Evolution of correlation functions in the hard sphere dynamics

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The series expansion for the evolution of the correlation functions of a finite system of hard spheres is derived from direct integration of the solution of the Liouville equation, with minimal regularity assumptions on the density of the initial measure. The usual BBGKY hierarchy of equations is then recovered. A graphical language based on the notion of collision history originally introduced by Spohn is developed, as a useful tool for the description of the expansion and of the elimination of degrees of freedom.

1. INTRODUCTION

In his famous derivation of the Boltzmann equation [8], O. E. Lanford makes use of a series expansion for the time–evolved correlation functions of a classical finite system of hard spheres in a box. This expresses the $n-$point correlation function at time t as a sum of integral terms involving all the higher order correlation functions at time zero. The expansion is derived, though not rigorously, from iteration of the BBGKY hierarchy of integro–differential equations, and is considered as a "series solution" of its Cauchy problem. A rigorous validation of the hierarchy (formally deduced first by Cercignani in [2]) and of the series has been given years later by H. Spohn in an unpublished note [13], and by R. Illner and M. Pulvirenti in [6] (see also the book [3]), using different methods.

In both the previous papers an assumption on the initial measure is made to derive the BBGKY hierarchy, that is the continuity along trajectories of the hard spheres flow. However, there is no physical reason to expect such a regularity property to hold, and it is worthwhile to notice that the final series expansion makes perfectly sense without assuming it. In fact, Spohn observes at the end of his note, by a density argument, that the expansion can be extended to a more general class of measures having no continuity properties. On the other hand, the interpretation of the BBGKY hierarchy as a family of partial differential equations is not at all easy, nor standard in any case, since it relies on the nontrivial properties of the operator T_t of the hard sphere dynamics. Hence, the series solution concept appears to be more appropriate for the description of the dynamics in terms of probability distributions, and one wonders whether it is possible to derive it without going through the usual hierarchy. The present paper is devoted to a derivation of the series expansion for the correlation functions, which is *not* based on the iteration of the BBGKY equations, and never requires continuity along trajectories. We rather construct a method of direct integration of the solution of the Liouville equation, that allows to establish the validity of the expansion in a sense even *stronger* than those obtained in the existing literature: the result holds for all times in a fixed full measure invariant subset of the phase space, exactly as it happens for the existence of the dynamics of the underlying system of particles. The hierarchy of integro–differential equations is then recovered by resummation of the series, *without* additional assumptions on the initial measure, thus strengthening an analogous result in [6].

Other rigorous discussions on the hard sphere dynamics and the associated BBGKY hierarchy are given in [15], [5], [10], [11] and [4].

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*This work is a revised version of part of the author's PhD thesis [12], written at the University of Rome "La Sapienza" under the direction of G. Gallavotti.

Let us recall the derivation of Lanford and state our main result in an informal way. Consider the vector of correlation functions $\boldsymbol{\rho} = \{\rho_n\}_{n\geq 1}$, where ρ_n is defined over the phase space of n hard spheres of mass m and diameter $a > 0$ in a box Λ. A point in this space is an $n-$tuple $(z_1, \cdots, z_n), z_j = (q_j, p_j)$, specifying position and momentum of the n particles. If N is the total number of particles, we set $\rho_n = 0$ for $n > N$. Then the BBGKY hierarchy for the evolution of ρ can be written

$$\frac{\partial}{\partial t}\boldsymbol{\rho}(t) = H\boldsymbol{\rho}(t) + Q\boldsymbol{\rho}(t) \ , \tag{1.1}$$

where

$$\left(H\boldsymbol{\rho}\right)_n(z_1, \cdots, z_n, t) \equiv \Big\{H_n, \rho_n\Big\}(z_1, \cdots, z_n, t) \tag{1.2}$$

is the $n-$particles Liouville operator acting on ρ_n (including the effects of elastic collisions) and the collision operator is defined by

$$\left(Q\boldsymbol{\rho}\right)_n(z_1, \cdots, z_n, t) = a^2 \sum_{j=1}^{n} \int d\hat{p}\ d\omega\ \omega \cdot \left(\frac{\hat{p} - p_j}{m}\right) \rho_{n+1}(z_1, \cdots, z_n, q_j + a\omega, \hat{p}, t) \ . \tag{1.3}$$

Here $\hat{p}$ is integrated over all $\mathbb{R}^3$, and ω runs over the unit sphere.

If $t \longrightarrow T_t(z_1, \cdots, z_n)$ is the flow of the dynamics, define the translation along trajectories of a vector of functions $\boldsymbol{f} = \{f_n\}_{n\geq 1}$ as

$$\left(S(t)\boldsymbol{f}\right)_n(z_1, \cdots, z_n) = f_n(T_{-t}(z_1, \cdots, z_n)) \ . \tag{1.4}$$

Then, integration and iteration of Equation (1.1) leads to the formal solution

$$\boldsymbol{\rho}(t) = S(t)\boldsymbol{\rho}(0) + \sum_{m=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m S(t - t_1) Q S(t_1 - t_2) \cdots Q S(t_m)\boldsymbol{\rho}(0) \ . \tag{1.5}$$

In this paper *we analyze in detail the structure of Eq.* (1.5) *and prove that it holds, for all times in a full measure subset of the phase space, for any absolutely continuous measure with density symmetric in the particle labels, and bounded by an equilibrium–like distribution.* The hierarchy (1.1) can be obtained then, in the mild sense of [6], by taking the derivative. No assumption of continuity is needed even for this last operation. We also allow the total number of particles N to be non fixed by the initial measure. The boundedness requirement is stronger than the necessary, and it is the same used by Lanford to control the convergence of the series in the Boltzmann–Grad limit. Here it is made to control easily through all the steps the integrals over momenta of the type (1.3), (1.5).

The main interest of the discussion is the method of the proof. For $n = N$ Eq. (1.5) reduces to the evolution of the density function, that is the solution of the Liouville equation:

$$\rho_N(z_1, \cdots, z_N, t) = \rho_N(T_{-t}(z_1, \cdots, z_N), 0) \ . \tag{1.6}$$

It is desirable that we can construct the series expansion for the ρ_n from *direct integration* of (1.6) over all the phase space of $N - n$ particles compatible with a fixed state $(z_1, \cdots, z_n)$. We show that in fact this can be done by eliminating the degrees of freedom particle by particle. To achieve the integration of the single particle state, it is important to understand the structure of the right hand side in (1.5). This has been widely studied since the work of Lanford [8], see for instance [7] or [14]. It results that the integrand function in the generic term of the formula, depends on the states assumed by certain clusters of particles following a fictitious evolution: this is constructed from the state $(z_1, \cdots, z_n)$ at time t, by suitably adding more and more particles as the time flows backwards. Following [14], we shall call *collision history* such an evolution.

The collision histories can be represented graphically in terms of special binary tree graphs. Therefore, a graphical picture of the series expansion (1.5) is obtained. This representation is our basic tool. In fact, it turns out that the integration of a particle state itself can be translated in graphical language, through appropriate *operations over tree graphs*. The graphical rules corresponding to the elimination of a particle state, clarify how the various terms of the expansion for ρ_n emerge from those for ρ_{n+1}, thus considerably simplifying the presentation of the proof. The analytical operations corresponding to these rules, are nothing but a suitable partitioning of the integration domain, and convenient representation (change of variables) of the subsets of the partition. Nevertheless, in order to establish the graphical rules, it is also essential to prove that some classes of collision histories give a net null contribution to the integration of the particle state: this is done again with the help of the tree graphs, by showing explicit one by one *cancellations* among the collision histories of these classes.

The paper is organised as follows. In Section 2 we define the model, we introduce our notations and state our assumptions on the initial measure. In Section 3 we introduce the concept of collision history, as well as the graphical rules for its representation, and explain how to represent formula (1.5) in terms of the tree graphs. In Section 4 we present our main results, while in Section 5 we discuss the proof of the main theorem, establishing the above mentioned graphical integration rules, and applying them to the generic inductive step. In Section 6 we present the conclusions. A discussion on the hard sphere dynamics is deferred to the Appendix.

2. THE HARD SPHERE SYSTEM

In this section we set model and notations, which we inherit essentially from [13], and state some preliminary result on the hard sphere dynamics (Section 2 A). In Section 2 B we introduce the class of measures on which we will work.

A. Model and notations

Let us consider a system of N hard spheres of unit mass and of diameter $a > 0$ moving in a box $\Lambda \subset \mathbb{R}^3$. Λ is bounded open and has a piecewise smooth elastically reflecting boundary $\partial\Lambda$. We will denote $z_i = (q_i, p_i) \in \Lambda \times \mathbb{R}^3$ the configuration of the i–th particle, $i = 1, \cdots, N$. For groups of particles we will use the short notations $\boldsymbol{z}_n = z_1, \cdots, z_n$, $\boldsymbol{z}_{n,j} = z_{n+1}, \cdots, z_{n+j}$. When there is no risk of confusion, we will simply call "particle i" a particle whose configuration is labelled by an index i.

We introduce the n particle phase space, $n = 1, \cdots, N$,

$$\Gamma_n = \left\{ \boldsymbol{z}_n \in (\Lambda \times \mathbb{R}^3)^n \;\middle|\; |q_i - q| \geq a/2 \text{ for every } q \in \partial\Lambda \text{ and } |q_i - q_j| \geq a,\ i \neq j \right\} . \tag{2.1}$$

A state of the system is given by a point in the full phase space Γ_N.

The equations of motion for the n particle system are defined as follows. Between collisions each particle moves on a straight line maintaining unchanged its velocity. In a collision of two hard spheres at positions q_i, q_j with $\omega = (q_i - q_j)/|q_i - q_j| = (q_i - q_j)/a \in S^2$ and with incoming momenta p_i', p_j' (that means $(p_i' - p_j') \cdot \omega < 0$), we have instantaneous transformation to the outgoing momenta p_i, p_j (with $(p_i - p_j) \cdot \omega > 0$) given by

$$\begin{aligned} p_i &= p_i' - \omega[\omega \cdot (p_i' - p_j')] , \\ p_j &= p_j' + \omega[\omega \cdot (p_i' - p_j')] . \end{aligned} \tag{2.2}$$

Finally, in a collision of a particle with momentum p_i' with the wall $\partial\Lambda$ at a point q which is regular (there is only one point of contact between the wall and the sphere, and the normal to the surface at that point is well defined), we have instantaneous transformation to the reflected outgoing momentum p_i given by

$$p_i = p_i' - 2n(q)(n(q) \cdot p_i') , \tag{2.3}$$

where $n(q)$ is the inner unit vector normal to $\partial\Lambda$ in q. It is easy to see that the collision transformations (2.2) and (2.3) are invertible and preserve Lebesgue measure on $\mathbb{R}^3 \times \mathbb{R}^3$ and $\mathbb{R}^3$ respectively.

The above prescription for the equations of motion does not cover all possible situations, e.g. triple collisions and collisions with corner points of the walls are excluded. Nevertheless, we have the following basic result:

Proposition 1. *[Existence of the dynamics (I)] In Γ_n there is a subset Γ_n^*, whose complement is a Lebesgue null set, such that for any $\boldsymbol{z}_n \in \Gamma_n^*$ there is a unique mapping*

$$t \mapsto T_t^{(n)} \boldsymbol{z}_n \in \Gamma_n^* \qquad t \in \mathbb{R} \tag{2.4}$$

which is a solution of the equations of motion having $T_0^{(n)} \boldsymbol{z}_n = \boldsymbol{z}_n$. Moreover, the shifts along trajectories $\boldsymbol{z}_n \mapsto T_t^{(n)} \boldsymbol{z}_n$ define a one–parameter group of Borel maps on Γ_n which leave Lebesgue measure invariant.

This has been stated and proved by Alexander in [1], pages 18–29, and it holds under few simple regularity assumptions on $\partial\Lambda$ (see pages 13–14 of [1] for the details on $\partial\Lambda$). We shall make the same assumptions in the present paper. The set Γ_n^* is shown to be a countable intersection of open sets with full measure. The operator (2.4) is called the flow of the n particle dynamics. Another analysis of the hard sphere dynamics may be found in [9], [3].

Observe that (unlike in [1]) we do not identify ingoing and outgoing momenta of a collision, but we regard them as corresponding to distinct points in phase space, so that the flow $T_t^{(n)}$ is only piecewise continuous in t. When necessary, we distinguish the limit from the future (+) and the limit from the past (−) writing

$$T_{t\pm}^{(n)} \boldsymbol{z}_n = \lim_{\varepsilon \to 0^+} T_{t\pm\varepsilon}^{(n)} \boldsymbol{z}_n \ . \tag{2.5}$$

For instance, in the statement of Proposition 1, when $\boldsymbol{z}_n \in \Gamma_n^* \cap \partial\Gamma_n$, it is understood that either $T_{0+}^{(n)} \boldsymbol{z}_n = \boldsymbol{z}_n$ or $T_{0-}^{(n)} \boldsymbol{z}_n = \boldsymbol{z}_n$. From now on, to be more definite we fix the (irrelevant) convention

$$T_t^{(n)} \boldsymbol{z}_n = T_{t+}^{(n)} \boldsymbol{z}_n \ . \tag{2.6}$$

The complement of Γ_n^* in Γ_n can be identified with the subset of points of Γ_n that evolved in time run into either:

- a "multiple" collision, that is (i) simultaneous contact of more than two hard spheres, (ii) simultaneous contact of two hard spheres with each other and at the same time with $\partial\Lambda$ or (iii) simultaneous contact of one hard sphere with two different points of $\partial\Lambda$;
- a grazing collision with the wall ($n(q) \cdot p_i' = 0$) or a grazing two–body collision ($(p_i' - p_j') \cdot \omega = 0$);
- a collision of a particle with a singular point $q \in \partial\Lambda$ where the normal vector $n(q)$ is not well defined;
- infinitely many collisions in finite time.

The flow through such situations will not be specified. We shall refer to them as the "singular configurations" (some examples in which a particle undergoes infinitely many collisions in a finite time are given in Sec. II.C of [1]).

We list some more notations that will be useful along the whole paper. For $\boldsymbol{z}_n \in \Gamma_n$, we set

$$\begin{aligned} \Gamma_k(\boldsymbol{z}_n) &= \left\{ \boldsymbol{y}_k \in (\Lambda \times \mathbb{R}^3)^k \ \middle| \ (\boldsymbol{z}_n, \boldsymbol{y}_k) \in \Gamma_{n+k} \right\} , \\ \Omega_i(\boldsymbol{z}_n) &= \left\{ \omega \in S^2 \ \middle| \ (\boldsymbol{z}_n, q_i + a\omega, p) \in \Gamma_{n+1} \ \ \forall p \in \mathbb{R}^3 \right\} , \qquad i = 1, \cdots, n \ . \end{aligned} \tag{2.7}$$

To conclude this section, we pursue a bit further the analysis on the dynamics of the system of particles. The following result will be used to study the properties of correlation functions:

Proposition 2. *[Existence of the dynamics (II)] In Γ_n there is a subset $\Gamma_n^\dagger$, whose complement is a Lebesgue null set, such that $\Gamma_n^\dagger \subseteq \Gamma_n^*$, $T_t^{(n)}\Gamma_n^\dagger = \Gamma_n^\dagger$ and, for $k = 1, \cdots, N-n$,*

$$\boldsymbol{z}_n \in \Gamma_n^\dagger \;\Rightarrow\; (\boldsymbol{z}_n, \boldsymbol{z}_{n,k}) \in \Gamma_{n+k}^* \textit{ for a.a. } \boldsymbol{z}_{n,k} \in \Gamma_k(\boldsymbol{z}_n) \; . \tag{2.8}$$

This statement is a consequence of Proposition 1. Its proof is given in the Appendix. Notice that $\boldsymbol{z}_n \in \Gamma_n^\dagger$ implies also $\boldsymbol{z}_{n+k} \in \Gamma_{n+k}^\dagger$ for a.a. $\boldsymbol{z}_{n,k}$. The set $\Gamma_n^\dagger$ can be identified with

$$\Gamma_n^\dagger = \left\{ \boldsymbol{z}_n \in \Gamma_n^* \;\middle|\; (T_s^{(n)}\boldsymbol{z}_n, \boldsymbol{z}_{n,k}) \in \Gamma_{n+k}^* \;\forall s \text{ and a.a. } \boldsymbol{z}_{n,k} \in \Gamma_k(T_s^{(n)}\boldsymbol{z}_n) \right\} , \tag{2.9}$$

which is also the maximal subset obeying the properties of Proposition 2.

B. Measures over the phase space

Since all the particles of the system are identical, we will work with the space $\mathcal{L}_N$ of Borel measurable functions $f_N : \Gamma_N \to \mathbb{R}$, symmetric in the particle labels ($f_N(\Pi(z_1, \ldots, z_N)) = f_N(z_1, \ldots, z_N)$ for any permutation Π). We also assume that the functions in $\mathcal{L}_N$ have a boundedness property on Γ_N of the type

$$|f_N(\boldsymbol{z}_N)| \leq A \prod_{j=1}^{N} h_\beta(p_j) \; , \qquad\qquad h_\beta(p) = \left(\frac{\beta}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{\beta}{2}p^2} \; , \tag{2.10}$$

for some $A, \beta > 0$.

In Eq. (2.10) we ignore the (possible) dependence on N of the constant A, being the total number of particles always fixed throughout the paper. In particular, we allow A to grow exponentially with N. It is worth to notice that this includes the states considered in the derivation of the Boltzmann equation [8].

Suppose to have an initial measure P on Γ_N with density $f_N^0 \in \mathcal{L}_N$ with respect to the Lebesgue measure $d\boldsymbol{z}_N = dz_1 \ldots dz_N$,

$$P(d\boldsymbol{z}_N) = f_N^0(\boldsymbol{z}_N) d\boldsymbol{z}_N \; . \tag{2.11}$$

Then, because the flow $T_t^{(N)}$ preserves the Lebesgue measure, the evolved measure at time t has a density $f_N(t)$ given by

$$f_N(\boldsymbol{z}_N, t) = f_N^0(T_{-t}^{(N)}\boldsymbol{z}_N) \tag{2.12}$$

almost everywhere in Γ_N, which is the Liouville equation in mild form. Points of $\Gamma_N \setminus \Gamma_N^*$ are removed from (2.12). Notice that estimate (2.10) is preserved in the time evolution by conservation of energy. In particular, $f_N(t) \in \mathcal{L}_N$. Of course since the flow $T_t^{(N)}$ is only well defined almost surely, even densities that are smooth at time zero will only be $\mathcal{L}_N-$functions at time t. Observe that, by our convention (2.6), f_N takes the same value in incoming and outgoing states of collision (a property not to be confused with the continuity along trajectories; see Eq. (4.1) below).

We define the correlation functions $\rho_n(t) \in \mathcal{L}_n, n = 1, 2, \ldots$ by

$$\begin{aligned} &\rho_n(\boldsymbol{z}_n, t) = N \ldots (N-n+1) \int_{\Gamma_{N-n}(\boldsymbol{z}_n)} dz_{n+1} \ldots dz_N f_N(\boldsymbol{z}_N, t) \; , \qquad n \leq N \; , \\ &\rho_n = 0 \; , \qquad\quad n > N \; , \\ &\rho_n^0(\boldsymbol{z}_n) \equiv \rho_n(\boldsymbol{z}_n, 0) \; . \end{aligned} \tag{2.13}$$

P can be, in general, any signed measure with density in $\mathcal{L}_N$. In the case P is a probability measure, the quantity

$$\frac{1}{N \cdots (N-n+1)} \int_{\mathcal{W}} dz_1 \cdots dz_n \rho_n(z_1, \cdots, z_n, t) \tag{2.14}$$

is the probability of finding particles $1, 2, \cdots, n$ at time t in the Borel set $\mathcal{W} \in \Gamma_n$.

3. COLLISION HISTORIES

In this section we analyze the structure of the expansion on the right hand side of (1.5) (Section 3 A). This is given in general by a large variety of terms. In order to have a clear picture of the many terms of the expansion and of the configurations of particles involved in them, we shall establish rules for their graphical representation (Section 3 B).

From now on, without loss of generality and to avoid overweight of notation, the time t will be always supposed to be positive.

A. The structure of formula (1.5): an integral over fictitious evolutions of particles

Let us look carefully at the explicit expression of the right hand side in Eq. (1.5). We compute it, say, in $\boldsymbol{z}_n$, taking into account the definitions (1.3) and (1.4). We see that, in the generic term, the integrand function contains one time–zero correlation function. This is evaluated in a configuration of particles which can be found by flowing backwards in time the configuration $\boldsymbol{z}_n$, and suitably adding new particles at the times t_1, t_2 etcetera. The new particles appear in a collision configuration with one of the pre–existent particles. This describes a special evolution that will be called "collision history", a name first used by Spohn in [13].

We want to stress since the beginning that the collision history is *not* a real trajectory of the particle system, and the associated collisions are not a sequence of real collisions. The correspondence between collision histories and sequences of real collisions is only very indirect ([13]).

We begin by explaining how to construct a *collision history*. The ingredients are the collection of variables (in parentheses we specify what will be their interpretation):

- $n \in \{1, 2, 3, \cdots\}$ (starting number of particles),
- $m \in \{0, 1, 2, \cdots\}$ (number of added particles),
- $\boldsymbol{z}_n \in \Gamma_n^*$ (starting configuration),
- $t > 0$ (total time span),
- $\boldsymbol{t}_m \in \mathbb{R}^m$ $(m \geq 1)$ with $t \equiv t_0 > t_1 > \cdots > t_m > t_{m+1} \equiv 0$ (times of creation of added particles),
- $\boldsymbol{j}_m \in \mathbb{N}^m$ $(m \geq 1)$ with $j_1 \in I_n, \cdots, j_m \in I_{n+m-1}$, where $I_k = \{1, \cdots, k\}$ (progenitors of added particles),
- $\hat{\boldsymbol{p}}_m \in \mathbb{R}^{3m}$ $(m \geq 1)$ (momenta of added particles at the time of their creation),
- $\boldsymbol{\omega}_m \in S^{2m}$ $(m \geq 1)$, with a constraint defined below (relative position, in units of a, of the added particles with respect to their progenitors).

To any choice of the variables in the list we associate a backwards evolution. We indicate with the greek letter

$$\zeta_i(s) = (\xi_i(s), \pi_i(s)) \in \Lambda \times \mathbb{R}^3 \tag{3.1}$$

the configuration of particle i (position and momentum) at time s in such evolution, defined as follows. Take the starting configuration $\boldsymbol{z}_n \in \Gamma_n^*$, put $(\zeta_1(t), \cdots, \zeta_n(t)) = \boldsymbol{z}_n$, and evolve it backwards in time as if there were no other particles in the space up to time t_1. This defines the piecewise continuous trajectory $(\zeta_1(s), \cdots, \zeta_n(s))$ for $t_1 < s < t$, that is $(\zeta_1(s), \cdots, \zeta_n(s)) = T^{(n)}_{-t+s}\boldsymbol{z}_n$. Set $(\zeta_1(t_1), \cdots, \zeta_n(t_1)) = T^{(n)}_{-t+t_1}\boldsymbol{z}_n$. If $m = 0$, put $t_1 = 0$: the construction is finished. Otherwise, at time t_1 stop your n particle system and add particle $n+1$ in a state $\zeta_{n+1}(t_1) = (\xi_{j_1}(t_1) + a\omega_1, \hat{p}_1)$, with $\omega_1 \in \Omega_{j_1}(\boldsymbol{\zeta}_n(t_1))$ and such that the dynamics of the obtained system of $n+1$ particles is well defined, i.e. $(\boldsymbol{\zeta}_n(t_1), \zeta_{n+1}(t_1)) \in \Gamma^*_{n+1}$. Observe that, at fixed $\boldsymbol{z}_n, t_1$, we will have either an incoming or an outgoing collision between particles j_1 and $n+1$, depending on the chosen values of $\omega_1, \hat{p}_1$. Now, evolve backwards in time particles $1, \ldots, n+1$ as if there were no other particles in the space up to time $t_2 < t_1$: this defines the piecewise continuous trajectory $(\zeta_1(s), \cdots, \zeta_{n+1}(s)) = T^{(n+1)}_{-t_1+s}\boldsymbol{\zeta}_{n+1}(t_1)$ for $t_2 < s < t_1$. Notice that, soon after t_1, particle j_1 in the backwards evolution will deviate from its free motion if and only if $\omega_1, \hat{p}_1$ correspond to an outgoing collision. Set $\boldsymbol{\zeta}_{n+1}(t_2) = T^{(n+1)}_{-t_1+t_2}\boldsymbol{\zeta}_{n+1}(t_1)$. If $m = 1$ $(t_2 = 0)$ the construction is finished. Otherwise, at time t_2 stop the system and add particle $n+2$ as above with momentum $\hat{p}_2$ and position at distance $a\omega_2$ from particle j_2, with $\omega_2 \in \Omega_{j_2}(\boldsymbol{\zeta}_{n+1}(t_2))$ and the constraint that the obtained system of $n+2$ particles is in Γ^*_{n+2}. Later on evolve your $n+2$ particles backwards in time up to time $t_3 < t_2$, and so on up to the final step, which is the evolution of particles $1, \ldots, n+m$ with the flow $T^{(n+m)}_{-t_m+s}, 0 \leq s < t_m$. We shall say in the future that particle $n+k$ is "created" by particle j_k, or that particle j_k is its "progenitor". An example is pictured in 1.

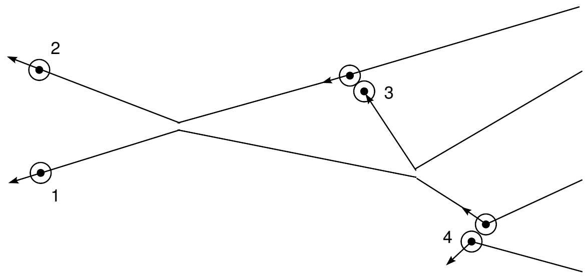

FIG. 1: Trajectory drawn by the particles in a collision history, in the case $n = 2, j_1 = 2, j_2 = 1$. Here particle 3 is added in an incoming collision configuration $(\omega_1 \cdot (\hat{p}_1 - \pi_{j_1}(t_1)) < 0)$, while particle 4 is added in an outgoing collision configuration $(\omega_2 \cdot (\hat{p}_2 - \pi_{j_2}(t_2)) > 0)$.

We will use always greek alphabet for collision histories. We will call $\boldsymbol{\zeta}(s)$ the configuration of all the particles of the history at time s. When no confusion arises, this symbol will have *no* subscript specifying the number of particles, which is actually variable in time, so that

$$\boldsymbol{\zeta}(s) = (\boldsymbol{\xi}(s), \boldsymbol{p}(s)) = (\xi_1(s), \cdots, \xi_{n+k}(s), \pi_1(s), \cdots, \pi_{n+k}(s)) \qquad \text{for } s \in (t_{k+1}, t_k] \ . \tag{3.2}$$

In particular, if s coincides with a time t_k, then $\boldsymbol{\zeta}(s)$ is the configuration of the particles of the evolution *after* having added the new particle $n+k$ (but *before* the related backwards collision, in the case the added particle is in outgoing configuration):

$$\boldsymbol{\zeta}(t_k) = (\boldsymbol{\zeta}_{n+k-1}(t_k), \xi_{j_k}(t_k) + a\omega_k, \hat{p}_k) \ . \tag{3.3}$$

Now we turn back to the description of the series expansion. A careful look of formula (1.5) leads to the following more explicit formal representation:

$$\rho_n(\boldsymbol{z}_n, t) = \sum_{m=0}^{\infty} \sum_{\substack{j_1, \cdots, j_m \\ j_k \in I_{n+k-1}}} \int_{\mathcal{C}_{j_m}(\boldsymbol{z}_n, t)} d\mu(\boldsymbol{t}_m, \boldsymbol{\omega}_m, \hat{\boldsymbol{p}}_m) \left(\prod_{k=1}^{m} B(\omega_k; \hat{p}_k - \pi_{j_k}(t_k)) \right) \rho^0_{n+m}(\boldsymbol{\zeta}(0)) \tag{3.4}$$

where

$$
\begin{aligned}
& d\mu(\boldsymbol{t}_m,\boldsymbol{\omega}_m,\hat{\boldsymbol{p}}_m) \equiv d\mu_m = dt_1\cdots dt_m d\omega_1\cdots d\omega_m d\hat{p}_1\cdots d\hat{p}_m \\
& \qquad\qquad\qquad\qquad (= \text{volume element over } \mathbb{R}^m\times S^{2m}\times\mathbb{R}^{3m})\ , \\
& B(\omega_k;\hat{p}_k-\pi_{j_k}(t_k)) \equiv B_k = a^2\omega_k\cdot(\hat{p}_k-\pi_{j_k}(t_k))\ , \\
& \mathcal{C}_{\boldsymbol{j}_m}(\boldsymbol{z}_n,t) = \Big\{(\boldsymbol{t}_m,\boldsymbol{\omega}_m,\hat{\boldsymbol{p}}_m)\in\mathbb{R}^m\times S^{2m}\times\mathbb{R}^{3m}\ \Big| \\
& \qquad\qquad t>t_1>\cdots>t_m>0,\ \omega_k\in\Omega_{j_k}(\boldsymbol{\zeta}_{n+k-1}(t_k))\Big\}\ . \qquad (3.5)
\end{aligned}
$$

The volume element $d\mu_m$ is the induced Lebesgue measure over $\mathbb{R}^m\times S^{2m}\times\mathbb{R}^{3m}$. The sum over m is extended to infinity by the convention in (2.13). The term $m=0$ must be interpreted as $\rho_n^0(T_{-t}^{(n)}\boldsymbol{z}_n)$.

In the following sections we will rigorously derive (3.4) from Liouville equation, starting with an initial density $f_N^0\in\mathcal{L}_N$. In particular, we will check its consistency, and this will require to prove that the collision histories involved in the integrand are well defined $d\mu_m-$a.e. in the domain of integration.

Observe that, since ρ_{n+m}^0 satisfies an estimate as in (2.10), applying conservation of energy at each creation of the collision history we find

$$
\rho_{n+m}^0(\boldsymbol{\zeta}(0)) \le A'\prod_{j=1}^n h_\beta(p_j)\prod_{k=1}^m h_\beta(\hat{p}_k) \qquad (3.6)
$$

for some $A'>0$. In particular, once we have proven that $(\prod_k B_k)\rho_{n+m}^0(\boldsymbol{\zeta}(0))$ is a well defined measurable function, it follows that its integral in any of the variables $\boldsymbol{z}_n,\boldsymbol{t}_m,\boldsymbol{\omega}_m,\hat{\boldsymbol{p}}_m$ is absolutely convergent. We will use this fact repeatedly during the proof of our results.

B. A graphical expression of (1.5)

The structure of the collision histories described in the previous section makes it quite natural a graphical representation of each term in the expansion (3.4) as a binary tree. Let us introduce at a formal level the useful family of graphs.

For fixed n, we define the *$m-$node, $n-$particle tree graph*, denoted $\mathcal{T}_{n,m}$, as the collection of integers $j_1,\cdots,j_m$ appearing in the right hand side of Eq. (3.4), i.e.

$$
j_1\in I_n, j_2\in I_{n+1},\cdots,j_m\in I_{n+m-1}\ , \qquad \text{with} \qquad I_k=\{1,2,\cdots,k\}, \qquad (3.7)
$$

so that we shall write

$$
\sum_{\substack{j_1,\cdots,j_m \\ j_k\in I_{n+k-1}}} = \sum_{\mathcal{T}_{n,m}}\ . \qquad (3.8)
$$

This has an equivalent graphical representation, given by the following simple procedure. First, draw n horizontal lines, all of them with the same length, stacked one above the other. Assign them the numbers $1,2,3,\cdots$, from the bottom upwards. We will refer to such lines as the "root lines" of the tree graph. Time will be thought as flowing from right to left along a horizontal axis, in such a way that the left extremum of the segments corresponds to time t while the right corresponds to time zero. Now, if $m\ge 1$, draw a heavy dot over the line j_1 (so that the line "crosses" the dot) and a new straight line with a certain slope (say, between 0 and $\pi/2$), having left extremum in the dot and right extremum at time zero. We shall call the dot "node 1" and the new added segment "line $n+1$". Node 1 will correspond to a time $t_1\in(0,t)$. If $m\ge 2$, draw a heavy dot ("node 2") over the line j_2, corresponding to a time $t_2\in(0,t_1)$ (hence on the right with respect to node 1), and a new straight line ("line 2") having left extremum in the dot and right extremum at time zero. This new line shall be horizontal if attached (through the node) to a sloped line, and sloped if attached to a horizontal line. Finally, iterate these operations until the last node ("m") and the last line ("$n + m$") are added. We may agree to avoid intersections between lines. Right extrema of the lines of a tree will be called "endpoints", while left extrema of the root lines will be called "roots". An example of tree graph is given in Figure 2.

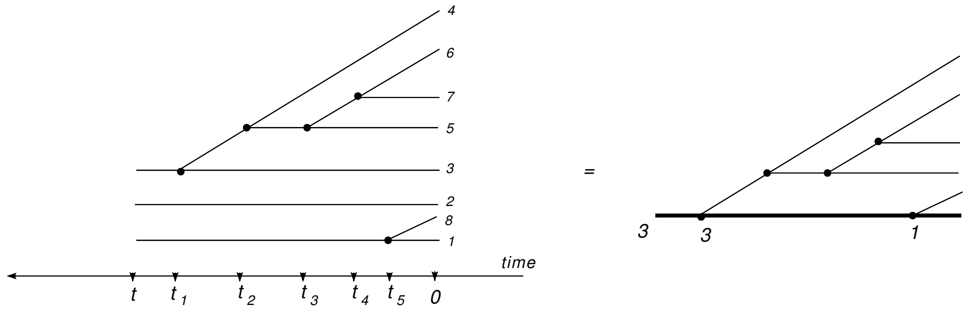

FIG. 2: Tree graph $\mathcal{T}_{3,5} = 3, 4, 5, 6, 1$. On the right, a second equivalent representation.

As shown in the figure, an alternative graphical representation can be given by superposing the n root lines. In this case the (only) root line of the graph is special: it will be drawn as a bold line and it will be decorated as follows: (i) a label n is attached to the root; (ii) if node k lies on the root line, a label j_k is attached to it. Though the graph in the left hand side of Figure 2 is perhaps more standard, we will sometimes use decorated trees as in the right, in order to avoid large diagrams. Furthermore, to simplify the notation, we will not add to the drawing the axis of time, and we will not indicate explicitly the names of the lines (see the right hand side of Figure 2).

Note that two tree graphs are "equivalent" if they can be superposed, together with their labels and without altering their topological structure neither the ordering of its nodes. In other words, the nodes of a tree are ordered along the time axis, so that the total number of different graphs $\mathcal{T}_{n,m}$ is $n(n+1)\cdots(n+m-1)$.

Since a collision history is identified by the collection of variables listed on page 6, we see that $\mathcal{T}_{n,m}$ can be associated to a *class* of collision histories. Namely, $\mathcal{T}_{n,m}$ represents all the collision histories with n particles at time t and m particles added during the backwards evolution, with progenitors specified by $j_1, \cdots, j_m$. In this sense, we have the dynamical interpretation of graphs:
– root lines: particles $1, \cdots, n$ of the history, leaving from time 0 to time t;
– line $n + k$: particle $n + k$ of the history, leaving from time 0 to time t_k;
– node k : binary collision in which particle $n + k$ is created from the progenitor j_k.
To have a precise correspondence between single collision histories and graphs it would be sufficient to add to the picture the following decorations: (i) labels $z_1, \cdots, z_n, t$ attached to the roots of the graph indicating the starting configuration and the time span; (ii) triples $(t_1, \omega_1, \hat{p}_1), \cdots, (t_m, \omega_m, \hat{p}_m)$ attached to the nodes $1, \cdots, m$, specifying the times of creation of the added particles and their position and momenta. Of course all these decorations should satisfy the constraints listed on page 6.

It is important to keep in mind that two particles of the collision history can interact many times during their common lifetime. In general, any couple of particles appearing in the graph at a given time can be in a collision configuration. The interactions which are *not* creations (and occur usually in the open time intervals (t_{k+1}, t_k)) will be called *recollisions*. In fact, they may generally involve particles that have already interacted at some creation time (in the future) with another particle of the history.

We conclude this section by rewriting Eq. (3.4) as

$$\rho_n(\boldsymbol{z}_n, t) = \sum_{m=0}^{\infty} \sum_{\mathcal{T}_{n,m}} V(\mathcal{T}_{n,m})(\boldsymbol{z}_n, t) \ , \tag{3.9}$$

where the *value of the tree* $V(\mathcal{T}_{n,m})$ is

$$V(\mathcal{T}_{n,m})(\boldsymbol{z}_n,t)=\int_{\mathcal{C}_{\mathcal{T}_{n,m}}(\boldsymbol{z}_n,t)} d\mu(\boldsymbol{t}_m,\boldsymbol{\omega}_m,\hat{\boldsymbol{p}}_m)\left(\prod_{k=1}^{m} B(\omega_k;\hat{p}_k-\pi_{j_k}(t_k))\right)\rho^0_{n+m}(\boldsymbol{\zeta}(0))\;, \tag{3.10}$$

i.e. the integral of the initial datum ρ^0_{n+m}, with a suitable weight, over all the possible time–zero states of the collision histories associated to $\mathcal{T}_{n,m}$. In the next section, to represent graphically $V(\mathcal{T}_{n,m})(\boldsymbol{z}_n,t)$, we will just draw the graph $\mathcal{T}_{n,m}$ as in Figure 2 and attach to the root a label $\boldsymbol{z}_n$.

4. THE EVOLUTION OF CORRELATION FUNCTIONS

In what follows we present our main theorem (Theorem 1). Then, we derive the usual BBGKY hierarchy of equations (Corollary 1). Finally, we present an extension of the result to measures of grand canonical type (Corollary 2).

Theorem 1. *Given an initial measure on Γ_N with density $f^0_N\in\mathcal{L}_N$, let $f_N(t)$ be the time–evolved density and $\rho_n(t)$ the associated correlation functions, as defined respectively in* (2.12) *and* (2.13)*. Then, the expansion* (3.9)–(3.10) *holds for any $t>0$, almost everywhere in Γ_n. If* (2.12) *and* (2.13) *are satisfied over the whole sets $\Gamma^\dagger_n$, then the expansion is valid for all $(\boldsymbol{z}_n,t)\in\Gamma^\dagger_n\times\mathbb{R}^+$.*

As mentioned in the introduction, unlike in [13] and in [6], we do not need f^0_N and ρ^0_n to be "continuous along trajectories", that is we do not need

$$\lim_{s\to 0} f^0_N(T^{(N)}_s(z_1,\cdots,z_N))=f^0_N(z_1,\cdots,z_N) \tag{4.1}$$

for a.a. $\boldsymbol{z}_N\in\Gamma_N$, where both the limits from the future and the past are understood. If the continuity along trajectories is assumed to be valid for f^0_N, then the Liouville Equation (2.12) together with some integrability bound on f^0_N imply that the same continuity property holds for $f_N(t)$ and for $\rho_n(t)$ at any time $t\geq 0$, and that the map $t\to\rho_n(\boldsymbol{z}_n,t)$ is also continuous for almost all $\boldsymbol{z}_n$ (see [13], where this is proved and used). All these properties, even if assumed, would be not helpful in the proof of Section 5.

As for the control on large momenta, assumption (2.10) could be substituted with a weaker one, since it will be actually needed just to ensure the absolute convergence of the integrals in the expansion: see (3.6) and the comment therein. Our choice of the decay behaviour for high momenta is the same used by Lanford in the careful estimates of [8] (see the details in [7]), necessary to perform the Boltzmann–Grad limit.

Finally, observe that our result is actually stronger than the one obtained in the previous literature [13], [6], [15]. We know by [1] that there exists a full measure subset of the phase space where the dynamics of the hard sphere system exists for all times (Proposition 1). (The last statement of) Theorem 1 recovers this property for the evolution of correlation functions. Unfortunately the subset $\Gamma^\dagger_n$, in which the expansion for the correlations is valid for all times, has not been characterized in a constructive manner (see Proposition 2): this would depend on details of the dynamics that have not been investigated. However, it will be clear from the proof, which method *is* instead constructive, that Eq. (2.9) defines the maximal subset of the phase space where the result can be derived for all times, as soon as Γ^*_n is given as the maximal subset on which the hard sphere dynamics is well defined. In particular, the last statement of our theorem will be still true if we replace $\Gamma^\dagger_n$ with any full measure invariant subset of it, say $\mathcal{H}_n$, satisfying the following "chain property": if $\boldsymbol{z}_n\in\mathcal{H}_n$, then $(\boldsymbol{z}_n,\boldsymbol{y}_k)\in\mathcal{H}_{n+k}$ for almost all $\boldsymbol{y}_k\in\Gamma_k(\boldsymbol{z}_n)$.

Let us turn now to the usual BBGKY hierarchy of integro–differential equations. The hierarchy can be recovered, though in a mild sense, from the expansion (3.9).

The *collision operator* Q acting on the time–evolved correlation function (abusing the notation used in the introduction) is defined by

$$(Q\rho_{n+1})(\boldsymbol{z}_n,t)=a^2\sum_{j=1}^{n}\int_{\mathbb{R}^3\times\Omega_j(\boldsymbol{z}_n)} d\hat{p}\,d\omega\;\omega\cdot(\hat{p}-p_j)\,\rho_{n+1}(\boldsymbol{z}_n,q_j+a\omega,\hat{p},t)\;. \tag{4.2}$$

This definition does not depend (almost surely) on values assumed by the initial measure on a set of measure zero. Suppose indeed that $f_N(0), \tilde{f}_N(0) \in \mathcal{L}_N$, with $f_N(0) = \tilde{f}_N(0)$ a.s. in Γ_N. Of course the Liouville equation implies that this remains true for any positive time. But, by the property in Remark 2 of page 26, the same is true also for almost all $(\boldsymbol{z}_N, t) \in \partial\Gamma_N \times \mathbb{R}$. This implies $\rho_n(t) = \tilde{\rho}_n(t)$ for a.a. $(\boldsymbol{z}_n, t) \in \partial\Gamma_n \times \mathbb{R}$, so that $Q\rho_{n+1} = Q\tilde{\rho}_{n+1}$ a.s. in $\Gamma_n \times \mathbb{R}$.

It holds

Corollary 1. *Given an initial measure on Γ_N with density $f_N^0 \in \mathcal{L}_N$, let $f_N(t), \rho_n(t)$ satisfy* (2.12), (2.13) *over $\Gamma_n^\dagger$. Then the function $t \longrightarrow (Q\rho_{n+1})\,(T_t^{(n)}\boldsymbol{z}_n, t)$ is $dt-$measurable and $t \longrightarrow \rho_n(T_t^{(n)}\boldsymbol{z}_n, t)$ is absolutely continuous, for all $\boldsymbol{z}_n \in \Gamma_n^\dagger$. The correlation functions satisfy*

$$\frac{d}{dt}\rho_n(T_t^{(n)}\boldsymbol{z}_n, t) = (Q\rho_{n+1})\,(T_t^{(n)}\boldsymbol{z}_n, t) \tag{4.3}$$

for all $\boldsymbol{z}_n \in \Gamma_n^\dagger$ and almost all $t > 0$.

The result, which strengthens the analogous in [3], is obtained by resummation of the series validated in Theorem 1 (see Section 5 D).

We stress that the mild continuity property stated in the corollary is a consequence of the only Liouville equation, and it does not imply the stronger continuity–along–trajectories of the correlation functions, which is in general not valid unless we assume Eq. (4.1) for the initial measure.

To gain regularity in the right hand side of the hierarchy, we need further assumptions. For instance, it can be checked that continuity in t of $(Q\rho_{n+1})\,(T_t^{(n)}\boldsymbol{z}_n, t)$ follows if the continuity–along–trajectories of $\rho_{n+1}(t)$ holds for a.a. values of the integration variables $\hat{p}, \omega$. This would be in turn ensured (at least for a.a. $\boldsymbol{z}_n, t$) by assumption (4.1), or also by the continuity of the initial density in a full measure subset of the phase space. We shall not pursue this further here.

It is worth to say that the proof of Theorem 1 extends easily to a more general class of measures with non definite (but finite) number of particles. Consider the grand canonical phase space

$$\Gamma = \cup_{n\geq 0}\Gamma_n\ . \tag{4.4}$$

There holds $\Gamma_n = \emptyset$ for n larger then $[3|\Lambda|/4\pi a^3]$, because of the hard core exclusion.

Call $\mathcal{L}$ the space of vectors of functions $\boldsymbol{f} : \Gamma \to \mathbb{R}, \boldsymbol{f} = \{f_n\}_{n\geq 0}$, with $f_n \in \mathcal{L}_n$. If P denotes a measure on Γ with density $\boldsymbol{f}^0 \in \mathcal{L}$ with respect to the Lebesgue measure, then the time–evolved measure at time t has a density $\boldsymbol{f}(t) \in \mathcal{L}$ given by

$$f_n(\boldsymbol{z}_n, t) = f_n^0(T_{-t}^{(n)}(\boldsymbol{z}_n))\ , \qquad n \geq 0 \tag{4.5}$$

almost everywhere in Γ_n.

We define the correlation function vector $\boldsymbol{\rho}(t) : \Gamma \longrightarrow \mathbb{R}, \boldsymbol{\rho} = \{\rho_n\}_{n\geq 0}$, by

$$\rho_n(\boldsymbol{z}_n, t) = \sum_{k=0}^{\infty}\frac{1}{k!}\int_{\Gamma_k(\boldsymbol{z}_n)} dz_{n+1}\cdots dz_{n+k} f_{n+k}(\boldsymbol{z}_{n+k}, t)\ . \tag{4.6}$$

It is easy to check that $\boldsymbol{\rho}(t) \in \mathcal{L}$ and that, furthermore, the map defined by (4.6) has the inverse

$$f_n(\boldsymbol{z}_n, t) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_{\Gamma_k(\boldsymbol{z}_n)} dz_{n+1}\cdots dz_{n+k}\rho_{n+k}(\boldsymbol{z}_{n+k}, t)\ . \tag{4.7}$$

We have the following

Corollary 2. *Given an initial measure on Γ with density $\boldsymbol{f}^0 \in \mathcal{L}$, let $\boldsymbol{f}(t)$ be the time–evolved density and $\boldsymbol{\rho}(t)$ the associated correlation functions, as defined respectively in* (4.5) *and* (4.6). *Then, the expansion* (3.9)–(3.10) *holds for any $t > 0$, almost everywhere in Γ_n. If* (2.12) *and* (2.13) *are satisfied over the whole sets $\Gamma_n^\dagger$, then the expansion is valid for all $(\boldsymbol{z}_n, t) \in \Gamma_n^\dagger \times \mathbb{R}^+$, and the results of Corollary 1 hold.*

Here $\Gamma_n^\dagger$ is defined as in Proposition 2, with $k \geq 1$. The (trivial) modifications of the proof of the main theorem leading to Corollary 2 will be discussed in Section 5 E.

5. PROOFS

To prove Theorem 1, we shall proceed by induction on n: supposing the claim true for the function ρ_{n+1}, we derive the expansion for the ρ_n by integrating out the state of a single selected particle. The proof is organised as follows. In Section 5 A we describe the generic step of the induction. In Proposition 3 we explain what is the result when one integrates out the one–particle state in a given term (tree) of the expansion. The proof of Proposition 3, which is our main task, is discussed in Section 5 B. After that, to conclude the proof of the main theorem we have to sum the result over all possible trees, which is done in Section 5 C. Finally, in the last two sections we prove Corollaries 1, 2.

The iterative integration rule and the technical steps of Sections 5 B,5 C, admit a quite simple graphical representation in terms of manipulations of tree graphs. This may help the reader to understand quickly the notations introduced along the proof.

The analytical operations leading to Proposition 3 consist in appropriate partitioning of the integration domain, and representation of its subsets via suitable changes of variables. Such parametrizations turn out to be rather simple, since they are constructed using only non–interacting one–particle trajectories. Nevertheless, as mentioned in the introduction, this is not enough: to prove the proposition it is also essential to notice that a certain class of collision histories gives a net null contribution to the integral, because of one by one cancellations. This will be the content of Lemma 3.

A. Integration of a particle state

For $n = N$ (and of course $n > N$) the statement of Theorem 1 is trivially implied by (2.13) and (2.12). Formula (3.9) gives

$$\rho_N(\boldsymbol{z}_N, t) = \quad \boldsymbol{z}_N \, \rule{3cm}{0.4pt} = \rho^0_N(T^{(N)}_{-t} \boldsymbol{z}_N) \; . \tag{5.1}$$

We proceed by induction on n. From (2.13) it follows

$$\rho_n(\boldsymbol{z}_n, t) = \frac{1}{N-n} \int_{\Gamma_1(\boldsymbol{z}_n)} dz_{n+1} \rho_{n+1}(\boldsymbol{z}_n, z_{n+1}, t) \; , \qquad 1 \leq n < N \; . \tag{5.2}$$

Let us assume that, for any $t > 0$, $V(\mathcal{T}_{n+1,m})$ is a Borel function over Γ_{n+1} with absolute value bounded by $A' \prod_{j=1}^{n+1} h_{\beta'}(p_j)$, for some $A', \beta' > 0$, and that Eq. (3.9) is valid for ρ_{n+1}. Then we can write

$$\rho_n(\boldsymbol{z}_n, t) = \frac{1}{N-n} \sum_{m=0}^{\infty} \sum_{\mathcal{T}_{n+1,m}} \mathcal{I}(\mathcal{T}_{n+1,m})(\boldsymbol{z}_n, t) \; , \tag{5.3}$$

$$\mathcal{I}(\mathcal{T}_{n+1,m})(\boldsymbol{z}_n, t) = \int_{\Gamma_1(\boldsymbol{z}_n)} dz_{n+1} V(\mathcal{T}_{n+1,m})(\boldsymbol{z}_{n+1}, t) \; , \tag{5.4}$$

a.e. in Γ_n.

The two last equations hold exactly in $\Gamma^\dagger_n$ if (2.12) and (2.13) are satisfied over the corresponding spaces. Unless where explicitly stated, we may assume that this is true from now on: f_N, ρ_n satisfy (2.12), (2.13) over the whole sets $\Gamma^\dagger_n$, and we fix $(\boldsymbol{z}_n, t) \in \Gamma^\dagger_n \times \mathbb{R}^+$. If this is not the case, it will be clear that each step of the proof that follows is still valid in some full measure, possibly $t-$dependent, subset of Γ_n.

In the rest of this section and in the next one, we will focus on the computation of (5.4).

Integration of a particle state in a single tree

Let us explain what is the result when we integrate a particle state in a given tree. The computation of (5.4) will be the main part of the proof, and the content of Section 5 B.

The bulk of Theorem 1 is contained in the following assertion.

Proposition 3. *Fix* $n, m, \mathcal{T} = \mathcal{T}_{n+1,m} = j_1, \cdots, j_m$. *Let* $\ell = m+1$ *if* $\{k \mid j_k = n+1\} = \emptyset$, *and* $\ell = \min\{k \mid j_k = n+1\}$ *otherwise. There holds*

$$\mathcal{I}(\mathcal{T}_{n+1,m}) = \delta_{\ell,m+1}\,(N-n-m)\,V(\mathcal{T}''_{n,m}) + \sum_{k=1}^{\ell}\sum_{i=1}^{n+k-1} V(\mathcal{T}'_{n,m+1})\,, \tag{5.5}$$

where $\mathcal{T}'' = \mathcal{T}''_{n,m} = \boldsymbol{j}''_m$ *and* $\mathcal{T}' = \mathcal{T}'_{n,m+1} = \boldsymbol{j}'_{m+1}$ *are the* $n-$*particle trees given by the rules*

$$\begin{aligned} \boldsymbol{j}''_m &= f''(j_1), \cdots, f''(j_m) \\ f''(j) &= \begin{cases} j & \text{if } j \le n \\ j-1 & \text{if } j \ge n+2 \end{cases} \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} \boldsymbol{j}'_{m+1} &= f'(j_1), \cdots, f'(j_{k-1}), i, f'(j_k), \cdots, f'(j_m) \\ f'(j) &= \begin{cases} j & \text{if } j \le n, j \ge n+k+1 \\ j-1 & \text{if } n+2 \le j \le n+k \\ n+k & \text{if } j = n+1 \end{cases} . \end{aligned} \tag{5.7}$$

All the terms on the r.h.s. of (5.5) *are Borel functions over* Γ_n *with absolute value bounded by* $A' \prod_{j=1}^{n} h_{\beta'}(p_j)$, *for some* $A', \beta' > 0$.

Here δ indicates the Kronecker delta. Notice that we drop the dependence on k, i of the trees $\mathcal{T}'$.

Representing the tree graph $\mathcal{I}(\mathcal{T}_{n+1,m})$ as made of $n+1$ distinct trees, as in the left hand side of Figure 2, we can give the following picture of Proposition 3. To compute $\mathcal{I}(\mathcal{T}_{n+1,m})$:

1. Consider the $(n+1)-$th tree graph in $\mathcal{T}_{n+1,m}$, i.e. the tree having line $n+1$ as root (note that ℓ is defined as the name of the first node of this tree, if any, going from left to right). *Attach* its root to the line i of $\mathcal{T}_{n+1,m}$, between node $k-1$ and node k, taking care to *preserve* the reciprocal ordering of the nodes of $\mathcal{T}_{n+1,m}$. The (only possible) resulting tree, $\mathcal{T}'_{n,m+1}$, will have the old m nodes of $\mathcal{T}_{n+1,m}$, plus one *new* node coming from this last operation. Compute now the value of the resulting tree.

2. Sum the result of the previous point over all possible choices of k and i.

3. If the $(n+1)-$th tree graph in $\mathcal{T}_{n+1,m}$ is trivial (i.e. it has no nodes), add to the result of point 2 the value of the $n-$particle tree obtained by *discarding* the trivial line, i.e. $\mathcal{T}''_{n,m}$, multiplied by a factor $(N-n-m)$.

See Figure 3 for an example. Several other examples are provided by Figure 4, in which the alternative

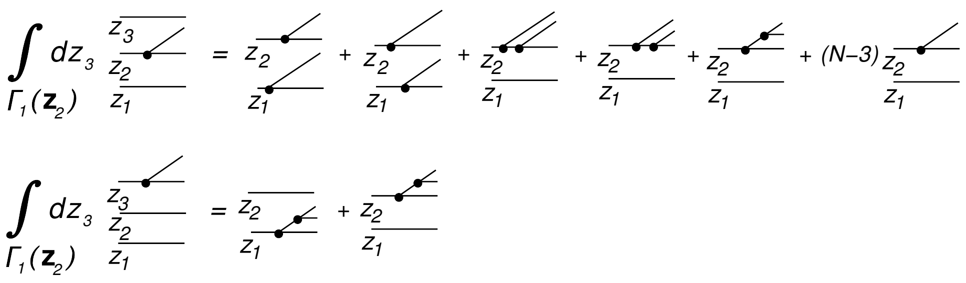

FIG. 3: Computation of the integral $\mathcal{I}$ for two tree graphs $\mathcal{T}_{n+1,m}$ with $n=2, m=1$. In the first line, a case with $\ell = 2$. In the second line, a case with $\ell = 1$. Notice that in the first line, on the right hand side, the third and the fourth graphs are equivalent, while the last graph is produced by operation 3 of the list above.

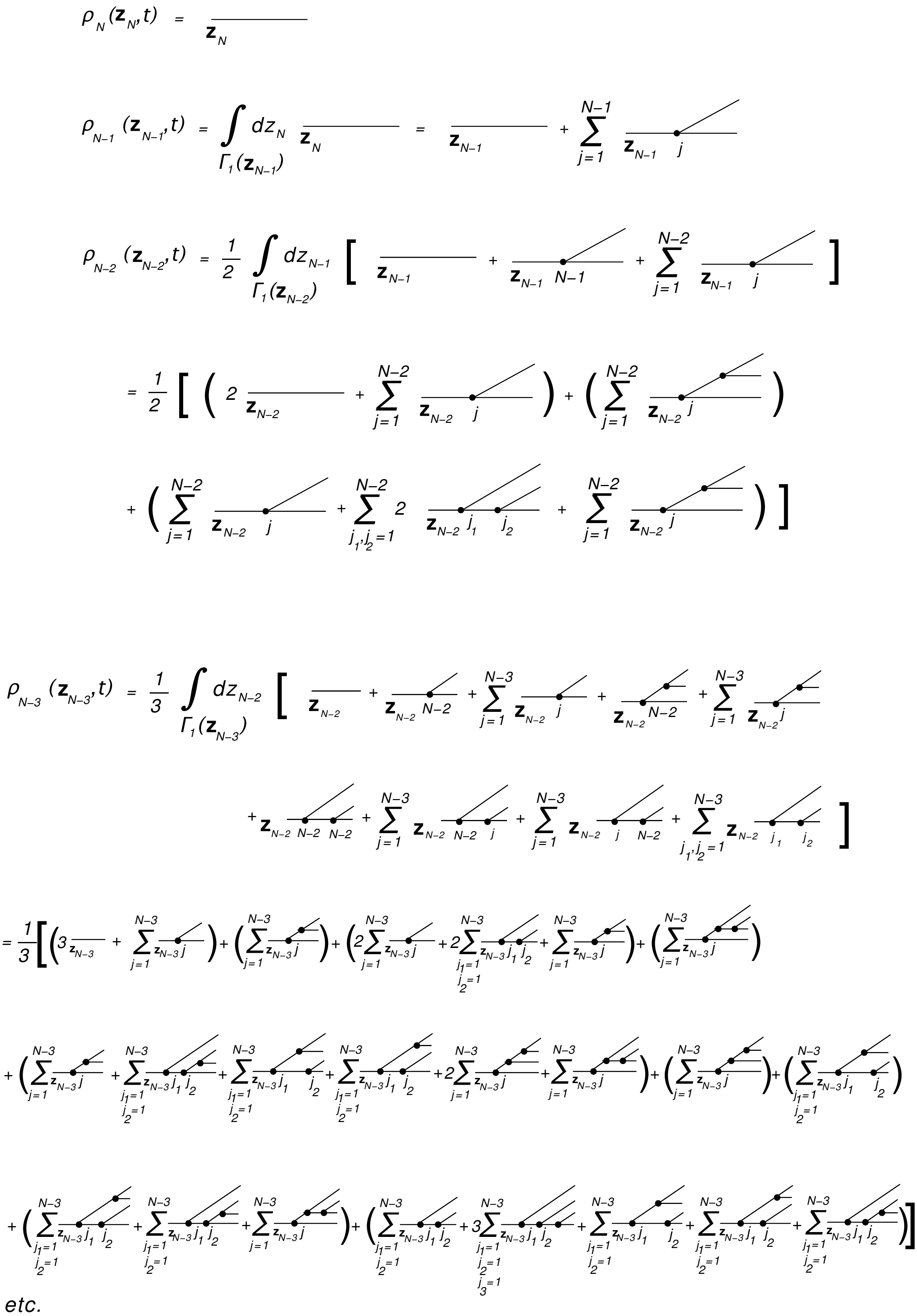

FIG. 4: Integration of degrees of freedom: from Liouville equation to BBGKY hierarchy.

graphical representation introduced on the right hand side of Figure 2 is used in order to avoid too large diagrams.

Observe that the new node described in point 1 of the list has number $k \leq \ell$ in the resulting tree $\mathcal{T}'_{n,m+1}$, while the other nodes have to be consequently renamed: those on its left conserve their name, while those on its right increase of a unit. In particular, if $\ell < m+1$, in the resulting tree the node $\ell + 1$ is the first one "crossed" by line $n + k$.

We close this section by giving an idea of how formula (5.5) emerges. To do so, let us discuss (briefly and somewhat loosely) the first nontrivial step, namely $n = N - 1$, or second line in Figure 4. In this case, m has to be 0 (and $\ell = 1$), and we only need to compute $\int_{\Gamma_1(\boldsymbol{z}_{N-1})} dz \; \rho_N^0(T_{-t}^{(N)}(\boldsymbol{z}_{N-1}, z))$.

Consider the backwards trajectory leading from $(\boldsymbol{z}_{N-1}, z)$ at time t, to $T_{-t}^{(N)}(\boldsymbol{z}_{N-1}, z)$ at time 0. Either the last particle ("particle N") goes freely, or interacts with one of the other $N-1$ particles. We make accordingly the partition $\Gamma_1(\boldsymbol{z}_{N-1}) = \Gamma'_0 \cup {\Gamma'_0}^c$, ${\Gamma'_0}^c = \cup_{j_1=1}^{N-1} \Gamma'_{j_1}$, with j_1 = index of the first particle encountered by particle N in its backwards motion. First, we reexpress $\int_{\Gamma'_{j_1}} dz$ through the change of variables $z \equiv (q,p) \to (t_1, \omega_1, \hat{p}_1)$, where t_1 is the time of the first (backwards) interaction between the particles N and j_1, $q - p(t-t_1) = q_{j_1}(t_1) + a\omega_1$, and $\hat{p}_1 = p$. Here $q_{j_1}(t_1), p_{j_1}(t_1)$ are position and momentum of particle j_1 at time t_1, evolved with the $(N-1)$−particle dynamics. The volume element transforms as $dz = a^2 \omega_1 \cdot (\hat{p}_1 - p_{j_1}(t_1)) dt_1 d\omega_1 d\hat{p}_1$. That is, using the notations of (3.4)–(3.5),

$$\int_{\Gamma_1(\boldsymbol{z}_{N-1})} dz \; \rho_N^0(T_{-t}^{(N)}(\boldsymbol{z}_{N-1}, z)) = \int_{\Gamma'_0} dz \; \rho_N^0(T_{-t}^{(N)}(\boldsymbol{z}_{N-1}, z)) + \sum_{j_1=1}^{N-1} \int_+ d\mu_1 \; \mathbb{1}_{B_1 > 0} \; B_1 \; \rho_N^0(\boldsymbol{\zeta}(0)) \,, \tag{5.8}$$

where $\int_+$ is restricted to trajectories such that particle N moves freely in the time interval (t_1, t).

Next, we observe that, if $\overline{z} = (q - pt, p)$, then $\int_{\Gamma'_0} dz \; \rho_N^0(T_{-t}^{(N)}(\boldsymbol{z}_{N-1}, z)) = \int_{\Gamma''_0} d\overline{z} \; \rho_N^0(T_{-t}^{(N-1)} \boldsymbol{z}_{N-1}, \overline{z})$, being Γ''_0 the subset of $\Gamma_1(T_{-t}^{(N-1)} \boldsymbol{z}_{N-1})$ such that particle N moves freely in the time interval $(0, t)$. To this result we add and subtract the integral over ${\Gamma''_0}^c = \Gamma_1(T_{-t}^{(N-1)} \boldsymbol{z}_{N-1}) \setminus \Gamma''_0 = \cup_{j_1=1}^{N-1} \Gamma''_{j_1}$, with j_1 = index of the first particle encountered by particle N in its forward motion. We find $\rho_{N-1}^0(T_{-t}^{(N-1)} \boldsymbol{z}_{N-1}) - \sum_{j_1=1}^{N-1} \int_{\Gamma''_{j_1}} d\overline{z} \; \rho_N^0(T_{-t}^{(N-1)} \boldsymbol{z}_{N-1}, \overline{z})$. Now we proceed as before, i.e. we change variables according to $\overline{z} \equiv (\overline{q}, \overline{p}) \to (t_1, \omega_1, \hat{p}_1)$, where t_1 is the time of the first (forward) interaction between particles N and j_1, $\overline{q} + \overline{p} t_1 = q_{j_1}(t_1) + a\omega_1$, and $\hat{p}_1 = \overline{p}$. The final result is

$$\begin{aligned} &\int_{\Gamma_1(\boldsymbol{z}_{N-1})} dz \; \rho_N^0(T_{-t}^{(N)}(\boldsymbol{z}_{N-1}, z)) \\ &= \rho_{N-1}^0(T_{-t}^{(N-1)} \boldsymbol{z}_{N-1}) + \sum_{j_1=1}^{N-1} \int_+ d\mu_1 \; \mathbb{1}_{B_1 > 0} \; B_1 \; \rho_N^0(\boldsymbol{\zeta}(0)) - \sum_{j_1=1}^{N-1} \int_- d\mu_1 \; \mathbb{1}_{B_1 < 0} \; |B_1| \; \rho_N^0(\boldsymbol{\zeta}(0)) \,. \end{aligned} \tag{5.9}$$

To reconstruct the left hand side of (5.5), we need finally to get rid of the restrictions $+/-$ under the signs of integral. Call respectively $+^c/-^c$ the complements of these restrictions. These are values of $(t_1, \omega_1, \hat{p}_1)$ such that particle N undergoes a collision in the forward / backwards evolution starting from $(\xi_{j_1}(t_1) + a\omega_1, \hat{p}_1)$. A one–to–one mapping is naturally established between $+^c$ and $-^c$, by looking at the first forward / backwards collision starting from $(\xi_{j_1}(t_1) + a\omega_1, \hat{p}_1)$ (see e.g. Figure 5 on page 22). For instance, we may rewrite the $\sum_{j_1=1}^{N-1} \int_{-^c} d\mu_1$ by applying the transformation $(j_1, t_1, \omega_1, \hat{p}_1) \to (j'_1, t'_1, \omega'_1, \hat{p}'_1)$, where t'_1 is the time of the first (backwards) interaction of particle N in $(0, t_1)$, j'_1 is the index of the particle involved in such collision, ω'_1 is the unit vector indicating the relative position of N with respect to j'_1 at time t'_1, and $\hat{p}_1 = \hat{p}'_1$. Since the volume element transforms as $-a^2 \omega_1 \cdot (\hat{p}_1 - p_{j_1}(t_1)) dt_1 d\omega_1 d\hat{p}_1 = a^2 \omega'_1 \cdot (\hat{p}'_1 - p_{j'_1}(t'_1)) dt'_1 d\omega'_1 d\hat{p}'_1$, we obtain that $\sum_{j_1=1}^{N-1} \int_{-^c} d\mu_1 \mathbb{1}_{B_1 > 0} B_1 \rho_N^0(\boldsymbol{\zeta}(0)) = \sum_{j_1=1}^{N-1} \int_{+^c} d\mu_1 \mathbb{1}_{B_1 < 0} |B_1| \rho_N^0(\boldsymbol{\zeta}(0))$. This concludes the proof of second line in Figure 4.

B. Proof of Proposition 3

Before starting the proof, we need some additional notation. First of all, in this section we shall drop the lower indices in the names of the trees, unless where stated, and use the symbols $\mathcal{T},\mathcal{T}',\mathcal{T}''$ introduced by Proposition 3. To avoid confusion, we will mark with a symbol $'$ (or $''$) the variables of the collision histories associated to $\mathcal{T}'$ (or $\mathcal{T}''$) of Proposition 3, and without that symbol those associated to the tree $\mathcal{T}$. More precisely, if the variables

$$\boldsymbol{z}_{n+1}, t, j_1, \cdots, j_m, t_1, \cdots, t_m, \hat{p}_1, \cdots, \hat{p}_m, \omega_1, \cdots, \omega_m \tag{5.10}$$

describe the collision histories $\boldsymbol{\zeta}$ associated to $\mathcal{T}$, then

$$\boldsymbol{z}_n, t, j'_1, \cdots, j'_{m+1}, t'_1, \cdots, t'_{m+1}, \hat{p}'_1, \cdots, \hat{p}'_{m+1}, \omega'_1, \cdots, \omega'_{m+1} \tag{5.11}$$

describe the collision histories $\boldsymbol{\zeta}'$ associated to $\mathcal{T}'$ (where the $\boldsymbol{j}'_{m+1}$ are given by (5.7)). A similar notation will be used for $\mathcal{T}''$. We recall also the notations $t_0 = t = t'_0 = t''_0$, $t_{m+1} = 0 = t'_{m+2} = t''_{m+1}$, that will be used in the sequel.

For generic $\boldsymbol{\zeta} = \zeta_1, \zeta_2, \cdots$, with $\zeta_i = (\xi_i, \pi_i)$ and $z = (q,p)$, we put

$$\text{dist}(\boldsymbol{\zeta}(s), z) = \min_i |\xi_i(s) - q| \;, \tag{5.12}$$

i.e. the minimum distance, in position space, of a particle in z from the cluster of particles of the collision history at time s. Similarly, we put

$$\text{dist}_k(\boldsymbol{\zeta}(s)) = \min_{i \neq k} |\xi_i(s) - \xi_k(s)| \;, \tag{5.13}$$

that is the minimum distance of particle k of the history from the other particles of the same history at time s.

When we need to specify positions and momenta of a generic configuration $z_1, \cdots, z_n$, with $z_i = (q_i, p_i)$, evolved at time s with the $n-$particle dynamics, we shall use the notation

$$(q_j^{(n)}(s), p_j^{(n)}(s)) \;, \qquad j = 1, \cdots, n \;. \tag{5.14}$$

Let us introduce (for the moment formally) some special subsets of the integration domains in computing the value of $\mathcal{T}'$. Call

$$\begin{aligned}
&\mathcal{F}^+_{k,i} = \Big\{ (\boldsymbol{t}'_{m+1}, \boldsymbol{\omega}'_{m+1}, \hat{\boldsymbol{p}}'_{m+1}) \;\text{ s.t. }\; \omega'_k \cdot (\hat{p}'_k - \pi'_{j'_k}(t'_k)) > 0 \;\text{ and} \\
&\text{dist}\left(\boldsymbol{\zeta}'(s), T^{(1)}_{-t'_k+s} \left(\xi'_{j'_k}(t'_k) + a\omega'_k, \hat{p}'_k \right) \right) > a \text{ for all } s \in (t'_k, t) \Big\} \;, \\
&\mathcal{F}^-_{k,i} = \Big\{ (\boldsymbol{t}'_{m+1}, \boldsymbol{\omega}'_{m+1}, \hat{\boldsymbol{p}}'_{m+1}) \;\text{ s.t. }\; \omega'_k \cdot (\hat{p}'_k - \pi'_{j'_k}(t'_k)) < 0 \;\text{ and} \\
&\text{dist}_{n+k}\left(\boldsymbol{\zeta}'(s) \right) > a \text{ for all } s \in (t'_{\ell+1}, t'_k) \Big\} \;.
\end{aligned} \tag{5.15}$$

In other words, $\mathcal{F}^+_{k,i}$ selects those collision histories associated to $\mathcal{T}'$ which satisfy the special property explained as follows. Consider particle $n+k$ of the collision history $\boldsymbol{\zeta}'$, i.e. the particle created in the "new" node of $\mathcal{T}'$ Assume that this particle is created in an outgoing collision configuration. Its state at the moment of creation is $(\xi'_{j'_k}(t'_k) + a\omega'_k, \hat{p}'_k)$. Then, if we evolve forward in time such a state up to time t, we do not see any interaction of the particle with any of the hard spheres appearing in the evolution $\boldsymbol{\zeta}'$. Similarly in $\mathcal{F}^-_{k,i}$, if we evolve particle $n+k$ (created in an incoming collision configuration) backwards in time up to the time in which it creates another particle of the history (if any; or up to zero otherwise), then we do not see any interaction of it with the hard spheres appearing in the evolution $\boldsymbol{\zeta}'$.

Abbreviating here $z^{(1)}_{n+k}(s) = (q^{(1)}_{n+k}(s), p^{(1)}_{n+k}(s)) = T^{(1)}_{-t'_k+s}(\xi'_{j'_k}(t'_k) + a\omega'_k, \hat{p}'_k)$, we shall complement the above definitions with

$$\begin{aligned}
&\mathcal{R}^+_{k,i} = \Big\{(\boldsymbol{t}'_{m+1}, \boldsymbol{\omega}'_{m+1}, \hat{\boldsymbol{p}}'_{m+1}) \ \text{ s.t. } \ \omega'_k \cdot (\hat{p}'_k - \pi'_{j'_k}(t'_k)) > 0 \ \text{ and } \exists\ i_+ \text{ and } s_+ \in (t'_k, t),\\
&\text{dist}\left(\boldsymbol{\zeta}'(s), z^{(1)}_{n+k}(s)\right) > a \ \ \forall s \in (t'_k, s_+), \ \ \left|q^{(1)}_{n+k}(s_+) - \xi'_{i_+}(s_+)\right| = a \ ,\\
&\left(q^{(1)}_{n+k}(s_+) - \xi'_{i_+}(s_+)\right) \cdot \left(p^{(1)}_{n+k}(s_+) - \pi'_{i_+}(s_+)\right) < 0\Big\} \ ,\\
&\mathcal{R}^-_{k,i} = \Big\{(\boldsymbol{t}'_{m+1}, \boldsymbol{\omega}'_{m+1}, \hat{\boldsymbol{p}}'_{m+1}) \ \text{ s.t. } \ \omega'_k \cdot (\hat{p}'_k - \pi'_{j'_k}(t'_k)) < 0 \ \text{ and } \exists\ i_- \text{ and } s_- \in (t'_{\ell+1}, t'_k),\\
&\text{dist}_{n+k}\left(\boldsymbol{\zeta}'(s)\right) > a \ \ \forall s \in (s_-, t'_k), \ \ \left|\xi'_{n+k}(s_-) - \xi'_{i_-}(s_-)\right| = a \ ,\\
&\left(\xi'_{n+k}(s_-) - \xi'_{i_-}(s_-)\right) \cdot \left(\pi'_{n+k}(s_-) - \pi'_i(i_-)\right) > 0\Big\} \ .
\end{aligned} \tag{5.16}$$

Finally, we denote the restriction of the integral defining $V(\mathcal{T}')$ to any subset A of the integration region as

$$V|_A(\mathcal{T}') = \int_{\mathcal{C}_{\mathcal{T}'}(\boldsymbol{z}_n,t)} d\mu(\boldsymbol{t}'_{m+1}, \boldsymbol{\omega}'_{m+1}, \hat{\boldsymbol{p}}'_{m+1}) \mathbb{1}_A \left(\prod_{r=1}^{m+1} B(\omega'_r; \hat{p}'_r - \pi'_{j'_r}(t'_r))\right) \rho^0_{n+m+1}(\boldsymbol{\zeta}'(0)) \ . \tag{5.17}$$

For the well–posedness of collision histories and of the integrals in which they are involved, we need the following result.

Lemma 1. *For any given $\mathcal{T}_{n,m}$ and any $t > 0$, if $\boldsymbol{z}_n$ varies in $\Gamma^\dagger_n$ and $(\boldsymbol{t}_m, \hat{\boldsymbol{p}}_m, \boldsymbol{\omega}_m)$ in $\mathcal{C}_{\mathcal{T}_{n,m}}(\boldsymbol{z}_n, t)$, the transformation $(\boldsymbol{z}_n, \boldsymbol{t}_m, \hat{\boldsymbol{p}}_m, \boldsymbol{\omega}_m) \longrightarrow \boldsymbol{\zeta}(0)$ described in Section 3 A defines $d\mu_m$−almost surely a map into $\Gamma^\dagger_{n+m}$. The integrand in* (3.10) *is a Borel function of $(\boldsymbol{z}_n, \boldsymbol{t}_m, \hat{\boldsymbol{p}}_m, \boldsymbol{\omega}_m)$ in the same domain.*

For $m = 0$ this is a trivial consequence of Proposition 1, since the transformation reduces to $T^{(n)}_{-t}$. Therefore, we may assume the statement to be valid for $n+1$ and $m \le N-n-1$, and prove it for n and $m \le N-n$, together with Proposition 3. Actually, at each step of the induction, we only need to prove the lemma for $m = N - n$. In fact, the proof of validity for given $\overline{m}, \overline{n}$, can be applied lexicographically to $\overline{m}, n$ with n arbitrary (just change the value of N).

Notice that Lemma 1 (for $m = 1$) implies immediately

Corollary 3. *If $\boldsymbol{z}_n \in \Gamma^\dagger_n$, then $(T^{(n)}_s \boldsymbol{z}_n, q^{(n)}_j(s) + a\omega, \hat{p}) \in \Gamma^\dagger_{n+1}$ for all $j = 1, \cdots, n$ and almost all $(s, \omega, \hat{p}) \in \mathbb{R} \times \Omega_j(T^{(n)}_s \boldsymbol{z}_n) \times \mathbb{R}^3$.*

The proof of Proposition 3 is made of two steps which we separate in the two lemmas that follow. We will first prove, in the next subsection,

Lemma 2. *Under the assumptions of Proposition 3, there holds*

$$\mathcal{I}(\mathcal{T}) = \delta_{\ell,m+1}\,(N-n-m)\,V(\mathcal{T}'') + \sum_{k=1}^{\ell}\sum_{i=1}^{n+k-1}\left(V|_{\mathcal{F}^+_{k,i}}(\mathcal{T}') + V|_{\mathcal{F}^-_{k,i}}(\mathcal{T}')\right) \ , \tag{5.18}$$

where the r.h.s. is a Borel function over Γ_n with absolute value bounded by $A' \prod_{j=1}^n h_{\beta'}(p_j)$, for some $A', \beta' > 0$.

The second step will consist in showing that the collision histories which have been eliminated by the cutoff in Eq. (5.18) give a net contribution equal to zero, i.e.

Lemma 3. *Under the assumptions of Proposition 3, there holds*

$$\sum_{k=1}^{\ell}\sum_{i=1}^{n+k-1}\left(V|_{\mathcal{R}^+_{k,i}}(\mathcal{T}') + V|_{\mathcal{R}^-_{k,i}}(\mathcal{T}')\right) = 0 \ . \tag{5.19}$$

This will be done in a subsequent subsection.

Proof of Lemma 2

Our task is to integrate out the variable z_{n+1} in the expression (see (5.4), (3.10))

$$\int_{\mathcal{C}_{\mathcal{T}}(\boldsymbol{z}_{n+1},t)} d\mu(\boldsymbol{t}_m,\boldsymbol{\omega}_m,\hat{\boldsymbol{p}}_m)\left(\prod_{r=1}^m B(\omega_r;\hat{p}_r-\pi_{j_r}(t_r))\right)\rho^0_{n+1+m}(\boldsymbol{\zeta}(0))\ , \tag{5.20}$$

where the collision history $\boldsymbol{\zeta}$ is the one associated to the tree $\mathcal{T}$. Since the claim in Lemma 1 is true for the considered tree and since $\rho^0_{n+1+m}\in\mathcal{L}_{n+1+m}$ (remember estimate (3.6)), the function $(z_{n+1},\boldsymbol{t}_m,\hat{\boldsymbol{p}}_m,\boldsymbol{\omega}_m)\longrightarrow(\prod_r B_r)\rho^0_{n+1+m}(\boldsymbol{\zeta}(0))$ is absolutely integrable over the space

$$\mathcal{C}^+=\{z_{n+1}\in\Gamma_1(\boldsymbol{z}_n),(\boldsymbol{t}_m,\hat{\boldsymbol{p}}_m,\boldsymbol{\omega}_m)\in\mathcal{C}_{\mathcal{T}}(\boldsymbol{z}_{n+1},t)\}\ . \tag{5.21}$$

By Fubini's theorem, we may rewrite $\mathcal{I}(\mathcal{T})$ as the $6(m+1)-$dimensional integral

$$\mathcal{I}(\mathcal{T})=\int_{\mathcal{C}^+}dz_{n+1}d\mu_m\left(\prod_{r=1}^m B_r\right)\rho^0_{n+1+m}(\boldsymbol{\zeta}(0))\ . \tag{5.22}$$

Almost surely over $\mathcal{C}^+$ we have the following partition:

$$1=\mathbb{1}_{\mathcal{F}_0^+}+\sum_{k=1}^{\ell}\sum_{i=1}^{n+k-1}\mathbb{1}_{\tilde{\mathcal{F}}^+_{k,i}}\ , \tag{5.23}$$

where

$$\begin{aligned}\mathcal{F}_0^+&=\left\{(z_{n+1},\boldsymbol{t}_m,\hat{\boldsymbol{p}}_m,\boldsymbol{\omega}_m)\in\mathcal{C}^+\ \text{ s.t. }\ \text{dist}\left(\boldsymbol{\zeta}(s),T^{(1)}_{-t+s}z_{n+1}\right)>a\ \forall s\in(t_\ell,t)\right\}\ ,\\ \tilde{\mathcal{F}}^+_{k,i}&=\left\{(z_{n+1},\boldsymbol{t}_m,\hat{\boldsymbol{p}}_m,\boldsymbol{\omega}_m)\in\mathcal{C}^+\ \text{ s.t. }\ \exists s_+\in(t_k,t_{k-1})\ ,\ \ \text{dist}\left(\boldsymbol{\zeta}(s),T^{(1)}_{-t+s}z_{n+1}\right)>a\ \forall s\in(s_+,t)\ ,\right.\\ &\left.\left|q^{(1)}_{n+1}(s_+)-\xi_i(s_+)\right|=a\ ,\ \left(q^{(1)}_{n+1}(s_+)-\xi_i(s_+)\right)\cdot\left(p^{(1)}_{n+1}(s_+)-\pi_i(s_+)\right)>0\right\}\ .\end{aligned} \tag{5.24}$$

The set $\mathcal{F}_0^+$ collects the histories $\boldsymbol{\zeta}$ such that particle $n+1$ flows backwards freely up to the time in which it creates another particle, or up to time zero if this never happens. The set $\tilde{\mathcal{F}}^+_{k,i}$ collects the histories such that particle $n+1$ flows backwards freely only up to a collision with the (pre–existent) particle i of the history, occurring in the time interval (t_k,t_{k-1}). The instant of this interaction is called s_+.

Consider the integral

$$\int_{\mathcal{C}^+}dz_{n+1}d\mu_m\mathbb{1}_{\tilde{\mathcal{F}}^+_{k,i}}\prod_{r=1}^m B_r\rho^0_{n+1+m}(\boldsymbol{\zeta}(0))\ . \tag{5.25}$$

Using only the free flow of particle $n+1$, we define the change of variables

$$(z_{n+1},\boldsymbol{t}_m,\hat{\boldsymbol{p}}_m,\boldsymbol{\omega}_m)\longrightarrow(s_+,\omega_+,p_+,\boldsymbol{t}_m,\hat{\boldsymbol{p}}_m,\boldsymbol{\omega}_m) \tag{5.26}$$

where s_+ is the time appearing in the definition of $\tilde{\mathcal{F}}^+_{k,i}$, and ω_+,p_+ describe the collision configuration at time s_+, that is

$$\omega_+=(q^{(1)}_{n+1}(s_+)-\xi_i(s_+))/a\ ,\qquad p_+=p^{(1)}_{n+1}(s_+)\ . \tag{5.27}$$

Introducing $j'_1,\cdots,j'_{m+1}$ as defined in (5.7) and renaming

$$\begin{aligned}&t'_1=t_1,\cdots,t'_{k-1}=t_{k-1},t'_k=s_+,t'_{k+1}=t_k,\cdots,t'_{m+1}=t_m,\\&\omega'_1=\omega_1,\cdots,\omega'_{k-1}=\omega_{k-1},\omega'_k=\omega_+,\omega'_{k+1}=\omega_k,\cdots,\omega'_{m+1}=\omega_m,\\&\hat{p}'_1=\hat{p}_1,\cdots,\hat{p}'_{k-1}=\hat{p}_{k-1},\hat{p}'_k=\hat{p}_+,\hat{p}'_{k+1}=\hat{p}_k,\cdots,\hat{p}'_{m+1}=\hat{p}_m,\end{aligned} \tag{5.28}$$

we see that (5.26) is an invertible transformation from $\tilde{\mathcal{F}}^+_{k,i}$ onto $\mathcal{F}^+_{k,i}$ (modulo exclusion of sets of measure zero), i.e. the change of variables is (partially) "generating" the node k of the tree $\mathcal{T}'$. Of course, the transformation introduced is a Borel map. Moreover, a simple computation shows that it has Jacobian determinant given by

$$dz_{n+1} = B(\omega'_k; \hat{p}'_k - \pi'_{j'_k}(t'_k)) dt'_k d\omega'_k d\hat{p}'_k \ . \tag{5.29}$$

This B factor is added to the product in (5.25), and reconstructs the factor associated to node k in the formula for $V(\mathcal{T}')$. Finally, notice that the collision histories $\boldsymbol{\zeta}$ (associated to $\mathcal{T}$) and $\boldsymbol{\zeta}'$ (associated to $\mathcal{T}'$ and given by the variables (5.11) defined above) *coincide* in the time interval $(0, t'_k)$, thanks to our initial restriction to $\tilde{\mathcal{F}}^+_{k,i}$. Summarising, we have found

$$\int_{\mathcal{C}^+} dz_{n+1} d\mu_m \mathbb{1}_{\tilde{\mathcal{F}}^+_{k,i}} \prod_{r=1}^{m} B_r \rho^0_{n+1+m}(\boldsymbol{\zeta}(0)) = V|_{\mathcal{F}^+_{k,i}}(\mathcal{T}') \ . \tag{5.30}$$

As the above change of variables can be done for any $\boldsymbol{z}_n \in \Gamma^\dagger_n$, the transformation $(\boldsymbol{z}_n, \boldsymbol{t}'_{m+1}, \hat{\boldsymbol{p}}'_{m+1}, \boldsymbol{\omega}'_{m+1}) \to \boldsymbol{\zeta}(0)$, $\boldsymbol{z}_n \in \Gamma^\dagger_n$, $(\boldsymbol{t}'_{m+1}, \hat{\boldsymbol{p}}'_{m+1}, \boldsymbol{\omega}'_{m+1})$ a.e. in $\mathcal{C}_{\mathcal{T}'}(\boldsymbol{z}_n, t) \cap \mathcal{F}^+_{k,i}$, associated to the tree $\mathcal{T}'$, is into $\Gamma^\dagger_{n+m+1}$ and measurable. Therefore Lemma 1 is true for all trees of the type $\mathcal{T}'$ when the variables are restricted to $\mathcal{F}^+_{k,i}$. Proceeding as in (3.6), we conclude that the integral in $V|_{\mathcal{F}^+_{k,i}}(\mathcal{T}')$ is absolutely convergent, and defines a Borel function over Γ_n satisfying the estimate

$$\left| V|_{\mathcal{F}^+_{k,i}}(\mathcal{T}') \right| \leq A' \prod_{j=1}^{n} h_{\beta'}(p_j) \ , \tag{5.31}$$

for suitable $A', \beta' > 0$.

Consider now the restriction to $\mathcal{F}^+_0$. Here particle $n+1$ flows freely up to t_ℓ, so that the first $n+\ell$ particles of the collision history are, at that time,

$$\zeta_1(t_\ell), \cdots, \zeta_n(t_\ell), T^{(1)}_{-t+t_\ell} z_{n+1}, \zeta_{n+2}(t_\ell), \cdots, \zeta_{n+\ell}(t_\ell) \ . \tag{5.32}$$

Hence, excluding particle $n+1$ and up to a renaming of variables, the collision history in the time interval (t_ℓ, t) is equally well described by $\mathcal{T}''$, see Eq. (5.6).

It is convenient to use first the measure–preserving change of variables

$$z_{n+1} \longrightarrow \overline{z} = (\overline{q}, \overline{p}) = T^{(1)}_{-t+t_\ell} z_{n+1} \ . \tag{5.33}$$

Furthermore, unlike in the case of $\tilde{\mathcal{F}}^+_{k,i}$ discussed above, we shall fix the order of integration and rewrite consequently the domains. To describe the collision history in (t_ℓ, t) we will use the set of variables associated to $\mathcal{T}''$,

$$\begin{aligned} \mathcal{C}_{\mathcal{T}''}(\boldsymbol{z}_n, (t_\ell, t)) = \Big\{ (\boldsymbol{t}''_{\ell-1}, \boldsymbol{\omega}''_{\ell-1}, \hat{\boldsymbol{p}}''_{\ell-1}) \in \mathbb{R}^{\ell-1} \times S^{2(\ell-1)} \times \mathbb{R}^{3(\ell-1)} \ \Big| \\ t = t''_0 > t''_1 > \cdots > t''_{\ell-1} > t''_\ell = t_\ell, \ \omega''_k \in \Omega_{j''_k}(\boldsymbol{\zeta}''_{n+k-1}(t''_k)) \Big\} \ , \end{aligned} \tag{5.34}$$

while, to describe the history in the time interval $(0, t_\ell)$, we will use the set of variables associated to the auxiliary tree $\mathcal{T}''' = \mathcal{T}'''_{n+\ell+1, m-\ell} = \boldsymbol{j}'''_{m-\ell}$,

$$\begin{aligned} \boldsymbol{j}'''_{m-\ell} &= f'''(j_{\ell+1}), \cdots, f'''(j_m) \\ f'''(j) &= \begin{cases} f''(j) & \text{if} \quad j \leq n+\ell, \ j \neq n+1 \\ n+\ell & \text{if} \quad j = n+1 \\ j & \text{if} \quad j \geq n+\ell+1 \end{cases} \ . \end{aligned} \tag{5.35}$$

With these notations we have

$$\int_{\mathcal{C}^+} dz_{n+1} d\mu_m \mathbb{1}_{\mathcal{F}_0^+} \left(\prod_{r=1}^m B_r\right) \rho^0_{n+1+m}(\boldsymbol{\zeta}(0)) = \int_{\mathcal{C}_{\mathcal{T}''}(\boldsymbol{z}_n,(t_\ell,t))} d\mu''_{\ell-1} \left(\prod_{r=1}^{\ell-1} B''_r\right) \int_0^{t''_{\ell-1}} dt_\ell$$
$$\cdot \int_{\Gamma_1(\boldsymbol{\zeta}''_{n+\ell-1}(t_\ell))} d\bar{z}\ \mathbb{1}_{\{\mathrm{dist}(\boldsymbol{\zeta}''(s),\bar{z}^{(1)}(s))>a\ \forall s\in(t_\ell,t)\}}$$
$$\cdot \int_{\{S^2\times\mathbb{R}^3,\ \boldsymbol{z}'''_{n+\ell+1}\in\Gamma_{n+\ell+1}\}} d\omega_\ell d\hat{p}_\ell\ B(\omega_\ell;\hat{p}_\ell-\bar{p})$$
$$\cdot \int_{\mathcal{C}_{\mathcal{T}'''}(\boldsymbol{z}'''_{n+\ell+1},t_\ell)} d\mu'''_{m-\ell} \left(\prod_{r=1}^{m-\ell} B'''_r\right) \rho^0_{n+m+1}(\boldsymbol{\zeta}'''(0))\ , \tag{5.36}$$

where $\bar{z}^{(1)}(s) = T^{(1)}_{-t_\ell+s}\bar{z}$,

$$\boldsymbol{z}'''_{n+\ell+1} = \boldsymbol{\zeta}''_{n+\ell-1}(t_\ell), \bar{z}, \bar{q}+a\omega_\ell, \hat{p}_\ell\ , \tag{5.37}$$

$\boldsymbol{\zeta}''$ is the history associated to $\mathcal{T}''$, $d\mu''_{\ell-1} = d\mu(\boldsymbol{t}''_{\ell-1}, \boldsymbol{\omega}''_{\ell-1}, \hat{\boldsymbol{p}}''_{\ell-1})$, $B''_r = B(\omega''_r;\hat{p}''_r - \pi''_{j''_r}(t''_r))$, etc. The last line of (5.36) is just $V(\mathcal{T}''')$ evaluated in the collision configuration $\boldsymbol{z}'''_{n+\ell+1}$ at time t_ℓ. Of course, in the case $\ell = m+1$, the expression is much simpler, since $t_\ell \equiv 0$ and there are no variables $\omega_\ell, \hat{p}_\ell, \cdots$ (see formula (5.46) below).

Call now

$$\mathcal{C}^- = \{(\boldsymbol{t}''_{\ell-1}, \hat{\boldsymbol{p}}''_{\ell-1}, \boldsymbol{\omega}''_{\ell-1}) \in \mathcal{C}_{\mathcal{T}''}(\boldsymbol{z}_n,(t_\ell,t)),\ \bar{z}\in\Gamma_1(\boldsymbol{\zeta}''_{n+\ell-1}(t_\ell))\}\ . \tag{5.38}$$

In formula (5.36) we can write

$$\mathbb{1}_{\{\mathrm{dist}(\boldsymbol{\zeta}''(s),\bar{z}^{(1)}(s))>a\ \forall s\in(t_\ell,t)\}} = 1 - \sum_{k=1}^{\ell}\sum_{i=1}^{n+k-1} \mathbb{1}_{\tilde{\mathcal{F}}^-_{k,i}} \tag{5.39}$$

with

$$\tilde{\mathcal{F}}^-_{k,i} = \Big\{(\bar{z}, \boldsymbol{t}''_{\ell-1}, \hat{\boldsymbol{p}}''_{\ell-1}, \boldsymbol{\omega}''_{\ell-1}) \in \mathcal{C}^-\ \ \text{s.t.}\ \exists s_- \in (t''_k, t''_{k-1})\ ,\ \ \mathrm{dist}\Big(\boldsymbol{\zeta}''(s), \bar{z}^{(1)}(s)\Big) > a\ \forall s\in(t_\ell, s_-)\ ,$$
$$\Big|\bar{q}^{(1)}(s_-) - \xi''_i(s_-)\Big| = a\ ,\ \Big(\bar{q}^{(1)}(s_-) - \xi''_i(s_-)\Big)\cdot\Big(\bar{p}^{(1)}(s_-) - \pi''_i(s_-)\Big) < 0\Big\}\ . \tag{5.40}$$

That is, we add and subtract the sets of variables such that a particle with state $\bar{z}$ collides, when evolved freely forward in time, with one of the particles of $\boldsymbol{\zeta}''$. We name s_- the instant of this interaction.

We shall see that in the (added and) subtracted restrictions to $\tilde{\mathcal{F}}^-_{k,i}$, the collision histories are well–defined. In fact, since $\mathcal{T}''$ has less than $m \leq N-n-1$ nodes, $\boldsymbol{\zeta}''_{n+\ell-1}(t_\ell) \in \Gamma^\dagger_{n+\ell-1}$ almost surely with respect to $d\mu''_{\ell-1}$ (apply Lemma 1). Of course an analogous property holds for $\boldsymbol{\zeta}''$ at different times. Thus, for any given t_ℓ and $d\mu''_{\ell-1}$–a.e., there holds $(\boldsymbol{\zeta}''_{n+\ell-1}(t_\ell), \bar{z}) \in \Gamma^\dagger_{n+\ell}$ for $\bar{z}$ in a full measure subset of $\Gamma_1(\boldsymbol{\zeta}''_{n+\ell-1}(t_\ell))$. If $n = N-1$, this is enough ($m=0, \ell=1$, no histories of type $\boldsymbol{\zeta}'''$). Otherwise, to deal with the case $\ell < m+1$, we apply Corollary 3 to any configuration $(\boldsymbol{\zeta}''_{n+\ell-1}(t_\ell), \bar{z}) \in \Gamma^\dagger_{n+\ell}$. This implies that $\boldsymbol{z}'''_{n+\ell+1} \in \Gamma^\dagger_{n+\ell+1}$ almost everywhere with respect to the measure $d\mu''_{\ell-1}dt_\ell d\bar{z} d\omega_\ell d\hat{p}_\ell$. Using Lemma 1 for the tree $\mathcal{T}'''$, we deduce that $\boldsymbol{\zeta}'''(0) \in \Gamma^\dagger_{n+m+1}$ almost everywhere in the domain of integration (and analogous property for $\boldsymbol{\zeta}'''$ at different times), the last line of (5.36) is well–defined and all the integrals are absolutely convergent.

Each restriction to $\tilde{\mathcal{F}}^-_{k,i}$ can be treated as we did for $\tilde{\mathcal{F}}^+_{k,i}$, i.e. with a change of variables

$$\bar{z} \longrightarrow (s_-, \omega_-, p_-) \tag{5.41}$$

where

$$\omega_- = (\bar{q}^{(1)}(s_-) - \xi''_i(s_-))/a\ , \qquad p_- = \bar{p}^{(1)}(s_-)\ . \tag{5.42}$$

Introducing $\boldsymbol{j}'_{m+1}$ as defined in (5.7), renaming

$$\begin{aligned}
&\boldsymbol{t}'_{k-1}=\boldsymbol{t}''_{k-1}, t'_k=s_-, t'_{k+1}=t''_k,\cdots, t'_\ell=t''_{\ell-1}, t'_{\ell+1}=t_\ell, t'_{\ell+2}=t'''_1,\cdots, t'_{m+1}=t'''_{m-\ell},\\
&\boldsymbol{\omega}'_{k-1}=\boldsymbol{\omega}''_{k-1}, \omega'_k=\omega_-, \omega'_{k+1}=\omega''_k,\cdots, \omega'_\ell=\omega''_{\ell-1}, \omega'_{\ell+1}=\omega_\ell, \omega'_{\ell+2}=\omega'''_1,\cdots, \omega'_{m+1}=\omega'''_{m-\ell},\\
&\hat{\boldsymbol{p}}'_{k-1}=\hat{\boldsymbol{p}}''_{k-1}, \hat{p}'_k=p_-, \hat{p}'_{k+1}=\hat{p}''_k,\cdots, \hat{p}'_\ell=\hat{p}''_{\ell-1}, \hat{p}'_{\ell+1}=\hat{p}_\ell, \hat{p}'_{\ell+2}=\hat{p}'''_1,\cdots, \hat{p}'_{m+1}=\hat{p}'''_{m-\ell},
\end{aligned} \tag{5.43}$$

and excluding sets of measure zero, we see that the above change of variables defines an invertible transformation from $\tilde{\mathcal{F}}^-_{k,i}$ onto $\mathcal{F}^-_{k,i}$, with

$$d\overline{z}=-B(\omega'_k;\hat{p}'_k-\pi'_{j'_k}(t'_k))dt'_k d\omega'_k d\hat{p}'_k\ . \tag{5.44}$$

Summarising, Lemma 1 is true for all trees of the type $\mathcal{T}'$ when the variables are restricted to $\mathcal{F}^-_{k,i}$, and

$$-\int_{\mathcal{C}^+} dz_{n+1}d\mu_m \mathbb{1}_{\tilde{\mathcal{F}}^-_{k,i}}\left(\prod_{r=1}^m B_r\right)\rho^0_{n+1+m}(\boldsymbol{\zeta}(0))=V|_{\mathcal{F}^-_{k,i}}(\mathcal{T}')\ , \tag{5.45}$$

with this satisfying the same estimate of $V|_{\mathcal{F}^+_{k,i}}$ in (5.31).

So far we have proved Lemma 1 for all trees $\mathcal{T}'$ with a restriction of the $k-$th node variables given by the definitions of $\mathcal{F}^+_{k,i},\mathcal{F}^-_{k,i}$. Observe that this restriction can be immediately eliminated by using the arbitrariness of the time interval $(0,t)$ and the invariance of the set $\Gamma^\dagger_{n+m}$. Varying $\mathcal{T}_{n+1,m}$ in the hypotheses of Proposition 3, we conclude that Lemma 1 holds for all $\mathcal{T}_{n,m}$ with $m\le N-n$.

To prove Lemma 2, we are left with the term "1" in (5.39). There are two cases.

Case $\ell=m+1$. Formula (5.36), without the cutoff $\mathbb{1}$, reduces to

$$\int_{\mathcal{C}_{\mathcal{T}''}(\boldsymbol{z}_n,t)} d\mu''_m\left(\prod_{r=1}^m B''_r\right)\int_{\Gamma_1(\boldsymbol{\zeta}''_{n+m}(0))} d\overline{z}\ \rho^0_{n+m+1}(\boldsymbol{\zeta}''(0),\overline{z})\ . \tag{5.46}$$

Using Eq. (5.2), this gives the term $(N-n-m)\,V(\mathcal{T}'')$.

Case $\ell<m+1$. Formula (5.36), without the cutoff $\mathbb{1}$, reduces to zero. Indeed, consider

$$\int_{\{\mathbb{R}^6\times S^2\times\mathbb{R}^3,\ \boldsymbol{z}'''_{n+\ell+1}\in\Gamma_{n+\ell+1}\}} d\overline{z}d\omega_\ell d\hat{p}_\ell\ B(\omega_\ell;\hat{p}_\ell-\overline{p})\ V(\mathcal{T}''')(\boldsymbol{z}'''_{n+\ell+1},t_\ell)\ . \tag{5.47}$$

Almost surely over the domain, the elastic scattering defines a one–to–one mapping between outgoing and incoming collision configurations $(\overline{z},\overline{q}+a\omega_\ell,\overline{p})$. Under this mapping the factor B in (5.47) changes sign, while $V(\mathcal{T}''')$ is preserved.

Summing all contributions, we obtain Eq. (5.18). This ends the proof of Lemma 2. □

Proof of Lemma 3

In what follows we shall indicate explicitly as $\mathcal{T}'(k,i)$ the dependence on k,i of the trees of type $\mathcal{T}'$.

Let us focus on $\mathcal{T}'(k,i),\mathcal{R}^-_{k,i}$. This collects the collision histories such that particle $n+k$, after having been generated by particle i in an incoming collision, "recollides" with some other particle of the history (see the comment before (3.9) about this terminology). Given one of such histories, let us *erase* the *free* flow of particle $n+k$ from the moment of generation (t'_k) to the moment of recollision, and think that the particle *appears* at the recollision time in an outgoing collision configuration. In other words, we transform the recollision in a creation. What we obtain is a new collision history, which will be associated to some $\mathcal{T}'(k^*,i^*)$ and will obey the constraint of $\mathcal{R}^+_{k^*,i*}$. Roughly speaking, the two related collision histories "cancel" each other in the computation of the left hand side of (5.19).

To make precise the last assertion, we decompose further the domains by specifying which particle recollides with $n+k$ and in which time interval the recollision occurs. Fixed a tree $\mathcal{T}'(k,i)$, we introduce a set $\mathcal{R}^-_{k,i;k^*,j}$

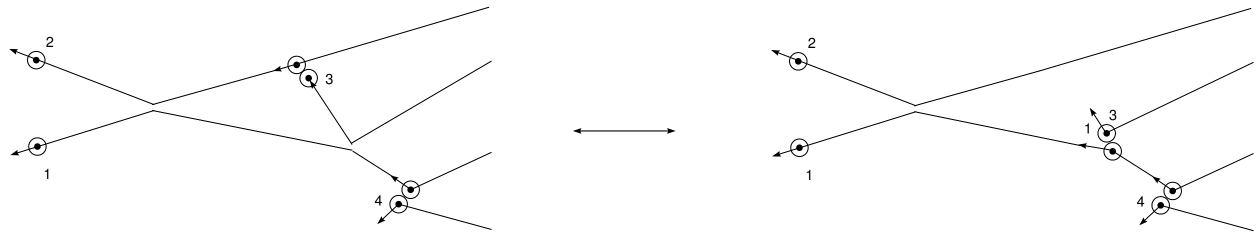

FIG. 5: Cancellations between collision histories. A case with $k = 1, i = 2, k^* = 1, i^* = 1$.

selecting the collision histories such that (i) particle $n+k$ is generated by particle i in an incoming collision; (ii) particle $n+k$ recollides with particle j of the history; (iii) such a recollision occurs in the time interval of the history (t'_{k^*+1}, t'_{k^*}). A similar notation is introduced for the + case. In formulas,

$$\begin{aligned}
&\sum_{k=1}^{\ell}\sum_{i=1}^{n+k-1}\Big(V|_{\mathcal{R}^+_{k,i}}(\mathcal{T}'(k,i)) + V|_{\mathcal{R}^-_{k,i}}(\mathcal{T}'(k,i))\Big) \\
&= \sum_{1\le k\le k^*\le \ell}\sum_{i=1}^{n+k-1}\Big[\Big(\sum_{j=1}^{n+k^*-1} V|_{\mathcal{R}^+_{k^*,j;k,i}}(\mathcal{T}'(k^*,j))\Big) + \Big(\sum_{\substack{j=1\\ j\neq n+k}}^{n+k^*} V|_{\mathcal{R}^-_{k,i;k^*,j}}(\mathcal{T}'(k,i))\Big)\Big]
\end{aligned} \tag{5.48}$$

where

$$\begin{aligned}
\mathcal{R}^+_{k^*,j;k,i} &= \Big\{(\boldsymbol{t}'_{m+1}, \boldsymbol{\omega}'_{m+1}, \hat{\boldsymbol{p}}'_{m+1}) \in \mathcal{R}^+_{k^*,j} \;\; \text{s.t.} \;\; s_+ \in (t'_k, t'_{k-1}),\; i_+ = i\Big\}\,, \\
\mathcal{R}^-_{k,i;k^*,j} &= \Big\{(\boldsymbol{t}'_{m+1}, \boldsymbol{\omega}'_{m+1}, \hat{\boldsymbol{p}}'_{m+1}) \in \mathcal{R}^-_{k,i} \;\; \text{s.t.} \;\; s_- \in (t'_{k^*+1}, t'_{k^*}),\; i_- = j\Big\}\,.
\end{aligned} \tag{5.49}$$

In (5.49) s_+, i_+, s_-, i_- are those appearing in the definition of $\mathcal{R}^+_{k^*,j}, \mathcal{R}^-_{k,i}$. Notice that, in the second sum over j of (5.48), the value $n+k$ is obviously missing, since particle $n+k$ cannot recollide with itself.

Fix an integral term $V|_{\mathcal{R}^-_{k,i;k^*,j}}$ of the above sum. Remember that this is an integral over a subset of $\mathcal{C}_{\mathcal{T}'(k,i)}(\boldsymbol{z}_n, t)$, i.e. the node–variables associated to the tree $\mathcal{T}'(k,i)$. We change the variables of integration according to

$$(t'_k, \omega'_k, \hat{p}'_k) \longrightarrow (s_-, \omega_-, p_-)\,, \tag{5.50}$$

where s_- is defined in (5.16), (5.49), and

$$\omega_- = (\xi'_{n+k}(s_-) - \xi'_j(s_-))/a\,, \qquad p_- = \pi'_{n+k}(s_-)\,. \tag{5.51}$$

With the renaming

$$\begin{aligned}
&\boldsymbol{t}'_{k-1} \to \boldsymbol{t}'_{k-1}, (t'_{k+1}, \cdots, t'_{k^*}) \to (t'_k, \cdots, t'_{k^*-1}), s_- = t'_{k^*}, (t'_{k^*+1}, \cdots, t'_{m+1}) \to (t'_{k^*+1}, \cdots, t'_{m+1}), \\
&\boldsymbol{\omega}'_{k-1} \to \boldsymbol{\omega}'_{k-1}, (\omega'_{k+1}, \cdots, \omega'_{k^*}) \to (\omega'_k, \cdots, \omega'_{k^*-1}), \omega_- = \omega'_{k^*}, (\omega'_{k^*+1}, \cdots, \omega'_{m+1}) \to (\omega'_{k^*+1}, \cdots, \omega'_{m+1}), \\
&\hat{\boldsymbol{p}}'_{k-1} \to \hat{\boldsymbol{p}}'_{k-1}, (\hat{p}'_{k+1}, \cdots, \hat{p}'_{k^*}) \to (\hat{p}'_k, \cdots, \hat{p}'_{k^*-1}), \hat{p}_- = \hat{p}'_{k^*}, (\hat{p}'_{k^*+1}, \cdots, \hat{p}'_{m+1}) \to (\hat{p}'_{k^*+1}, \cdots, \hat{p}'_{m+1}),
\end{aligned} \tag{5.52}$$

we obtain an invertible map (modulo sets of measure zero) onto $\mathcal{R}^+_{k^*,i^*;k,i}$, that is a subset of the node–variables associated to the tree $\mathcal{T}'(k^*, i^*)$, with

$$i^* = \begin{cases} j & \text{if} \quad j < n+k \\ j-1 & \text{if} \quad j > n+k \end{cases}\,. \tag{5.53}$$

Now, observe that the Jacobian determinant is given by the relation

$$-B(\omega'_k; \hat{p}'_k - \pi'_{j'_k}(t'_k))dt'_k d\omega'_k d\hat{p}'_k = B(\omega_-; p_- - \pi'_j(s_-))ds_- d\omega_- dp_-\,, \tag{5.54}$$

the minus sign coming from the fact that the variables appearing in the l.h.s. describe an incoming collision, while the variables appearing in the r.h.s. describe an outgoing collision. Therefore, the net effect of the transformation is

$$V|_{\mathcal{R}^-_{k,i;k^*,j}}(\mathcal{T}'(k,i)) = -V|_{\mathcal{R}^+_{k^*,i^*;k,i}}(\mathcal{T}'(k^*,i^*)) \ . \tag{5.55}$$

Inserting this into Eq. (5.48), we obtain Lemma 3. □

C. The sum over trees

To prove Theorem 1, it remains to substitute Eq. (5.5) into Eq. (5.3) and perform the sum over trees. This can be achieved conveniently by working directly on the graphs, as shown in Figure 4. The rules given by the list on page 13 tell us which trees appear on the right hand side of (5.5). (Notice that, by applying the rule in step 1 with different values of k, i, a tree $\mathcal{T}_{n+1,m}$ can even produce more copies of the same tree $\mathcal{T}'_{n,m+1}$.) Hence, it is sufficient to check that any given $n-$particle, $m-$node tree $\mathcal{T}_{n,m}$ can be produced in exactly $N-n$ copies, by applying the rules to different $(n+1)-$particle trees. This follows from the remarks: (i) $\mathcal{T}_{n,m}$ is produced in $N-n-m$ copies by operation 3 of the list; (ii) $\mathcal{T}_{n,m}$ is produced by creating its node k, applying operation 1 of the list to a suitable $(n+1)-$particle tree. Summing up, we have $N-n-m+m = N-n$ copies.

The analogous algebraic proof is as follows:

$$\begin{aligned}
\rho_n(\boldsymbol{z}_n,t) &= \frac{1}{N-n}\sum_{m=0}^{\infty}\sum_{\mathcal{T}_{n+1,m}}\Big[\delta_{\ell,m+1}\,(N-n-m)\,V(\mathcal{T}''_{n,m}) + \sum_{k=1}^{\ell}\sum_{i=1}^{n+k-1} V(\mathcal{T}'_{n,m+1})\Big]\\
&= \frac{1}{N-n}\sum_{m=0}^{\infty}(N-n-m)\sum_{\substack{j_1,\cdots,j_m\\ j_r\in I_{n+r}\\ j_r\neq n+1}} V(\mathcal{T}''_{n,m}) + \frac{1}{N-n}\sum_{m=1}^{\infty}\sum_{\substack{j_1,\cdots,j_{m-1}\\ j_r\in I_{n+r}}}\sum_{k=1}^{\ell}\sum_{i=1}^{n+k-1} V(\mathcal{T}'_{n,m})\\
&= \frac{1}{N-n}\sum_{m=0}^{\infty}(N-n-m)\sum_{\substack{j''_1,\cdots,j''_m\\ j''_r\in I_{n+r-1}}} V(\mathcal{T}''_{n,m}) + \frac{1}{N-n}\sum_{m=1}^{\infty}\sum_{k=1}^{m}\sum_{\substack{j_1,\cdots,j_{m-1}\\ j_r\in I_{n+r}\\ j_1,\cdots,j_{k-1}\neq n+1}}\sum_{i=1}^{n+k-1} V(\mathcal{T}'_{n,m})\\
&= \frac{1}{N-n}\sum_{m=0}^{\infty}(N-n-m)\sum_{\mathcal{T}''_{n,m}} V(\mathcal{T}''_{n,m}) + \frac{1}{N-n}\sum_{m=1}^{\infty}\sum_{k=1}^{m}\sum_{\substack{j'_1,\cdots,j'_m\\ j'_r\in I_{n+r-1}}} V(\mathcal{T}'_{n,m})\\
&= \frac{1}{N-n}\sum_{m=0}^{\infty}(N-n-m)\sum_{\mathcal{T}''_{n,m}} V(\mathcal{T}''_{n,m}) + \frac{1}{N-n}\sum_{m=1}^{\infty} m\sum_{\mathcal{T}'_{n,m}} V(\mathcal{T}'_{n,m})\\
&= \sum_{m=0}^{\infty}\sum_{\mathcal{T}_{n,m}} V(\mathcal{T}_{n,m}) \ ,
\end{aligned} \tag{5.56}$$

where in the third and in the fourth line we have used respectively the definitions (5.6) and (5.7). □

D. The BBGKY hierarchy. Proof of Corollary 1

Let us rewrite the expansion (3.9) in a resummed form, which is convenient to obtain informations about the derivative.

We have proven that the integrals (3.10) are absolutely convergent, so that the integration order can be exchanged freely. Then, in the hypotheses of Corollary 1, fixed $\boldsymbol{z}_n \in \Gamma_n^\dagger$ and $t > 0$, we have

$$\begin{aligned}
\rho_n(\boldsymbol{z}_n,t) &= \mathcal{T}_{n,0}(\boldsymbol{z}_n,t) + \sum_{\mathcal{T}_{n,1}} V(\mathcal{T}_{n,1})(\boldsymbol{z}_n,t) + \sum_{m>1}\sum_{\mathcal{T}_{n,m}} V(\mathcal{T}_{n,m})(\boldsymbol{z}_n,t) \\
&= \rho_n^0(T_{-t}^{(n)}\boldsymbol{z}_n) + \sum_{j_1=1}^{n} \int_{(0,t)\times\mathbb{R}^3\times\Omega_{j_1}(T_{-t+t_1}^{(n)}\boldsymbol{z}_n)} dt_1 d\hat{p}_1 d\omega_1 \; a^2\omega_1\cdot\left(\hat{p}_1 - p_{j_1}^{(n)}(t_1)\right) \\
&\qquad \cdot\Bigg[\rho_{n+1}^0\left(T_{-t_1}^{(n+1)}\left(T_{-t+t_1}^{(n)}\boldsymbol{z}_n, q_{j_1}^{(n)}(t_1)+a\omega_1,\hat{p}_1\right)\right) \\
&\qquad + \sum_{m\geq 1}\sum_{\mathcal{T}_{n+1,m}} V\left(\mathcal{T}_{n+1,m}\right)\left(T_{-t+t_1}^{(n)}\boldsymbol{z}_n, q_{j_1}^{(n)}(t_1)+a\omega_1,\hat{p}_1,t_1\right)\Bigg] , \qquad (5.57)
\end{aligned}$$

where $q_{j_1}^{(n)}(t_1), p_{j_1}^{(n)}(t_1)$ are position and momentum of particle j_1 in $T_{-t+t_1}^{(n)}\boldsymbol{z}_n$. In the last term we have put together the one–node trees and the higher order trees, $dt_1 d\hat{p}_1 d\omega_1$ being the integration associated to the first node.

By Corollary 3, we may use again Equation (3.9) to identify the term in the square brackets with a $\rho_{n+1}(\cdot,t_1)$, that is

$$\begin{aligned}
\rho_n(\boldsymbol{z}_n,t) &= \rho_n^0(T_{-t}^{(n)}\boldsymbol{z}_n) + \sum_{j_1=1}^{n} \int_{(0,t)\times\mathbb{R}^3\times\Omega_{j_1}(T_{-t+t_1}^{(n)}\boldsymbol{z}_n)} dt_1 d\hat{p}_1 d\omega_1 \; a^2\omega_1\cdot\left(\hat{p}_1 - p_{j_1}^{(n)}(t_1)\right) \\
&\qquad \cdot\rho_{n+1}\left(T_{-t+t_1}^{(n)}\boldsymbol{z}_n, q_{j_1}^{(n)}(t_1)+a\omega_1,\hat{p}_1,t_1\right) . \qquad (5.58)
\end{aligned}$$

This formula is the *resummed* form of the expansion for the correlation functions, in the sense that iterating the equation $N-n$ times we are back to the Equation (3.9).

Remind now that the Liouville equation can be also written as $f_N(T_t^{(N)}\boldsymbol{z}_N,t) = f_N(\boldsymbol{z}_N)$ and that, being $\Gamma_n^\dagger$ invariant, $\rho_n(T_t^{(n)}\boldsymbol{z}_n,t) = N\ldots(N-n+1)\int_{\Gamma_{N-n}(T_t^{(n)}\boldsymbol{z}_n)} dz_{n+1}\ldots dz_N f_N(T_t^{(n)}\boldsymbol{z}_n, z_{n+1},\cdots,z_N,t)$ for all $\boldsymbol{z}_n \in \Gamma_n^\dagger$. In particular, we may substitute $\boldsymbol{z}_n \to T_t^{(n)}\boldsymbol{z}_n$ in (5.58). Recalling (4.2), we obtain

$$\rho_n(T_t^{(n)}\boldsymbol{z}_n,t) = \rho_n(\boldsymbol{z}_n,0) + \int_0^t dt_1\left(Q\rho_{n+1}\right)(T_{t_1}^{(n)}(\boldsymbol{z}_n),t_1) \qquad (5.59)$$

where, by Fubini's theorem, the integral in dt_1 is well defined. Eq. (5.59) shows that, for all $\boldsymbol{z}_n \in \Gamma_n^\dagger$, the function $t \to (Q\rho_{n+1})(T_t^{(n)}(\boldsymbol{z}_n),t)$ is absolutely continous, with derivative satisfying (4.3) for almost all times. □

E. Indefinite number of particles. Proof of Corollary 2

Each term in the sum in Equation (4.6) may be dealed with the procedure explained in the previous sections. This leads directly to a tree expansion like the one in the right hand side of (3.9), in which the value of the tree, say $\tilde{V}(\mathcal{T}_{n,m})$, must be computed in a slightly different way. Namely, ρ_{n+m}^0 in (3.10) is replaced by

$$\frac{1}{(k-m)!}\int_{\Gamma_{k-m}(\boldsymbol{\zeta}(0))} dz_{n+m+1}\cdots dz_{n+k} f_{n+k}^0(\boldsymbol{\zeta}(0), z_{n+m+1},\cdots,z_{n+k}) . \qquad (5.60)$$

Performing the sum over k, that is $\sum_{k\geq m}^{\infty}$, and using (4.6), we recover Eq. (3.9). □

6. CONCLUSIONS

In this work we discussed a derivation of the series expansion used by Lanford [8] to perform the Boltzmann–Grad limit, expressing the time–evolved $n-$point correlation function in terms of the higher order correlation functions at time zero for a system of N hard spheres in a finite volume. We established a method of construction of the series based on step by step direct integration of degrees of freedom from the solution of Liouville equation, rather than the usual iteration of the BBGKY equations. Each term of the expansion was written in the form of integral over a class of special evolutions of particles called "collision histories", for which we could introduce a convenient graphical representation. These graphs are useful to control the integration procedure leading from the expansion for ρ_{n+1} to the expansion for ρ_n. Mutual cancellations between collision histories showing special "recollision properties" were exhibited as an important part of the proof.

The method provides a construction of the series expansion in a fixed full measure subset of the phase space, under the only hypotheses of some integrability bound for the density of the initial measure, and symmetry in the particle labels. This strengthens results previously obtained in literature. Furthermore, without assuming continuity along trajectories of the initial measure, we could resum the final expansion and recover the BBGKY hierarchy of integro–differential equations for hard spheres. Finally, we stated an extension of the results to initial measures with non definite number of particles.

Acknowledgements. The author thanks J. L. Lebowitz for his invitation at Rutgers University, where the idea of this work was conceived, and acknowledges G. Gallavotti for proposing the work and for fundamental suggestions. The author thanks also G. Genovese, G. Gentile, A. Giuliani, A. Pellegrinotti, M. Pulvirenti, C. Saffirio and, in particular, H. Spohn for helpful discussions and encouragement.

Appendix. On the dynamics of hard spheres

In this appendix we prove Proposition 2. Unfortunately, it is not clear whether the two sets $\Gamma_n^\dagger$ and Γ_n^* coincide. Therefore, we will deduce Proposition 2 from Proposition 1 by using an abstract argument.

It is sufficient to prove the assertion for any finite bound on the energy. A little abuse of notation will be used in this section: we indicate with the usual symbols $\Gamma_n, \Gamma_n^*, \Gamma_n^\dagger$... the bounded sets corresponding to an energy of the (whole) system not larger than $E > 0$. We denote with $|\cdot|$ the Lebesgue measure on $\mathbb{R}^{6n}$ and with $|\cdot|_{\text{ext}}$ the associated outer measure, defined as $|A|_{\text{ext}} = \inf_{\{C_n\}_{n\geq 1}} \sum_n |C_n|$ where the infimum is taken over all possible collections of boxes such that $A \subset \cup_n C_n$. The proof will make use of two simple properties of the outer measure: first, the flow preserves outer measure, i.e.

$$|A|_{\text{ext}} = |T_t^{(n)} A|_{\text{ext}} \;, \qquad\qquad A \subset \Gamma_n^* \;, \tag{6.1}$$

which follows from the fact that the flow is an invertible and measure preserving transformation; second, if $B_{\boldsymbol{z}_n}$ is a collection of sets in $\mathbb{R}^{6k}$ indexed by $\boldsymbol{z}_n \in A \subset \mathbb{R}^{6n}$ and such that $|B_{\boldsymbol{z}_n}| > 0$ uniformly in A, then

$$|A|_{\text{ext}} \neq 0 \implies \Big|\Big\{ (\boldsymbol{z}_n, \boldsymbol{y}_k) \;\Big|\; \boldsymbol{z}_n \in A, \boldsymbol{y}_k \in B_{\boldsymbol{z}_n} \Big\}\Big|_{\text{ext}} \neq 0 \;. \tag{6.2}$$

Let us define the "bad sets of adjoint points"

$$B_{k,\boldsymbol{z}_n} = \Big\{ \boldsymbol{y}_k \in \Gamma_k(\boldsymbol{z}_n) \;\Big|\; (\boldsymbol{z}_n, \boldsymbol{y}_k) \in \Gamma_{n+k} \setminus \Gamma_{n+k}^* \Big\} \;. \tag{6.3}$$

As a consequence of Proposition 1, the following subset of Γ_n must be null:

$$Z = \bigcup_{k=1}^{N-n} Z_k \;, \qquad Z_k = \Big\{ \boldsymbol{z}_n \in \Gamma_n^* \;\Big|\; |B_{k,\boldsymbol{z}_n}|_{\text{ext}} > 0 \Big\} \tag{6.4}$$

(otherwise, by (6.2) we could find a subset of $\Gamma_{n+k} \setminus \Gamma_{n+k}^*$ of positive outer measure).

We do not know if Z is invariant under the flow. Nevertheless, to conclude the proof, it is enough to show that

$$\Big| \bigcup_{s\in\mathbb{R}} T_s^{(n)} Z \Big|_{\text{ext}} = 0 \;, \tag{6.5}$$

since then the complement of this set in Γ^*_n would satisfy all the properties stated in the proposition. Given any sequence of positive numbers $\varepsilon_q \to 0$, it is thus sufficient to prove that

$$\Big|\bigcup_s T_s^{(n)} Z_{k,q}\Big|_{\text{ext}} = 0 \;, \qquad Z_{k,q} = \Big\{ \boldsymbol{z}_n \in \Gamma^*_n \;\Big|\; |B_{k,\boldsymbol{z}_n}|_{\text{ext}} > \varepsilon_q \Big\} \;. \tag{6.6}$$

For $\boldsymbol{z}_n \in \Gamma^*_n,\ \boldsymbol{y}_k \in \Gamma^*_k \cap \Gamma_k(\boldsymbol{z}_n)$, we define the time of first forward interaction between $\boldsymbol{z}_n$ and $\boldsymbol{y}_k$

$$\tau(\boldsymbol{z}_n;\boldsymbol{y}_k) = \inf\Big\{ t>0 \;\Big|\; T_t^{(n+k)}(\boldsymbol{z}_n,\boldsymbol{y}_k) = \Big(T_t^{(n)}\boldsymbol{z}_n, T_t^{(k)}\boldsymbol{y}_k\Big) \Big\} \;, \tag{6.7}$$

and we call

$$B^{(\delta)}_{k,\boldsymbol{z}_n} = B_{k,\boldsymbol{z}_n} \bigcap \Big\{ \boldsymbol{y}_k \in \Gamma^*_k \cap \Gamma_k(\boldsymbol{z}_n) \;\Big|\; \tau(\boldsymbol{z}_n;\boldsymbol{y}_k) > \delta \Big\} \;, \qquad \delta > 0 \;. \tag{6.8}$$

Observe that, by the bound on the energy, the set of values of $\boldsymbol{y}_k$ in the domain of τ such that $\tau(\boldsymbol{z}_n;\boldsymbol{y}_k) \leq \delta$ has a measure that goes to zero with δ, uniformly in $\boldsymbol{z}_n$. Hence we can find a $\delta_q > 0$ such that $|B^{(\delta_q)}_{k,\boldsymbol{z}_n}|_{\text{ext}} > \varepsilon_q/2$ for all $\boldsymbol{z}_n \in Z_{k,q}$. For such a choice we deduce that

$$0 = \Big| \bigcup_{s\in[0,\delta_q]} \bigcup_{\boldsymbol{z}_n \in Z_{k,q}} T_s^{(n+k)} \Big(\boldsymbol{z}_n, B^{(\delta_q)}_{k,\boldsymbol{z}_n}\Big) \Big|_{\text{ext}} = \Big| \bigcup_{s\in[0,\delta_q]} \bigcup_{\boldsymbol{z}_n \in Z_{k,q}} \Big(T_s^{(n)}\boldsymbol{z}_n, T_s^{(k)} B^{(\delta_q)}_{k,\boldsymbol{z}_n}\Big) \Big|_{\text{ext}} \;, \tag{6.9}$$

where the first equality is true because the set is contained in $\Gamma_{n+k} \setminus \Gamma^*_{n+k}$ (applying again Proposition 1). By (6.1), $|T_s^{(k)} B^{(\delta_q)}_{k,\boldsymbol{z}_n}|_{\text{ext}} > \varepsilon_q/2$. Therefore by (6.2) we have that

$$\Big| \bigcup_{s\in[0,\delta_q]} T_s^{(n)} Z_{k,q} \Big|_{\text{ext}} = 0 \;. \tag{6.10}$$

Since $\bigcup_s T_s^{(n)} Z_{k,q} = \bigcup_{j\in\mathbb{Z}} T^{(n)}_{j\delta_q} \bigcup_{s\in[0,\delta_q]} T_s^{(n)} Z_{k,q}$, Eq. (6.6) follows. The proof of Proposition 2 is complete. $\square$

We add now some other useful remark concerning the dynamics of hard spheres. Consider the set of "collision surfaces", i.e. the boundary of the phase space $\partial\Gamma_n$. On it we define the induced Lebesgue measure $d\sigma(\boldsymbol{z}_n)$. The restriction of $d\sigma(\boldsymbol{z}_n)$ to the set where particles i and j are colliding, with $q_j = q_i + a\omega$, is $dz_1 \cdots dz_i \cdots dz_{j-1} dp_j d\omega dz_{j+1} \cdots dz_n$, while the restriction to the set in which particle i is colliding with the wall, $q_i = q + (a/2)n(q), q \in \partial\Lambda$, is $dz_1 \cdots dz_{i-1} dq dp_i dz_{i+1} \cdots dz_n$. Of course the prescription assigns measure zero to the set of multiple collisions, grazing collisions and singular collisions with the wall. Let us call $\partial\Gamma^+_n$ ($\partial\Gamma^-_n$) the subset of points that can be reached continuously from the interior of Γ_n through the backwards (forward) free flow. $\partial\Gamma^+_n$ ($\partial\Gamma^-_n$) includes all the regular outgoing (incoming) collisions, plus some singular configuration. Excluding the singular points, the collision rule establish an invertible and measure preserving transformation between $\partial\Gamma^+_n$ and $\partial\Gamma^-_n$. Let $\tau_\pm(\boldsymbol{z}_n) = \inf\{t > 0 \text{ s.t. } T^{(n)}_{\pm t}\boldsymbol{z}_n \in \partial\Gamma_n\}$, i.e. the first forward (+) or backwards (−) collision time after zero. The connection of $d\sigma$ with the measure $d\boldsymbol{z}_n$ over Γ_n is made through the map $\boldsymbol{z}_n \to (\boldsymbol{z}'_n = T^{(n)}_{-\tau_-(\boldsymbol{z}_n)}\boldsymbol{z}_n, t' = \tau_-(\boldsymbol{z}_n))$, which is one–to–one from $\Gamma_n \setminus \partial\Gamma_n$ to the set $\{(\boldsymbol{z}'_n, t') \text{ s.t. } \boldsymbol{z}'_n \in \partial\Gamma^+_n, t' \in (0, \tau_+(\boldsymbol{z}'_n))\}$. Namely, we have $d\boldsymbol{z}_n = d\tilde\sigma(\boldsymbol{z}'_n)dt'$, where $d\tilde\sigma(\boldsymbol{z}'_n) = a^2\omega\cdot(p_j - p_i)d\sigma(\boldsymbol{z}'_n)$ if particle i and j are colliding, or $d\tilde\sigma(\boldsymbol{z}'_n) = p_i \cdot n(q) d\sigma(\boldsymbol{z}'_n)$ if particle i is colliding with the wall.

Remark (1). *Any full measure, invariant subset of Γ_n intersects $\partial\Gamma_n$ in a set which is full with respect to the induced Lebesgue measure.* In fact, if $A \subset \Gamma_n$ is full measure and invariant and A^c is its complement, then $0 = \int_{A^c} d\boldsymbol{z}_n = \int_{A^c\cap\partial\Gamma^+_n} d\tilde\sigma(\boldsymbol{z}'_n)\tau_+(\boldsymbol{z}'_n)$. Since the integrand is a.e. strictly positive, the statement follows.

Remark (2). *Any null measure subset of Γ_n is avoided by the $n-$particle flow $T_t^{(n)}\boldsymbol{z}_n$, for a.a. $(\boldsymbol{z}_n, t) \in \partial\Gamma_n \times \mathbb{R}$.* To prove this, we essentially follow [15]. Let now $A \subset \Gamma_n$ be a null measure subset. By the previous remark, points outside Γ^*_n are avoided for a.a. $\boldsymbol{z}_n \in \partial\Gamma_n$ and all t. Hence we may suppose $A \subset \Gamma^*_n$. For any given t, $T_t^{(n)}A$ exists and it is still null measure. But $\int_{T_t^{(n)}A} d\boldsymbol{z}_n =$

$\int_{\partial\Gamma_n^+\times\mathbb{R}^+} d\tilde{\sigma}(\boldsymbol{z}'_n)dt'\mathbb{1}_{\{t'<\tau_+(\boldsymbol{z}'_n)\}}\mathbb{1}_{\{T_{t'}^{(n)}\boldsymbol{z}'_n\in T_t^{(n)}A\}}$. This proves the assertion for t restricted to $\mathbb{R}^+$. The case $t\in\mathbb{R}^-$ is dealed in the same way.

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