x}_{2j-1},\sigma_j,\rho_{2j-1},\omega_{2j-1}} \psi^{(\leq 0)-}_{\mathbf{x}_{2j},\sigma_j,\rho_{2j},\omega_{2j}}\Big] W^{(0)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2n})
\end{aligned}$$

with:

1) $\underline{\omega} = (\omega_1,\ldots,\omega_{2n})$, $\underline{\sigma} = (\sigma_1,\ldots,\sigma_n)$ and $\mathbf{p}_F^{\omega} = (0,\vec p_F^{\,\omega})$;
2) $\hat W^{(0)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{k}'_1,\ldots,\mathbf{k}'_{2n-1}) = \hat W_{M,2n,\underline{\rho}}(\mathbf{k}'_1+\mathbf{p}_F^{\omega_j},\ldots,\mathbf{k}'_{2n-1}+\mathbf{p}_F^{\omega_{2n-1}})$, see (6.8);
3) the kernels $W^{(0)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2n})$ are defined as:

$$\begin{aligned}
&W^{(0)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2n}) = \qquad\qquad (7.7)\\
&= (\beta|\Lambda|)^{-2n} \sum_{\mathbf{k}'_1,\ldots,\mathbf{k}'_{2n}} e^{i\sum_{j=1}^{2n}(-1)^j \mathbf{k}_j\mathbf{x}_j} \hat W^{(0)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{k}'_1,\ldots,\mathbf{k}'_{2n-1})\, \delta\Big(\sum_{j=1}^{2n}(-1)^j(\mathbf{p}_F^{\omega_j}+\mathbf{k}'_j)\Big)\;.
\end{aligned}$$

Moreover, $P_{\chi_0,A_0}(d\psi^{(\leq 0)})$ is defined as

$$\begin{aligned}
P_{\chi_0,A_0}(d\psi^{(\leq 0)}) &= \mathcal{N}_0^{-1}\Big[\prod_{\mathbf{k}'\in\mathcal{B}^{\omega}_{\beta,L}}^{\chi_0(|\mathbf{k}'|)>0} \prod_{\sigma,\omega,\rho} d\hat\psi^{(\leq 0)+}_{\mathbf{k}',\sigma,\rho,\omega} d\hat\psi^{(\leq 0)-}_{\mathbf{k}',\sigma,\rho,\omega}\Big]\cdot \qquad (7.8)\\
&\cdot \exp\Big\{-(\beta|\Lambda|)^{-1} \sum_{\omega=\pm}^{\sigma=\uparrow\downarrow} \sum_{\mathbf{k}'\in\mathcal{B}^{\omega}_{\beta,L}}^{\chi_0(|\mathbf{k}'|)>0} \chi_0^{-1}(|\mathbf{k}'|) \hat\psi^{(\leq 0)+}_{\mathbf{k}',\sigma,\cdot,\omega} A_{0,\omega}(\mathbf{k}') \hat\psi^{(\leq 0)-}_{\mathbf{k}',\sigma,\cdot,\omega}\Big\}\;,
\end{aligned}$$

where:

$$\begin{aligned}
A_{0,\omega}(\mathbf{k}') &= \begin{pmatrix} -ik_0 & -v_0\Omega^*(\vec k' + \vec p_F^{\,\omega}) \\ -v_0\Omega(\vec k' + \vec p_F^{\,\omega}) & -ik_0 \end{pmatrix} =\\
&= \begin{pmatrix} -i\zeta_0 k_0 + s_0(\mathbf{k}') & v_0(ik'_1 - \omega k'_2) + t_{0,\omega}(\mathbf{k}') \\ v_0(-ik'_1 - \omega k'_2) + t^*_{0,\omega}(\mathbf{k}') & -i\zeta_0 k_0 + s_0(\mathbf{k}') \end{pmatrix}\;,
\end{aligned}$$

$\mathcal{N}_0$ is chosen in such a way that $\int P_{\chi_0,A_0}(d\psi^{(\leq 0)}) = 1$, $\zeta_0 = 1$, $s_0 = 0$ and $|t_{0,\omega}(\mathbf{k}')| \leq C|\mathbf{k}'|^2$.

It is apparent that the $\psi^{(\leq 0)}$ field has zero mass (i.e., its propagator decays polynomially at large distances in $\mathbf{x}$-space). Therefore, its integration requires an infrared multiscale analysis. As in the analysis of the ultraviolet problem, we define a sequence of geometrically decreasing momentum scales 2^h, with $h = 0, -1, -2, \ldots$ Correspondingly we introduce compact support functions $f_h(\mathbf{k}') = \chi_0(2^{-h}|\mathbf{k}'|) - \chi_0(2^{-h+1}|\mathbf{k}'|)$ and we rewrite

$$\chi_0(|\mathbf{k}'|) = \sum_{h=h_\beta}^{0} f_h(\mathbf{k}') \;, \tag{7.9}$$

with $h_\beta := \lfloor \log_2 \left(\frac{3\pi}{4\beta}\right) \rfloor$ (the reason why the sum in Eq.(7.9) runs over $h \geq h_\beta$ is that, if $\mathbf{k}' \in \mathcal{B}^{\omega}_{\beta,L}$ and $h < h_\beta$, then $f_h(\mathbf{k}')$, simply because $|\mathbf{k}'| \geq \pi/\beta$). The purpose is to perform the integration of (7.5) in an iterative way. We step by step decompose the propagator into a sum of two propagators, the first supported on momenta $\sim 2^h$, $h \leq 0$, the second supported on momenta smaller than 2^h. Correspondingly we rewrite the Grassmann field as a sum of two independent fields: $\psi^{(\leq h)} = \psi^{(h)} + \psi^{(\leq h-1)}$ and we integrate the field $\psi^{(h)}$. In this way we inductively prove that, for any $h \leq 0$, Eq.(7.5) can be rewritten as

$$\Xi_{M,\beta,\Lambda} = e^{-\beta|\Lambda| F_h} \int P_{\chi_h, A_h}(d\psi^{(\leq h)}) e^{-\mathcal{V}^{(h)}(\psi^{(\leq h)})} \;, \tag{7.10}$$

where $F_h, A_h, \mathcal{V}^{(h)}$ will be defined recursively, $\chi_h(|\mathbf{k}'|) = \sum_{k=h_\beta}^{h} f_k(\mathbf{k}')$ and $P_{\chi_h,A_h}(d\psi^{(\leq h)})$ is defined in the same way as $P_{\chi_0,A_0}(d\psi^{(\leq 0)})$ with $\psi^{(\leq 0)}, \chi_0$, $A_{0,\omega}, \zeta_0$, $v_0, s_0, t_{0,\omega}$ replaced by $\psi^{(\leq h)}, \chi_h, A_{h,\omega}, \zeta_h, v_h, s_h, t_{h,\omega}$, respectively. Moreover $\mathcal{V}^{(h)}(0) = 0$ and

$$\begin{aligned}\mathcal{V}^{(h)}(\psi) = \sum_{n=1}^{\infty} (\beta|\Lambda|)^{-2n} \sum_{\underline{\sigma},\underline{\rho},\underline{\omega}} \sum_{\mathbf{k}'_1,\ldots,\mathbf{k}'_{2n}} \Big[\prod_{j=1}^{n} \hat{\psi}^{(\leq h)+}_{\mathbf{k}'_{2j-1},\sigma_j,\rho_{2j-1},\omega_{2j-1}} \hat{\psi}^{(\leq h)-}_{\mathbf{k}'_{2j},\sigma_j,\rho_{2j},\omega_{2j}} \Big] \cdot \\ \cdot \hat{W}^{(h)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{k}'_1,\ldots,\mathbf{k}'_{2n-1})\, \delta(\sum_{j=1}^{2n} (-1)^j (\mathbf{p}_F^{\omega_j} + \mathbf{k}'_j)) = \\ = \sum_{\substack{n\geq 1\\ \underline{\sigma},\underline{\rho},\underline{\omega}}} \int d\mathbf{x}_1 \cdots d\mathbf{x}_{2n} \Big[\prod_{j=1}^{n} \psi^{(\leq h)+}_{\mathbf{x}_{2j-1},\sigma_j,\rho_{2j-1},\omega_{2j-1}} \psi^{(\leq h)-}_{\mathbf{x}_{2j},\sigma_j,\rho_{2j},\omega_{2j}} \Big] W^{(h)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2n}) \;.\end{aligned} \tag{7.11}$$

Note that the field $\psi^{(\leq h)}_{\mathbf{k}',\sigma,\omega}$, whose propagator is given by $\chi_h(|\mathbf{k}'|)[A^{(h)}_\omega(\mathbf{k}')]^{-1}$, has the same support as χ_h, that is on a neighborood of size 2^h around the singularity $\mathbf{k}' = \mathbf{0}$ (that, in the original variables, corresponds to the Dirac point $\mathbf{k} = \mathbf{p}_F^{\omega}$). It is important for the following to think $\hat{W}^{(h)}_{2n,\underline{\rho},\underline{\omega}}$, $h \leq 0$, as functions of the variables $\{\zeta_k, v_k\}_{h<k\leq 0}$. The iterative construction below will inductively imply that the dependence on these variables is well defined. The iteration continues up to the scale h_β and the result of the last iteration is $\Xi_{M,\beta,\Lambda} = e^{-\beta|\Lambda| F^{(M)}_{\beta,\Lambda}}$.

Localization and renormalization. In order to inductively prove Eq.(7.10) we write

$$\mathcal{V}^{(h)} = \mathcal{L}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)} \tag{7.12}$$

where

$$\mathcal{L}\mathcal{V}^{(h)} = \frac{1}{\beta|\Lambda|} \sum_{\sigma=\uparrow\downarrow}^{\omega=\pm} \sum_{\mathbf{k}'}^{\chi_h(|\mathbf{k}'|)>0} \hat{\psi}^{(\leq h)+}_{\mathbf{k}',\sigma,\omega} \hat{W}^{(h)}_{2,(\omega,\omega)}(\mathbf{k}') \hat{\psi}^{(\leq h)-}_{\mathbf{k}',\sigma,\omega} \ , \tag{7.13}$$

and $\mathcal{R}\mathcal{V}^{(h)}$ is given by Eq.(7.11) with $\sum_{n=1}^{\infty}$ replaced by $\sum_{n=2}^{\infty}$, that is it contains only the monomials with four fermionic fields or more. The symmetries of the action, which are described in Appendix C and are preserved by the iterative integration procedure, imply that, in the zero temperature and thermodynamic limit,

$$\hat{W}^{(h)}_{2,(\omega,\omega)}(\mathbf{k}') = \begin{pmatrix} -iz_h k_0 & \delta_h(ik'_1 - \omega k'_2) \\ \delta_h(-ik'_1 - \omega k'_2) & -iz_h k_0 \end{pmatrix} + O(|\mathbf{k}'|^2) \ , \tag{7.14}$$

for suitable real constants z_h, δ_h. Eq.(7.14) is the analogue of Eq.(6.12); its proof is completely analogous to the proof of Lemma 6.2 and will not be belabored here.

Once that the above definitions are given, we can describe our iterative integration procedure for $h \leq 0$. We start from Eq.(7.10) and we rewrite it as

$$\int P_{\chi_h, A_h}(d\psi^{(\leq h)})\, e^{-\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\leq h)}) - \mathcal{R}\mathcal{V}^{(h)}(\psi^{(\leq h)}) - \beta|\Lambda| F_h} \ , \tag{7.15}$$

with

$$\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\leq h)}) = (\beta|\Lambda|)^{-1} \sum_{\omega,\sigma} \sum_{\mathbf{k}'}^{\chi_h(|\mathbf{k}'|)>0} \cdot \tag{7.16}$$

$$\cdot \hat{\psi}^{(\leq h)+}_{\mathbf{k}',\sigma,\omega} \begin{pmatrix} -iz_h k_0 + \sigma_h(\mathbf{k}') & \delta_h(ik'_1 - \omega k'_2) + \tau_{h,\omega}(\mathbf{k}') \\ \delta_h(-ik'_1 - \omega k_2) + \tau^*_{h,\omega}(\mathbf{k}') & -iz_h k_0 + \sigma_h(\mathbf{k}') \end{pmatrix} \hat{\psi}^{(\leq h)-}_{\mathbf{k}',\sigma,\omega} \ .$$

Then we include $\mathcal{L}\mathcal{V}^{(h)}$ in the fermionic integration, so obtaining

$$\int P_{\chi_h, \overline{A}_{h-1}}(d\psi^{(\leq h)})\, e^{-\mathcal{R}\mathcal{V}^{(h)}(\psi^{(\leq h)}) - \beta|\Lambda|(F_h + e_h)} \ , \tag{7.17}$$

where

$$e_h = \frac{1}{\beta|\Lambda|} \sum_{\omega,\sigma} \sum_{\mathbf{k}'} \sum_{n\geq 1} \frac{(-1)^n}{n} \mathrm{Tr}\Big\{ \big[\chi_h(\mathbf{k}') A^{-1}_{h,\omega}(\mathbf{k}') W^{(h)}_{2,(\omega,\omega)}(\mathbf{k}')\big]^n \Big\} \tag{7.18}$$

is a constant taking into account the change in the normalization factor of the measure and

$$\overline{A}_{h-1,\omega}(\mathbf{k}') = \begin{pmatrix} -i\overline{\zeta}_{h-1}k_0 + \overline{s}_{h-1}(\mathbf{k}') & \overline{v}_{h-1}(ik'_1 - \omega k'_2) + \overline{t}_{h-1,\omega}(\mathbf{k}') \\ \overline{v}_{h-1}(-ik'_1 - \omega k'_2) + \overline{t}^*_{h-1,\omega}(\mathbf{k}') & -i\overline{\zeta}_{h-1}k_0 + \overline{s}_{h-1}(\mathbf{k}') \end{pmatrix} \tag{7.19}$$

with:

$$\begin{aligned} \overline{\zeta}_{h-1}(\mathbf{k}') &= \zeta_h + z_h\chi_h(\mathbf{k}') \;, & \overline{v}_{h-1}(\mathbf{k}') &= v_h + \delta_h\chi_h(\mathbf{k}') \;, \\ \overline{s}_{h-1}(\mathbf{k}') &= s_h(\mathbf{k}') + \sigma_h(\mathbf{k}')\chi_h(\mathbf{k}') \;, & \overline{t}_{h-1,\omega}(\mathbf{k}') &= t_{h,\omega}(\mathbf{k}') + \tau_{h,\omega}(\mathbf{k}')\chi_h(\mathbf{k}') \;. \end{aligned} \tag{7.20}$$

Now we can perform the integration of the $\psi^{(h)}$ field. We rewrite the Grassmann field $\psi^{(\le h)}$ as a sum of two independent Grassmann fields $\psi^{(\le h-1)} + \psi^{(h)}$ and correspondingly we rewrite Eq.(7.17) as

$$e^{-\beta|\Lambda|(F_h+e_h)} \int P_{\chi_{h-1},A_{h-1}}(d\psi^{(\le h-1)}) \int P_{f_h,\overline{A}_{h-1}}(d\psi^{(h)})\, e^{-\mathcal{R}\mathcal{V}^{(h)}(\psi^{(\le h-1)}+\psi^{(h)})} \;, \tag{7.21}$$

where

$$A_{h-1,\omega}(\mathbf{k}') = \begin{pmatrix} -i\zeta_{h-1}k_0 + s_{h-1}(\mathbf{k}') & v_{h-1}(ik'_1 - \omega k'_2) + t_{h-1,\omega}(\mathbf{k}') \\ v_{h-1}(-ik'_1 - \omega k'_2) + t^*_{h-1,\omega}(\mathbf{k}') & -i\zeta_{h-1}k_0 + s_{h-1}(\mathbf{k}') \end{pmatrix} \tag{7.22}$$

with:

$$\begin{aligned} \zeta_{h-1} &= \zeta_h + z_h \;, & v_{h-1} &= v_h + \delta_h \;, \\ s_{h-1}(\mathbf{k}') &= s_h(\mathbf{k}') + \sigma_h(\mathbf{k}') \;, & t_{h-1,\omega}(\mathbf{k}') &= t_{h,\omega}(\mathbf{k}') + \tau_{h,\omega}(\mathbf{k}') \;. \end{aligned} \tag{7.23}$$

The single scale propagator is

$$\int P_{f_h,\overline{A}_{h-1}}(d\psi^{(h)})\psi^{(h)-}_{\mathbf{x}_1,\sigma_1,\omega_1}\psi^{(h)+}_{\mathbf{x}_2,\sigma_2,\omega_2} = \delta_{\sigma_1,\sigma_2}\delta_{\omega_1,\omega_2}g^{(h)}_\omega(\mathbf{x}_1 - \mathbf{x}_2) \;, \tag{7.24}$$

where

$$g^{(h)}_\omega(\mathbf{x}_1 - \mathbf{x}_2) = \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}'\in\mathcal{D}^\omega_{\beta,L}} e^{-i\mathbf{k}'(\mathbf{x}_1-\mathbf{x}_2)} f_h(\mathbf{k}')\Big[\overline{A}_{h-1,\omega}(\mathbf{k}')\Big]^{-1} \;. \tag{7.25}$$

After the integration of the field on scale h we are left with an integral involving the fields $\psi^{(\le h-1)}$ and the new effective interaction $\mathcal{V}^{(h-1)}$, defined as

$$e^{-\mathcal{V}^{(h-1)}(\psi^{(\le h-1)})-\overline{e}_h\beta|\Lambda|} = \int P_{f_h,\overline{A}_{h-1}}(d\psi^{(h)})\, e^{-\mathcal{R}\mathcal{V}^{(h)}(\psi^{(\le h-1)}+\psi^{(h)})} \;, \tag{7.26}$$

with $\mathcal{V}^{(h-1)}(0) = 0$. It is easy to see that $\mathcal{V}^{(h-1)}$ is of the form Eq.(7.11) and that $F_{h-1} = F_h + e_h + \overline{e}_h$. It is sufficient to use the identity

$$\overline{e}_h + \mathcal{V}^{(h-1)}(\psi^{(\le h-1)}) = \sum_{n\ge1}\frac{1}{n!}(-1)^{n+1}\mathcal{E}^T_h\big(\mathcal{R}\mathcal{V}^{(h)}\big(\psi^{(\le h-1)} + \psi^{(h)}\big); n\big) \;, \tag{7.27}$$

where $\mathcal{E}^T_h(X(\psi^{(h)});n)$ is the truncated expectation of order n w.r.t. the propagator $g^{(h)}_\omega$, which is the analogue of Eq.(6.7) with $\psi^{(u.v.)}$ replaced by $\psi^{(h)}$ and with $P(d\psi^{(u.v.)})$ replaced by $P_{f_h,\overline{A}_{h-1}}(d\psi^{(h)})$.

Note that the above procedure allows us to write the *effective constants* (ζ_h, v_h), $h \le 0$, in terms of (ζ_k, v_k), $h < k \le 0$, namely $\zeta_{h-1} = \beta^\zeta_h\big((\zeta_h, v_h), \ldots, (\zeta_0, v_0)\big)$ and $v_{h-1} = \beta^v_h\big((\zeta_h, v_h), \ldots, (\zeta_0, v_0)\big)$, where $\beta^\#_h$ is the so–called *Beta function*.

An iterative implementation of Eq.(7.27) leads to a representation of $\mathcal{V}^{(h)}$, $h<0$, in terms of a new tree expansion. The set of trees of order n contributing to $\mathcal{V}^{(h)}(\psi^{(\le h)})$ is denoted by $\mathcal{T}_{h,n}$. The trees in $\mathcal{T}_{h,n}$ are defined in a way very similar to those in $\widetilde{\mathcal{T}}_{M;h,n}$, but for the following differences: (i) all the endpoints are on scale 1; (ii) the scale labels of the vertices of $\tau \in \mathcal{T}_{h,n}$ are between $h+1$ and 1; (iii) with each endpoint v we associate one of the monomials with four or more Grassmann fields contributing to $\mathcal{R}\mathcal{V}^{(0)}(\psi^{(\le 0)})$, corresponding to the terms with $n \ge 2$ in the r.h.s. of Eq.(7.6) (here $\mathcal{R}$ is the linear operator acting as the identity on Grassmann monomials of order 4 or more, and acting as 0 on Grassmann monomials of order 0 or 2). In terms of these trees, the effective potential $\mathcal{V}^{(h)}$, $h \le -1$, can be written as

$$\mathcal{V}^{(h)}(\psi^{(\le h)}) + \beta|\Lambda|\overline{e}_{k+1} = \sum_{n=1}^\infty \sum_{\tau\in\mathcal{T}_{h,n}} \mathcal{V}^{(h)}(\tau, \psi^{(\le h)}) \ , \tag{7.28}$$

where, if v_0 is the first vertex of τ and $\tau_1, \ldots, \tau_s$ ($s = s_{v_0}$) are the subtrees of τ with root v_0,

$$\mathcal{V}^{(h)}(\tau, \psi^{(\le h)}) = \frac{(-1)^{s+1}}{s!}\mathcal{E}^T_{h+1}\big[\mathcal{R}\mathcal{V}^{(h+1)}(\tau_1, \psi^{(\le h+1)}); \ldots ; \mathcal{R}\mathcal{V}^{(h+1)}(\tau_s, \psi^{(\le h+1)})\big] \ , \tag{7.29}$$

where, if τ_i is trivial, then $h+1=0$ and $\mathcal{R}\mathcal{V}^{(0)}(\tau_i, \psi^{(\le h+1)}) = \mathcal{R}\mathcal{V}^{(0)}(\psi^{(\le h+1)})$. Repeating step by step the discussion leading to Eqs.(6.32),(6.35) and (6.36), and using analogous definitions, we find that

$$\mathcal{V}^{(h)}(\tau, \psi^{(\le h)}) = \sum_{\mathbf{P}\in\mathcal{P}_\tau} \sum_{T\in\mathbf{T}} \int d\mathbf{x}_{v_0} \psi^{(\le h)}_{P_{v_0}} W^{(h)}_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0}) \ , \tag{7.30}$$

with

$$\begin{aligned} W_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0}) = &\left[\prod_{i=1}^n K^{(h_i)}_{v_i^*}(\mathbf{x}_{v_i^*})\right] \Bigg\{ \prod_{\substack{v \text{ not} \\ \text{e.p.}}} \frac{1}{s_v!} \int dP_{T_v}(\mathbf{t}_v) \det G^{h_v,T_v}(\mathbf{t}_v) \cdot \\ &\cdot \left[\prod_{\ell\in T_v} \delta_{\omega(\ell),\omega'(\ell)} \delta_{\sigma(\ell),\sigma'(\ell)} \left[g^{(h_v)}_{\omega(\ell)}(\mathbf{x}(\ell) - \mathbf{x}'(\ell))\right]_{\rho(\ell),\rho'(\ell)}\right]\Bigg\} \ . \end{aligned} \tag{7.31}$$

and: v_i^*, $i = 1, \ldots, n$, are the endpoints of τ; $K_{v_i^*}^{(h_i)}(\mathbf{x}_{v_i^*})$ is the kernel of one of the monomials contributing to $\mathcal{RV}^{(0)}(\psi^{(\le h_v)})$; $G^{h,T}$ is a matrix with elements

$$G_{f,f'}^{h,T} = t_{i(f),i(f')}\delta_{\omega(f),\omega(f')}\delta_{\sigma(f),\sigma(f')}\left[g_{\omega(f)}^{(h)}(\mathbf{x}(f) - \mathbf{x}(f'))\right]_{\rho(f),\rho(f')}, \tag{7.32}$$

Once again, it is important to note that $G^{h,T}$ is a Gram matrix, i.e., defining $\mathbf{e}_+ = \mathbf{e}_\uparrow = (1,0)$ and $\mathbf{e}_- = \mathbf{e}_\downarrow = (0,1)$, the matrix elements in Eq.(7.32), using a notation similar to Eq.(5.7), can be written in terms of scalar products:

$$G_{f,f'}^{h,T} = \tag{7.33}$$
$$= \langle \mathbf{u}_{i(f)} \otimes \mathbf{e}_{\omega(f)} \otimes \mathbf{e}_{\sigma(f)} \otimes \mathbf{A}_{\mathbf{x}(f),\rho(f),\omega(f)}, \mathbf{u}_{i(f')} \otimes \mathbf{e}_{\omega(f')} \otimes \mathbf{e}_{\sigma(f')} \otimes \mathbf{B}_{\mathbf{x}(f'),\rho(f'),\omega(f')}\rangle ,$$

where

$$[\mathbf{A}_{\mathbf{x},\rho,\omega}(\mathbf{z})]_i = \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}'\in\mathcal{B}_{\beta,L}^{\omega}} \frac{e^{-i\mathbf{k}'(\mathbf{z}-\mathbf{x})}\sqrt{f_h(\mathbf{k}')}}{|\det \overline{A}_{h-1,\omega}(\mathbf{k}')|^{1/4}}\,\delta_{\rho,i} ,$$

$$[\mathbf{B}_{\mathbf{x},\rho,\omega}(\mathbf{z})]_i = \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}'\in\mathcal{B}_{\beta,L}^{\omega}} \frac{e^{-i\mathbf{k}'(\mathbf{z}-\mathbf{x})}\sqrt{f_h(\mathbf{k}')}}{|\det \overline{A}_{h-1,\omega}(\mathbf{k}')|^{3/4}} \cdot \tag{7.34}$$
$$\cdot \begin{pmatrix} i\overline{\zeta}_{h-1}k_0 - \overline{s}_{h-1}(\mathbf{k}') & \overline{v}_{h-1}(ik_1' - \omega k_2') + \overline{t}_{h-1,\omega}(\mathbf{k}') \\ \overline{v}_{h-1}(-ik_1' - \omega k_2') + \overline{t}_{h-1,\omega}^*(\mathbf{k}') & i\overline{\zeta}_{h-1}k_0 - \overline{s}_{h-1}(\mathbf{k}') \end{pmatrix}_{i,\rho} .$$

so that, using that $\overline{s}_{h-1}$ is purely imaginary,

$$||\mathbf{A}_{\mathbf{x},\rho,\omega}||^2 = ||\mathbf{B}_{\mathbf{x},\rho,\omega}||^2 = \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}'\in\mathcal{B}_{\beta,L}^{\omega}} \frac{f_h(\mathbf{k}')}{|\det \overline{A}_{h-1,\omega}(\mathbf{k}')|^{1/2}} \le C2^{2h} , \tag{7.35}$$

for a suitable constant C.

Using the representation Eqs.(7.28),(7.29),(7.31) and proceeding as in the proof of Lemma 6.1, we can get a bound on the kernels of the effective potentials, which is the key ingredient for the proof of Theorem 2.1 and is summarized in the following theorem.

Theorem 7.1 *There exists a constants $U_0 > 0$, independent of M, β and Λ, such that the kernels $W_{2l,\underline{\rho},\underline{\omega}}^{(h)}(\mathbf{x}_1, \ldots, \mathbf{x}_{2l})$ in Eq.(7.11), $h \le -1$, are analytic functions of U in the complex domain $|U| \le U_0$, satisfying, for any $0 \le \theta < 1$ and a suitable constant $C_\theta > 0$ (independent of M, β, Λ), the following estimates:*

$$\frac{1}{\beta|\Lambda|}\int d\mathbf{x}_1 \cdots d\mathbf{x}_{2l}|W_{2l,\underline{\rho},\underline{\omega}}^{(h)}(\mathbf{x}_1, \ldots, \mathbf{x}_{2l})| \le 2^{h(3-2l+\theta)}\left(C_\theta\,|U|\right)^{max(1,l-1)} . \tag{7.36}$$

Moreover, the constants e_h and $\overline{e}_h$ defined by Eq.(7.18) and Eq.(7.27) are analytic functions of U in the same domain $|U| \le U_0$, uniformly bounded as $|e_h| + |\overline{e}_h| \le C_\theta|U|2^{h(3+\theta)}$. Both the kernels $W_{2l,\underline{\rho},\underline{\omega}}^{(h)}(\mathbf{x}_1, \ldots, \mathbf{x}_{2l})$ and the constants $e_h, \overline{e}_h$ admit well defined limits as $M, \beta, |\Lambda| \to \infty$, which are reached uniformly (and, with some abuse of notation, will be denoted by the same symbols).

Before we present its proof, let us show how Theorem 7.1 implies, as a corollary, Theorem 2.1. It is enough to observe that, by Proposition 4.1 and by the multiscale integration procedure described above, $f_\beta(U) = F_0 + \sum_{h=h_\beta}^{0}(e_h + \overline{e}_h)$, with $F_0, e_h, \overline{e}_h$ analytic functions of U for U small enough (see the discussion at the end of Section 6 and the statement of Theorem 7.1). Since $|e_h| + |\overline{e}_h| \le C_\theta |U| 2^{h(3+\theta)}$, the sum $\sum_{h=h_\beta}^{0}(e_h + \overline{e}_h)$ is absolutely convergent, uniformly in h_β; therefore, both $f_\beta(U)$ and its $\beta \to \infty$ limit are analytic functions of U for U small enough. This concludes the proof of Theorem 2.1.

Proof of Theorem 7.1. Let us preliminarily assume that, for $h' \le h \le -1$, and for suitable constants c, c_n, the corrections $z_h, \delta_h, \sigma_h(\mathbf{k}')$ and $\tau_h(\mathbf{k}')$ defined in Eq.(7.14) and Eq.(7.16), satisfy the following estimates:

$$\begin{aligned} &\max\{|z_h|, |\delta_h|\} \le c|U| 2^{\theta h} , \\ &\sup_{2^{h'-1} \le |\mathbf{k}'| \le 2^{h'+1}} \{||\partial^n_{\mathbf{k}'} \sigma_h(\mathbf{k}')||, ||\partial^n_{\mathbf{k}'} \tau_{h,\omega}(\mathbf{k}')||\} \le c_n |U| 2^{2(h'-h)} 2^{(1+\theta-n)h} . \end{aligned} \tag{7.37}$$

Using Eq.(7.37) we inductively see that the running coupling functions ζ_h, v_h, $s_h(\mathbf{k}')$ and $t_h(\mathbf{k}')$ satisfy similar estimates:

$$\begin{aligned} &\max\{|\zeta_h - 1|, |v_h - v_0|\} \le c|U| , \\ &\sup_{2^{h'-1} \le |\mathbf{k}'| \le 2^{h'+1}} \{||\partial^n_{\mathbf{k}'} s_h(\mathbf{k}')||, ||\partial^n_{\mathbf{k}'} (t_{h,\omega}(\mathbf{k}') - t_{0,\omega}(\mathbf{k}'))||\} \le c_n |U| 2^{2(h'-h)} 2^{(1+\theta-n)h} . \end{aligned} \tag{7.38}$$

Now, using the definition of $g^{(h)}_\omega$, see Eq.(7.25) and Eq.(7.19), and the bounds Eq.(7.38), we get (proceeding as in the proof of Eq.(A.5) in Appendix A),

$$||g^{(h)}_\omega(\mathbf{x}_1 - \mathbf{x}_2)|| \le C_K \frac{2^{2h}}{1 + (2^h ||\mathbf{x}_1 - \mathbf{x}_2||)^K} , \tag{7.39}$$

for all $K \ge 0$ and a suitable constant C_K. Using the tree expansion described above and, in particular, Eqs.(7.28),(7.30),(7.31), we find that the l.h.s. of Eq.(7.36) can be bounded from above by

$$\begin{aligned} &\sum_{n\ge 1} \sum_{\tau \in \mathcal{T}_{h,n}} \sum_{\substack{\mathbf{P} \in \mathcal{P}_\tau \\ |P_{v_0}| = 2l}} \sum_{T \in \mathbf{T}} \int \prod_{\ell \in T^*} d(\mathbf{x}(\ell) - \mathbf{x}'(\ell)) \left[\prod_{i=1}^{n} |K^{(h_i)}_{v_i^*}(\mathbf{x}_{v_i^*})| \right] \cdot \\ &\qquad \cdot \left[\prod_{\substack{v \text{ not} \\ \text{e.p.}}} \frac{1}{s_v!} \max_{\mathbf{t}_v} \left| \det G^{h_v, T_v}(\mathbf{t}_v) \right| \prod_{\ell \in T_v} \left|\left| g^{(h_v)}_{\omega(\ell)}(\mathbf{x}(\ell) - \mathbf{x}'(\ell)) \right|\right| \right] \end{aligned} \tag{7.40}$$

where T^* is a tree graph obtained from $T = \cup_v T_v$, by adding in a suitable (obvious) way, for each endpoint v_i^*, $i = 1, \dots, n$, one or more lines connecting the space-time points belonging to $\mathbf{x}_{v_i^*}$.

An application of the Gram–Hadamard inequality Eq.(5.6), combined with the representation Eq.(7.33) and the dimensional bound Eq.(7.35), implies that

$$|\mathrm{det} G^{h_v,T_v}(\mathbf{t}_v)| \leq (\mathrm{const.})^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-2(s_v-1)} \cdot 2^{h_v\left(\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-2(s_v-1)\right)} . \tag{7.41}$$

By the decay properties of $g^{(h)}_\omega(\mathbf{x})$, Eq.(7.39), it also follows that

$$\prod_{\substack{v \text{ not}\\ \text{e.p.}}} \frac{1}{s_v!} \int \prod_{\ell\in T_v} d(\mathbf{x}(\ell)-\mathbf{x}'(\ell))\, ||g^{(h_v)}_{\omega(\ell)}(\mathbf{x}(\ell)-\mathbf{x}'(\ell))|| \leq c^n \prod_{\substack{v \text{ not}\\ \text{e.p.}}} \frac{1}{s_v!} 2^{-h_v(s_v-1)} . \tag{7.42}$$

The bound Eq.(6.10) on the kernels produced by the ultraviolet integration implies that

$$\int \prod_{\ell\in T^*\setminus\cup_v T_v} d(\mathbf{x}(\ell)-\mathbf{x}'(\ell)) \prod_{i=1}^{n} |K^{(h_i)}_{v_i^*}(\mathbf{x}_{v_i^*})| \leq \prod_{i=1}^{n} C^{p_i}|U|^{\frac{p_i}{2}-1} , \tag{7.43}$$

where $p_i = |P_{v_i^*}|$. Combining the previous bounds, we find that Eq.(7.40) can be bounded from above by

$$\sum_{n\geq 1}\sum_{\tau\in\mathcal{T}_{h,n}} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\ |P_{v_0}|=2l}} \sum_{T\in\mathbf{T}} C^n \Big[\prod_{\substack{v \text{ not}\\ \text{e.p.}}} \frac{1}{s_v!} 2^{h_v\left(\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-3(s_v-1)\right)}\Big]\Big[\prod_{i=1}^{n} C^{p_i}|U|^{\frac{p_i}{2}-1}\Big] \tag{7.44}$$

Let us recall that $n(v)=\sum_{i:v_i^*>v} 1$ is the number of endpoints following v on τ, that v' is the vertex immediately preceding v on τ and that $|I_v|$ is the number of field labels associated to the endpoints following v on τ (note that $|I_v|\geq 4n(v)$). Using the following relations, which can be easily proved by induction,

$$\begin{aligned} &\sum_{v \text{ not e.p.}} h_v\Big[\Big(\sum_{i=1}^{s_v}|P_{v_i}|\Big)-|P_v|\Big] = h(|I_{v_0}|-|P_{v_0}|) + \sum_{v \text{ not e.p.}} (h_v-h_{v'})(|I_v|-|P_v|) ,\\ &\sum_{v \text{ not e.p.}} h_v(s_v-1) = h(n-1) + \sum_{v \text{ not e.p.}} (h_v-h_{v'})(s_v-1) , \end{aligned} \tag{7.45}$$

we find that Eq.(7.44) can be bounded above by

$$\begin{aligned} &\sum_{n\geq 1}\sum_{\tau\in\mathcal{T}_{h,n}} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\ |P_{v_0}|=2l}} \sum_{T\in\mathbf{T}} C^n 2^{h(3-|P_{v_0}|+|I_{v_0}|-3n)} \cdot\\ &\qquad \cdot\Big[\prod_{\substack{v \text{ not}\\ \text{e.p.}}} \frac{1}{s_v!} 2^{(h_v-h_{v'})(3-|P_v|+|I_v|-3n(v))}\Big]\Big[\prod_{i=1}^{n} C^{p_i}|U|^{\frac{p_i}{2}-1}\Big] . \end{aligned} \tag{7.46}$$

Using the following identities

$$2^{hn}\prod_{\substack{v \text{ not}\\ \text{e.p.}}} 2^{(h_v-h_{v'})n(v)} = \prod_{v \text{ e.p.}} 2^{h_{v'}} , \tag{7.47}$$

$$2^{h|I_{v_0}|}\prod_{\substack{v \text{ not}\\ \text{e.p.}}} 2^{(h_v-h_{v'})|I_v|} = \prod_{v \text{ e.p.}} 2^{h_{v'}|I_v|} , \tag{7.48}$$

combined with remark that all the endpoints are on scale 1 (so that the right hand sides of Eqs.(7.47)-(7.48) are equal to 1), we obtain

$$\frac{1}{\beta|\Lambda|}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2l}|W^{(h)}_{2l,\underline{\rho},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2l})| \le \sum_{n\ge1}\sum_{\tau\in\mathcal{T}_{h,n}}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\ |P_{v_0}|=2l}}\sum_{T\in\mathbf{T}} C^n 2^{h(3-|P_{v_0}|)}\cdot$$
$$\cdot\Big[\prod_{\substack{v\text{ not}\\ \text{e.p.}}}\frac{1}{s_v!}2^{-(h_v-h_{v'})(|P_v|-3)}\Big]\Big[\prod_{i=1}^n C^{p_i}|U|^{\frac{p_i}{2}-1}\Big]\,. \tag{7.49}$$

Note that, if v is not an endpoint, $|P_v|-3\ge1$ by the definition of $\mathcal{R}$. Now, note that the number of terms in $\sum_{T\in\mathbf{T}}$ can be bounded by $C^n\prod_{v\text{ not e.p.}}s_v!$. Using also that $|P_v|-3\ge1$ and $|P_v|-3\ge|P_v|/4$, we find that the l.h.s. of Eq.(7.49) can be bounded as

$$\frac{1}{\beta|\Lambda|}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2l}|W^{(h)}_{2l,\underline{\rho},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2l})| \le 2^{h(3-|P_{v_0}|)}\sum_{n\ge1}C^n\sum_{\tau\in\mathcal{T}_{h,n}}\cdot \tag{7.50}$$
$$\cdot\Big(\prod_{\substack{v\text{ not}\\ \text{e.p.}}}2^{-\theta(h_v-h_{v'})}2^{-(1-\theta)(h_v-h_{v'})/2}\Big)\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\ |P_{v_0}|=2l}}\Big(\prod_{\substack{v\text{ not}\\ \text{e.p.}}}2^{-(1-\theta)|P_v|/8}\Big)\prod_{i=1}^n C^{p_i}|U|^{\frac{p_i}{2}-1}\,.$$

Proceeding as in the previous section, we get the analogue of Eq.(6.48):

$$\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau\\ |P_{v_0}|=2l}}\Big(\prod_{\substack{v\text{ not}\\ \text{e.p.}}}2^{-(1-\theta)|P_v|/8}\Big)\prod_{i=1}^n C^{p_i}|U|^{\frac{p_i}{2}-1}\le C_\theta^n|U|^n\,.$$

Finally, using that $\prod_{v\text{ not e.p.}}2^{-\theta(h_v-h_{v'})}\le 2^{\theta h}$ and that, for $0<\theta<1$ (in analogy with Eq.(6.49)),

$$\sum_{\tau\in\mathcal{T}_{h,n}}\prod_{v\text{ not e.p.}}2^{-(1-\theta)(h_v-h_{v'})/2}\le C^n\,;$$

and collecting all the previous bounds, we obtain

$$\frac{1}{\beta|\Lambda|}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2l}|W^{(h)}_{2l,\underline{\rho},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2l})| \le 2^{h(3-|P_{v_0}|+\theta)}\sum_{n\ge1}C^n|U|^n\,, \tag{7.51}$$

which is the desired result.

We now need to prove the assumption Eq.(7.37). We proceed by induction. It is easy to see that the assumption is valid for $h=0$; in fact,

$$\begin{aligned}
&|z_0| = |\partial_{k_0}\hat W^{(0)}_{2,(1,1),(\omega,\omega)}(\mathbf{p}_F^\omega)|\,, \qquad |\delta_0| = |\partial_{k_1}\hat W_{2,(1,2),(\omega,\omega)}(\mathbf{p}_F^\omega)|\,,\\
&|\sigma_0(\mathbf{k}')| = \Big|\sum_{\mu,\nu=0}^{2}\int_0^1 ds\int_0^s ds' k'_\mu k'_\nu\partial_\mu\partial_\nu W_{2,(1,1),(\omega,\omega)}(s'\mathbf{k}')\Big|\,,\\
&|\tau_0(\mathbf{k}')| = \Big|\sum_{\mu,\nu=0^2}\int_0^1 ds\int_0^s ds' k'_\mu k'_\nu\partial_\mu\partial_\nu W_{2,(1,2),(\omega,\omega)}(s'\mathbf{k}')\Big|\,,
\end{aligned} \tag{7.52}$$

so that,

$$\max\{|z_0|,|\delta_0|\} \le \frac{1}{\beta|\Lambda|}\int d\mathbf{x}d\mathbf{y}\,||\mathbf{x}-\mathbf{y}||\,||W_{2,(\omega,\omega)}(\mathbf{x}-\mathbf{y})||\;, \tag{7.53}$$

$$\sup_{|\mathbf{k}'|\sim 2^{h'}}\{||\partial^n_{\mathbf{k}'}\sigma_0(\mathbf{k}')||,||\partial^n_{\mathbf{k}'}\tau_0(\mathbf{k}')||\} \le \frac{C2^{2h'}}{\beta|\Lambda|}\int d\mathbf{x}d\mathbf{y}\,||\mathbf{x}-\mathbf{y}||^{n+2}\,||W_{2,(\omega,\omega)}(\mathbf{x}-\mathbf{y})||\;,$$

which can be bounded via the same strategy as the proof of Eq.(6.10), yielding Eq.(7.37) for $h=0$. Similarly, assuming that Eq.(7.37) is valid for all $h\ge k+1$, the quantities of interest for $h=k$ can be bounded by

$$\max\{|z_k|,|\delta_k|\} \le \frac{1}{\beta|\Lambda|}\int d\mathbf{x}d\mathbf{y}\,||\mathbf{x}-\mathbf{y}||\,||W^{(k)}_{2,(\omega,\omega)}(\mathbf{x},\mathbf{y})||\;, \tag{7.54}$$

and

$$\sup_{2^{h'-1}\le|\mathbf{k}'|\le 2^{h'+1}}\{||\partial^n_{\mathbf{k}'}\sigma_k(\mathbf{k}')||,||\partial^n_{\mathbf{k}'}\tau_{k,\omega}(\mathbf{k}')||\} \le$$
$$\le \frac{C2^{2h'}}{\beta|\Lambda|}\int d\mathbf{x}d\mathbf{y}\,||\mathbf{x}-\mathbf{y}||^{n+2}||W^{(k)}_{2,(\omega,\omega)}(\mathbf{x},\mathbf{y})||\;. \tag{7.55}$$

The same proof leading to Eq.(7.51) shows that the r.h.s. of Eq.(7.54) can be bounded by the r.h.s. of Eq.(7.51) times 2^{-k} (that is the dimensional estimate for $||\mathbf{x}-\mathbf{y}||$), and that the r.h.s. of Eq.(7.54) can be bounded by the r.h.s. of Eq.(7.51) times $(\text{const.})^n 2^{2h'}2^{-(n+2)k}$ (where $2^{-k(n+2)}$ is the dimensional estimate for $||\mathbf{x}-\mathbf{y}||^{n+2}$).

It remains to prove the estimates on $e_h,\overline{e}_h$. The bound on $\overline{e}_h$ is an immediate corollary of the discussion above, simply because $\overline{e}_h$ can be bounded by Eq.(7.40) with $l=0$. Finally, remember that e_h is given by Eq.(7.18): an explicit computation of $A^{-1}_{h,\omega}(\mathbf{k}')W^{(h)}_{2,(\omega,\omega)}(\mathbf{k}')$ and the use of Eqs.(7.37)-(7.38) imply that $||A^{-1}_{h,\omega}(\mathbf{k}')W^{(h)}_{2,(\omega,\omega)}(\mathbf{k}')||\le C|U|2^{\theta h}$, from which: $|e_h|\le C'2^{3h}\sum_{n\ge1}(C|U|2^{\theta h})^n$, as desired. This concludes the proof of Theorem 7.1 and, therefore, as discussed after the statement of Theorem 7.1, it also concludes the proof of analyticity of the specific free energy and ground state energy. ∎

8. CONCLUSIONS

In conclusion, I presented a self-contained proof of the analyticity of the specific free energy and ground state energy of the 2D Hubbard model on the honeycomb lattice, at half-filling and weak coupling. The proof is based on rigorous fermionic RG methods and can be extended to the construction of the interacting correlations, i.e., the off-diagonal elements of the reduced density matrices of the system [24], and to the computation of the universal optical conductivity [28]. Such construction shows that the interacting correlations decay to zero at infinity with the same decay exponents as those of the non-interacting case. The "only" effect of the interactions is to change by a finite amount the quasi-particle weight Z^{-1} at the Fermi surface and the Fermi velocity v.

The example presented here is the only known example of a realistic 2D interacting Fermi system for which the ground state (including the correlations) can be constructed. The main difference with respect to other more standard 2D Fermi systems is the fact that here, at half-filling, the Fermi surface reduces to a set of two isolated points. This fact dramatically improves the infrared scaling properties of the theory: the four-fermions interaction, rather than being marginal, as in many other similar cases, is irrelevant; this is the technical point that makes the construction of the ground state possible and "relatively easy".

It is natural to ask how the system behaves in the presence of electromagnetic interactions among the electrons, which is the case of interest for applications to clean graphene samples. In this case, the system has many analogies with (2+1)-dimensional QED. The four-fermions interaction, rather than being irrelevant, is marginal, and the fixed point of the theory is expected to be non-trivial. The long distance decay of correlations is expected to be described in terms of anomalous critical exponents and the effective Fermi velocity is expected to grow up to the maximal possible value, i.e., the speed of light. The specific values of the critical exponents suggest that local distortions of the lattice (in the form of the so-called Kekulé pattern [30]) are amplified by electromagnetic interactions: this led us to propose a possible mechanism for the spontaneous formation of the Kekulé pattern, via a mechanism analogous to the 1D Peierls' instability [27]. All these claims have been proved so far only order by order in renormalized perturbation theory [26, 27]; proving them in a non-perturbative fashion is an important open problem, whose solution would represent a corner stone in the mathematical theory of quantum Coulomb systems.

Appendix A: Dimensional estimates of the propagators

In this Appendix we prove the dimensional bounds on the propagators used in Sections 4-5-6-7 and, in particular, the bounds $||g||_\infty \le CM$ and $||g||_1 \le C\beta$, stated right before Eq.(4.26), and the estimates Eqs.(6.15),(7.39). The basic idea is to decompose the propagator in Eq.(4.5),

$$g(\mathbf{x}) = \frac{1}{\beta|\Lambda|} \sum_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}} e^{-i\mathbf{k}\mathbf{x}} \frac{\chi_0(2^{-M}|k_0|)}{k_0^2 + v_0^2|\Omega(\vec{k})|^2} \begin{pmatrix} ik_0 & -v_0\Omega^*(\vec{k}) \\ -v_0\Omega(\vec{k}) & ik_0 \end{pmatrix} , \tag{A.1}$$

as a sum of single scale propagators:

$$g(\mathbf{x}) = \sum_{h=1}^{M} g^{(h)}(\mathbf{x}) + \sum_{\omega=\pm}\sum_{h=h_\beta}^{0} \overline{g}^{(h)}_\omega(\mathbf{x}) \;, \tag{A.2}$$

where: (i) if $1 \le h \le M$, then $g^{(h)}(\mathbf{x})$ is defined as in Eq.(6.14); (ii) if $h_\beta := \lfloor \log_2\left(\frac{3\pi}{4\beta}\right)\rfloor$ and $h_\beta \le h \le 0$, then

$$\overline{g}^{(h)}_\omega(\mathbf{x}) = \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}} e^{-i\mathbf{k}\mathbf{x}} \frac{f_h(k_0,\vec k-\vec p_F^{\,\omega})}{k_0^2+v_0^2|\Omega(\vec k)|^2}\begin{pmatrix} ik_0 & -v_0\Omega^*(\vec k)\\ -v_0\Omega(\vec k) & ik_0\end{pmatrix} \;, \tag{A.3}$$

with $f_h(\mathbf{k}') = \chi_0(2^{-h}|\mathbf{k}'|) - \chi_0(2^{-h+1}|\mathbf{k}'|)$. The main issue is to prove Eq.(6.15), i.e.,

$$||g^{(h)}(\mathbf{x})|| \le \frac{C_K}{1+(2^h|x_0|_\beta + |\vec x|_\Lambda)^K} \;, \tag{A.4}$$

and the analogue of Eq.(7.39), i.e.,

$$||g^{(h)}_\omega(\mathbf{x})|| \le \frac{C_K 2^{2h}}{1+(2^h||\mathbf{x}||)^K} \;, \tag{A.5}$$

where $||\mathbf{x}||^2 = |x_0|^2_\beta + |\vec x|^2_\Lambda$, for all $K \ge 0$. Note that $||g||_\infty \le CM$ and $||g||_1 \le C\beta$ are immediate corollaries of Eqs.(A.4)-(A.5). In fact, plugging these bounds into Eq.(A.2), we find:

$$||g||_\infty \le \sum_{h=1}^{M} C_0 + \sum_{\omega=\pm}\sum_{h=h_\beta}^{0} C_0 2^{2h} \le C_0(M+\frac{8}{3}) \;, \tag{A.6}$$

$$\begin{aligned}||g||_1 &\le C_4\sum_{h=1}^{M}\int_{(\beta,\Lambda)} d\mathbf{x}\frac{1}{1+(2^h|x_0|_\beta+|\vec x|_\Lambda)^K} + 2C_4\sum_{h=h_\beta}^{0}\int_{(\beta,\Lambda)} d\mathbf{x}\frac{2^{2h}}{1+(2^h||\mathbf{x}||)^4} \le\\ &\le C_4'\sum_{h=h_\beta}^{M} 2^{-h} \le C_4' 2^{-h_\beta+1} \;,\end{aligned} \tag{A.7}$$

which are the desired bounds.

So, let us start by proving Eq.(A.4). We denote by $\partial_{\mathbf{k}}$ the discrete derivative, and by $\tilde\partial_\mu = \tilde e_\mu\cdot\partial_{\mathbf{k}}$, with $\mu\in\{0,1,2\}$, the discrete derivative in the direction $\tilde e_\mu$ (here $\tilde e_0 = (1,0,0)$, $\tilde e_1 = (0,\frac12,\frac{\sqrt3}{2})$ and $\tilde e_2 = (0,\frac12,-\frac{\sqrt3}{2})$), defined as follows: given a compact support function $\hat F(\mathbf{k})$ with $\mathbf{k}\in\mathcal{B}^{(\infty)}_{\beta,L}$, we let

$$\tilde\partial_\mu \hat F(\mathbf{k}) = \tilde e_\mu\cdot\partial_{\mathbf{k}}\hat F(\mathbf{k}) = \frac{\hat F(\mathbf{k}+\tilde e_\mu\Delta k_\mu)-\hat F(\mathbf{k})}{\Delta k_\mu} \;, \tag{A.8}$$

with $\Delta k_0 = \frac{2\pi}{\beta}$ and $\Delta k_1 = \Delta k_2 = \frac{4\pi}{3L}$. Note that, if

$$F(\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{B}^{(\infty)}_{\beta,L}} e^{-i\mathbf{k}\cdot\mathbf{x}}\hat F(\mathbf{k}) \;, \tag{A.9}$$

then, defining $\tilde{x}_\mu = \mathbf{x} \cdot \tilde{e}_\mu$,

$$\sum_{\mathbf{k} \in \mathcal{B}^{(\infty)}_{\beta,L}} e^{-i\mathbf{k}\cdot\mathbf{x}} \tilde{\partial}_\mu \hat{F}(\mathbf{k}) = \left(\frac{e^{i\Delta k_\mu \tilde{x}_\mu} - 1}{\Delta k_\mu} \right) \sum_{\mathbf{k} \in \mathcal{B}^{(\infty)}_{\beta,L}} e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{F}(\mathbf{k}) \; , \tag{A.10}$$

so that, using the fact that $|x_0|_\beta \le \frac{\pi}{2} d_\beta(x_0)$ and $|\vec{x}|_\Lambda \le \frac{\pi}{\sqrt{2}} (d_L^2(\tilde{x}_1) + d_L^2(\tilde{x}_2))^{1/2}$ (here $d_\beta(x_0) = \frac{\sin(\pi x_0/\beta)}{\pi/\beta}$ and $d_L(\tilde{x}_i) = \frac{\sin(2\pi \tilde{x}_i/3L)}{2\pi/3L}$), we find

$$\begin{aligned} |x_0|^2_\beta |F(\mathbf{x})| &\le \frac{\pi^2}{4} d^2_\beta(x_0) |F(\mathbf{x})| = \frac{\pi^2}{4} \left| \frac{e^{i\Delta k_0 x_0} - 1}{\Delta k_0} \right|^2 \cdot \Big| \sum_{\mathbf{k} \in \mathcal{B}^{(\infty)}_{\beta,L}} e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{F}(\mathbf{k}) \Big| = \\ &= \frac{\pi^2}{4} \Big| \sum_{\mathbf{k} \in \mathcal{B}^{(\infty)}_{\beta,L}} e^{-i\mathbf{k}\cdot\mathbf{x}} \partial^2_{k_0} \hat{F}(\mathbf{k}) \Big| \le \frac{\pi^2}{4} \sum_{\mathbf{k} \in \mathcal{B}^{(\infty)}_{\beta,L}} \big| \partial^2_{k_0} \hat{F}(\mathbf{k}) \big| \; , \end{aligned} \tag{A.11}$$

and, similarly,

$$\begin{aligned} |\vec{x}|^2_\Lambda |F(\mathbf{x})| &\le \frac{\pi^2}{2} \Big(d_L^2(\tilde{x}_1) + d_L^2(\tilde{x}_2) \Big) |F(\mathbf{x})| = \\ &= \frac{\pi^2}{2} \sum_{\mu=1}^2 \left| \frac{e^{i\Delta k_\mu \tilde{x}_\mu} - 1}{\Delta k_\mu} \right|^2 \cdot \Big| \sum_{\mathbf{k} \in \mathcal{B}^{(\infty)}_{\beta,L}} e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{F}(\mathbf{k}) \Big| = \\ &= \frac{\pi^2}{2} \sum_{\mu=1}^2 \Big| \sum_{\mathbf{k} \in \mathcal{B}^{(\infty)}_{\beta,L}} e^{-i\mathbf{k}\cdot\mathbf{x}} \tilde{\partial}^2_\mu \hat{F}(\mathbf{k}) \Big| \le \frac{3\pi^2}{8} \sum_{\mathbf{k} \in \mathcal{B}^{(\infty)}_{\beta,L}} \big| (\partial^2_{k_1} + \partial^2_{k_2}) \hat{F}(\mathbf{k}) \big| \; . \end{aligned} \tag{A.12}$$

Coming back to Eq.(A.4), recalling that

$$g^{(h)}(\mathbf{x}) = \frac{1}{\beta|\Lambda|} \sum_{\mathbf{k} \in \mathcal{B}^{(M)}_{\beta,L}} e^{-i\mathbf{k}\mathbf{x}} \frac{f_{u.v.}(\mathbf{k}) H_h(k_0)}{k_0^2 + v_0^2 |\Omega(\vec{k})|^2} \begin{pmatrix} ik_0 & -v_0 \Omega^*(\vec{k}) \\ -v_0 \Omega(\vec{k}) & ik_0 \end{pmatrix} \; , \qquad h \ge 2 \; . \tag{A.13}$$

Therefore,

$$||g^{(h)}(\mathbf{x})|| \le \frac{1}{\beta|\Lambda|} \sum_{\mathbf{k} \in \mathcal{B}^{(M)}_{\beta,L}} \frac{f_{u.v.}(\mathbf{k}) H_h(k_0)}{\big[k_0^2 + v_0^2 |\Omega(\vec{k})|^2 \big]^{1/2}} \le (\text{const.}) 2^h \, 2^{-h} \; , \tag{A.14}$$

where in the last inequality we used the fact that, on the support of $f_{u.v.}(\mathbf{k}) H_h(k_0)$, $k_0^2 + v_0^2 |\Omega(\vec{k})|^2 \sim 2^{2h}$ (here $\sim$ means that the ratio of the two sides is bounded above and below by universal constants) and that, moreover, the measure of the support of $f_{u.v.}(\mathbf{k}) H_h(k_0)$ is itself of order 2^h, that is $(\beta|\Lambda|)^{-1} \sum_{\mathbf{k}} f_{u.v.}(\mathbf{k}) H_h(k_0) \sim 2^h$.

Moreover, using Eqs.(A.11)-(A.12), we have that for all $N \ge 0$ and a suitable constant C,

$$|x_0|_\beta^{2N}||g^{(h)}(\mathbf{x})|| \leq \frac{C^N}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}^{(M)}}\Big\|\partial_{k_0}^{2N}\Big[\frac{f_{u.v.}(\mathbf{k})H_h(k_0)}{k_0^2+v_0^2|\Omega(\vec{k})|^2}\begin{pmatrix} ik_0 & -v_0\Omega^*(\vec{k}) \\ -v_0\Omega(\vec{k}) & ik_0\end{pmatrix}\Big]\Big\|\,,$$

$$|\vec{x}|_\Lambda^{2N}||g^{(h)}(\mathbf{x})|| \leq \frac{C^N}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}^{(M)}}\Big\|(\vec{\partial}_{\vec{k}}\cdot\vec{\partial}_{\vec{k}})^N\Big[\frac{f_{u.v.}(\mathbf{k})H_h(k_0)}{k_0^2+v_0^2|\Omega(\vec{k})|^2}\begin{pmatrix} ik_0 & -v_0\Omega^*(\vec{k}) \\ -v_0\Omega(\vec{k}) & ik_0\end{pmatrix}\Big]\Big\|$$

Now, in the first line, every derivative with respect to k_0 can act either on the denominator $k_0^2+v_0^2|\Omega(\vec{k})|^2$, or on $f_{u.v.}(\mathbf{k})H_h(k_0)$, or on the diagonal elements of the matrix, ik_0; in all these cases, recalling that, on the support of $f_{u.v.}H_h$, $|k_0|\sim 2^h$, every derivative ∂_{k_0} can be estimated dimensionally by a factor proportional to 2^{-h}. This leads to the bound

$$|x_0|_\beta^{2N}||g^{(h)}(\mathbf{x})|| \leq (\text{const.})^N 2^h\, 2^{-2Nh}\, 2^{-h}\,, \tag{A.15}$$

which differs from Eq.(A.14) precisely by the factor 2^{-2Nh}. Similarly, in the second line, the derivative with respect to $\vec{k}$ can act either on the denominator $k_0^2+v_0^2|\Omega(\vec{k})|^2$, or on $f_{u.v.}(\mathbf{k})H_h(k_0)$, or on the off-diagonal elements of the matrix, $-v_0\Omega(\vec{k})$ or $-v_0\Omega^*(\vec{k})$; in the first case, $\vec{\partial}_{\vec{k}}$ can be estimated dimensionally by a factor proportional to 2^{-h} while, in the other cases, by an order 1 factor. This leads to the bound

$$|\vec{x}|_\Lambda^{2N}||g^{(h)}(\mathbf{x})|| \leq (\text{const.})^N 2^h\, 2^{-h}\,, \tag{A.16}$$

which, if combined with Eqs.(A.14)-(A.15), finally implies Eq.(A.4).

The proof of Eq.(A.4) is very similar. In fact, for all $N\geq 0$,

$$||\mathbf{x}||^{2N}\,||\overline{g}_\omega^{(h)}(\mathbf{x})|| \leq \frac{C^N}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}^{(M)}}\Big\|\partial_{\mathbf{k}}^{2N}\Big[\frac{f_h(k_0,\vec{k}-\vec{p}_F^{\,\omega})}{k_0^2+v_0^2|\Omega(\vec{k})|^2}\begin{pmatrix} ik_0 & -v_0\Omega^*(\vec{k}) \\ -v_0\Omega(\vec{k}) & ik_0\end{pmatrix}\Big]\Big\| \tag{A.17}$$

Now, using the fact that, on the support of $f_h(k_0,\vec{k}-\vec{p}_F^{\,\omega})$, $|k_0|, |\vec{k}-\vec{p}_F^{\,\omega}|, |\Omega(\vec{k})|\sim 2^h$, we see that every derivative $\partial_{\mathbf{k}}$, when acting either on f_h, or on the denominator $k_0^2+v_0^2|\Omega(\vec{k})|^2$, or on the elements of the matrix, can be estimated dimensionally by a factor proportional to 2^{-h}; moreover, the measure of the support of f_h is itself of order 2^{3h}, that is $(\beta|\Lambda|)^{-1}\sum_{\mathbf{k}} f_h(k_0,\vec{k}-\vec{p}_F^{\,\omega})\sim 2^{3h}$. This leads to the bound

$$||\mathbf{x}||^{2N}\,||\overline{g}_\omega^{(h)}(\mathbf{x})|| \leq (\text{const.})^N 2^{3h}\, 2^{-2Nh}\, 2^{-h}\,, \tag{A.18}$$

which finally implies Eq.(A.4). ∎

Appendix B: Truncated expectations and determinants

In this Appendix we prove Eq.(5.3), following [23, Appendix A.3.2]. Given s set of indices $P_1, \ldots, P_s$, consider the quantity $\mathcal{E}^T(\psi_{P_1}, \ldots, \psi_{P_s})$. Define

$$P_j^{\pm} = \{f \in P_j \ : \ \varepsilon(f) = \pm\} \tag{B.1}$$

and set $f = (j,i)$ for $f \in P_j^{\pm}$, with $i = 1, \ldots, |P_j^{\pm}|$. Note that $\sum_{j=1}^s |P_j^+| = \sum_{j=1}^s |P_j^-|$, otherwise the considered truncated expectation is vanishing. Define

$$\mathcal{D}\psi = \prod_{j=1}^s \Big[\prod_{f \in P_j^+} \mathrm{d}\psi^+_{\mathbf{x}(f),\sigma(f),\rho(f)} \Big] \Big[\prod_{f \in P_j^-} \mathrm{d}\psi^-_{\mathbf{x}(f),\sigma(f),\rho(f)} \Big] \ , \tag{B.2}$$

$$(\psi^+, G\psi^-) = \sum_{j,j'=1}^s \sum_{i=1}^{|P_j^-|} \sum_{i'=1}^{|P_{j'}^+|} \psi^+_{(j',i')} G_{(j,i),(j',i')} \psi^-_{(j,i)} \ , \tag{B.3}$$

where $\psi^{\pm}_{(j,i)} := \psi^{\pm}_{\mathbf{x}(j,i),\sigma(j,i),\rho(j,i)}$ and, if $n = \sum_{j=1}^s \left|P_j^+\right| = \sum_{j=1}^s \left|P_j^-\right|$, then G is the $n \times n$ matrix with entries

$$G_{(j,i),(j',i')} := \delta_{\sigma(j,i),\sigma(j',i')} g_{\rho(j,i),\rho(j',i')}(\mathbf{x}(j,i) - \mathbf{x}(j',i')) \ , \tag{B.4}$$

so that

$$\mathcal{E}\left(\prod_{j=1}^s \psi_{P_j}\right) = \det G = \int \mathcal{D}\psi \ \exp\left[-\left(\psi^+, G\psi^-\right)\right] \ . \tag{B.5}$$

Setting $X := \{1, \ldots, s\}$ and

$$\overline{V}_{jj'} = \sum_{i=1}^{|P_j^-|} \sum_{i'=1}^{|P_{j'}^+|} \psi^+_{(j',i')} G_{(j,i),(j',i')} \psi^-_{(j,i)} \ , \tag{B.6}$$

we write

$$V(X) = \sum_{j,j' \in X} \overline{V}_{jj'} = \sum_{j \le j'} V_{jj'} \ , \tag{B.7}$$

where

$$V_{jj'} = \begin{cases} \overline{V}_{jj'} \ , & \text{if } j = j' \ , \\ \overline{V}_{jj'} + \overline{V}_{j'j} \ , & \text{if } j < j' \ . \end{cases} \tag{B.8}$$

In terms of these definitions, Eq.(B.5) can be rewritten as

$$\mathcal{E}\left(\prod_{j=1}^s \psi_{P_j}\right) = \det G = \int \mathcal{D}\psi \, e^{-V(X)} \ . \tag{B.9}$$

We now want to express the last expression in terms of the functions W_X, defined as follows:

$$W_X(X_1, \ldots, X_r; t_1, \ldots, t_r) = \sum_{\ell \in L(X)} \prod_{k=1}^r t_k(\ell) \, V_\ell \ , \tag{B.10}$$

where:

1. X_k are subsets of X with $|X_k| = k$ and such that

$$\begin{cases} X_1 = \{1\} \ , \\ X_{k+1} \supset X_k \ ; \end{cases} \tag{B.11}$$

2. $L(X)$ is the set of unordered pairs in X, i.e., the set of pairs (j, j') with $j, j' \in X$, such that (j, j') is identified with (j', j) and, possibly, $j = j'$; we shall say that $\ell = (j, j')$ is *non tivial* if $j \neq j'$, and *trivial* otherwise;

3. the functions $t_k(\ell)$ are defined as follows:

$$t_k(\ell) = \begin{cases} t_k \ , & \text{if} \quad \ell \sim \partial X_k \ , \\ 1 \ , & \text{otherwise} \ , \end{cases} \tag{B.12}$$

where $\ell \sim X_k$ means that $\ell = (j, j')$ "intersects the boundary" of X_k, i.e., it connects $j \in X_k$ with $j' \notin X_k$, or viceversa. See Fig. 7.

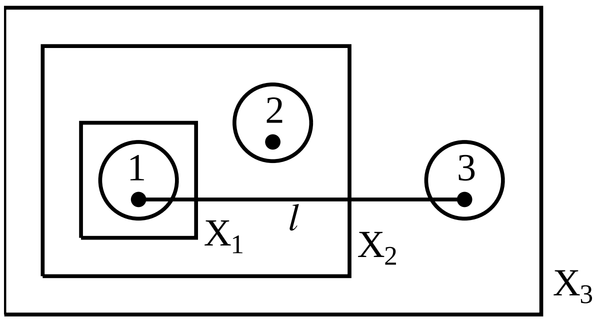

FIG. 7. Graphical representation of the sets X_k, $k = 1, 2, 3$. In the example $X_1 = \{1\}$, $X_2 = \{1, 2\}$, $X_3 = \{1, 2, 3\}$ and $\ell = (1, 3)$. The line ℓ intersects both the boundary of X_1 and of X_2, i.e., $\ell \sim \partial X_1$ and $\ell \sim \partial X_2$.

Let us show how to re-express $e^{-V(X)}$ in terms of the W_X's. The basic step is the following: using the definition Eq.(B.10), we rewrite

$$W_X(X_1; t_1) = t_1 V(X) + (1 - t_1) \left[V(X_1) + V(X \backslash X_1) \right] \ ; \tag{B.13}$$

that is, we recognize that $W_X(X_1; t_1)$ interpolates between the full $V(X)$ and two of its "proper subsets", $V(X_1)$, $V(X \backslash X_1)$. In this way,

$$\begin{aligned} e^{-V(X)} &= e^{-W_X(X_1;0)} + \int_0^1 \mathrm{d}t_1 \left[\frac{\partial}{\partial t_1} e^{-W_X(X_1;t_1)} \right] \\ &= e^{-W_X(X_1;0)} - \sum_{\ell_1 \sim \partial X_1} V_{\ell_1} \int_0^1 \mathrm{d}t_1 \, e^{-W_X(X_1;t_1)} \ . \end{aligned} \tag{B.14}$$

Let us now iterate this construction: let us consider one of the terms in the summation in the r.h.s. of Eq.(B.14); if $\ell_1 = (1, j^*)$, we let $X_2 := X_1 \cup \{j^*\}$ and we note that, by definition,

$$W_X(X_1, X_2; t_1, t_2) = t_2 W_X(X_1; t_1) + (1 - t_2) \left[W_{X_2}(X_1; t_1) + V(X \backslash X_2)\right] \ , \quad \text{(B.15)}$$

so that

$$\begin{aligned} e^{-W_X(X_1;t_1)} &= e^{-W_X(X_1,X_2;t_1,0)} + \int_0^1 \mathrm{d}t_2 \left[\frac{\partial}{\partial t_2} e^{-W_X(X_1,X_2;t_1,t_2)}\right] \qquad \text{(B.16)} \\ &= e^{-W_X(X_1,X_2;t_1,0)} - \sum_{\ell_2 \sim \partial X_2} V_{\ell_2} \int_0^1 \mathrm{d}t_2\, t_1(\ell_2)\, e^{-W_X(X_1,X_2;t_1,t_2)} \ . \end{aligned}$$

Substituting Eq.(B.16) into Eq.(B.14) we get:

$$\begin{aligned} e^{-V(X)} = e^{-W_X(X_1;0)} &+ \sum_{\ell_1 \sim \partial X_1} \int_0^1 \mathrm{d}t_1\, (-1)\, V_{\ell_1}\, e^{-W_X(X_1,X_2;t_1,0)} + \qquad \text{(B.17)} \\ &+ \sum_{\ell_1 \sim \partial X_1} \sum_{\ell_2 \sim \partial X_2} \int_0^1 \mathrm{d}t_1 \int_0^1 \mathrm{d}t_2\, (-1)^2\, V_{\ell_1}\, V_{\ell_2}\, t_1(\ell_2)\, e^{-W_X(X_1,X_2;t_1,t_2)} \ . \end{aligned}$$

Using the fact that

$$\begin{aligned} &W_X(X_1, \ldots, X_{p+1}; t_1, \ldots, t_{p+1}) = t_{p+1} W_X(X_1, \ldots, X_p; t_1, \ldots, t_p) + \\ &(1 - t_{p+1}) \left[W_{X_{p+1}}(X_1, \ldots, X_p; t_1, \ldots, t_p) + V(X \setminus X_{p+1})\right] \qquad \text{(B.18)} \end{aligned}$$

and iterating we find

$$\begin{aligned} e^{-V(X)} = &\sum_{r=0}^{s-1} \sum_{\ell_1 \sim \partial X_1} \cdots \sum_{\ell_r \sim \partial X_r} \int_0^1 \mathrm{d}t_1 \ldots \int_0^1 \mathrm{d}t_r\, (-1)^r\, V_{\ell_1} \ldots V_{\ell_r} \\ &\left(\prod_{k=1}^{r-1} t_1(\ell_{k+1}) \ldots t_k(\ell_{k+1})\right) e^{-W_X(X_1,\ldots,X_{r+1};t_1,\ldots,t_r,0)} \ , \qquad \text{(B.19)} \end{aligned}$$

where, if $r = 0$, the summand should be interpreted as equal to $e^{-W_X(X_1;0)}$. Moreover, by the definition of the W_X's, if $r > 1$,

$$W_X(X_1, \ldots, X_r; t_1, \ldots, t_{r-1}, 0) = W_{X_r}(X_1, \ldots, X_{r-1}; t_1, \ldots, t_{r-1}) + V(X \setminus X_r) \ . \qquad \text{(B.20)}$$

Using this remark and the notion of *anchored tree*, which will be defined in a moment, Eq.(B.19) can be rewritten in a more suggestive and convenient way. Let $\mathcal{T}(X)$ be the set of tree graphs on X, i.e., the set of $(|X| - 1)$-ples of lines in $L(X)$ connecting (in a minimal way) all the elements of X. Given a sequence of subsets $X_1 \subset \cdots \subset X_r$ as above, we shall say that $T \in \mathcal{T}(X_r)$ is an *anchored tree on* $(X_1, \ldots, X_r)$ if its lines can be ordered in such a way that $\ell_1 \sim \partial X_1, \ell_2 \sim$

$\partial X_2, \ldots, \ell_{r-1} \sim \partial X_{r-1}$. Moreover, given a sequence $X_1 \subset \cdots \subset X_s$ as above and a non-trivial line $\ell \in L(X)$, we let

$$n(\ell) = \max\{k : \ell \sim \partial X_k\} \ , \qquad n'(\ell) = \min\{k : \ell \sim \partial X_k\} \ ; \tag{B.21}$$

if ℓ is trivial, we let $n(\ell) = n'(\ell) = 0$. Using these definitions, we rewrite Eq.(B.19) as:

$$e^{-V(X)} = \sum_{r=1}^{s} \sum_{X_r \subset X} \sum_{X_2,\ldots,X_{r-1}} \sum_{T \text{ on } (X_1,\ldots,X_r)} (-1)^{r-1} \prod_{\ell \in T} V_\ell \cdot \tag{B.22}$$
$$\cdot \int_0^1 \mathrm{d}t_1 \ldots \int_0^1 \mathrm{d}t_{r-1} \left(\prod_{\ell \in T} \frac{\prod_{k=1}^{r-1} t_k(\ell)}{t_{n(\ell)}} \right) e^{-W_{X_r}(X_1,\ldots,X_{r-1};t_1,\ldots,t_{r-1})}\, e^{-V(X \setminus X_r)} \ ,$$

where "T on $(X_1, \ldots, X_r)$" means that T is an anchored tree on $(X_1, \ldots, X_r)$. Defining

$$K(X_r) = \sum_{X_2,\ldots,X_{r-1}} \sum_{T \text{ on } (X_1,\ldots,X_r)} \prod_{\ell \in T} V_\ell \tag{B.23}$$
$$\int_0^1 \mathrm{d}t_1 \ldots \int_0^1 \mathrm{d}t_{r-1} \left(\prod_{\ell \in T} \frac{\prod_{k=1}^{r-1} t_k(\ell)}{t_{n(\ell)}} \right) e^{-W_{X_r}(X_1,\ldots,X_{r-1};t_1,\ldots,t_{r-1})} \ ,$$

Eq.(B.22) can be further rewritten as

$$e^{-V(X)} = \sum_{\substack{Y \subset X \\ Y \ni \{1\}}} (-1)^{|Y|-1} K(Y)\, e^{-V(X \setminus Y)} \ , \tag{B.24}$$

and, iterating,

$$e^{-V(X)} = \sum_{m=1}^{s} \sum_{(Y^1,\ldots,Y^m)} (-1)^s (-1)^m \prod_{i=1}^{m} K(Y^i) \ , \tag{B.25}$$

where the second summation runs over partitions of X of multiplicity m, i.e., over m-ples of disjoint sets $Y^1, \ldots, Y^m$ such that $\cup_{i=1}^m Y^i = X$. Plugging Eq.(B.25) into Eq.(B.9) gives

$$\mathcal{E}\left(\prod_{j=1}^{s} \psi_{P_j} \right) = \sum_{m=1}^{s} \sum_{(Y^1,\ldots,Y^m)} (-1)^{s+m} (-1)^\sigma \int \mathcal{D}\psi_{Y^i} \prod_{i=1}^{m} K(Y^i) \ , \tag{B.26}$$

where $\mathcal{D}\psi_{Y^i} = \prod_{j \in Y^i} \left[\prod_{f \in P_j^+} \mathrm{d}\psi^+_{\mathbf{x}(f),\sigma(f),\rho(f)} \right] \left[\prod_{f \in P_j^-} \mathrm{d}\psi^-_{\mathbf{x}(f),\sigma(f),\rho(f)} \right]$ and $(-1)^\sigma$ is the sign of the permutation leading from the ordering of the fields in $\mathcal{D}\psi$ to the ones in $\prod_i \mathcal{D}\psi_{Y^i}$. In Eq.(B.26), each factor $K(Y^i)$, after small manipulations of its definition, Eq.(B.23), can be rewritten as

$$K(Y^i) = \sum_{T \in \mathcal{T}(Y^i)} \sum_{\substack{Y^i_2,\ldots,Y^i_{|Y^i|-1} \\ \text{compatible with } T}} \prod_{\ell \in T} V_\ell \int_0^1 dt_1 \cdots \int_0^1 dt_{|Y^i|-1} \cdot$$
$$\cdot \prod_{\ell \in T} (t_{n'(\ell)} \cdots t_{n(\ell)-1}) e^{-\sum_{\ell \in L(Y^i)} t_{n'(\ell)} \cdots t_{n(\ell)} V_\ell} \ , \tag{B.27}$$

where: (i) $Y_1^i := \{\min\{j : j \in Y^i\}\}$; (ii) the second summations is over sequences of subsets Y_q^i of Y^i with $|Y_q^i| = q$ and such that $Y_1^i \subset Y_2^i \subset \cdots \subset Y_{|Y^i|-1}^i \subset Y_{|Y^i|}^i \equiv Y^i$ and T is anchored on $(Y_1^i, \ldots, Y_{|Y^i|}^i)$; (iii) in the second line, if $n(\ell) = n'(\ell)$, the factor $t_{n'(\ell)} \cdots t_{n(\ell)-1}$ should be interpreted as equal to 1; (iv) in the exponent, if ℓ is trivial (and, therefore, $n'(\ell) = n(\ell) = 0$), t_0 should be interpreted as equal to 1. Now, using the analogue of Eq.(4.20) in a case, like the one at hand, where the monomials ψ do not necessarily all commute among each other, we have that (denoting the elements of Y^i as $Y^i = \{j_1^i, \ldots, j_{|Y_i|}^i\}$)

$$\mathcal{E}\left(\prod_{j=1}^{s} \psi_{P_j}\right) = \sum_{m=1}^{s} \sum_{(Y^1,\ldots,Y^m)} (-1)^\sigma \mathcal{E}^T(\psi_{P_{j_1^1}}, \ldots, \psi_{P_{j^1_{|Y^1|}}}) \cdots \mathcal{E}^T(\psi_{P_{j_1^m}}, \ldots, \psi_{P_{j^m_{|Y^m|}}}) \tag{B.28}$$

where $(-1)^\sigma$ is the parity of the permutation leading to the ordering on the r.h.s. from the one on the l.h.s.; note that σ is the same as in eqn(B.26). Comparing Eq.(B.28) with Eq.(B.26) and using Eq.(B.27), we get:

$$\mathcal{E}^T(\psi_{P_1}, \ldots, \psi_{P_s}) = (-1)^{s+1} \sum_{T \in \mathcal{T}(X)} \int \mathcal{D}\psi \prod_{\ell \in T} V_\ell \int dP_T(\mathbf{t}) e^{-V(\mathbf{t})} , \tag{B.29}$$

where we defined:

$$dP_T(\mathbf{t}) = \sum_{\substack{X_2,\ldots,X_{s-1} \\ \text{compatible with } T}} \prod_{\ell \in T} \left(t_{n'(\ell)} \ldots t_{n(\ell)-1}\right) \prod_{q=1}^{s-1} \mathrm{d}t_q \tag{B.30}$$

and

$$V(\mathbf{t}) \equiv \sum_{\ell \in L(X)} t_{n'(\ell)} \ldots t_{n(\ell)} V_\ell . \tag{B.31}$$

Finally, if we use the definitions Eqs.(B.6),(B.8) and explicitly integrate the Grassmann variables along the lines of the anchored tree, we end up with

$$\mathcal{E}^T\left(\psi_{P_1}, \ldots, \psi_{P_s}\right) = \sum_{T \in \mathbf{T}(\mathcal{P})} \alpha_T \prod_{\ell \in T} G_{f_\ell^1, f_\ell^2} \int \mathcal{D}^*(\mathrm{d}\psi) \int \mathrm{d}P_T(\mathbf{t})\, e^{-V^*(\mathbf{t})} , \tag{B.32}$$

where $\mathcal{P} = \{P_1, \ldots, P_s\}$ and $\mathbf{T}(\mathcal{P})$ is the set of anchored trees between the "boxes" $P_1, \ldots, P_s$, i.e., the graphs that become trees if one identifies all the points in the same "clusters" P_i (note that now the lines of an anchored tree $T \in \mathbf{T}(\mathcal{P})$ are pairs of field variables (f, f'), rather than pairs of indices (j, j')). Moreover, α_T is a sign (irrelevant to the purpose of the bounds performed in this paper), f_ℓ^1, f_ℓ^2 are the two field labels associated to the two (entering and exiting) half-lines contracted into ℓ, and

$$\mathcal{D}^*(\mathrm{d}\psi) = \prod_{\substack{f \in P \\ f \notin T}} d\psi^{\varepsilon(f)}_{\mathbf{x}(f),\sigma(f),\rho(f)} , \qquad V^*(\mathbf{t}) = \sum_{\ell \in L(X)} t_{n'(\ell)} \ldots t_{n(\ell)} V_\ell^T \tag{B.33}$$

where

$$V^T_{jj'} = \begin{cases} \overline{V}^T_{jj'} \ , & \text{if } j = j' \ , \\ \overline{V}^T_{jj'} + \overline{V}^T_{j'j} \ , & \text{if } j < j' \ . \end{cases} \tag{B.34}$$

and, if $\big((j,i),(j',i')\big)$ is the line obtained by contracting $\psi^-_{(j,i)}$ with $\psi^+_{(j',i')}$,

$$\overline{V}^T_{jj'} = \sum_{i=1}^{|P_j^-|} \sum_{i'=1}^{|P_{j'}^+|} \psi^+_{(j',i')} G_{(j,i),(j',i')} \psi^-_{(j,i)} \, \chi\Big(\big((j,i),(j',i')\big) \notin T\Big) \ . \tag{B.35}$$

The term $\int \mathcal{D}^*(\mathrm{d}\psi)\, e^{-V^*(\mathbf{t})}$ in Eq.(B.32) is the determinant of the $(n-s+1)\times(n-s+1)$ matrix $G^T(\mathbf{t})$ (here $2n = \sum_{i=1}^s |P_i|$), with elements $G^T_{f,f'}(\mathbf{t}) = t_{j(f),j(f')} G_{f,f'}$, where $f, f' \notin T$, $j(f) \in \{1,\dots,s\}$ and $t_{j,j'} = t_{n'((j,j'))} \dots t_{n((j,j'))}$:

$$\int \mathcal{D}^*(\mathrm{d}\psi)\, e^{-V^*(\mathbf{t})} = \det G^T(\mathbf{t}) \ . \tag{B.36}$$

Plugging this into Eq.(B.32) finally gives Eq.(5.3). In order to complete the proof of the claims following Eq.(5.3) we are left with proving the following Lemma.

Lemma B.1 *$dP_T(\mathbf{t})$ is a normalized, positive and σ–additive measure on the natural σ–algebra of $[0,1]^{s-1}$. Moreover there exists a set of unit vectors $\mathbf{u}_j \in \mathbb{R}^s$, $j = 1,\dots,s$, such that $t_{j,j'} = \mathbf{u}_j \cdot \mathbf{u}_{j'}$.*

Proof. Let us denote by b_k the number of lines $\ell \in T$ exiting from the points $x(j,i)$, $j \in X_k$, such that $\ell \sim X_k$. Let us consider the integral

$$\sum_{\substack{X_2,\dots,X_{s-1} \\ \text{compatible with } T}} \int_0^1 \mathrm{d}t_1 \dots \int_0^1 \mathrm{d}t_{s-1} \prod_{\ell\in T} \big(t_{n'(\ell)} \dots t_{n(\ell)-1}\big) = 1 \ , \tag{B.37}$$

and note that, by construction, the parameter t_k inside the integral in the l.h.s. appears at the power $b_k - 1$. In fact any line intersecting ∂X_k contributes by a factor t_k, except for the line connecting X_k with the point in $X_{k+1}\backslash X_k$. See Fig. 8.

Then

$$\prod_{\ell\in T} \big(t_{n'(\ell)} \dots t_{n(\ell)-1}\big) = \prod_{k=1}^{s-1} t_k^{b_k - 1} \ , \tag{B.38}$$

and in Eq.(B.37) the $s-1$ integrations are independent. One has:

$$\int_0^1 \mathrm{d}t_1 \dots \int_0^1 \mathrm{d}t_{s-1} \prod_{\ell\in T} \big(t_{n'(\ell)} \dots t_{n(\ell)-1}\big) = \prod_{k=1}^{s-1} \left(\int_0^1 \mathrm{d}t_k \, t_k^{b_k-1}\right) = \prod_{k=1}^{s-1} \frac{1}{b_k} \ , \tag{B.39}$$

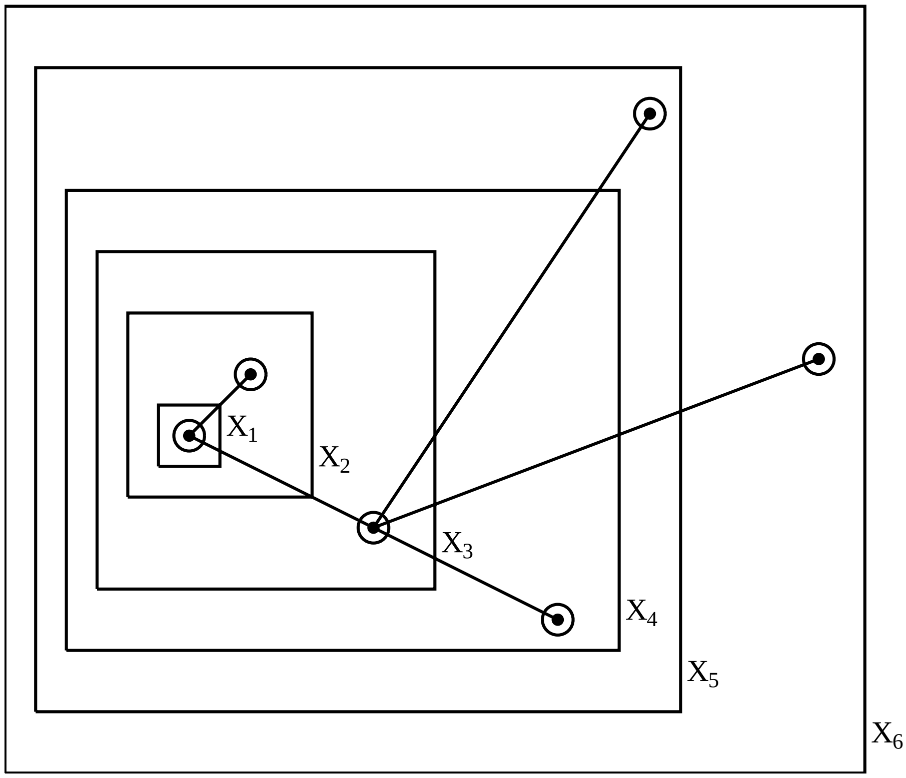

FIG. 8. The sets $X_1, \ldots, X_6$, the anchored tree T and the lines $\ell_1, \ldots, \ell_5$ belonging to T. In the example, the coefficients $b_1, \ldots, b_5$ are respectively equal to: $2, 1, 3, 2, 1$.

which is well defined, since $b_k \geq 1$. Moreover we can write:

$$\sum_{\substack{X_2,\ldots,X_{s-1} \\ \text{compatible with } T}} = \sum^{*}_{\substack{X_2 \\ X_1 \text{ fixed}}} \sum^{*}_{\substack{X_3 \\ X_1,X_2 \text{ fixed}}} \cdots \sum^{*}_{\substack{X_{s-1} \\ X_1,\ldots,X_{s-2} \text{ fixed}}} , \tag{B.40}$$

where the $*$'s over the sums remind that all the summations are subject to the constraint that the subsets $X_1, X_2, \ldots, X_s$ must be compatible with the structure of T. Now, the number of terms in the sum over X_k, once that T and the sets $X_1, \ldots, X_{k-1}$ are fixed, is exactly b_{k-1}, so that

$$\sum_{\substack{X_2,\ldots,X_{s-1} \\ \text{compatible with } T}} 1 = b_1 \ldots b_{s-2} , \tag{B.41}$$

and, recalling that $b_{s-1} = 1$,

$$\sum_{\substack{X_2,\ldots,X_{s-1} \\ \text{compatible with } T}} \int_0^1 \mathrm{d}t_1 \ldots \int_0^1 \mathrm{d}t_{s-1} \prod_{\ell \in T} \left(t_{n'(\ell)} \ldots t_{n(\ell)-1}\right) = \prod_{k=1}^{s-2} \frac{b_k}{b_k} , \tag{B.42}$$

yielding to $\int dP_T(\mathbf{t}) = 1$. The positivity and σ–addivity of $dP_T(\mathbf{t})$ is obvious by definition.

We are left with proving that, for any given sequence of subsets $X_1 \subset X_2 \subset \cdots \subset X_s$ compatible with T, we can find unit vectors $\mathbf{u}_j \in \mathbb{R}^s$ such that $t_{j,j'} = \mathbf{u}_j \cdot \mathbf{u}_{j'}$. With no losso of generality, we can assume that $X_1 = \{1\}$, $X_2 = \{1,2\}$, ..., $X_{s-1} = \{1, \ldots, s-1\}$. We introduce a family of unit vectors in $\mathbb{R}^s$ defined as follows:

$$\begin{cases} \mathbf{u}_1 = \mathbf{v}_1 \;, \\ \mathbf{u}_j = t_{j-1}\mathbf{u}_{j-1} + \mathbf{v}_j\sqrt{1-t_{j-1}^2} \;, \quad j = 2, \ldots, s \;, \end{cases} \tag{B.43}$$

where $\{\mathbf{v}_i\}_{i=1}^s$ is an orthonormal basis. From this equation, as desired:

$$\mathbf{u}_j \cdot \mathbf{u}_{j'} = t_j \ldots t_{j'-1} \;. \tag{B.44}$$

This concludes the proof of the determinant formula. ∎

Appendix C: Symmetry properties

In this Appendix we prove Lemma 6.2. The key remark underlying the nice properties stated in the Lemma is that both the Gaussian integration $P_M(d\psi)$ and the interaction $\mathcal{V}(\psi)$ are invariant under the action of a number of remarkable symmetry transformations, which are also preserved by the multiscale integration. In the following, we denote by $\sigma_1, \sigma_2, \sigma_3$ the standard Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;, \qquad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;. \tag{C.1}$$

Moreover, in order to avoid confusion with the Pauli matrices, we shall use the symbol τ as spin index rather than σ. As a preliminary result, let us start by listing all the transformation properties under which our theory is invariant.

Lemma C.1 *For any choice of M, β, L, the fermionic Gaussian integration $P(d\psi)$ and the interaction $\mathcal{V}(\psi)$ are separately invariant under the following transformations:*

(1) Spin flip: $\hat\psi^{\varepsilon}_{\mathbf{k},\tau} \leftrightarrow \hat\psi^{\varepsilon}_{\mathbf{k},-\tau}$;

(2) Global $U(1)$: $\hat\psi^{\varepsilon}_{\mathbf{k},\tau} \to e^{i\varepsilon\alpha_\tau}\hat\psi^{\varepsilon}_{\mathbf{k},\tau}$, with $\alpha_\tau \in \mathbb{R}/2\pi\mathbb{Z}$ independent of $\mathbf{k}$;

(3) Spin $SO(2)$: $\begin{pmatrix} \hat\psi^{\varepsilon}_{\mathbf{k},\uparrow,\cdot} \\ \hat\psi^{\varepsilon}_{\mathbf{k},\downarrow,\cdot} \end{pmatrix} \to e^{-i\theta\sigma_2}\begin{pmatrix} \hat\psi^{\varepsilon}_{\mathbf{k},\uparrow,\cdot} \\ \hat\psi^{\varepsilon}_{\mathbf{k},\downarrow,\cdot} \end{pmatrix}$, with $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ independent of $\mathbf{k}$;

(4) Discrete rotations: $\hat\psi^{-}_{\mathbf{k},\tau} \to e^{-i\vec{k}(\vec\delta_3-\vec\delta_1)\frac{\sigma_3}{2}}\hat\psi^{-}_{T\mathbf{k},\tau}$ and $\hat\psi^{+}_{\mathbf{k},\tau} \to \hat\psi^{+}_{T\mathbf{k},\tau}e^{i\vec{k}(\vec\delta_3-\vec\delta_1)\frac{\sigma_3}{2}}$, with $T\mathbf{k} = (k_0, e^{-i\frac{2\pi}{3}\sigma_2}\vec{k})$;

(5) *Complex conjugation:* $\hat\psi^{\varepsilon}_{\mathbf{k},\tau} \to \hat\psi^{\varepsilon}_{-\mathbf{k},\tau}$ *and* $c \to c^*$*, where* c *is generic constant appearing in* $P(d\Psi)$ *or in* $\mathcal{V}(\Psi)$*;*

(6.a) *Horizontal reflections:* $\hat\psi^{-}_{\mathbf{k},\tau} \to \sigma_1 \hat\psi^{-}_{R_h\mathbf{k},\tau}$ *and* $\hat\psi^{+}_{\mathbf{k},\tau} \to \hat\psi^{+}_{R_h\mathbf{k},\tau}\sigma_1$*, with* $R_h\mathbf{k} = (k_0, -k_1, k_2)$*;*

(6.b) *Vertical reflections:* $\hat\psi^{\varepsilon}_{\mathbf{k},\tau} \to \hat\psi^{\varepsilon}_{R_v\mathbf{k},\tau}$*, with* $R_v\mathbf{k} = (k_0, k_1, -k_2)$*;*

(7) *Particle-hole:* $\hat\psi^{-}_{\mathbf{k},\tau} \to i\hat\psi^{+,T}_{P\mathbf{k},\tau}$*,* $\hat\psi^{+}_{\mathbf{k},\tau} \to i\hat\psi^{-,T}_{P\mathbf{k},\tau}$*, with* $P\mathbf{k} = (k_0, -k_1, -k_2)$*;*

(8) *Inversion:* $\hat\psi^{-}_{\mathbf{k},\tau} \to -i\sigma_3 \hat\psi^{-}_{I\mathbf{k},\tau}$*,* $\hat\psi^{+}_{\mathbf{k},\tau} \to -i\hat\psi^{+}_{I\mathbf{k},\tau}\sigma_3$*, with* $I\mathbf{k} = (-k_0, k_1, k_2)$*.*

Proof of Lemma C.1. Let us first recall the definitions of $P_M(d\psi)$ and $\mathcal{V}(\psi)$:

$$P_M(d\psi) = \frac{1}{\mathcal{N}} \prod_{\mathbf{k}\in\mathcal{B}^{(M)}_{\beta,L}}^{\tau=\uparrow\downarrow} d\hat\psi^{+}_{\mathbf{k},\tau,1} d\hat\psi^{-}_{\mathbf{k},\tau,1} d\hat\psi^{+}_{\mathbf{k},\tau,2} d\hat\psi^{-}_{\mathbf{k},\tau,2}\, e^{-\frac{1}{\beta|\Lambda|}\hat\psi^{+}_{\mathbf{k},\tau}\hat g_{\mathbf{k}}^{-1}\hat\psi^{-}_{\mathbf{k},\tau}} \;,$$

$$\mathcal{V}(\psi) = \frac{U}{(\beta|\Lambda|)^3} \sum_{\mathbf{k},\mathbf{k}',\mathbf{p}}^{\alpha=\pm} \big(\hat\psi^{+}_{\mathbf{k}+\mathbf{p},\uparrow} n_\alpha \hat\psi^{-}_{\mathbf{k},\uparrow}\big)\big(\hat\psi^{+}_{\mathbf{k}'-\mathbf{p},\downarrow} n_\alpha \hat\psi^{-}_{\mathbf{k}',\downarrow}\big) \;,$$

where $\mathcal{N}$ is the normalization constant in Eq.(4.8) and $n_\pm = (1 \pm \sigma_3)/2$. Given these definitions and recalling the definition of $\hat g_{\mathbf{k}}^{-1}$, we see that the invariance of $P_M(d\psi)$ and $\mathcal{V}(\psi)$ are equivalent to the invariance of the following combinations:

$$(*) := \sum_{\mathbf{k},\tau} \hat\psi^{+}_{\mathbf{k},\tau} \begin{pmatrix} ik_0 & v_0\Omega^*(\vec k) \\ v_0\Omega(\vec k) & ik_0 \end{pmatrix} \hat\psi^{-}_{\mathbf{k},\tau} \;, \tag{C.2}$$

$$(**) := \sum_{\mathbf{k},\mathbf{k}',\mathbf{p}}^{\alpha=\pm} \big(\hat\psi^{+}_{\mathbf{k}+\mathbf{p},\uparrow} n_\alpha \hat\psi^{-}_{\mathbf{k},\uparrow}\big)\big(\hat\psi^{+}_{\mathbf{k}'-\mathbf{p},\downarrow} n_\alpha \hat\psi^{-}_{\mathbf{k}',\downarrow}\big) \;. \tag{C.3}$$

Now, the invariance of $(*)$ and $(**)$ under symmetries (1) and (2) is completely apparent. Let us check one by one the invariance under the other symmetries.

Symmetry (3). Note that $(*)$ and $(**)$ can be rewritten as

$$(*) = \sum_{\mathbf{k},\rho,\rho'} (\hat g_{\mathbf{k}}^{-1})_{\rho\rho'} \sum_{\tau} \hat\psi^{+}_{\mathbf{k},\tau,\rho} \hat\psi^{-}_{\mathbf{k}',\tau,\rho'} \;, \tag{C.4}$$

$$(**) = \frac{1}{2} \sum_{\substack{\mathbf{k},\mathbf{k}',\mathbf{p}\\ \alpha,\rho}} (n_\alpha)_{\rho\rho} \sum_{\tau,\tau'} \big(\hat\psi^{+}_{\mathbf{k}+\mathbf{p},\tau,\rho} \hat\psi^{-}_{\mathbf{k},\tau,\rho}\big)\big(\hat\psi^{+}_{\mathbf{k}'-\mathbf{p},\tau',\rho} \hat\psi^{-}_{\mathbf{k}',\tau',\rho}\big) \tag{C.5}$$

Invariance of these expression under symmetry (3) follows from the invariance of the combination $\sum_\tau \hat\psi^{+}_{\mathbf{k},\tau,\rho}\hat\psi^{-}_{\mathbf{k}',\tau,\rho'}$: in fact, denoting by R^θ the matrix $e^{-i\theta\sigma_2}$, we see that under (3)

$$\sum_\tau \hat\psi^{+}_{\mathbf{k},\tau,\rho}\hat\psi^{-}_{\mathbf{k}',\tau,\rho'} \to \sum_{\tau,\tau_1,\tau_2} \hat\psi^{+}_{\mathbf{k},\tau_1,\rho} R^{\theta,T}_{\tau_1,\tau} R^{\theta}_{\tau,\tau_2} \hat\psi^{-}_{\mathbf{k}',\tau_2,\rho'} \;. \tag{C.6}$$

which is invariant, simply by the fact that R^θ is orthogonal.

Symmetry (4). Under the action of (4),

$$(*) \to \sum_{\mathbf{k},\tau} \hat{\psi}^+_{T\mathbf{k},\tau} e^{i\vec{k}(\vec{\delta}_3-\vec{\delta}_1)\frac{\sigma_3}{2}} \begin{pmatrix} ik_0 & v_0\Omega^*(\vec{k}) \\ v_0\Omega(\vec{k}) & ik_0 \end{pmatrix} e^{-i\vec{k}(\vec{\delta}_3-\vec{\delta}_1)\frac{\sigma_3}{2}} \hat{\psi}^-_{T\mathbf{k},\tau} , \tag{C.7}$$

with $T\mathbf{k} = (k_0, R^{\frac{2\pi}{3}}\vec{k})$. The r.h.s. of Eq.(C.7) can be rewritten as

$$\sum_{\mathbf{k},\tau} \hat{\psi}^+_{T\mathbf{k},\tau} \begin{pmatrix} ik_0 & v_0\Omega^*(\vec{k})e^{i\vec{k}(\vec{\delta}_3-\vec{\delta}_1)} \\ v_0\Omega(\vec{k})e^{-i\vec{k}(\vec{\delta}_3-\vec{\delta}_1)} & ik_0 \end{pmatrix} \hat{\psi}^-_{T\mathbf{k},\tau} , \tag{C.8}$$

which is the same as $(*)$, as it follows by the remark that $\Omega(\vec{k})e^{-i\vec{k}(\vec{\delta}_3-\vec{\delta}_1)} = \Omega(R^{\frac{2\pi}{3}}\vec{k})$. Regarding the interaction term, note that

$$\left[e^{i(\vec{k}+\vec{p})(\vec{\delta}_3-\vec{\delta}_1)\frac{\sigma_3}{2}} n_\alpha e^{-i\vec{k}(\vec{\delta}_3-\vec{\delta}_1)\frac{\sigma_3}{2}}\right] = e^{i\alpha\frac{\vec{p}}{2}(\vec{\delta}_3-\vec{\delta}_1)} n_\alpha , \tag{C.9}$$

which immediately shows that $(**)$ is invariant under (4).

Symmetry (5). The term $(*)$ is changed under (5) as:

$$(*) \to \sum_{\mathbf{k},\tau} \hat{\psi}^+_{-\mathbf{k},\tau} \begin{pmatrix} -ik_0 & v_0\Omega(\vec{k}) \\ v_0\Omega^*(\vec{k}) & -ik_0 \end{pmatrix} \hat{\psi}^-_{-\mathbf{k},\tau} , \tag{C.10}$$

which is the same as $(*)$ (simply because $\Omega^*(\vec{k}) = \Omega(-\vec{k})$). Similarly, the term $(**)$ is changed under (5) as:

$$(**) \to \sum_{\mathbf{k},\mathbf{k}',\mathbf{p}}^{\alpha=\pm} \left(\hat{\psi}^+_{-\mathbf{k}-\mathbf{p},\uparrow} n_\alpha \hat{\psi}^-_{-\mathbf{k},\uparrow}\right)\left(\hat{\psi}^+_{-\mathbf{k}'+\mathbf{p},\downarrow} n_\alpha \hat{\psi}^-_{-\mathbf{k}',\downarrow}\right) , \tag{C.11}$$

which is the same as $(**)$.

Symmetry (6.a). The term $(*)$ is changed under (6.a) as:

$$\begin{aligned}(*) &\to \sum_{\mathbf{k},\tau} \hat{\psi}^+_{R_h\mathbf{k},\tau} \sigma_1 \begin{pmatrix} ik_0 & v_0\Omega^*(\vec{k}) \\ v_0\Omega(\vec{k}) & ik_0 \end{pmatrix} \sigma_1 \hat{\psi}^-_{R_h\mathbf{k},\tau} = \\ &= \sum_{\mathbf{k},\tau} \hat{\psi}^+_{R_h\mathbf{k},\tau} \begin{pmatrix} ik_0 & v_0\Omega(\vec{k}) \\ v_0\Omega^*(\vec{k}) & ik_0 \end{pmatrix} \hat{\psi}^-_{R_h\mathbf{k},\tau} .\end{aligned} \tag{C.12}$$

Using the fact that $\Omega^*(\vec{k}) = \Omega((-k_1, k_2))$, we see that this term is invariant under (6.a). Regarding the interaction term, if we note that $\sigma_1 n_\alpha \sigma_1 = n_{-\alpha}$, we immediately see that it is invariant under (6.a).

Symmetry (6.b). Invariance of the term $(*)$ follows from the remark that $\Omega^*(\vec{k}) = \Omega((k_1, -k_2))$; invariance of $(**)$ is trivial.

Symmetry (7). The term $(*)$ is changed under (7) as:

$$(*) \to \sum_{\mathbf{k},\tau} \hat{\psi}^+_{P\mathbf{k},\tau} \begin{pmatrix} ik_0 & v_0\Omega^*(\vec{k}) \\ v_0\Omega(\vec{k}) & ik_0 \end{pmatrix}^T \hat{\psi}^-_{P\mathbf{k},\tau} \ , \tag{C.13}$$

which is the same as $(*)$ (simply because $\Omega^*(\vec{k}) = \Omega(-\vec{k})$). Similarly, using the fact that $n_\alpha^T = n_\alpha$, the term $(**)$ is changed under (7) as:

$$(**) \to \sum_{\mathbf{k},\mathbf{k}',\mathbf{p}}^{\alpha=\pm} \big(\hat{\psi}^+_{P\mathbf{k},\uparrow} n_\alpha \hat{\psi}^-_{P(\mathbf{k}+\mathbf{p}),\uparrow}\big)\big(\hat{\psi}^+_{P\mathbf{k}',\downarrow} n_\alpha \hat{\psi}^-_{P(\mathbf{k}'-\mathbf{p}),\downarrow}\big) \ , \tag{C.14}$$

which is the same as $(**)$.

Symmetry (8). The term $(*)$ is changed under (8) as:

$$\begin{aligned}(*) \to -\sum_{\mathbf{k},\tau} \hat{\psi}^+_{I\mathbf{k},\tau}\sigma_3 \begin{pmatrix} ik_0 & v_0\Omega^*(\vec{k}) \\ v_0\Omega(\vec{k}) & ik_0 \end{pmatrix} \sigma_3 \hat{\psi}^-_{I\mathbf{k},\tau} = \\ = \sum_{\mathbf{k},\tau} \hat{\psi}^+_{I\mathbf{k},\tau}\sigma_3 \begin{pmatrix} -ik_0 & v_0\Omega^*(\vec{k}) \\ v_0\Omega(\vec{k}) & -ik_0 \end{pmatrix} \sigma_3 \hat{\psi}^-_{I\mathbf{k},\tau} \ ,\end{aligned} \tag{C.15}$$

which is the same as $(*)$. Moreover, note that $\sigma_3 n_\alpha \sigma_3 = n_\alpha$, so that the term $(**)$ is obviously invariant under (8). This concludes the proof of Lemma C.1. ■

Now, we are ready to prove Lemma 6.2.

Proof of Lemma 6.2. The key remark is that, in addition to the fact that $P_M(d\psi)$ is invariant under the transformations (1)–(8), the ultraviolet and infrared integrations $P(d\psi^{(u.v.)})$ and $P(d\psi^{(i.r.)})$ are separately invariant under the analogous transformations of the fields $\psi^{(u.v.)}$ and $\psi^{(i.r.)}$. Therefore, the effective potential $\mathcal{V}_0(\psi^{(i.r.)})$ is also invariant under the same transformations. Let us restrict our attention to the quadratic contribution to $\mathcal{V}_0$, i.e.,

$$\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k},\tau} \hat{\psi}^{(i.r.)+}_{\mathbf{k},\tau} \hat{W}_2(\mathbf{k}) \hat{\psi}^{(i.r.)-}_{\mathbf{k},\tau} \ , \tag{C.16}$$

which, as we said, must be invariant under the symmetries (1)–(8) listed in Lemma C.1. The very fact that we can write the quadratic term in the form Eq.(C.16), with $\hat{W}_2(\mathbf{k})$ independent of the spin index τ, is a consequence of symmetries (1)–(3). It is straightforward to check that Eq.(6.11) follows from the invariance of Eq.(C.16) under (4)–(8).

So, we are left with proving Eq.(6.12). Let us start by showing that $\hat W(\mathbf{p}_F^\omega)=0$. Writing $\hat W(\mathbf{p}_F^\omega)=a_0^\omega I+a_1^\omega\sigma_1+a_2^\omega\sigma_2+a_3^\omega\sigma_3$ and using the fact that, by Eq.(6.11), $\hat W(\mathbf{p}_F^\omega)=\sigma_1\hat W(\mathbf{p}_F^\omega)\sigma_1=-\sigma_3\hat W(\mathbf{p}_F^\omega)\sigma_3$, we immediately see that $a_0^\omega=a_2^\omega=a_3^\omega=0$. Moreover, using the fact that (still by Eq.(6.11)) $\hat W(\mathbf{p}_F^\omega)=e^{i\omega\frac{2\pi}{3}\frac{\sigma_3}{2}}\hat W(\mathbf{p}_F^\omega)e^{-i\omega\frac{2\pi}{3}\frac{\sigma_3}{2}}$, we get $a_1^\omega\sigma_1=a_1^\omega(\cos(2\pi/3)\sigma_1-\omega\sin(2\pi/3)\sigma_2)$, which implies $a_1^\omega=0$.

Let us now look at $A_\mu^\omega:=\partial_{k_\mu}\hat W(\mathbf{p}_F^\omega)$. Writing $A_0^\omega=b_0^\omega I+b_1^\omega\sigma_1+b_2^\omega\sigma_2+b_3^\omega\sigma_3$ and using that, by Eq.(6.11), $A_0^\omega=-(A_0^{-\omega})^*=A_0^{-\omega}=\sigma_1A_0^\omega\sigma_1=\sigma_3A_0^\omega\sigma_3$, we see that $A_0^\omega=-iz_0I$, with z_0 real and independent of ω. In a similar way, writing $A_1^\omega=c_0^\omega I+c_1^\omega\sigma_1+c_2^\omega\sigma_2+c_3^\omega\sigma_3$ and $A_2^\omega=d_0^\omega I+d_1^\omega\sigma_1+d_2^\omega\sigma_2+d_3^\omega\sigma_3$, using the fact that $A_1^\omega=-(A_1^{-\omega})^*=A_1^{-\omega}=-\sigma_1A_1^\omega\sigma_1=-(A_1^{-\omega})^T=-\sigma_3A_1^\omega\sigma_3$ and $A_2^\omega=-(A_2^{-\omega})^*=-A_2^{-\omega}=\sigma_1A_2^\omega\sigma_1=-(A_2^{-\omega})^T=-\sigma_3A_2^\omega\sigma_3$, we see that $A_1^\omega=c_2\sigma_2$ and $A_2^\omega=\omega d_1\sigma_1$, with c_2 and d_1 real and independent of ω. Finally, using the fact that, again by Eq.(6.11), $A_i^\omega=\sum_{j=1}^2 e^{i\omega\frac{2\pi}{3}\sigma_3}(R^{\frac{2\pi}{3}})_{ij}A_j^\omega$, we get $c_2=d_1=:-\delta_0$ that, if combined with our previous findings, implies Eq.(6.12). This concludes the proof of Lemma 6.2. ∎

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[48] Of course, there is some arbitrariness in the definition of $\hat{a}^{\pm}_{\vec{k},\sigma}, \hat{b}^{\pm}_{\vec{k},\sigma}$: we could change the two sets of operators by multiplying them by two $\vec{k}$-dependent phase factors, $e^{\mp i\vec{k}\vec{A}}, e^{\mp i\vec{k}\vec{B}}$, with $\vec{A}, \vec{B}$ two arbitrary constant vectors, and get a completely equivalent theory in momentum space. The freedom in the choice of these phase factors corresponds to the freedom in the choice of the origins of the two sublattices Λ_A, Λ_B; this symmetry is sometimes referred to as Berry-gauge invariance and it can be thought as a local gauge symmetry in momentum space. Note that, by changing the Berry phase, the boundary conditions on $\hat{a}^{\pm}_{\vec{k},\sigma}, \hat{b}^{\pm}_{\vec{k},\sigma}$ at the boundaries of the first Brillouin zone change; our explicit choice of the Berry phase has the "advantage" of making $\hat{a}^{\pm}_{\vec{k},\sigma}, \hat{b}^{\pm}_{\vec{k},\sigma}$ periodic over Λ^* rather than quasi-periodic.