Universal finite size corrections and the central charge in non solvable Ising models

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Abstract

We investigate a non solvable two-dimensional ferromagnetic Ising model with nearest neighbor plus weak finite range interactions of strength λ. We rigorously establish one of the predictions of Conformal Field Theory (CFT), namely the fact that at the critical temperature the finite size corrections to the free energy are universal, in the sense that they are exactly independent of the interaction. The corresponding central charge, defined in terms of the coefficient of the first subleading term to the free energy, as proposed by Affleck and Blote-Cardy-Nightingale, is constant and equal to 1/2 for all $0 \leq \lambda \leq \lambda_0$ and λ_0 a small but finite convergence radius. This is one of the very few cases where the predictions of CFT can be rigorously verified starting from a microscopic non solvable statistical model. The proof uses a combination of rigorous renormalization group methods with a novel partition function inequality, valid for ferromagnetic interactions.

Contents

1 Introduction and main results

The applications of Conformal Field Theory (CFT) to statistical mechanics are based on the assumption that a statistical model at the critical point admits a non-trivial, conformally invariant, scaling limit, as suggested by the renormalization theory of critical phenomena. The two-dimensional (2D) local scale invariance strongly constraints the structure of the critical theory, as understood by Belavin, Polyakov and Zamolodchikov [2]. They recognized that the theory is characterized by a dimensionless constant c, the central charge, which is associated with an anomaly term in the commutation relations of the stress energy tensor. The central charge can be also defined in terms of the finite size corrections to the free energy at criticality [1, 6]. In some cases, the critical theory is fully characterized by the value of c, which takes the form $c = 1 - 6/m(m+1)$, $m = 2, 3, 4, \ldots$ Once c is fixed to one of these special values, the critical exponents are all explicitly known in terms of the Kac formula [11].

In practice, the identification of the critical theory associated with a given microscopic lattice model is done by inspection, by trying to match the known informations about the lattice model's exponents with the Kac formula. Once a correspondence is established or guessed, a large number of non trivial predictions on the model's correlation functions at criticality can be inferred, which in general cannot be analytically derived by other means. It is therefore important to check these predictions in specific models, which could serve as benchmarks for this scheme. Unfortunately, there are just a few cases, based on exactly solvable lattice models, in which so far this correspondence could be rigorously established. A remarkable example is the nearest neighbor Ising model at the critical point, whose understanding in the scaling limit improved substantially in the last few months [7, 8, 9]. More in general, it is very hard to rigorously compute critical exponents, correlation functions or finite size corrections to thermodynamic functions at the critical point. In recent times new methods for the analysis of non-integrable 2D spin systems, based on the Renormalization Group, have been developed, starting from [24] and [21], where the authors computed the critical exponent of the so-called energy field operator in a class of perturbed Ising models (such as the one considered in this paper) and in a class of two stacked interacting Ising models (including the 8 vertex and the Ashkin-Teller models), respectively. By these methods one can try to verify some of the CFT predictions in the context of non solvable lattice models.

In particular in this paper we consider an Ising model with a generic ferromagnetic short range interaction, with Hamiltonian

$$H = -J \sum_{\mathbf{x}\in\Lambda_{\ell,L}} \sum_{j=1,2} \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\hat{\mathbf{e}}_j} - \lambda \sum_{\{\mathbf{x},\mathbf{y}\}} \sigma_{\mathbf{x}} v(\mathbf{x}-\mathbf{y})\sigma_{\mathbf{y}} \ , \tag{1.1}$$

where J and λ are positive constants, $\Lambda_{\ell,L} \subset \mathbb{Z}^2$ is a finite rectangular box of sides L and ℓ with periodic boundary conditions, $\sigma_{\mathbf{x}} \in \{\pm 1\}$, and $\hat{\mathbf{e}}_j$ are the two unit coordinate vectors on $\mathbb{Z}^2$. The sum in the second term of Eq.(1.1) is over all unordered pairs of sites in $\Lambda_{\ell,L}$; the interaction potential $v(\mathbf{x}-\mathbf{y})$ is rotation invariant, positive and has finite range, namely: $v(\mathbf{x}-\mathbf{y}) = 0$, $\forall |\mathbf{x}| > R_0 := M_0$, for a suitable positive integer M_0. With no loss of generality, we can assume that $v(\mathbf{0}) = v(\mathbf{e}_j) = 0$. The case $\lambda = 0$ corresponds to the nearest-neighbor Ising model which is exactly solvable [23, 19, 20, 27, 22]; in the case $\lambda \neq 0$ no solution is known but in [24] the exponent of the energy-energy correlation was computed and shown to be *universal* (i.e., equal to 2 as in the $\lambda = 0$ case), in contrast with the critical temperature or the amplitude of the correlations which are model dependent.

A number of key informations on the system are encoded in the *partition function*:

$$\mathcal{Z}_\beta(\Lambda_{\ell,L}) = \sum_{\underline{\sigma}\in\Lambda_{\ell,L}} e^{-\beta H(\underline{\sigma})} \ . \tag{1.2}$$

At all temperatures, the thermodynamic limit for the *pressure* is well-defined and independent of the speed at which ℓ and L are sent to infinity:

$$p_\beta = \lim_{\ell,L\to\infty} \frac{1}{\ell L} \log \mathcal{Z}_\beta(\Lambda_{\ell,L}) \ . \tag{1.3}$$

While the mere existence of the limit can be proved very generally, based on convexity and subadditivity arguments [25], the explicit form of p_β can be computed either by the Onsager's solution (for $\lambda = 0$) or by cluster expansion and Renormalization Group analysis (for λ sufficiently small, see below). It turns out that p_β depends explicitly on λ, at all temperatures β. Moreover, at all temperatures but the critical one, the limit is reached exponentially fast. At $\beta = \beta_c(\lambda)$ the limit is reached polynomially, and we shall define $f_\infty := p_{\beta_c(\lambda)}$. Remarkably, according to the ideas and methods of CFT, the finite size corrections to f_∞ are expected to be universal, in particular independent of λ and v. More precisely, in the presence of periodic boundary conditions, Ref.[1, 6] predicted the validity of the following formula, asymptotically for large ℓ:

$$\lim_{L\to\infty} \frac{\log \mathcal{Z}_{\beta_c(\lambda)}(\Lambda_{\ell,L})}{\ell L} = f_\infty + \frac{c\pi}{6}\frac{1}{\ell^2} + o(\frac{1}{\ell^2}) \ . \tag{1.4}$$

where c is the central charge of the critical theory. For the nearest neighbor Ising model $c = 1/2$ and supposedly the same should be true for perturbed Ising models of the form (1.1). Our main results is a rigorous confirmation of this expectation.

Theorem 1.1. *Given the model (1.1) with λ positive and small enough, there exists a critical temperature $\beta_c(\lambda)$ such that Eq.(1.4) holds with*

$$c = 1/2 \tag{1.5}$$

Recall that the critical exponent of the energy field operator is known to be universal and equal to 2 [24, 14], as expected from the use of the Kac formula at $c = 1/2$ (see e.g. [11]): therefore, our result says not only that the finite size corrections to the free energy are universal at criticality and that the corresponding central charge is constant, but also that such central charge has the *right value*, i.e. the one matching with the one guessed from the critical exponents and the Kac formula. This is a very non trivial connection, predicted by CFT, between critical exponents and finite size corrections to the thermodynamic functions, and our results is the first proof of the correctness of this prediction in a non-integrable statistical model. It would be very interesting to extend the connection to the critical exponent of the spin field operator, but this seems a much harder problem than the one solved here; on the basis of the above correspondence, such exponent is expected to be $1/4$, but its rigorous computation at $\lambda \neq 0$ is still beyond reach of the current techniques. Another very interesting extension would be to prove an analogous universality result for statistical models with $c = 1$, such as the 8 vertex or the Ashkin-Teller model. In these cases several critical exponents have been computed and proved to be λ-dependent [21, 15, 4, 5], as expected from CFT; at the same time, the finite size correction to the free energy are expected to be independent of λ and given by Eq.(1.4) with $c = 1$, but this fact is unproved so far. We hope to come back to this question in a future publication.

In the case of the nearest-neighbor Ising model, $\lambda = 0$, the result Eq.(1.5) was proved by [10] and follows from the exact solution. If $\lambda \neq 0$, the proof of our main theorem uses a Renormalization Group (RG) analysis first introduced in [24] for the computation of the two-point energy correlation function, and recently extended in [14] to the analysis of the n-point energy correlations. In addition to the ideas of [24, 14], we use here some novel partition function inequalities, which we can only prove for ferromagnetic interactions, $\lambda \geq 0$. The assumption of positivity of the interaction is expected to be technical and we believe that the analogue of Theorem 1.1 should hold also for λ small and negative.

More in detail, our proof proceeds as follows. We start by writing the partition function of the interacting Ising model as a sum of four Grassmann integrals with different boundary conditions, in a way that naturally extends the analogous representation for the nearest neighbor model. Using a partition function inequality, we reduce ourselves to the study of just one out of these four Grassmann partition functions, namely the one with antiperiodic boundary conditions; we prove that the other three terms are subleading and do not contribute to the central charge. On the other side, the Grassmann partition function with antiperiodic boundary conditions can be rewritten as the product of two terms: the first is equal to the non-interacting Grassmann partition function with renormalized parameters (this is the contribution from the "infrared fixed point"), while the rest includes all the corrections coming from finite infrared scales and from the irrelevant terms. Now, remarkably:

1. the dependence of the first factor upon the renormalized parameters can be scaled out by a simple change of variables, after which the factor takes the form $Z^{2\times Volume}\times$(free partition function), with Z the "wave function renormalization", which is volume-independent; therefore, the presence of the renormalized parameters changes the bulk pressure, but

not the finite size corrections;

2. the rest can be studied by a multiscale analysis, which requires the introduction of two running coupling constants, playing the role of wave function and critical temperature counterterms; these running coupling constant go to zero exponentially fast in the infrared limit, thanks to a dimensional improvement in the dimensional bounds following from the fact that the theory is super-renormalizable, in the sense that all the field operators with more than two fields are irrelevant in the RG sense; correspondingly, the finite size corrections to the pressure coming from these term can be shown to go to zero faster than ℓ^{-2} as the infrared cutoff ℓ is removed, namely like $\ell^{-2-\theta}$, for $0 < \theta < 1$.

The strategy resembles closely the one used in [16, 17] to prove the universality of the optical conductivity in interacting graphene. An extra difficulty that we have to face in our case is the definition of the localization and renormalization procedure at finite volume, which uses and extends the strategy proposed in [3].

The rest of the paper is organized as follows. In Section 2 we review the proof of Theorem 1.1 in the non-interacting case $\lambda = 0$. In Section 3 we prove the main theorem in the $\lambda \neq 0$ case: we first review the Grassmann representation of the interacting partition function (Section 3.1); then we state the partition function inequality that we use to effectively eliminate three out of the four Grassmann partition function (Section 3.2); next we describe the RG computation of the antiperiodic Grassmann partition function (Section 3.3), and we use it to compute the bulk and subleading contributions to the pressure (Section 3.4). In Section 4 we prove the partition function inequality stated in Section 3.2. Some technical issues are deferred to the Appendices: in Appendix A we give the details of the computation of the Grassmann partition functions in the nearest neighbor Ising model; in Appendix B, we study a subleading correction to the pressure coming from the ratio of the Grassmann partition functions, both in the non-interacting (Appendix B.1) and in the interacting case (Appendix B.2).

2 The nearest-neighbor Ising model

In this section we review the proof of our main theorem in the case of the nearest neighbor (n.n.) Ising model, $\lambda = 0$. The proof can be found in [10], but is reproduced here for the reader's convenience. From now on we shall drop the dependence on β in the symbol used for the partition function, in order to avoid a too cumbersome notation. We shall be as explicit as possible in distinguishing the formulas where the value of β is generic from those where β is fixed to be the critical one. We shall also assume that ℓ and L are even, in order to simplify the signs appearing in some formulas.

The starting point is the representation of the n.n. Ising model's partition function $\mathcal{Z}^0(\{J_b\};\Lambda_{\ell,L})$ with periodic boundary conditions and bond-dependent link variables in terms of a sum over multipolygons (see e.g. [12]):

$$\mathcal{Z}^0(\{J_b\};\Lambda_{\ell,L}) = 2^{\ell L}\Big[\prod_{b\in\mathcal{B}_{\ell,L}}\cosh(\beta J_b)\Big]\sum_{\Gamma\subseteq\Lambda_{\ell,L}}\prod_{\gamma\in\Gamma}\prod_{b\in\gamma}\tanh(\beta J_b)\ , \qquad (2.1)$$

where $\mathcal{B}_{\ell,L}$ is the set of n.n. bonds in $\Lambda_{\ell,L}$ and Γ is a collection of disjoint polygons in $\Lambda_{\ell,L}$, each of which is a closed connected collection of bonds; here closed means that at each of the vertices covered by the polygon there is an even number (either 2 or 4) of incident bonds. Note that some of the polygons in Γ can wind up the torus $\Lambda_{\ell,L}$, due to the periodic boundary conditions. Note also that the case $\Gamma=\emptyset$ is included in the sum, in which case the corresponding contribution is equal to 1.

A convenient (for computational purposes) way of re-expressing the partition sum Eq.(2.1) is by writing it in terms of Grassmann integrals, see e.g. [26, 15, 14]. Define

$$\begin{aligned}
&\mathcal{Z}^0_{\boldsymbol{\alpha}}(\{J_b\};\Lambda_{\ell,L}) = 2^{\ell L}\Big[\prod_{b\in\mathcal{B}_{\ell,L}}\cosh(\beta J_b)\Big]\int\mathcal{D}\Phi\, e^{S(\{J_b\};\Phi)}\ ,\\
&S(\{J_b\};\Phi) = \sum_{\mathbf{x}\in\Lambda}\Big[\tanh(\beta J_{(\mathbf{x},\mathbf{x}+\hat{\mathbf{e}}_1)})\overline{H}_{\mathbf{x}}H_{\mathbf{x}+\hat{\mathbf{e}}_1}+\tanh(\beta J_{(\mathbf{x},\mathbf{x}+\hat{\mathbf{e}}_2)})\overline{V}_{\mathbf{x}}V_{\mathbf{x}+\hat{\mathbf{e}}_2}\\
&+\overline{H}_{\mathbf{x}}H_{\mathbf{x}}+\overline{V}_{\mathbf{x}}V_{\mathbf{x}}+\overline{V}_{\mathbf{x}}\overline{H}_{\mathbf{x}}+V_{\mathbf{x}}\overline{H}_{\mathbf{x}}+H_{\mathbf{x}}\overline{V}_{\mathbf{x}}+V_{\mathbf{x}}H_{\mathbf{x}}\Big]\ . \qquad (2.2)
\end{aligned}$$

Here $\overline{H}_{\mathbf{x}},H_{\mathbf{x}},\overline{V}_{\mathbf{x}},V_{\mathbf{x}}$ are independent *Grassmann variables*, four for each lattice site, $\Phi=\{\overline{H}_{\mathbf{x}},H_{\mathbf{x}},\overline{V}_{\mathbf{x}},\ V_{\mathbf{x}}\}_{\mathbf{x}\in\Lambda}$ denotes the collection of all of these Grassmann symbols and $\mathcal{D}\Phi$ is a shorthand for $\prod_{\mathbf{x}}d\overline{H}_{\mathbf{x}}dH_{\mathbf{x}}d\overline{V}_{\mathbf{x}}dV_{\mathbf{x}}$. The label $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$, with $\alpha_1,\alpha_2\in\{\pm\}$, refers to the boundary conditions, which are periodic or antiperiodic in the horizontal (resp. vertical) direction, depending on whether α_1 (resp. α_2) is equal to $+$ or $-$. The connection between the Grassmann integral and the mutlipolygon representation can be made apparent by expanding the exponential inside the Grassmann integral and by integrating term by term. The result is

$$\begin{aligned}
\mathcal{Z}^0_{\boldsymbol{\alpha}}(\{J_b\};\Lambda_{\ell,L}) = 2^{\ell L}\Big[\prod_{b\in\mathcal{B}_{\ell,L}}\cosh(\beta J_b)\Big]\cdot \qquad (2.3)\\
\cdot\sum_{\Gamma\subseteq\Lambda_{\ell,L}}(-\alpha_1)^{h(\Gamma)}(-\alpha_2)^{v(\Gamma)}(-1)^{h(\Gamma)v(\Gamma)}\prod_{\gamma\in\Gamma}\prod_{b\in\gamma}\tanh(\beta J_b)\ ,
\end{aligned}$$

where $h(\Gamma)$ and $v(\Gamma)$ are the number of windings of Γ on the torus, in the horizontal and vertical directions, respectively. The r.h.s. of Eq.(2.3) is very similar to the multipolygon representation Eq.(2.1), modulo the sign $(-\alpha_1)^{h(\Gamma)}(-\alpha_2)^{v(\Gamma)}\ (-1)^{h(\Gamma)v(\Gamma)}$, which depends on whether the parity of the number of windings in the horizontal-vertical directions are even-even, or even-odd, or odd-even, or odd-odd. The value taken by this sign with different boundary conditions and different winding parities can be conveniently summarized in the following table.

	even-even	even-odd	odd-even	odd-odd
$\boldsymbol{\alpha} = (+,+)$	$+$	$-$	$-$	$-$
$\boldsymbol{\alpha} = (+,-)$	$+$	$+$	$-$	$+$
$\boldsymbol{\alpha} = (-,+)$	$+$	$-$	$+$	$+$
$\boldsymbol{\alpha} = (-,-)$	$+$	$+$	$+$	$-$

More explicitly, we can write:

$$\begin{aligned}\mathcal{Z}^0_{++}(\{J_b\};\Lambda_{\ell,L}) &= \mathcal{Z}^0_{e-e}(\{J_b\};\Lambda_{\ell,L}) - \\ &-\mathcal{Z}^0_{e-o}(\{J_b\};\Lambda_{\ell,L}) - \mathcal{Z}^0_{o-e}(\{J_b\};\Lambda_{\ell,L}) - \mathcal{Z}^0_{o-o}(\{J_b\};\Lambda_{\ell,L})\ ,\end{aligned} \tag{2.4}$$

$$\begin{aligned}\mathcal{Z}^0_{+-}(\{J_b\};\Lambda_{\ell,L}) &= \mathcal{Z}^0_{e-e}(\{J_b\};\Lambda_{\ell,L}) + \\ &+\mathcal{Z}^0_{e-o}(\{J_b\};\Lambda_{\ell,L}) - \mathcal{Z}^0_{o-e}(\{J_b\};\Lambda_{\ell,L}) + \mathcal{Z}^0_{o-o}(\{J_b\};\Lambda_{\ell,L})\ ,\end{aligned} \tag{2.5}$$

$$\begin{aligned}\mathcal{Z}^0_{-+}(\{J_b\};\Lambda_{\ell,L}) &= \mathcal{Z}^0_{e-e}(\{J_b\};\Lambda_{\ell,L}) - \\ &-\mathcal{Z}^0_{e-o}(\{J_b\};\Lambda_{\ell,L}) + \mathcal{Z}^0_{o-e}(\{J_b\};\Lambda_{\ell,L}) + \mathcal{Z}^0_{o-o}(\{J_b\};\Lambda_{\ell,L})\ ,\end{aligned} \tag{2.6}$$

$$\begin{aligned}\mathcal{Z}^0_{--}(\{J_b\};\Lambda_{\ell,L}) &= \mathcal{Z}^0_{e-e}(\{J_b\};\Lambda_{\ell,L}) + \\ &+\mathcal{Z}^0_{e-o}(\{J_b\};\Lambda_{\ell,L}) + \mathcal{Z}^0_{o-e}(\{J_b\};\Lambda_{\ell,L}) - \mathcal{Z}^0_{o-o}(\{J_b\};\Lambda_{\ell,L})\ ,\end{aligned} \tag{2.7}$$

where

$$\mathcal{Z}^0_{e-e}(\{J_b\};\Lambda_{\ell,L}) = 2^{\ell L}\Big[\prod_{b\in\mathcal{B}_{\ell,L}}\cosh(\beta J_b)\Big]\sum\nolimits^{(e-e)}_{\Gamma\subseteq\Lambda_{\ell,L}}\prod_{\gamma\in\Gamma}\prod_{b\in\gamma}\tanh(\beta J_b)\ , \tag{2.8}$$

and the superscript $(e-e)$ on the sum $\sum^{(e-e)}_{\Gamma\subseteq\Lambda_{\ell,L}}$ indicates the constraint that Γ winds over $\Lambda_{\ell,L}$ an even/even number of times in the horizontal/vertical direction, *including the case that* Γ *does not wind over the torus*; in other words, when we say "even number of windings", we include the case of zero windings. Of course, the other partition functions, namely $\mathcal{Z}^0_{e-o}(\{J_b\};\Lambda_{\ell,L})$, $\mathcal{Z}^0_{o-e}(\{J_b\};\Lambda_{\ell,L})$, $\mathcal{Z}^0_{o-o}(\{J_b\};\Lambda_{\ell,L})$, are defined similarly, with the constraint that Γ winds up over the torus an even/even number of times in the horizontal/vertical direction replaced by the one that Γ winds up an even/odd, odd/even, odd/odd number of times, respectively. By definition, the total partition function in Eq.(2.1) is

$$\begin{aligned} \mathcal{Z}^0(\{J_b\};\Lambda_{\ell,L}) = \mathcal{Z}^0_{e-e}(\{J_b\};\Lambda_{\ell,L}) + \\ +\mathcal{Z}^0_{e-o}(\{J_b\};\Lambda_{\ell,L}) + \mathcal{Z}^0_{o-e}(\{J_b\};\Lambda_{\ell,L}) + \mathcal{Z}^0_{o-o}(\{J_b\};\Lambda_{\ell,L}) \ . \end{aligned} \tag{2.9}$$

Alternatively, using Eqs.(2.4)–(2.7), we can also write

$$\begin{aligned} \mathcal{Z}^0(\{J_b\};\Lambda_{\ell,L}) = \frac{1}{2}\big(\mathcal{Z}^0_{--}(\{J_b\};\Lambda_{\ell,L}) + \\ +\mathcal{Z}^0_{-+}(\{J_b\};\Lambda_{\ell,L}) + \mathcal{Z}^0_{+-}(\{J_b\};\Lambda_{\ell,L}) - \mathcal{Z}^0_{++}(\{J_b\};\Lambda_{\ell,L})\big) \ , \end{aligned} \tag{2.10}$$

which is the desired connection between the multipolygon and the Grassmann representations. This relation is valid for all $\{J_b\}_{b\in\mathcal{B}_{\ell,L}}$ and all inverse temperatures β. On the other hand, if J_b is independent of b, the identity Eq.(2.10) gives us a mean to compute the partition function in closed form, simply because the Grassmann integrals $\mathcal{Z}^0_{\boldsymbol{\alpha}}(\Lambda_{\ell,L}) := \mathcal{Z}^0_{\boldsymbol{\alpha}}(\{J_b\};\Lambda_{\ell,L})\big|_{J_b\equiv J}$ are gaussian and translation invariant. The computation is based on the following elementary properties of Grassmann integrals. If ψ_i, $i\in\{1,\ldots,2n\}$, are Grassmann variables and A_{ij} is a $2n\times 2n$ antisymmetric matrix, one has

$$\int d\psi_1\cdots d\psi_{2n}\, e^{-\frac{1}{2}(\psi,A\psi)} = \frac{1}{2^n n!}\sum_\pi (-1)^\pi A_{\pi(1),\pi(2)}\ldots A_{\pi(2n-1),\pi(2n)} =: \mathrm{Pf}A \tag{2.11}$$

where π is a permutation of $(1,\ldots,2n)$, $(-1)^\pi$ is its signature and $\mathrm{Pf}A$ is by definition the *Pfaffian* of A. Moreover, if $\bar\psi_i,\psi_i$, with $i\in\{1,\ldots,m\}$ are two sets of Grassmann variables and B is an $m\times m$ matrix,

$$\int d\bar\psi_1 d\psi_1\cdots d\bar\psi_m d\psi_m\, e^{-(\bar\psi,B\psi)} = \det B \ . \tag{2.12}$$

Using these relations, we can obtain an explicit formula for $Z^0_{\alpha_1,\alpha_2}(\{J_b\};\Lambda_{\ell,L})$, $\alpha_i=\pm$, by proceeding in the following way: one first goes to Fourier space, thus block-diagonalizing the quadratic action appearing in the Grassmann integral; each block one is left with involves the degrees of freedom associated with the Fourier modes $\mathbf{k}$ and $-\mathbf{k}$, with $\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}$, and

$$\mathcal{D}_{\boldsymbol{\alpha}} = \Big\{\mathbf{k} = \Big(\frac{2\pi}{\ell}(r+\frac{1-\alpha_1}{4}),\frac{2\pi}{L}(n+\frac{1-\alpha_2}{4})\Big) : r=0,\ldots,\ell-1;\ n=0,\ldots,L-1\Big\} \ . \tag{2.13}$$

The computation of the Grassmann integral of the variables associated with each block is elementary and leads, using (2.11) or (2.13), to a determinant or to a Pfaffian, depending on whether $\mathbf{k}$ differs from $-\mathbf{k}$ or not. The result is the following. If $\boldsymbol{\alpha} \neq (+,+)$, then, defining $t = \tanh(\beta J)$ and $S_t(\Phi) := S(\{J_b\};\Phi)\big|_{J_b\equiv J}$,

$$\frac{\mathcal{Z}^0_{\boldsymbol{\alpha}}(\Lambda_{\ell,L})}{\big(2\cosh^2(\beta J)\big)^{\ell L}} = \int \mathcal{D}\Phi e^{S_t(\Phi)} = \prod_{\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}} [(1+t^2)^2 - 2t(1-t^2)(\cos k_1 + \cos k_2)]^{1/2} \,. \tag{2.14}$$

If, on the contrary, $\boldsymbol{\alpha} = (+,+)$,

$$\begin{aligned}\frac{\mathcal{Z}^0_{++}(\Lambda_{\ell,L})}{\big(2\cosh^2(\beta J)\big)^{\ell L}} &= \int \mathcal{D}\Phi e^{S_t(\Phi)} = (2-(1+t)^2)(2-(1-t)^2)\cdot \\ &\cdot \prod_{\substack{\mathbf{k}\in\mathcal{D}_{++}:\\ \mathbf{k}\neq\mathbf{0},(\pi,\pi)}} [(1+t^2)^2 - 2t(1-t^2)(\cos k_1 + \cos k_2)]^{1/2} \,,\end{aligned} \tag{2.15}$$

which is positive for $\beta < \beta_c$, negative for $\beta > \beta_c$ and vanishes at the critical point (the critical point β_c is defined by the condition that $t = \sqrt{2}-1$). The difference in the results obtained for $\boldsymbol{\alpha} \neq (+,+)$ or $\boldsymbol{\alpha} = (+,+)$ is due to the fact that in the first case all the modes $\mathbf{k} \in \mathcal{D}_{\boldsymbol{\alpha}}$ can be grouped into pairs $(\mathbf{k}, -\mathbf{k})$ and, correspondingly, the evaluation of the gaussian Grassmann integral reduces purely to a product over determinants (each determinant being the integral over the variables of the modes $\mathbf{k}$ and $-\mathbf{k}$). In the second case, all modes but two can be grouped into pairs, the two special momenta being $\mathbf{k} = \mathbf{0}$ and $\mathbf{k} = (\pi,\pi)$; therefore the result is equal to the product of the determinants associated with the paired momenta times the two Pfaffians coming from the modes $\mathbf{0}$ and (π,π), which give the factor $(2-(1+t)^2)(2-(1-t)^2)$ in the r.h.s. of Eq.(2.15). See Appendix A for details.

In conclusion, evaluating Eq.(2.10) at β_c in the translation invariant case and using the fact that the $(+,+)$ Grassmann partition function vanishes at criticality, we find:

$$\begin{aligned}\mathcal{Z}^0(\Lambda_{\ell,L})\Big|_{\beta=\beta_c} &= \frac{1}{2}\big(\mathcal{Z}^0_{--}(\Lambda_{\ell,L}) + \mathcal{Z}^0_{-+}(\Lambda_{\ell,L}) + \mathcal{Z}^0_{+-}(\Lambda_{\ell,L})\big)\Big|_{\beta=\beta_c} \\ &= \mathcal{Z}^0_{--}(\Lambda_{\ell,L})\Big[\frac{1}{2}\Big(1 + \frac{\mathcal{Z}^0_{-+}(\Lambda_{\ell,L})}{\mathcal{Z}^0_{--}(\Lambda_{\ell,L})} + \frac{\mathcal{Z}^0_{+-}(\Lambda_{\ell,L})}{\mathcal{Z}^0_{--}(\Lambda_{\ell,L})}\Big)\Big]\Big|_{\beta=\beta_c} \,,\end{aligned} \tag{2.16}$$

with

$$
\begin{aligned}
\mathcal{Z}^0_{--}(\Lambda_{\ell,L})\Big|_{\beta=\beta_c} &= (\sqrt{2})^{\ell L} \prod_{\mathbf{k}\in\mathcal{D}_{--}} (4-2\cos k_1 - 2\cos k_2)^{1/2} \\
\mathcal{Z}^0_{-+}(\Lambda_{\ell,L})\Big|_{\beta=\beta_c} &= (\sqrt{2})^{\ell L} \prod_{\mathbf{k}\in\mathcal{D}_{-+}} (4-2\cos k_1 - 2\cos k_2)^{1/2} \qquad (2.17)\\
\mathcal{Z}^0_{+-}(\Lambda_{\ell,L})\Big|_{\beta=\beta_c} &= (\sqrt{2})^{\ell L} \prod_{\mathbf{k}\in\mathcal{D}_{+-}} (4-2\cos k_1 - 2\cos k_2)^{1/2} \; .
\end{aligned}
$$

Taking the logarithm at both sides of Eq.(2.16), dividing by the volume, and taking the infinite volume limit, we get the bulk term f_∞, which is given by Onsager's formula:

$$
\begin{aligned}
f_\infty &= \lim_{\ell,L\to\infty} \frac{1}{\ell L} \log \mathcal{Z}^0(\Lambda_{\ell,L})\Big|_{\beta=\beta_c} \qquad (2.18)\\
&= \frac{1}{2}\log 2 + \frac{1}{2} \int_{[-\pi,\pi]^2} \frac{d\mathbf{k}}{(2\pi)^2} \log(4-2\cos k_1 - 2\cos k_2) \; .
\end{aligned}
$$

Using the notation of Eq.(1.4), we write the first finite volume correction to the critical pressure in the form $\frac{c\pi}{6\ell^2}$, with

$$
\begin{aligned}
\frac{c\pi}{6} &= \lim_{\ell\to\infty}\lim_{L\to\infty} \Big[\frac{\ell}{2L} \sum_{\mathbf{k}\in\mathcal{D}_{--}} \log(4-2\cos k_1 - 2\cos k_2) - \ell^2\big(f_\infty - \frac{1}{2}\log 2\big)\Big] \\
&+ \lim_{\ell\to\infty}\lim_{L\to\infty} \frac{\ell}{L} \log\Big[\frac{1}{2}\Big(1 + \frac{\mathcal{Z}^0_{-+}(\Lambda_{\ell,L})}{\mathcal{Z}^0_{--}(\Lambda_{\ell,L})} + \frac{\mathcal{Z}^0_{+-}(\Lambda_{\ell,L})}{\mathcal{Z}^0_{--}(\Lambda_{\ell,L})}\Big)\Big|_{\beta=\beta_c}\Big] \; . \qquad (2.19)
\end{aligned}
$$

provided this limit exists and is finite. In Appendix B.1 we show that the limit in the second line is equal to zero. On the contrary, the one in the first line is non trivial and can be explicitly computed as follows. Taking the limit $L\to\infty$ first, we can rewrite the first line as

$$
\lim_{\ell\to\infty} \ell^2 \sum_{n=0}^{\ell/2-1} \int_{\xi_n-\frac{\pi}{\ell}}^{\xi_n+\frac{\pi}{\ell}} \frac{dk_1}{2\pi} \int_{-\pi}^{\pi} \frac{dk_2}{2\pi} \big[\log(4-2\cos\xi_n-2\cos k_2) - \log(4-2\cos k_1 - 2\cos k_2)\big] \; , \qquad (2.20)
$$

where $\xi_n = \frac{2\pi}{\ell}(n+\frac{1}{2})$. The integral over k_2 can be performed explicitly [18, Formula 4.224(9)], leading to

$$
\lim_{\ell\to\infty} \ell^2 \sum_{n=0}^{\ell/2-1} \int_{\xi_n-\frac{\pi}{\ell}}^{\xi_n+\frac{\pi}{\ell}} \frac{dk_1}{2\pi} \big[\gamma(\xi_n)-\gamma(k_1)\big] = \lim_{\ell\to\infty} \ell^2 \sum_{n=0}^{\ell/2-1} \int_{-\frac{\pi}{\ell}}^{\frac{\pi}{\ell}} \frac{dk'}{2\pi} \big[\gamma(\xi_n)-\gamma(\xi_n+k')\big] \; , \qquad (2.21)
$$

with $\gamma(k) := \cosh^{-1}(2-\cos k)$. Expanding in Taylor series $\gamma(\xi_n + k')$ around ξ_n up to second order included, and using the fact that $\gamma(k)$ is a C^∞ function on $[0,\pi]$, we find that Eq.(2.21) can be rewritten as

$$\lim_{\ell\to\infty}\Big[-\frac{\ell^2}{2}\sum_{n=0}^{\ell/2-1}\gamma''(\xi_n)\int_{-\frac{\pi}{\ell}}^{\frac{\pi}{\ell}}\frac{dk'}{2\pi}(k')^2 + O(\frac{1}{\ell})\Big] = \tag{2.22}$$
$$= -\frac{\pi}{12}\int_0^\pi dk\,\gamma''(k) = \frac{\pi}{12}(\gamma'(0^+)-\gamma'(\pi)) = \frac{\pi}{12} \equiv \frac{c\pi}{6}\ ,$$

which corresponds to $c = 1/2$, as desired.

3 The interacting case

We now attack the problem of computing the first non trivial finite volume correction to the critical pressure in the interacting, $\lambda \neq 0$, case. While in nearest neighbor case the partition function is expressed in terms of Grassamann integrals quadratinc in the exponent (that is, *Gaussian* Grassmann integrals), in the next to nearest neighbor case the Grassman integrals are not quadratic anymore, and they cannot be explicitely computed. In the following We make use of the results and methods of [14], which we refer to for the proof of numerous relations used in the following. As in the previous section, we drop the dependence on β in the symbol used for the partition function, and we assume that ℓ and L are even.

3.1 Grassmann representation of the interacting partition function

The interacting partition function Eq.(1.2) can be written in a form analogous to Eq.(2.10), for all temperatures β:

$$\mathcal{Z}(\Lambda_{\ell,L}) = \frac{1}{2}\big(\mathcal{Z}_{--}(\Lambda_{\ell,L}) + \mathcal{Z}_{-+}(\Lambda_{\ell,L}) + \mathcal{Z}_{+-}(\Lambda_{\ell,L}) - \mathcal{Z}_{++}(\Lambda_{\ell,L})\big)\ , \tag{3.1}$$

with $\mathcal{Z}_{\boldsymbol{\alpha}}(\Lambda_{\ell,L})$ given by (see [14, Proposition 1]):

$$\mathcal{Z}_{\boldsymbol{\alpha}}(\Lambda_{\ell,L}) = C_{\ell,L}\int \mathcal{D}\Phi\, e^{S_t(\Phi)+\mathcal{V}(\Phi)}\ , \tag{3.2}$$

where:

- $C_{\ell,L}$ is a normalization constant, defined as

$$C_{\ell,L} = (2\cosh^2(\beta J))^{\ell L} e^{V_{\ell,L}(\lambda)} \prod_{\{\mathbf{x},\mathbf{y}\}} \cosh^2\big(\frac{\beta\lambda}{2} v(\mathbf{x}-\mathbf{y})\big) \tag{3.3}$$

with $V_{\ell,L}(\lambda)$ an analytic function of λ, defined as (using the notation of [14], see the proof of [14, Proposition 1] and, in particular, [14, Eq.(2.29)])

$$V_{\ell,L}(\lambda) = 2\ell L \sum_{\substack{\Gamma\subseteq\Lambda_{\ell,L}:\\ \operatorname{supp}\Gamma\ni b_0}} \frac{\varphi^T(\Gamma)}{|\operatorname{supp}\Gamma|} \prod_{\gamma\in\Gamma} \zeta(\emptyset,\emptyset;\gamma)\ . \tag{3.4}$$

Here b_0 is an arbitrary n.n. bond of $\Lambda_{\ell,L}$, $\operatorname{supp}\Gamma = \cup_{\gamma\in\Gamma}\gamma$ and $|\operatorname{supp}\Gamma|$ is the number of bonds in $\operatorname{supp}\Gamma$. The sum in Eq.(3.4) is independent of b_0, by translation invariance; the *activity* $\zeta(\emptyset,\emptyset;\gamma)$ (defined in [14, Eq.(2.18)]) is a translation invariant exponentially decaying function, satisfying the bound (see [14, Eq.(2.28)])

$$|\zeta(\emptyset,\emptyset;\gamma)| \le \nu^{|\gamma|}\ , \qquad \nu = 4e^{1+\beta|\lambda|/2}\big(\frac{\beta|\lambda|}{2}\big)^{1/M_0}\ . \tag{3.5}$$

- If we define $E_{\mathbf{x},1} = \overline{H}_{\mathbf{x}} H_{\mathbf{x}+a\hat{\mathbf{e}}_1}$ and $E_{\mathbf{x},2} = \overline{V}_{\mathbf{x}} V_{\mathbf{x}+a\hat{\mathbf{e}}_2}$, then $\mathcal{V}(\Phi)$ is a polynomial in $\{E_{\mathbf{x},j}\}_{\mathbf{x}\in\Lambda_{\ell,L}}^{j=1,2}$, which can be expressed as

$$\mathcal{V}(\Phi) = \sum_{n\ge1}\sum_{j_1,\dots,j_n}\sum_{\mathbf{x}_1,\dots,\mathbf{x}_n} W_{\underline{j}}(\mathbf{x}_1,\dots,\mathbf{x}_n)\prod_{i=1}^n E_{\mathbf{x}_i,j_i} \tag{3.6}$$

where $\underline{j} = (j_1,\dots,j_n)$ and the kernel $W_{\underline{j}}(\mathbf{x}_1,\dots,\mathbf{x}_n)$ is translation invariant and satisfies the following decay bound:

$$|W_{\underline{j}}(\mathbf{x}_1,\dots,\mathbf{x}_n)| \le C^n(\beta|\lambda|)^{\max\{1,cn\}} e^{-\kappa\,\delta(\mathbf{x}_1,\dots,\mathbf{x}_n)} \tag{3.7}$$

for suitable constants $C, c, \kappa > 0$ depending only on M_0 (the range of the interaction); here $\delta(\mathbf{x}_1,\dots,\mathbf{x}_n)$ is the tree distance of the set $X = \{\mathbf{x}_1,\dots,\mathbf{x}_n\}$, that is the length of the shortest tree graph composed of bonds in $\mathcal{B}_{\ell,L}$ which connects all the elements of X.

The Grassmann integral Eq.(3.2) and the correlation functions of the Φ field induced by the "measure" $\mathcal{D}\Phi\, e^{S_t(\Phi)+\mathcal{V}(\Phi)}$ have been studied in great detail in [14]. In particular, part of the main result of [14] can be reinterpreted by saying that there is a critical temperature $\beta_c(\lambda)$ such that, if we fix $t = t_c =$

$\tanh(\beta_c(\lambda)J)$, then the two-point function of the Φ field decays polynomially (like distance^{-1}) at large distances: if $\boldsymbol{\alpha} \neq (+,+)$ and we perform the unitary change of variables from Φ to the *critical modes* ψ, χ defined as

$$\begin{pmatrix} \psi_{\mathbf{x},+} \\ \psi_{\mathbf{x},-} \\ \chi_{\mathbf{x},+} \\ \chi_{\mathbf{x},-} \end{pmatrix} = U \begin{pmatrix} \overline{H}_{\mathbf{x}} \\ H_{\mathbf{x}} \\ \overline{V}_{\mathbf{x}} \\ V_{\mathbf{x}} \end{pmatrix} , \qquad U = \frac{1}{2} \begin{pmatrix} e^{i\frac{\pi}{4}} & e^{-i\frac{\pi}{4}} & 1 & -i \\ e^{-i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} & 1 & i \\ -e^{i\frac{\pi}{4}} & -e^{-i\frac{\pi}{4}} & 1 & -i \\ -e^{-i\frac{\pi}{4}} & -e^{i\frac{\pi}{4}} & 1 & i \end{pmatrix} , \tag{3.8}$$

then, asymptotically for large distances,

$$\langle \psi_{\mathbf{x},\omega} \psi_{\mathbf{y},\omega'} \rangle_{t_c} = \frac{\int \mathcal{D}\Phi e^{S_{t_c}(\Phi)+\mathcal{V}(\Phi)} \psi_{\mathbf{x},\omega} \psi_{\mathbf{y},\omega'}}{\int \mathcal{D}\Phi e^{S_{t_c}(\Phi)+\mathcal{V}(\Phi)}} \simeq \frac{\bar{Z}(\lambda)}{\pi t_c} \frac{\delta_{\omega,\omega'}}{(y_1 - x_1) + i\omega(y_2 - x_2)} . \tag{3.9}$$

Here the symbol "$\simeq$" means "up to faster decaying terms as $|\mathbf{x} - \mathbf{y}| \to \infty$" and $\bar{Z}(\lambda)$ is the analytic function appearing in [14, Theorem 1.1]. An infinite volume limit $\ell, L \to \infty$, performed while keeping the sites $\mathbf{x}, \mathbf{y}$ fixed, is implicit in Eq.(3.9). Eq.(3.9) can be read by saying that the asymptotic behavior of $\langle \psi_{\mathbf{x},\omega} \psi_{\mathbf{y},\omega'} \rangle_{t_c}$ is the same as that of $\langle \psi_{\mathbf{x},\omega} \psi_{\mathbf{y},\omega'} \rangle^0_{t_c^0, Z}$, where $t_c^0 = \sqrt{2} - 1$, $Z = \frac{1}{\bar{Z}(\lambda)} \frac{t_c}{t_c^0}$ and $\langle \cdot \rangle^0_{Z,t_c^0}$ is the average with respect to a properly renormalized massless gaussian integration:

$$\langle \, \cdot \, \rangle^0_{Z,t_c^0} = \frac{\int \mathcal{D}\Phi e^{Z S_{t_c^0}(\Phi)} \, \cdot}{\int \mathcal{D}\Phi e^{Z S_{t_c^0}(\Phi)}} . \tag{3.10}$$

A way of computing Z, similar to but slightly different from the one proposed in [14], will be described below. The integration $\mathcal{D}\Phi e^{Z S_{t_c^0}(\Phi)}$ is the right reference measure, around which to perform the perturbation analysis of Eq.(3.2) at $t = t_c$, along the lines of [14].

However, before we start describing the renormalization group computation of $\mathcal{Z}_{\boldsymbol{\alpha}}(\Lambda_{\ell,L})$ at $t = t_c$, let us discuss how to deal with the sign problem in Eq.(3.1): it is in fact apparent that the expression in the r.h.s. involves a difference between Grassmann partition functions, which may in principle produce dangerous cancellations between the different terms. This has to be contrasted with the computation in the $\lambda = 0$ where, as we saw above, at criticality $\mathcal{Z}^0_{++}(\Lambda_{\ell,L}) = 0$ and the other three terms are all positive. This may even be true at $\lambda \neq 0$ and $t = t_c$, but we do not know how to prove it. Nevertheless, we can prove an a priori partition function inequality, which can be thought of as a weak version of this claim, and is actually enough to the purpose of computing the pressure up to the first non trivial finite volume correction. This is discussed in the next subsection.

3.2 A partition function inequality

As we said above, our goal is to compute Eq.(3.1) at criticality, by using the representation Eq.(3.2) and a renormalization group analysis for $\int \mathcal{D}\Phi\, e^{S_t(\Phi)+\mathcal{V}(\Phi)}$, along the lines of [14]. However, as we will see below, our renormalization group computation of $\int \mathcal{D}\Phi\, e^{S_t(\Phi)+\mathcal{V}(\Phi)}$ only works for $\boldsymbol{\alpha} \neq (+,+)$, due to the possible vanishing of $\mathcal{Z}_{++}(\Lambda_{\ell,L})$ at $t = t_c$; the problem is that at the unperturbed level $\mathcal{Z}^0_{++}(\Lambda_{\ell,L}) = 0$ at criticality and, therefore, in order to perturbatively compute $\mathcal{Z}_{++}(\Lambda_{\ell,L})$, we do not know where to perturb around. Luckily enough, in order to compute the pressure up to the first non trivial finite volume correction, we do not really need to prove that $\mathcal{Z}_{++}(\Lambda_{\ell,L})$ is zero or much smaller than the other three partition functions: a weaker statement, summarized in the following Lemma, is actually enough for our purposes.

Lemma 1. *Under the stated assumptions on the potential $v(\mathbf{x})$ in Eq.(1.1), for $\lambda \geq 0$ and all inverse temperatures $\beta > 0$, the following inequalities hold:*

$$\frac{1}{3} \leq \frac{\mathcal{Z}(\Lambda_{\ell,L})}{\mathcal{Z}_{--}(\Lambda_{\ell,L}) + \mathcal{Z}_{-+}(\Lambda_{\ell,L}) + \mathcal{Z}_{+-}(\Lambda_{\ell,L})} \leq 1 \; , \tag{3.11}$$

$$\mathcal{Z}_{-+}(\Lambda_{\ell,L}) + \mathcal{Z}_{+-}(\Lambda_{\ell,L}) \geq 0 \; . \tag{3.12}$$

The proof of this lemma is postponed to Section 4. In order to use this result, we have to combine it with the information that

$$\mathcal{Z}_{--}(\Lambda_{\ell,L}) > 0 \; , \tag{3.13}$$

which is proved in the next subsection for λ small enough. Putting Eqs.(3.11)–(3.13) together we get:

$$\frac{1}{3}\mathcal{Z}_{--}(\Lambda_{\ell,L}) \leq \mathcal{Z}(\Lambda_{\ell,L}) \leq \mathcal{Z}_{--}(\Lambda_{\ell,L})\Big(1 + \frac{|\mathcal{Z}_{-+}(\Lambda_{\ell,L})|}{\mathcal{Z}_{--}(\Lambda_{\ell,L})} + \frac{|\mathcal{Z}_{+-}(\Lambda_{\ell,L})|}{\mathcal{Z}_{--}(\Lambda_{\ell,L})}\Big) \; . \tag{3.14}$$

Let us now put ourselves at criticality, $t = t_c$. In Appendix B.2 we prove that

$$\lim_{\ell\to\infty}\lim_{L\to\infty} \frac{\ell}{L} \log\Big(1 + \frac{|\mathcal{Z}_{-+}(\Lambda_{\ell,L})|}{\mathcal{Z}_{--}(\Lambda_{\ell,L})} + \frac{|\mathcal{Z}_{+-}(\Lambda_{\ell,L})|}{\mathcal{Z}_{--}(\Lambda_{\ell,L})}\Big) = 0 \tag{3.15}$$

and, therefore, using the notation of Eq.(1.4)

$$f_\infty = \lim_{\ell,L\to\infty} \frac{1}{\ell L} \log \mathcal{Z}(\Lambda_{\ell,L}) = \lim_{\ell,L\to\infty} \frac{1}{\ell L} \log \mathcal{Z}_{--}(\Lambda_{\ell,L}) \; , \tag{3.16}$$

$$\frac{c\pi}{6} = \lim_{\ell\to\infty}\lim_{L\to\infty}\Big[\frac{\ell}{L}\log\mathcal{Z}(\Lambda_{\ell,L}) - \ell^2 f_\infty\Big] = \lim_{\ell\to\infty}\lim_{L\to\infty}\Big[\frac{\ell}{L}\log\mathcal{Z}_{--}(\Lambda_{\ell,L}) - \ell^2 f_\infty\Big] \; , \tag{3.17}$$

which reduces the computation of the central charge to the evaluation of the finite volume corrections to $\mathcal{Z}_{--}(\Lambda_{\ell,L})$. These are computed in the next two subsections.

3.3 Renormalization group analysis of $\mathcal{Z}_{--}(\Lambda_{\ell,L})$

We start from Eq.(3.2) with $\boldsymbol{\alpha} = (-,-)$ and $t = t_c$, and we rewrite it as:

$$\mathcal{Z}_{--}(\Lambda_{\ell,L}) = C_{\ell,L} \int \mathcal{D}\Phi \, e^{ZS_{t_c^0}(\Phi)+\overline{\mathcal{V}}(\Phi)} , \tag{3.18}$$

where

$$\overline{\mathcal{V}}(\Phi) = \mathcal{V}(\Phi) + S_{t_c}(\Phi) - ZS_{t_c^0}(\Phi) . \tag{3.19}$$

In the following, t_c and Z will be constructed in such a way that, asymptotically for $|\mathbf{x}| \to \infty$,

$$\langle \psi_{\mathbf{0},\omega}\psi_{\mathbf{x},\omega'}\rangle_{t_c} \simeq \langle \psi_{\mathbf{0},\omega}\psi_{\mathbf{x},\omega'}\rangle^0_{t_c^0,Z} \simeq \frac{1}{Z}\frac{1}{\pi t_c^0}\frac{\delta_{\omega,\omega'}}{x_1 + i\omega x_2} \tag{3.20}$$

that is the theory is critical (i.e. massless) and $Z\pi t_c$ is the dressed wave function renormalization, see the discussion after Eq.(3.9). Equivalently, the t_c and Z will be chosen in such a way that the flow of the running coupling constants defined in the iterative construction of $\mathcal{Z}_{--}(\Lambda_{\ell,L})$ described below remain bounded at all scales; we refer to the following for the definition of running coupling constants and for the study of their flow; see in particular the neighborood of Eqs.(3.74)) and (3.76 for a discussion about how to fix t_c, Z in terms of an implicit function theorem.

It is convenient to multiply and divide Eq.(3.18) by the proper normalization, $\tilde{\mathcal{Z}}^0_{--}(\Lambda_{\ell,L}) = \int \mathcal{D}\Phi e^{ZS_{t_c^0}(\Phi)}$, thus finding

$$\mathcal{Z}_{--}(\Lambda_{\ell,L}) = C_{\ell,L}\tilde{\mathcal{Z}}^0_{--}(\Lambda_{\ell,L}) \int P(d\Phi)\, e^{\overline{\mathcal{V}}(\Phi)} , \tag{3.21}$$

where $P(d\Phi) = \left[\tilde{\mathcal{Z}}^0_{--}(\Lambda_{\ell,L})\right]^{-1}\mathcal{D}\Phi\, e^{ZS_{t_c^0}(\Phi)}$ is the normalized gaussian reference measure. The normalization constants $C_{\ell,L}$ and $\tilde{\mathcal{Z}}^0_{--}(\Lambda_{\ell,L})$ are explicit and, therefore, the only non trivial part to deal with is

$$\Xi_{--}(\Lambda_{\ell,L}) := \int P(d\Phi)\, e^{\overline{\mathcal{V}}(\Phi)} . \tag{3.22}$$

In order to compute this integral we proceed by following essentially the same strategy of [14], modulo a few small modifications described below. As a first step, we pass to the critical modes, already introduced in Eq.(3.8); in order to fix the normalizations as in [14] we also rescale the variables as (see [14, Eq.(2.38)]) $\psi_\omega \to -i\omega\sqrt{\pi t_c^0}\,\psi_\omega$, $\chi_\omega \to -i\omega\sqrt{\pi t_c^0}\,\chi_\omega$, and next we perform the following linear change of variables (see [14, Eq.(2.52)]))

$$\hat\chi_{\mathbf{k}} \to \hat\chi_{\mathbf{k}} + C_\chi^{-1}(\mathbf{k})Q(\mathbf{k})\hat\psi_{\mathbf{k}}\ , \tag{3.23}$$

where, if $\sigma_\chi(\mathbf{k}) = \cos k_1 + \cos k_2 + 2\frac{\sqrt{2}+1}{t_c^0}$,

$$C_\chi(\mathbf{k}) = \begin{pmatrix} -i\sin k_1 + \sin k_2 & i\sigma_\chi(\mathbf{k}) \\ -i\sigma_\chi(\mathbf{k}) & -i\sin k_1 - \sin k_2 \end{pmatrix}\ , \tag{3.24}$$

$$Q(\mathbf{k}) = \begin{pmatrix} -i\sin k_1 - \sin k_2 & i\cos k_1 - i\cos k_2 \\ -i\cos k_1 + i\cos k_2 & -i\sin k_1 + \sin k_2 \end{pmatrix}\ . \tag{3.25}$$

After these changes of variables we get the analogue of [14, Eq.(2.53)], namely

$$\Xi_{--}(\Lambda_{\ell,L}) = \int P(d\psi)P(d\chi)e^{\overline{\mathcal{V}}(\psi,\chi)}\ , \tag{3.26}$$

where:

- If, for $\mathbf{k}\in\mathcal{D}_{--}$, we define

  $$\hat\psi_{\mathbf{k},\omega} = \sum_{\mathbf{x}\in\Lambda_{\ell,L}} e^{i\mathbf{k}\mathbf{x}}\psi_{\mathbf{x},\omega}\ , \qquad \hat\chi_{\mathbf{k},\omega} = \sum_{\mathbf{x}\in\Lambda_{\ell,L}} e^{i\mathbf{k}\mathbf{x}}\chi_{\mathbf{x},\omega}\ ,$$

  then the Grassmann gaussian integrations $P(d\psi)$, $P(d\chi)$ can be written as

  $$P(d\psi) := \frac{1}{\mathcal{N}_{\psi,\boldsymbol{\alpha}}}\Big[\prod_{\mathbf{k}\in\mathcal{D}_{--}}\prod_{\omega=\pm} d\hat\psi_{\mathbf{k},\omega}\Big]\exp\Big\{-\frac{Z}{4\pi L\ell}\sum_{\mathbf{k}\in\mathcal{D}_{--}}\hat\psi^T_{-\mathbf{k}}C_\psi(\mathbf{k})\hat\psi_{\mathbf{k}}\Big\}\ ,$$

  $$P(d\chi) := \frac{1}{\mathcal{N}_{\chi,\boldsymbol{\alpha}}}\Big[\prod_{\mathbf{k}\in\mathcal{D}_{--}}\prod_{\omega=\pm} d\hat\chi_{\mathbf{k},\omega}\Big]\exp\Big\{-\frac{Z}{4\pi L\ell}\sum_{\mathbf{k}\in\mathcal{D}_{--}}\hat\chi^T_{-\mathbf{k}}C_\chi(\mathbf{k})\hat\chi_{\mathbf{k}}\Big\}\ ,$$

  where, letting $\sigma_\psi(\mathbf{k}) := \cos k_1 + \cos k_2 - 2$,

  $$C_\psi(\mathbf{k}) = \begin{pmatrix} -i\sin k_1 + \sin k_2 & i\sigma_\psi(\mathbf{k}) \\ -i\sigma_\psi(\mathbf{k}) & -i\sin k_1 - \sin k_2 \end{pmatrix} - Q(\mathbf{k})C_\chi^{-1}(\mathbf{k})Q(\mathbf{k}) \tag{3.27}$$

  and $\mathcal{N}_\psi, \mathcal{N}_\chi$ two normalizations, such that $\int P(d\psi) = \int P(d\chi) = 1$.

- $\overline{\mathcal{V}}(\psi,\chi)$ is the rewriting of $\overline{\mathcal{V}}(\Phi)$ in terms of the new variables. It is easy to check that its kernels satisfy the same decay estimates as those of $\overline{\mathcal{V}}(\Phi)$, see Eq.(3.7).

The propagators of the ψ and χ fields are, respectively,

$$\begin{aligned} g^{\psi}_{\omega,\omega'}(\mathbf{x}-\mathbf{y}) &= \int P(d\psi)\psi_{\mathbf{x},\omega}\psi_{\mathbf{y},\omega'} = \frac{1}{Z}\frac{2\pi}{L\ell}\sum_{\mathbf{k}\in\mathcal{D}_{--}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\big(\big[C_\psi(\mathbf{k})\big]^{-1}\big)_{\omega,\omega'}\ , \\ g^{\chi}_{\omega,\omega'}(\mathbf{x}-\mathbf{y}) &= \int P(d\chi)\chi_{\mathbf{x},\omega}\chi_{\mathbf{y},\omega'} = \frac{1}{Z}\frac{2\pi}{L\ell}\sum_{\mathbf{k}\in\mathcal{D}_{--}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\big(\big[C_\chi(\mathbf{k})\big]^{-1}\big)_{\omega,\omega'}\ . \end{aligned}$$

A simple explicit computation shows that asymptotically as $|\mathbf{x}-\mathbf{y}|\to\infty$ the propagator of the ψ field behaves like $\frac{1}{Z}\frac{\delta_{\omega,\omega'}}{(x_1-y_1)+i\omega(x_2-y_2)}$, as it should, while the one of the χ field is exponentially decaying. Therefore, we can integrate out the χ field, by proceeding as described in detail in [14, Section 3.1], and we get the analogue of [14, Eq.(3.1)]:

$$\Xi_{--}(\Lambda_{\ell,L}) = e^{L\ell E_0}\int P(d\psi)e^{\mathcal{V}^{(0)}(\psi)} \tag{3.28}$$

where E_0 is an analytic function of λ, vanishing at $\lambda=0$ and weakly depending on the volume; namely, if we denote by E_0^∞ its infinite volume limit, then $|E_0-E_0^\infty|\le C|\lambda|e^{-c\ell}$ for two suitable constants $C,c>0$; see below for a more detailed discussion of the finite volume corrections to the thermodynamic quantities of the theory.

Multiscale analysis. We now need to integrate out the ψ field. However, since the ψ field is massless, we cannot do it trivially in one step. A convenient way to proceed is by an iterative procedure, described in detail in [14, Section 3.2]. We define a sequence of geometrically decreasing momentum scales 2^h, with $h=1,0,\dots$ Correspondingly we define a sequence of cutoff functions $f_h(\mathbf{k})$ in the following way. Let $\chi(t)$ be a smooth compact support function that is 1 for $t\le 1$ and 0 for $t\ge 2$. We choose $f_0(\mathbf{k})=1-\chi(|\mathbf{k}|)$ and $f_h(\mathbf{k}')=\chi(2^{-h}|\mathbf{k}|)-\chi(2^{-h+1}|\mathbf{k}|)$ $\forall h<0$, so that f_h for $h<0$ is non zero only if $2^{h-1}\le|\mathbf{k}|\le 2^{h+1}$, and

$$1=\sum_{h\le 0} f_h(\mathbf{k})\ . \tag{3.29}$$

The resolution of the identity Eq.(3.29) induces a rewriting of the propagator of $\psi =: \psi^{(\le 0)}$ as a sum of propagators concentrated on smaller and smaller momentum scales and an iterative procedure to compute Z. At each step we decompose the propagator into a sum of two propagators, the first approximately supported on momenta $\sim 2^h$ (i.e. with a Fourier transform proportional to $f_h(\mathbf{k})$), $h\le 0$, the second approximately supported on momenta smaller than 2^h. Correspondingly we rewrite the Grassmann field as a sum of two independent fields: $\psi^{(\le h)} = \psi^{(h)} + \psi^{(\le h-1)}$ and we integrate out the field $\psi^{(h)}$ in the same way as we did for χ. The result is that, for any $h \le 0$, we can rewrite

$$\Xi_{--}(\Lambda_{\ell,L}) = e^{L\ell E_h} \int P(d\psi^{(\le h)}) e^{\mathcal{V}^{(h)}(\psi^{(\le h)})} \tag{3.30}$$

where $E_h, \mathcal{V}^{(h)}$ are defined recursively, and $P(d\psi^{(\le h)})$ is the gaussian integration with propagator given in momentum space by $\sum_{h' \le h} \hat{g}^{(h')}(\mathbf{k})$, with

$$\hat{g}^{(h)}(\mathbf{k}) = \frac{2\pi}{Z} f_h(\mathbf{k}) \big[C_\psi(\mathbf{k}) \big]^{-1} . \tag{3.31}$$

Note that the direct space counterpart of $g^{(h)}$ decays to zero faster than any power:

$$|g^{(h)}(\mathbf{x})| \le \frac{C_p}{(1 + 2^h |\boldsymbol{\delta}(\mathbf{x})|)^p} , \quad \forall p \ge 0 . \tag{3.32}$$

where

$$\boldsymbol{\delta}(\mathbf{x}) = \Big(\frac{\ell}{\pi} \sin\big(\frac{\pi x_1}{\ell}\big), \frac{L}{\pi} \sin\big(\frac{\pi x_2}{L}\big) \Big) .$$

The outcome of the iterative construction is that, in particular, the *effective potential* $\mathcal{V}^{(h)}$ has the following structure:

$$\mathcal{V}^{(h)}(\psi) = \sum_{n \ge 1} \sum_{\underline{\omega}, \underline{\mathbf{x}}} W^{(h)}_{2n;\underline{\omega}}(\underline{\mathbf{x}}) \Big[\prod_{i=1}^{2n} \psi_{\mathbf{x}_i, \omega_i} \Big] , \tag{3.33}$$

where $\underline{\mathbf{x}} = (\mathbf{x}_1, \dots, \mathbf{x}_{2n})$, $\underline{\omega} = (\omega_1, \dots, \omega_{2n})$ and

$$\frac{1}{L\ell} \sum_{\mathbf{x}_1, \dots, \mathbf{x}_{2n}} |W^{(h)}_{2n;\underline{\omega}}(\mathbf{x}_1, \dots, \mathbf{x}_{2n})| \le C^n 2^{h(2-n)} |\lambda|^{\max\{1, cn\}} \tag{3.34}$$

for suitable $C, c > 0$. For future reference, let us also rewrite Eq.(3.33) in momentum space:

$$\mathcal{V}^{(h)}(\psi) = \sum_{n \ge 1} \frac{1}{(L\ell)^{2n-1}} \sum_{\substack{\mathbf{k}_1, \dots, \mathbf{k}_{2n} \\ \underline{\omega}}} \hat{W}^{(h)}_{2n;\underline{\omega}}(\mathbf{k}_1, \dots, \mathbf{k}_{2n-1}) \Big[\prod_{i=1}^{2n} \hat{\psi}_{\mathbf{k}_i, \omega_i} \Big] \delta(\mathbf{k}_1 + \dots + \mathbf{k}_{2n}) \tag{3.35}$$

where $\delta(\mathbf{k})$ is a Kronecker's delta, periodic over the torus $\mathbb{R}^2 / 2\pi\mathbb{Z}^2$.

The iteration continues until the scale $h^* := \lfloor \log_2(\pi/\ell) \rfloor$ is reached. At that point, the left-over propagator, $g^{(\le h^*)}$ is massive on the "right scale" (i.e. on the very same scale 2^{h^*}), so that the associated degrees of freedom can be integrated in one step. The result is the desired partition function.

Localization and renormalization. In order to inductively prove Eq.(3.30) we write

$$\mathcal{V}^{(h)}(\psi) = \mathcal{L}\mathcal{V}^{(h)}(\psi) + \mathcal{R}\mathcal{V}^{(h)}(\psi) \;, \tag{3.36}$$

where $\mathcal{L}\mathcal{V}^{(h)}(\psi)$ and $\mathcal{R}\mathcal{V}^{(h)}(\psi)$ are the so-called *local* and *irrelevant* part of the effective potential, defined in the next few formulas. We use a definition of localization operator on the lattice at finite volume, analogous to the one used in [3] where the finite volume effects of the renormalization procedure are discussed in great detail. To be fair, the definitions below are a bit more complicated than those in [3] (see the slightly cumbersome definitions (3.38)-(3.39)), the reason being that we want to make sure that the relative finite size errors induced by the localization procedure are of the order $O(\ell^{-4})$ rather than $O(\ell^{-2})$, which would not be enough to our purposes; see Eqs.(3.42)–(3.44) and (3.48) below. Let us now come back to the definition of $\mathcal{L}\mathcal{V}^{(h)}(\psi)$ and $\mathcal{R}\mathcal{V}^{(h)}(\psi)$. If we think of the kernel $\hat W^{(h)}_{2,0;(\omega_1,\omega_2)}$ as a 2×2 matrix with matrix indices ω_1, ω_2, we let

$$\begin{aligned}\mathcal{L}\mathcal{V}^{(h)}(\psi) &= \frac{1}{L\ell}\sum_{\mathbf{k}} \hat\psi^T_{\mathbf{k}}\big[\mathcal{L}\hat W^{(h)}_2(\mathbf{k})\big]\hat\psi_{-\mathbf{k}} \\ &+\frac{1}{(L\ell)^3}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}\big[\mathcal{L}\hat W^{(h)}_{4;\underline{\omega}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\big]\hat\psi^{(\le h)}_{\mathbf{k}_1,\omega_1}\hat\psi^{(\le h)}_{\mathbf{k}_2,\omega_2}\hat\psi^{(\le h)}_{\mathbf{k}_3,\omega_3}\hat\psi^{(\le h)}_{-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3,\omega_4}\;,\end{aligned} \tag{3.37}$$

where, setting $\bar{\mathbf{k}}_{\eta\eta'} = (\eta\frac{\pi}{\ell}, \eta'\frac{\pi}{L})$,

$$\begin{aligned}\mathcal{L}\hat W^{(h)}_2(\mathbf{k}) = \frac14\sum_{\eta,\eta'=\pm}\bigg[&\frac98 \hat W^{(h)}_2(\bar{\mathbf{k}}_{\eta\eta'})\Big(1+\eta\frac{\sin k_1}{\sin(\pi/\ell)}+\eta'\frac{\sin k_2}{\sin(\pi/L)}\Big)\\ &-\frac18\hat W^{(h)}_2(3\bar{\mathbf{k}}_{\eta\eta'})\Big(1+\eta\frac{\sin k_1}{\sin(3\pi/\ell)}+\eta'\frac{\sin k_2}{\sin(3\pi/L)}\Big)\bigg]\;,\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}\mathcal{L}\hat W^{(h)}_{4;\underline{\omega}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \frac{1}{64}\sum_{\substack{\eta_1,\eta_2,\eta_3,\\ \eta'_1,\eta'_2,\eta'_3}}\bigg[&\frac98\hat W^{(h)}_{4;\underline{\omega}}(\bar{\mathbf{k}}_{\eta_1\eta'_1},\bar{\mathbf{k}}_{\eta_2\eta'_2},\bar{\mathbf{k}}_{\eta_3\eta'_3})\\ &-\frac18\hat W^{(h)}_{4;\underline{\omega}}(3\bar{\mathbf{k}}_{\eta_1\eta'_1},3\bar{\mathbf{k}}_{\eta_2\eta'_2},3\bar{\mathbf{k}}_{\eta_3\eta'_3})\bigg]\;.\end{aligned} \tag{3.39}$$

Note that in the limit $L,\ell\to\infty$, the action of the localization operator reduces to:

$$\mathcal{L}\hat W^{(h)}_2(\mathbf{k}) = \hat W^{(h)}_2(\mathbf{0}) + (\sin k_1\partial_{k_1} + \sin k_2\partial_{k_2})\hat W^{(h)}_2(\mathbf{0})$$

and, similarly,

$$\mathcal{L}\hat{W}^{(h)}_{4;\underline{\omega}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \hat{W}^{(h)}_{4;\underline{\omega}}(\mathbf{0},\mathbf{0},\mathbf{0}) \; .$$

In other words, at finite L and ℓ, $\mathcal{L}\hat{W}^{(h)}_2(\mathbf{k})$ and $\mathcal{L}\hat{W}^{(h)}_{4;\underline{\omega}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ have to be understood as finite volume, lattice versions of the Taylor expansion up to order 0 or 1, respectively. The coefficients $9/8$ and $-1/8$ have been fixed in such a way that the difference between the finite and infinite volume localization operators goes to zero as $[\max\{L,\ell\}]^{-4}$. Note that there is some freedom in the choice of the localization operator at finite volume, the various choices differing among themselves by terms that vanish in the infinite volume limit. We could further modify the definition of localization in such a way that the operator remains the same in the infinite volume limit, but the difference with its finite volume counterpart goes to zero faster than any power of $\max\{L,\ell\}$. However, this alternative definition would be more cumbersome than the one given above, without giving us any relevant improvement in the bounds, since the decay $[\max\{L,\ell\}]^{-4}$ of the finite volume terms is enough to our purposes. Note also that our definition of $\mathcal{L}$ has the nice feature of being a projection operator: $\mathcal{L}^2 = \mathcal{L}$.

The direct-space counterparts of Eqs.(3.38)-(3.39) read as follows:

$$\mathcal{L}\sum_{\mathbf{x},\mathbf{y}} \psi^{(\le h)}_{\mathbf{x}} W^{(h)}_2(\mathbf{x},\mathbf{y})\psi^{(\le h)}_{\mathbf{y}} = \tag{3.40}$$

$$= \sum_{\mathbf{x},\mathbf{y}} \psi^{(\le h)}_{\mathbf{x}} W^{(h)}_2(\mathbf{x},\mathbf{y}) \Big[G_{\mathbf{y},\mathbf{x}}\psi^{(\le h)}_{\mathbf{x}} + \sum_{i=1}^{2} d_i(\mathbf{y},\mathbf{x})\bar{\partial}_i \psi^{(\le h)}_{\mathbf{x}} \Big] \; ,$$

and

$$\mathcal{L} \sum_{\substack{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4 \\ \omega_1,\omega_2,\omega_3,\omega_4}} W^{(h)}_{4;\underline{\omega}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) \prod_{i=1}^{4} \psi^{(\le h)}_{\mathbf{x}_i,\omega_i} = \tag{3.41}$$

$$= \sum_{\substack{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4 \\ \omega_1,\omega_2,\omega_3,\omega_4}} W^{(h)}_{4;\underline{\omega}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) \prod_{i=1}^{4} \Big[G_{\mathbf{x}_i,\mathbf{x}_4} \psi^{(\le h)}_{\mathbf{x}_4,\omega_i} \Big] \; ,$$

where $G_{\mathbf{y},\mathbf{x}}$ and $d_i(\mathbf{y},\mathbf{x})$ are translation invariant and

$$G_{\mathbf{x},\mathbf{0}} = \frac{9}{8}\cos\big(\frac{\pi x_1}{\ell}\big)\cos\big(\frac{\pi x_2}{L}\big) - \frac{1}{8}\cos\big(\frac{3\pi x_1}{\ell}\big)\cos\big(\frac{3\pi x_2}{L}\big) \; , \tag{3.42}$$

$$d_1(\mathbf{x},\mathbf{0}) = \frac{9}{8}\frac{\sin\big(\frac{\pi x_1}{\ell}\big)}{\sin\big(\frac{\pi}{\ell}\big)}\cos\big(\frac{\pi x_2}{L}\big) - \frac{1}{8}\frac{\sin\big(\frac{3\pi x_1}{\ell}\big)}{\sin\big(\frac{3\pi}{\ell}\big)}\cos\big(\frac{3\pi x_2}{L}\big) \; , \tag{3.43}$$

$$d_2(\mathbf{x},\mathbf{0}) = \frac{9}{8}\cos\big(\frac{\pi x_1}{\ell}\big)\frac{\sin\big(\frac{\pi x_2}{L}\big)}{\sin\big(\frac{\pi}{L}\big)} - \frac{1}{8}\cos\big(\frac{3\pi x_1}{\ell}\big)\frac{\sin\big(\frac{3\pi x_2}{L}\big)}{\sin\big(\frac{3\pi}{L}\big)} \; , \tag{3.44}$$

and $\bar\partial_i$ is the symmetric discrete derivative w.r.t. x_i, i.e., $\bar\partial_1 f(x_1,x_2) = \frac{1}{2}\big[f(x_1+1,x_2) - f(x_1-1,x_2)\big]$, and similarly for $\bar\partial_2$. A few comments are now in order.

1) The action of $\mathcal{L}$ in direct space can be seen as an action on the fields, as indicated by Eqs.(3.40)-(3.41). The action of $\mathcal{R} = 1-\mathcal{L}$ can be interpreted in the same way, too: it takes the form

$$\mathcal{R}\sum_{\mathbf{x},\mathbf{y}} \psi^{(\le h)}_{\mathbf{x}} W_2^{(h)}(\mathbf{x},\mathbf{y})\psi^{(\le h)}_{\mathbf{y}} = \sum_{\mathbf{x},\mathbf{y}} \psi^{(\le h)}_{\mathbf{x}} W_2^{(h)}(\mathbf{x},\mathbf{y}) T^{(\le h)}_{\mathbf{y},\mathbf{x}} \tag{3.45}$$

$$\mathcal{R}\sum_{\underline{\mathbf{x}},\underline{\omega}} W^{(h)}_{4;\underline{\omega}}(\underline{\mathbf{x}})\prod_{i=1}^4 \psi^{(\le h)}_{\mathbf{x}_i,\omega_i} = \sum_{\underline{\mathbf{x}},\underline{\omega}} W^{(h)}_{4;\underline{\omega}}(\underline{\mathbf{x}})\Big[D^{(\le h)}_{\mathbf{x}_1,\mathbf{x}_4;\omega_1}\psi^{(\le h)}_{\mathbf{x}_2,\omega_2}\psi^{(\le h)}_{\mathbf{x}_3,\omega_3}\psi^{(\le h)}_{\mathbf{x}_4,\omega_4} +$$
$$+ G_{\mathbf{x}_1,\mathbf{x}_4}\psi^{(\le h)}_{\mathbf{x}_4,\omega_1} D^{(\le h)}_{\mathbf{x}_2,\mathbf{x}_4;\omega_2}\psi^{(\le h)}_{\mathbf{x}_3,\omega_3}\psi^{(\le h)}_{\mathbf{x}_4,\omega_4} + G_{\mathbf{x}_1,\mathbf{x}_4}\psi^{(\le h)}_{\mathbf{x}_4,\omega_1} G_{\mathbf{x}_2,\mathbf{x}_4}\psi^{(\le h)}_{\mathbf{x}_4,\omega_2} D^{(\le h)}_{\mathbf{x}_3,\mathbf{x}_4;\omega_3}\psi^{(\le h)}_{\mathbf{x}_4,\omega_4}\Big]$$

where

$$T^{(\le h)}_{\mathbf{y},\mathbf{x};\omega} = \psi^{(\le h)}_{\mathbf{y},\omega} - G_{\mathbf{y},\mathbf{x}}\psi^{(\le h)}_{\mathbf{x}} - \sum_{i=1}^2 d_i(\mathbf{y},\mathbf{x})\bar\partial_i\psi^{(\le h)}_{\mathbf{x}}\ , \tag{3.46}$$

$$D^{(\le h)}_{\mathbf{y},\mathbf{x};\omega} = \psi^{(\le h)}_{\mathbf{y},\omega} - G_{\mathbf{y},\mathbf{x}}\psi^{(\le h)}_{\mathbf{x}} \tag{3.47}$$

2) The functions $G_{\mathbf{y},\mathbf{x}}$ and $d_i(\mathbf{y},\mathbf{x})$ are antiperiodic over $\Lambda_{\ell,L}$ in both their arguments. This implies that the fields $T^{(\le h)}_{\mathbf{y},\mathbf{x};\omega}$ and $D^{(\le h)}_{\mathbf{y},\mathbf{x};\omega}$ are antiperiodic in $\mathbf{y}$ and periodic in $\mathbf{x}$. Therefore, the summands in Eqs.(3.40)-(3.41) are periodic both in $\mathbf{y}$ and $\mathbf{x}$.

3) For fixed $\mathbf{x}$, asymptotically as $L \gg \ell \gg |\mathbf{x}|$, we can rewrite

$$G_{\mathbf{x},\mathbf{0}} = 1 + O\big((|\mathbf{x}|/\ell)^4\big)\ , \qquad d_i(\mathbf{x},\mathbf{0}) = x_i\big[1 + O\big((|\mathbf{x}|/\ell)^4\big)\big]\ . \tag{3.48}$$

which is useful for bounding dimensionally the effect of the finite volume on the localization procedure.

4) The action of $\mathcal{R}$ produces a dimensional gain on the Grassmann monomials which it acts on non-trivially. This can be seen as follows. As discussed in item (1), the action of $\mathcal{R}$ can be thought of as a replacement of a field $\psi^{(\le)}_{\mathbf{y}}$ either by $T^{(\le h)}_{\mathbf{y},\mathbf{x}}$ or by $D^{(\le h)}_{\mathbf{y},\mathbf{x}}$, with $\mathbf{x},\mathbf{y}\in\Lambda_{\ell,L}$ (recall that the box is topologically a torus); since both $\mathbf{x}$ and $\mathbf{y}$ are represented by infinitely many equivalent images on $\mathbb{Z}^2$, it is always possible to pick two such images, to be called $\bar{\mathbf{x}},\bar{\mathbf{y}}$, so that their euclidean distance on $\mathbb{Z}^2$ is the same as the distance between $\mathbf{x}$ and $\mathbf{y}$ on the torus $\Lambda_{\ell,L}$. We have $D^{(\le h)}_{\mathbf{y},\mathbf{x}} = (-1)^{(y_1-\bar y_1)/\ell + (y_2-\bar y_2)/L} D^{(\le h)}_{\bar{\mathbf{y}},\bar{\mathbf{x}}}$, where $D^{(\le h)}_{\bar{\mathbf{y}},\bar{\mathbf{x}}} = \psi^{(\le h)}_{\bar{\mathbf{y}}} - G_{\bar{\mathbf{y}},\bar{\mathbf{x}}}\psi^{(\le h)}_{\bar{\mathbf{x}}}$ can be conveniently written as:

$$D^{(\le h)}_{\bar{\mathbf{y}},\bar{\mathbf{x}};\omega} = (\bar{\mathbf{y}}-\bar{\mathbf{x}})\cdot\int_0^1 ds\,\partial_{\mathbf{x}'}\psi^{(\le h)}_{\mathbf{x}'}\big|_{\mathbf{x}'=\mathbf{x}+s(\mathbf{y}-\mathbf{x})} + (1-G_{\bar{\mathbf{y}},\bar{\mathbf{x}}})\psi^{(\le h)}_{\bar{\mathbf{x}}} \tag{3.49}$$

where $(1 - G_{\bar{\mathbf{y}},\bar{\mathbf{x}}})$ is of the order $|\bar{\mathbf{y}} - \bar{\mathbf{x}}|^4\ell^{-4}$. Note that when we plug this formula in the last two lines of Eq.(3.45), the factors $(\bar{\mathbf{y}} - \bar{\mathbf{x}})$ or $(1 - G_{\bar{\mathbf{y}},\bar{\mathbf{x}}})$ multiply the kernel $W_4^{(h)}$, whose value is obtained by integrating the degrees of freedom on scales strictly larger than h, i.e., it is a sum over Feynman diagrams with propagators $g^{(h_i)}(\mathbf{x}_i - \mathbf{y}_i)$ all of scale $h_i > h$. Consider the contribution from the interpolated term first. When we decompose the factor $(\bar{\mathbf{y}} - \bar{\mathbf{x}})$ along the diagrams contributing to $W_4^{(h)}$, some of the propagators $g^{(h_i)}(\mathbf{x}_i - \mathbf{y}_i)$ are multiplied by $(\mathbf{x}_i - \mathbf{y}_i)$, which is dimensionally equivalent to a factor 2^{-h_i} (see Eq.(3.32)). On the other hand, the derivative $\partial_{\mathbf{x}'}$ inside the interpolation integral, when acting on $\psi_{\mathbf{x}'}^{(\le h)}$, is dimensionally equivalent to a factor 2^h, simply because $\psi_{\mathbf{x}'}^{(\le h)}$ is a field that is smooth on scale 2^{-h}. Similarly, the contribution from the second term in Eq.(3.49) can be bounded by first replacing $(1 - G_{\bar{\mathbf{y}},\bar{\mathbf{x}}})$ by (const.)$|\bar{\mathbf{y}} - \bar{\mathbf{x}}|^4\ell^{-4}$ and then by decomposing the factor $|\bar{\mathbf{y}} - \bar{\mathbf{x}}|^4$ along the diagrams contributing to $W_4^{(h)}$, so obtaining $2^{-4h_i}\ell^{-4} = 2^{4(h^*-h_i)}$ for some $h_i > h$. In conclusion, the action of $\mathcal{R}$ on the four-legged kernels is dimensionally equivalent to a dimensional gain 2^{h-h_i}, with $h_i > h$, if we pick the first term in Eq.(3.49), or to $2^{4(h^*-h_i)}$ if we pick the second term, which is due to the finite volume corrections to the localization procedure. A similar discussion applies to the action of $\mathcal{R}$ on the two-legged kernels, in which case $\mathcal{R}$ is equivalent to either $2^{2(h-h_i)}$, with $h_i > h$, if we pick the analogue of the first term in Eq.(3.49), or to $2^{4(h^*-h_i)}$, if we pick the analogue of the second term.

5) A key fact which makes the theory at hand treatable (and asymptotically free) is that the action of $\mathcal{L}$ on the quartic kernels is zero "by the Pauli principle", i.e., simply by the Grassmann rule $\psi^2_{\mathbf{x},\omega} = 0$. In fact, note that in the second line of Eq.(3.41) at least two of the four ω indices must be equal among each other. Therefore the expression in the second line of Eq.(3.37) is identically zero. This property can be diagramatically interpreted by saying that the fermionic nature of the theory automatically renormalizes the four-field interaction, which is dimensionally marginal (see below) but effectively irrelevant thanks to the cancellation that we just mentioned.

6) By the previous comment, the only non vanishing contribution to the local part of the effective action is the one in the first line of Eq.(3.37). By its very definition, it is apparent that $\mathcal{L}\hat{W}_2^{(h)}$ is invariant under the discrete symmetries of the theory (reflections, discrete rotations, etc, see [14, Section II.D] for a thorough discussion of this point). Therefore, its most general form is (see [14, (2.68)])

$$\mathcal{L}\hat{W}_2^{(h)}(\mathbf{k}) = \begin{pmatrix} z_h(i\sin k_1 - \sin k_2) & i2^h\nu_h \\ -i2^h\nu_h & z_h(i\sin k_1 + \sin k_2) \end{pmatrix} \tag{3.50}$$

for two *real* constants z_h, ν_h, which will be called the *running coupling constants* of our theory. The initial values of these constants z_0, ν_0, are induced by the choices of t_c, Z in Eq.(3.19), as well as by the effects of the change of variables described after Eq.(3.22) and of the integration of the massive fields in Eq.(3.28). It is straightforward (if lengthy) to keep track of this series of transformations and to check that z_0, ν_0 are analytically invertible functions of t_c, Z in a neighborood of $t_c = t_c^0, Z = 1, \lambda = 0$:

$$(z_0, \nu_0) = \big(F_0(\lambda, t_c, Z), N_0(\lambda, t_c, Z)\big) \quad \Leftrightarrow \quad \begin{cases} Z = 1 + \zeta(\lambda, z_0, \nu_0) \\ t_c = t_c^0 + \tau(\lambda, z_0, \nu_0) \end{cases} \tag{3.51}$$

where F_0, N_0, ζ, τ are analytic functions of their arguments. It is straightforward (if lengthy) to check that, if $|\lambda| + |\nu_0| + |z_0| \le \varepsilon_0$,

$$\frac{\partial \zeta(\lambda, z_0, \nu_0)}{\partial z_0} = a + O(\varepsilon_0) \ , \qquad \frac{\partial \zeta(\lambda, z_0, \nu_0)}{\partial \nu_0} = O(\varepsilon_0) \ , \tag{3.52}$$

$$\frac{\partial \tau(\lambda, z_0, \nu_0)}{\partial z_0} = O(\varepsilon_0) \ , \qquad \frac{\partial \tau(\lambda, z_0, \nu_0)}{\partial \nu_0} = b + O(\varepsilon_0) \ , \tag{3.53}$$

for two *non vanishing* constants a and b. This property will play an important role in the following, in the choice of the parameter Z.

Tree expansion. Going back to the inductive proof of Eq.(3.30), we note that representation is valid at the first step, see Eq.(3.28). Assuming the representation to be valid at scale h, let us show that the same structure is preserved at the following step. By using the rewriting Eq.(3.36) and the *addition principle* (see e.g. [13, Section 4]), we rewrite Eq.(3.30) as

$$e^{L\ell E_h} \int P(d\psi^{(\le h-1)}) \int P(d\psi^{(h)}) e^{\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\le h-1)} + \psi^{(h)}) + \mathcal{R}\mathcal{V}^{(h)}(\psi^{(\le h-1)} + \psi^{(h)})} \tag{3.54}$$

where $P(d\psi^{(h)})$ is the quadratic Grassmann integration with propagator $\hat{g}^{(h)}(\mathbf{k})$, see Eq.(3.31). By integrating out the degrees of freedom on scale h, which are massive with mass of the order of 2^h, we get:

$$e^{L\ell(E_h + e_h)} \int P(d\psi^{(\le h-1)}) e^{\mathcal{V}^{(h-1)}(\psi^{(\le h-1)})} \ , \tag{3.55}$$

$$L\ell e_h + \mathcal{V}^{(h-1)}(\psi) = \log \int P(d\psi^{(h)}) e^{\mathcal{L}\mathcal{V}^{(h)}(\psi + \psi^{(h)}) + \mathcal{R}\mathcal{V}^{(h)}(\psi + \psi^{(h)})} \ ,$$

which proves the inductive hypothesis Eq.(3.30), provided that E_{h-1} is fixed as

$$E_{h-1} = E_h + e_h = E_0 + \sum_{h \le j \le 0} e_j \ . \tag{3.56}$$

Note that the above procedure allows us to write the running coupling constants z_h, ν_h with $h \leq 0$, in terms of z_k, ν_k with $h < k \leq 0$:

$$z_{h-1} = z_h + \beta_h^z \;, \qquad \nu_{h-1} = 2\nu_h + \beta_h^\nu \;, \tag{3.57}$$

where $\beta_h^\# = \beta_h^\#\big((z_h, \nu_h), \ldots, (z_0, \nu_0)\big)$ is the so–called *Beta function.*

By the very definition of *truncated expectation* (see e.g. [14, (2.57)]), the second line of Eq.(3.55) can be rewritten as

$$\begin{aligned} & L\ell e_h + \mathcal{V}^{(h-1)}(\psi) = \\ & = \sum_{n\geq 1} \mathcal{E}_h^T\big(\underbrace{\mathcal{L}\mathcal{V}^{(h)}(\psi+\cdot)+\mathcal{R}\mathcal{V}^{(h)}(\psi+\cdot);\cdots;\mathcal{L}\mathcal{V}^{(h)}(\psi+\cdot)+\mathcal{R}\mathcal{V}^{(h)}(\psi+\cdot)}_{n \text{ times}}\big) \end{aligned} \tag{3.58}$$

where $\mathcal{E}_h^T$ is the truncated expectation associated with the gaussian integration $P(d\psi^{(h)})$. Iterating this relation, we are led to a tree expansion for the effective potential, as described in detail in e.g. [14, Sections III.A- III.D]. The resulting trees are defined in a way that is very similar to those in [14, Sections III.D], with the following minor differences (we refer for comparison to the description of trees in items 1 to 5 in [14, Section III.D], to be called GGM.1-GGM.5, as well as to the discussion following that item list):
1) The trees we consider have only normal endpoints: $m = 0$ in the notation of item GGM.1.
2) The ultraviolet scale N (in the notation of GGM.2) is replaced by 0 and we denote by $\mathcal{T}_n^{(h)}$ the set of labeled trees with root on scale h and n endpoints.
3) The first four lines of GGM.5 are replaced by "With each normal endpoint v on scale 2 we associate a factor $\overline{\mathcal{V}}(\psi^{(\leq 0)}, \chi)$; here $\chi := \psi^{(1)}$ should be thought of as the field on scale 1. With the endpoints on scale $h_v \leq 1$ we associate either a factor $2^h \nu_h \frac{1}{L\ell}\sum_{\mathbf{k},\omega} i\omega \hat{\psi}_{\mathbf{k},\omega}\hat{\psi}_{-\mathbf{k},-\omega}$ or $z_h \frac{1}{L\ell}\sum_{\mathbf{k},\omega}\hat{\psi}_{\mathbf{k},\omega}(i\sin k_1 - \omega \sin k_2)\hat{\psi}_{-\mathbf{k},\omega}$, in which cases we shall refer to the endpoint as being of type ν or z, respectively."

In terms of the definitions of trees, the effective potential $\mathcal{V}^{(h)}$ can be written, in analogy with [14, (3.44)] as

$$L\ell e_h + \mathcal{V}^{(h)}(\psi^{(\leq h)}) = \sum_{n\geq 1}\sum_{\tau\in\mathcal{T}_n^{(h)}} \mathcal{V}^{(h)}(\tau, \psi^{(\leq h)}) \;, \tag{3.59}$$

where, if v_0 is the first vertex of τ, if $\tau_1, \ldots, \tau_s$ ($s = s_{v_0}$) are the subtrees of τ with root v_0, and if $\mathcal{E}_{h+1}^T$ is the truncated expectation associated to the propagator $\hat{g}^{(h)}(\mathbf{k})$,

$$\mathcal{V}^{(h)}(\tau, \psi^{(\leq h)}) = \frac{1}{s!}\mathcal{E}_{h+1}^T\big(\overline{V}^{(h+1)}(\tau_1, \psi^{(\leq h+1)}); \ldots; \overline{V}^{(h+1)}(\tau_s, \psi^{(\leq h+1)})\big) \;, \tag{3.60}$$

and $\overline{V}^{(h+1)}(\tau_i, \psi^{(\leq h+1)})$:

- is equal to $\mathcal{R}\mathcal{V}^{(h+1)}(\tau_i, \psi^{(\leq h+1)}))$ if τ_i is non trivial;
- is equal to $2^{h+1}\nu_{h+1}\frac{1}{L\ell}\sum_{\mathbf{k},\omega} i\omega\hat{\psi}_{\mathbf{k},\omega}\hat{\psi}_{-\mathbf{k},-\omega}$ if τ_i is trivial, $h < N$ and the endpoint of τ_i is of type ν;
- is equal to $z_{h+1}\frac{1}{L\ell}\sum_{\mathbf{k},\omega}\hat{\psi}_{\mathbf{k},\omega}(i\sin k_1 - \omega\sin k_2)\hat{\psi}_{-\mathbf{k},\omega}$ if τ_i is trivial, $h < N$ and the endpoint of τ_i is of type z;
- is equal to $\overline{\mathcal{V}}(\psi^{(\leq N+1)})$ if τ_i is trivial and $h = 0$.

The values of the trees can be estimated dimensionally as described in detail in [14, Sections III.D.1-III.D.2]. In a notation analogous to the one introduced in those sections, we get the analogue of [14, (3.35)], namely, if $W_{\tau,\mathbf{P},\mathbf{T},\beta}$ is the kernel of the renormalized effective potential labelled by a tree $\tau \in \mathcal{T}_n^{(h)}$, a set of field labels $\mathbf{P}$, a spanning tree T and a set of interpolation parameters β,

$$\frac{1}{L\ell}\sum_{\beta\in B_T}\sum_{\mathbf{x}_{v_0}}|W_{\tau,\mathbf{P},T,\beta}(\mathbf{x}_{v_0})| \leq C^n\Big[\prod_{v\ \text{e.p.}}(C|\lambda|)^{\max\{1,c|P_v|\}}\Big]\cdot$$
$$\cdot 2^{h(2-\frac{1}{2}|P_{v_0}|)}\Big[\prod_{v\,\text{not e.p.}} 2^{(h_v-h_{v'})(2-\frac{1}{2}|P_v|-z(P_v))}\Big], \tag{3.61}$$

with $z(P_v) = 2\delta_{|P_v|,2} + \delta_{|P_v|,4}$.

Remark. The bound Eq.(3.61) is essentially dimensional, i.e., it can be understood (modulo the combinatorics) by replacing all the propagators and the integrations involved in the definition of the kernel by their dimensional estimates, that is $|g^{(k)}(\mathbf{x})| \leq (\text{const.})2^k$ and $\int d\mathbf{x}|g^{(k)}(\mathbf{x})| \leq (\text{const.})2^{-k}$, see Eq.(3.32). To be fair, the dimensional estimate that we would obtain by these simple replacements would be similar to Eq.(3.61), but with $z(P_v)$ replaced by zero. In order to justify the presence of the dimensional factors $\prod_{v\,\text{not e.p.}} 2^{-(h_v-h_{v'})z(P_v)}$, one needs to take into account the action of $\mathcal{R}$, see the discussion in item (4) above in the lines preceding and following Eq.(3.49). It is important to note that in order for the bound in Eq.(3.61) to be summable over the scale labels, it is not really necessary to have $z(P_v) = 2\delta_{|P_v|,2} + \delta_{|P_v|,4}$: a gaining factor $\tilde{z}(P_v) = (1+\varepsilon)\delta_{|P_v|,2} + \varepsilon\delta_{|P_v|,4}$ with $\varepsilon \in (0,1)$ would make the job, too. For this reason, in order to regularize the kernels it is actually enough to use a portion of the gaining factors described in item (4) above; if desired, we can keep some of them on a side. E.g., concerning the factors $2^{4(h^*-h_i)}$ coming from the finite volume corrections to the definition of $\mathcal{R}$, we can use a fraction $2^{(1+\varepsilon)(h^*-h_i)}$ with $\varepsilon \in (0,1)$ to regularize the kernels, and we can keep $2^{(3-\varepsilon)(h^*-h_i)}$ on a side.

The bound Eq.(3.61) is valid *provided that the running coupling constants remain of order* λ, for all scales between h and 0. Under this assumption, Eq.(3.61) implies the analyticity of the kernels of $\mathcal{V}^{(h)}$ and the decay bounds Eq.(3.34).

An immediate corollary of the bound Eq.(3.61) is that contributions from trees $\tau \in \mathcal{T}_n^{(h)}$ with a vertex v on scale $h_v = k > h$ admit an improved bound with respect to Eq.(3.34), with an extra dimensional factor $2^{\theta(h-k)}$, $0 < \theta < 1$, which can be thought of as a dimensional gain with respect to the "basic" dimensional bound in Eq.(3.34). This improved bound is usually referred to as the *short memory* property (i.e., long trees are exponentially suppressed); it is due to the fact that the renormalized scaling dimensions $d_v = 2 - \frac{1}{2}|P_v| - z(P_v)$ in Eq.(3.61) are all ≤ -1, and can be obtained by taking a fraction of the factors $2^{(h_v - h_{v'})d_v}$ associated to the branches of the tree τ on the path connecting the vertex on scale k to the one on scale h.

Under the same assumptions, the beta function itself, $\beta_h^{\#}$, is analytic and dimensionally bounded by a constant independent of h. Moreover, the contributions to it from trees that have at least one node on scale $k > h$ is dimensional bounded proportionally to $2^{\theta(h-k)}$, with $0 < \theta < 1$. It is remarkable that thanks to these bounds, the dynamical system induced by the beta function can be fully studied and shown to lead to a bounded flow of the running coupling constants.

The flow of the running coupling constants. As announced above, the flow equations for the running coupling constants are

$$z_{h-1} = z_h + \beta_h^z \, , \qquad \nu_{h-1} = 2\nu_h + \beta_h^\nu \, , \tag{3.62}$$

where $\beta_h^{\#} = \beta_h^{\#}\big((z_h, \nu_h), \ldots, (z_0, \nu_0)\big)$ is an analytic function of its argument, with an analyticity domain bounded by: $|z_k| + |\nu_k| \leq \varepsilon_0$, for all $h \leq k \leq 0$, where ε_0 is a suitable (small) positive constant.

Note that both β_h^z and β_h^ν can be expressed as sums over trees with at least one endpoint on scale 2, the reason being that the local part of the trees with only endpoints of scale ≤ 1 is zero by the support properties of the single-scale propagators that enter the definition of $\beta_h^{\#}$. Therefore, by the short memory property, $|\beta_h^z|, |\beta_h^\nu| \leq C_\theta |\lambda| 2^{\theta h}$, for $\theta \in (0,1)$, uniformly in L, ℓ. The idea is to first solve the flow equations in the $L, \ell \to \infty$ limit, by properly choosing the initial data in such a way that the sequence $\{(z_h, \nu_h)\}_{h \leq 0}$ remains bounded. Then we will use the same initial data as the $L, \ell = \infty$ case in the finite volume equations and we will show that the resulting flow remains bounded and close to the infinite volume one, with explicit bounds on the error terms.

Let us then consider the case $L, \ell = \infty$ first. We denote by $\beta_h^{\infty,\#}$ the corresponding beta function. We define $\mathfrak{M}_{K,\theta}$ to be the space of sequences

$\underline{v} = (\underline{z}, \underline{\nu}) = \{(z_h, \nu_h)\}_{h\le 0}$ such that $|z_h| + |\nu_h| \le K|\lambda|2^{\theta h}$, $\forall h \le 0$; we shall think of $\mathfrak{M}_{K,\theta}$ as a Banach space with norm $||\cdot||_\theta$, where $||\underline{v}||_\theta = \sup_{k\le 0}(|z_h| + |\nu_h|)2^{-\theta k}$. Note that every exponentially decaying solution to the beta function equations (if any) can be looked for as a fixed point of the map $\mathbf{T} : \mathfrak{M}_{K,\theta} \to \mathfrak{M}_{K,\theta}$ defined by

$$(\mathbf{T}\underline{z})_h = -\sum_{j\le h} \beta_j^{\infty,z}(\underline{v}) \ , \qquad (\mathbf{T}\underline{\nu})_h = -\sum_{j\le h} 2^{j-h-1}\beta_j^{\infty,\nu}(\underline{v}) \ . \tag{3.63}$$

The fact that, for K sufficiently large, $\mathbf{T}$ is a map from $\mathfrak{M}_{N;K,\theta}$ to itself is a simple consequence of the bound $|\beta_h^{\infty,\#}| \le C_\theta|\lambda|2^{\theta h}$. Moreover, if $\underline{v}, \underline{v}' \in \mathfrak{M}_{K,\theta}$, then using the short memory property:

$$|\beta_j^{\infty,\#}(\underline{v}) - \beta_j^{\infty,\#}(\underline{v}')| \le C'_\theta|\lambda|2^{j\theta}\sum_{k\ge j}|v_k - v'_k| \le C''_\theta|\lambda|2^{j\theta}||\underline{v} - \underline{v}'||_\theta \ , \tag{3.64}$$

which implies that

$$||\mathbf{T}\underline{v} - \mathbf{T}\underline{v}'||_\theta \le C'''_\theta|\lambda|\|\underline{v} - \underline{v}'\|_\theta \ , \tag{3.65}$$

i.e. $\mathbf{T}$ is a contraction for $|\lambda|$ sufficiently small. Then the Banach fixed point theorem implies that $\mathbf{T}$ has a unique fixed point $\underline{v}^*$ in $\mathfrak{M}_{K,\theta}$, which represents an exponentially decaying solution to the flow equations, with initial data of order λ and given explicitly by the following expressions:

$$z_0^* = z_0^*(\lambda, Z) := -\sum_{j\le 0}\beta_j^{\infty,z}(\underline{v}^*) \ , \qquad \nu_0^* = \nu_0^*(\lambda, Z) := -\sum_{j\le 0} 2^{j-1}\beta_j^{\infty,\nu}(\underline{v}^*) \ . \tag{3.66}$$

By the previous construction, the functions $z_0^*(\lambda, Z)$ and $\nu_0^*(\lambda, Z)$ are analytic in their arguments in a neighborood of $\lambda = 0$, $Z = 1$.

We now consider the case of finite L, ℓ and we pick the same initial datum (z_0, ν_0) for the flow equation as for the infinite volume case:

$$z_0 = z_0^*(\lambda, Z) \ , \qquad \nu_0 = \nu_0^*(\lambda, Z) \ . \tag{3.67}$$

Denoting by $\underline{\bar{v}} = \{(\bar{z}_h, \bar{\nu}_h)\}_{h^*\le h\le 0}$ the sequence of running coupling constants generated at finite volume by this initial datum (using the notation above, the infinite volume counterpart of this sequence is denoted by $\underline{v}^* = \{(z_h^*, \nu_h^*)\}_{h\le 0}$), we get

$$\begin{cases} \bar{z}_h - z_h^* = \sum_{h<j\le 0}[\beta_j^z(\underline{\bar{v}}) - \beta_j^{\infty,z}(\underline{v}^*)] \ , \\ \bar{\nu}_h - \nu_h^\infty = \sum_{h<j\le 0} 2^{j-h-1}[\beta_j^\nu(\underline{\bar{v}}) - \beta_j^{\infty,\nu}(\underline{v}^*)] \ . \end{cases} \tag{3.68}$$

Based on this equation, and thinking of $\underline{\bar{v}}$ as an infinite sequence (obtained e.g. by posing $\bar{v}_h = 0$, $\forall h < h^*$), we can prove that, for any $\theta < \varepsilon < 1$,

$$|\bar{z}_h - z_h^*| + |\bar{\nu}_h - \nu_h^*| \leq C_{\theta,\varepsilon} 2^{\theta h} \Big(\frac{2^{-h}}{\ell}\Big)^{3-\varepsilon} . \tag{3.69}$$

The proof is by induction in h. If $h = 0$ the claim is obviously true, simply because the l.h.s. is zero. For $h < 0$, assuming the estimate to be valid for all the scales $h < k \leq 0$, we use Eq.(3.68), by rewriting the expressions in square brackets as

$$[\beta_j^{\#}(\underline{\bar{v}}) - \beta_j^{\infty,\#}(\underline{\bar{v}})] + [\beta_j^{\infty,\#}(\underline{\bar{v}}) - \beta_j^{\infty,\#}(\underline{v}^*)] . \tag{3.70}$$

Now, the second term can be bounded as in Eq.(3.64), $|\beta_j^{\infty,\#}(\underline{\bar{v}}) - \beta_j^{\infty,\#}(\underline{v}^*)| \leq C'_\theta |\lambda| 2^{j\theta} \sum_{k\geq j} |\bar{v}_k - v_k^*|$, so that by using the inductive assumption we get that the corresponding contribution, $\sum_{h<j\leq 0} |\beta_j^{\infty,\#}(\underline{\bar{v}}) - \beta_j^{\infty,\#}(\underline{v}^*)|$, is bounded from above by $C''_\theta |\lambda| 2^{2h\theta} \big(2^{-h}\ell^{-1}\big)^{3-\varepsilon}$, as desired.

The first term in Eq.(3.70) is due to the finite volume corrections. Remember that, by using the tree construction explained above, $\beta_j^{\#}(\underline{\bar{v}})$ can be written as a sum of the form $\frac{1}{L\ell} \sum_{\tau,\alpha} \sum_{\mathbf{x}_1,\ldots,\mathbf{x}_{n_\alpha}} \tilde{W}_j^{\#}(\mathbf{x}_1, \ldots, \mathbf{x}_{n_\alpha})$, where α is a suitable multi-index (collecting the indices $\mathbf{P}, T, \beta$ indicated in Eq.(3.61)) and $\tilde{W}$ is periodic over $\Lambda_{\ell,L}$ in all its coordinates. By construction, $\tilde{W}_j^{\#}(\underline{\mathbf{x}})$ is a combination of propagators on the scales indexed by the tree labels, as well as of the functions G, d_i in Eqs.(3.42)–(3.44) resulting from the action of $\mathcal{R}$ on the nodes of the tree. Moreover $\tilde{W}_j^{\#}(\mathbf{x}_1, \ldots, \mathbf{x}_{n_\alpha})$ is translation invariant, so that we can fix one variable to $\mathbf{0}$, for instance $\mathbf{x}_1$, and write:

$$\begin{aligned} &\beta_j^{\#}(\underline{\bar{v}}) - \beta_j^{\infty,\#}(\underline{\bar{v}}) = \\ &= \sum_{\tau,\alpha} \Big[\sum_{\mathbf{x}_2,\ldots,\mathbf{x}_{n_\alpha} \in \Lambda_{\ell,L}} \tilde{W}_j^{\#}(\mathbf{0}, \mathbf{x}_2 \ldots, \mathbf{x}_{n_\alpha}) - \sum_{\mathbf{x}_2,\ldots,\mathbf{x}_{n_\alpha} \in \mathbb{Z}^2} \tilde{W}_j^{\infty,\#}(\mathbf{0}, \mathbf{x}_2 \ldots, \mathbf{x}_{n_\alpha}) \Big] , \end{aligned} \tag{3.71}$$

where $\tilde{W}_j^{\infty,\#}$ is the infinite volume limit of $\tilde{W}_j^{\#}$, which differs from the latter because of the replacement of the factors G and d_i by 1 and x_i, respectively, and for the replacement of the propagators $g^{(k)}$ by their infinite volume limit.

Now, the two sums $\sum_{\mathbf{x}_2,\ldots,\mathbf{x}_{n_\alpha}}$ in Eq.(3.71) can be written as $\sum^*_{\mathbf{x}_2,\ldots,\mathbf{x}_{n_\alpha}} + \sum^{**}_{\mathbf{x}_2,\ldots,\mathbf{x}_{n_\alpha}}$, where $\sum^*_{\mathbf{x}_2,\ldots,\mathbf{x}_{n_\alpha}}$ is over the set $|\mathbf{x}_i| \leq \ell/4$, $\forall i = 2, \ldots, n_\alpha$, while $\sum^{**}_{\mathbf{x}_2,\ldots,\mathbf{x}_{n_\alpha}}$ involves at least one coordinate outside the ball $B_{\ell/4} = \{\mathbf{x} : |\mathbf{x}| \leq \ell/4\}$. The easiest terms to bound are

$$\sum^{**}_{\mathbf{x}_2,\ldots,\mathbf{x}_{n_\alpha}} \big[|\tilde{W}_j^{\#}(\mathbf{0}, \mathbf{x}_2 \ldots, \mathbf{x}_{n_\alpha})| + |\tilde{W}_j^{\infty,\#}(\mathbf{0}, \mathbf{x}_2 \ldots, \mathbf{x}_{n_\alpha})|\big] ,$$

which are of the order $|\lambda|2^{j\theta}(2^{-j}/\ell)^p$ for an arbitrary $p \geq 0$, simply because $\tilde{W}_j^{\#}$ contains a chain of propagators (each decaying faster than any power on a scale $h_i \geq j$, see Eq.(3.32)) connecting $\mathbf{0}$ with a coordinate $\bar{\mathbf{x}}$ outside the ball $B_{\ell,4}$. We are left with bounding

$$\sum_{\tau,\alpha} \sum_{\mathbf{x}_2,\ldots,\mathbf{x}_{n_\alpha} \in B_{\ell/4}} \left[\tilde{W}_j^{\#}(\mathbf{0}, \mathbf{x}_2 \ldots, \mathbf{x}_{n_\alpha}) - \tilde{W}_j^{\infty,\#}(\mathbf{0}, \mathbf{x}_2 \ldots, \mathbf{x}_{n_\alpha})\right] , \qquad (3.72)$$

where the differences in square brackets can be written as a sum of terms each of which involves either the difference between G and 1 (or, similarly, d_i and x_i), or the difference between a propagator $g^{(k)}$ and its infinite volume limit. Regarding the first class of terms, remember that the relative difference between G, d_i and their infinite volume limit is bounded dimensionally by $2^{4(h^*-h_i)}$, see the discussion in item (4) above, in particular after Eq.(3.49). Part of this factor (at least a portion $2^{(1+\varepsilon)(h^*-h_i)}$) is needed in order to renormalize the kernels of the effective potential, see Remark following Eq.(3.61). What we are left with is exactly a factor $2^{(3-\varepsilon)(h^*-h_i)}$, as commented at the end of that Remark. Summing these contributions over the scales $j > h$ gives the desired bound Eq.(3.69). Finally, regarding the terms involving a difference between the finite and infinite volume propagators, we observe that by the Poisson summation formula, the finite volume propagator $g^{(h)}(\mathbf{x})$, which is antiperiodic in its argument, can be written as a sum over images:

$$g^{(h)}(x_1, x_2) = \sum_{\mathbf{n} \in \mathbb{Z}^2} (-1)^{n_1+n_2} g_\infty^{(h)}(x_1 + n_1\ell, x_2 + n_2 L) =: g_\infty^{(h)}(\mathbf{x}) + \delta g^{(h)}(\mathbf{x}) , \qquad (3.73)$$

where $\delta g^{(h)}(\mathbf{x})$ is smaller than any power of $(2^{-h}/\ell)$, namely, if $|\mathbf{x}| \leq \ell/2$, $|\delta g^{(h)}(\mathbf{x})| \leq C_p 2^h (2^{-h}/\ell)^p$, $\forall p \geq 0$. Therefore, the terms involving a difference between a finite and an infinite volume propagator are smaller than any power in $(2^{-j}/\ell)$, which is more than enough to the purpose of deriving the desired bound Eq.(3.69). This concludes the proof of that bound.

In order to complete the discussion related to the choice of the initial data z_0, ν_0, we are left with inverting the relation for Z, which is obtained by combining Eq.(3.67) with the infinite volume limit of Eq.(3.51), namely:

$$Z = 1 + \zeta\big(\lambda, z_0^*(\lambda, Z), \nu_0^*(\lambda, Z)\big) . \qquad (3.74)$$

The key ingredients to be used are the derivative estimates Eq.(3.52) together with the observation that the derivatives with respect to Z of the propagators obey to the same decay bound as the propagators themselves, so that

$$\frac{\partial z_0^*}{\partial Z} = -\sum_{j \leq 0} \frac{\partial \beta_j^{\infty,z}(\underline{v}^*)}{\partial Z} = O(\lambda) , \qquad (3.75)$$

simply because $\partial_Z \beta_j^{\infty,z}(\underline{v}^*) = O(\lambda 2^{j\theta'})$ for some $\theta' < \theta$. Therefore we can apply the implicit function theorem to invert Eq.(3.74). This concludes the discussion about the choice of the initial data for the flow equation and, correspondingly, of the parameter Z in Eq.(3.18). Note that once Z is fixed, the critical temperature t_c is given by Eq.(3.51):

$$t_c = t_c^0 + \tau\big(\lambda, z_0^*(\lambda, Z), \nu_0^*(\lambda, Z)\big) \ . \tag{3.76}$$

3.4 Explicit computation of the bulk and finite volume corrections to the pressure

In this section we compute the bulk and correction terms from the free energy at the critical temperature, on the basis of Eqs.(3.16)-(3.17) and of the construction of $\mathcal{Z}_{--}(\Lambda_{\ell,L})$ in the previous sections. Using Eq.(3.21), we write

$$\log \mathcal{Z}_{--}(\Lambda_{\ell,L}) = \log C_{\ell,L} + \log \tilde{\mathcal{Z}}^0_{--}(\Lambda_{\ell,L}) + \log \int P(d\Phi)\, e^{\overline{\mathcal{V}}(\Phi)} \tag{3.77}$$

and we compute separately the contributions from the three terms.

The term $\log C_{\ell,L}$. Using Eq.(3.3) and the properties of the potential $v(\mathbf{x})$ spelled after Eq.(1.1), as well as the definition of $V_{\ell,L}(\lambda)$ in Eq.(3.4), we can write

$$\begin{aligned}\frac{1}{\ell L}\log C_{\ell,L} &= \log\big(2\cosh^2(\beta_c J)\big) + \sum_{\mathbf{x}\in\mathbb{Z}^2}\log\cosh\big(\frac{\beta_c\lambda}{2}v(\mathbf{x})\big) + \\ &+ 2\sum_{\substack{\Gamma\subseteq\Lambda_{\ell,L}:\\ \operatorname{supp}\Gamma\ni b_0}} \frac{\varphi^T(\Gamma)}{|\operatorname{supp}\Gamma|}\prod_{\gamma\in\Gamma}\zeta(\emptyset,\emptyset;\gamma)\ ,\end{aligned} \tag{3.78}$$

where β_c is the interacting critical temperature (such that $t_c = \tanh(\beta_c J)$, with t_c fixed as in Eq.(3.76)) and b_0 can be chosen aribitrarily, e.g., it can be fixed to be the bond connecting $\mathbf{0}$ with $\mathbf{e}_1$. Note that the last term in the first line is independent of ℓ, L, because $v(\mathbf{x})$ has finite range, while the term in the second line differs from its infinite volume limit by exponentially small terms, which correspond to the contributions from multipolygons Γ that either wind up over the torus $\Lambda_{\ell,L}$ or touch the complement of $\Lambda_{\ell,L}$ on $\mathbb{Z}^2$ (their exponential smallness follows from the decay bound Eq.(3.5)). In conclusion, the term $\log C_{\ell,L}$ contributes to the bulk term of the free energy

Eq.(3.16),

$$f_{\infty;1} := \lim_{\ell,L\to\infty} \frac{1}{\ell L} \log C_{\ell,L} = \log\big(2\cosh^2(\beta_c J)\big) + \tag{3.79}$$
$$+ \sum_{\mathbf{x}\in\mathbb{Z}^2} \log\cosh\Big(\frac{\beta_c\lambda}{2} v(\mathbf{x})\Big) + 2 \sum_{\substack{\Gamma\subseteq\mathbb{Z}^2:\\ \operatorname{supp}\Gamma\ni b_0}} \frac{\varphi^T(\Gamma)}{|\operatorname{supp}\Gamma|} \prod_{\gamma\in\Gamma} \zeta(\emptyset,\emptyset;\gamma)\ ,$$

but not to the finite volume correction Eq.(3.17).

The term $\log \tilde{\mathcal{Z}}^0_{--}(\Lambda_{\ell,L})$. By its very definition, see the lines preceding Eq.(3.21), we can write

$$\frac{1}{\ell L} \log \tilde{\mathcal{Z}}^0_{--}(\Lambda_{\ell,L}) = \frac{1}{\ell L} \log \int \mathcal{D}\Phi e^{Z S_{t_c^0}(\Phi)} = 2\log Z + \frac{1}{\ell L} \log \int \mathcal{D}\Phi e^{S_{t_c^0}(\Phi)}\ , \tag{3.80}$$

where in the last identity we performed the Grassmann change of variables $\Phi \to \sqrt{Z}\Phi$. The last term, $\frac{1}{\ell L}\log\int \mathcal{D}\Phi e^{S_{t_c^0}(\Phi)}$, is the non interacting pressure evaluated at the critical point (up to an additive constant $\frac{1}{\ell L}\log C_{\ell,L}\big|_{\lambda=0,\beta=\beta_c^0} = \sqrt{2}+1$), whose bulk and finite volume corrections have been evaluated in great detail in Section 2. Putting things together we get:

$$f_{\infty;2} := \lim_{\ell,L\to\infty} \frac{1}{\ell L} \log \tilde{\mathcal{Z}}^0_{--}(\Lambda_{\ell,L}) = \tag{3.81}$$
$$= \log \frac{\sqrt{2}Z^2}{\sqrt{2}+1} + \frac{1}{2} \int_{[-\pi,\pi]^2} \frac{d\mathbf{k}}{(2\pi)^2} \log(4 - 2\cos k_1 - 2\cos k_2)$$

and

$$\lim_{\ell\to\infty}\lim_{L\to\infty} \Big[\frac{\ell}{L} \log \tilde{\mathcal{Z}}^0_{--}(\Lambda_{\ell,L}) - \ell^2 f_{\infty;2}\Big] = \frac{\pi}{12}\ . \tag{3.82}$$

The term $\log \int P(d\Phi)\, e^{\overline{\mathcal{V}}(\Phi)}$. By the Renormalization Group analysis of $\int P(d\Phi)\, e^{\overline{\mathcal{V}}(\Phi)}$ described above, we can write:

$$\frac{1}{\ell L} \log \int P(d\Phi)\, e^{\overline{\mathcal{V}}(\Phi)} = E_{h^*} = E_0 + \sum_{h^*\le h\le 0} e_h \tag{3.83}$$

with E_0 and e_h defined by Eqs.(3.28) and (3.55). Using the fact that $|z_k| + |\nu_k| \le C|\lambda| 2^{\theta k}$ and the tree expansion explained above, we can bound

$$|e_h| \le C|\lambda| 2^{2h} 2^{\frac{\theta}{2}h}\ , \tag{3.84}$$

uniformly in ℓ, L, for all $\theta \in (0,1)$. Denoting by E_0^∞ and e_h^∞ the infinite volume limits of E_0, e_h, we can write the contribution to the bulk free energy under consideration as:

$$f_{\infty;3} := \lim_{\ell, L\to\infty} \log \int P(d\Phi)\, e^{\overline{\mathcal{V}}(\Phi)} = E_0^\infty + \sum_{h\le 0} e_h^\infty \;, \tag{3.85}$$

which is an exponentially convergent series, whose sum is of order λ. The finite volume correction of interest can then be written as

$$\begin{aligned}
&\lim_{\ell\to\infty}\lim_{L\to\infty}\Big[\frac{\ell}{L}\log\int P(d\Phi)\, e^{\overline{\mathcal{V}}(\Phi)} - \ell^2 f_{\infty;3}\Big] = \\
&\qquad = \lim_{\ell\to\infty} \ell^2 \big[(\tilde E_0 - E_0^\infty) + \sum_{h^*\le h\le 0}(\tilde e_h - e_h^\infty) - \sum_{h<h^*} e_h^\infty\big]\;,
\end{aligned} \tag{3.86}$$

where $\tilde E_0 = \lim_{L\to\infty} E_0$ and $\tilde e_h = \lim_{L\to\infty} e_h$. Now, $(\tilde E_0 - E_0^\infty)$ is exponentially small in ℓ, as it follows from the fact that it can be written as a sum of terms that involve at least one difference between the finite and infinite volume propagator of the χ field, which is exponentially small in ℓ; therefore $\lim_{\ell\to\infty}\ell^2(\tilde E_0 - E_0^\infty) = 0$. The last contribution to the r.h.s. of Eq.(3.86) is also easy to estimate; in fact, using Eq.(3.84) and the definition of $h^* = \lfloor \log_2(\pi/\ell)\rfloor$, we get:

$$\sum_{h<h^*} |e_h^\infty| \le (\mathrm{const.})|\lambda| 2^{2h^*} 2^{\frac{\theta}{2}h^*} \le (\mathrm{const.})|\lambda| \ell^{-2-\frac{\theta}{2}}\;, \tag{3.87}$$

which implies that $\lim_{\ell\to\infty}\ell^2\sum_{h<h^*} e_h^\infty = 0$. We are left with

$$\lim_{\ell\to\infty}\ell^2 \sum_{h^*\le h\le 0}(\tilde e_h - e_h^\infty)\;. \tag{3.88}$$

Remember that e_h (as well as $\tilde e_h$ and e_h^∞) is a function of the whole sequence of coupling constants $\underline{\bar v}$ (see the lines following Eq.(3.67) for a definition of $\underline{\bar v}$ and of its infinite volume counterpart $\underline{v}^*$); we shall also indicate by $\underline{\tilde v}$ the $L\to\infty$ limit of $\underline{\bar v}$. We can then write Eq.(3.88) as

$$\lim_{\ell\to\infty}\ell^2\sum_{h^*\le h\le 0}\big[(\tilde e_h(\underline{\tilde v}) - e_h^\infty(\underline{\tilde v})) + (e_h^\infty(\underline{\tilde v}) - e_h^\infty(\underline{v}^*))\big]\;. \tag{3.89}$$

Thanks to the bound Eq.(3.69) on the difference between the finite and infinite volume running coupling constants, we immediately see that the second term in square brackets can be bounded as:

$$|e_h^\infty(\underline{\tilde v}) - e_h^\infty(\underline{v}^*)| \le (\mathrm{const.})|\lambda| 2^{(2+\frac{\theta}{2})h}\Big(\frac{2^{-h}}{\ell}\Big)^{3-\varepsilon}\;. \tag{3.90}$$

Picking $\theta, \varepsilon > 0$ small enough, we get

$$\sum_{h^* \le h \le 0} |e_h^\infty(\tilde{\underline{v}}) - e_h^\infty(\underline{v}^*)| \le (\text{const.})|\lambda| 2^{(2+\frac{\theta}{2})h^*} \Big(\frac{2^{-h^*}}{\ell}\Big)^{3-\varepsilon} \le (\text{const.})|\lambda| \ell^{-2-\frac{\theta}{2}} , \tag{3.91}$$

which implies $\lim_{\ell\to\infty} \ell^2 \big(e_h^\infty(\tilde{\underline{v}}) - e_h^\infty(\underline{v}^*)\big) = 0$. We are left with the first term in square brackets in Eq.(3.89), which can be studied in a way similar to Eq.(3.71). By repeating a discussion completely analogous to the one following Eq.(3.71), we find that $|\tilde{e}_h(\tilde{\underline{v}}) - e_h^\infty(\tilde{\underline{v}})|$ admits the same bound Eq.(3.90) as $|e_h^\infty(\tilde{\underline{v}}) - e_h^\infty(\underline{v}^*)|$, so that also $\lim_{\ell\to\infty} \ell^2 \big(\tilde{e}_h(\tilde{\underline{v}}) - e_h^\infty(\tilde{\underline{v}})\big) = 0$. In conclusion, the finite volume corrections Eq.(3.86) are exactly zero. Putting all the contributions together, we find that the central charge c defined by Eq.(3.17) is independent of λ and equal to 1/2, as desired.

4 Proof of the partition function inequality.

In this section we prove Lemma 1. We start by proving it in the case $\lambda = 0$ with generic couplings $\{J_b\}$. In this case, using Eqs.(2.4)–(2.7), we immediately find

$$\mathcal{Z}^0_{+-}(\{J_b\};\Lambda_{\ell,L}) + \mathcal{Z}^0_{-+}(\{J_b\};\Lambda_{\ell,L}) = 2\mathcal{Z}^0_{e-e}(\{J_b\};\Lambda_{\ell,L}) + 2\mathcal{Z}^0_{o-o}(\{J_b\};\Lambda_{\ell,L}) \tag{4.1}$$

and

$$\begin{aligned}\mathcal{Z}^0_{+-}(\{J_b\};\Lambda_{\ell,L}) + \mathcal{Z}^0_{-+}(\{J_b\};\Lambda_{\ell,L}) + \mathcal{Z}^0_{--}(\{J_b\};\Lambda_{\ell,L}) = 3\mathcal{Z}^0_{e-e}(\{J_b\};\Lambda_{\ell,L}) +\\ +\mathcal{Z}^0_{e-o}(\{J_b\};\Lambda_{\ell,L}) + \mathcal{Z}^0_{o-e}(\{J_b\};\Lambda_{\ell,L}) + \mathcal{Z}^0_{o-o}(\{J_b\};\Lambda_{\ell,L}) \ .\end{aligned} \tag{4.2}$$

Remember that the four partition functions $\mathcal{Z}^0_{e-e}(\{J_b\};\Lambda_{\ell,L})$, $\mathcal{Z}^0_{e-o}(\{J_b\};\Lambda_{\ell,L})$, $\mathcal{Z}^0_{o-e}(\{J_b\};\Lambda_{\ell,L})$ and $\mathcal{Z}^0_{o-o}(\{J_b\};\Lambda_{\ell,L})$ are all positive, see the definitions Eq. (2.8) and following lines. Therefore, the right hand side of Eq.(4.1) is ≥ 0, which proves Eq.(3.12) for $\lambda = 0$ and bond-dependent couplings. Moreover, the r.h.s. of Eq.(4.2) is bounded from above by

$$\begin{aligned}3\mathcal{Z}^0_{e-e}(\{J_b\};\Lambda_{\ell,L}) + 3\mathcal{Z}^0_{e-o}(\{J_b\};\Lambda_{\ell,L}) + 3\mathcal{Z}^0_{o-e}(\{J_b\};\Lambda_{\ell,L}) + 3\mathcal{Z}^0_{o-o}(\{J_b\};\Lambda_{\ell,L})\\ \equiv 3\mathcal{Z}^0(\{J_b\};\Lambda_{\ell,L}) \ ,\end{aligned} \tag{4.3}$$

where we used Eq.(2.9). This proves the lower bound in Eq.(3.11) for $\lambda = 0$ and bond-dependent couplings. The upper bound in Eq.(3.11) for this case is proved analogously: it is enough to observe that the r.h.s. of Eq.(4.2) can be bounded from below by

$$\begin{aligned}\mathcal{Z}^0_{e-e}(\{J_b\};\Lambda_{\ell,L}) + \mathcal{Z}^0_{e-o}(\{J_b\};\Lambda_{\ell,L}) + \mathcal{Z}^0_{o-e}(\{J_b\};\Lambda_{\ell,L}) + \mathcal{Z}^0_{o-o}(\{J_b\};\Lambda_{\ell,L})\\ \equiv \mathcal{Z}^0(\{J_b\};\Lambda_{\ell,L}) \ ,\end{aligned} \tag{4.4}$$

which leads to the desired bound.

Let us now turn to the interacting case. The key issue is to obtain a representation of the four Grassmann partition functions $\mathcal{Z}_{++}(\Lambda_{\ell,L})$, $\mathcal{Z}_{+-}(\Lambda_{\ell,L})$, $\mathcal{Z}_{-+}(\Lambda_{\ell,L})$, $\mathcal{Z}_{--}(\Lambda_{\ell,L})$ in terms of (positive!) multipolygons partition sums. Such a representation is implicitly derived in [14, Section II.A]. Let us make it explicit here. The starting point is the representation for the partition function in terms of disconnected polymers (see [14, Eqs.(2.8)-(2.9)-(2.10)]):

$$\begin{aligned}\mathcal{Z}(\Lambda_{\ell,L}) &= \Big[\prod_{\{\mathbf{x},\mathbf{y}\}} \cosh^2\big(\tfrac{1}{2}\beta\lambda v(\mathbf{x}-\mathbf{y})\big)\Big] \sum_{\underline{\sigma}} e^{\beta J \sum_b \tilde{\sigma}_b} \cdot \\ &\quad \cdot \sum_{n\geq 0} \sum_{\{\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n\}\subseteq \Lambda_{\ell,L}} \varphi(\{\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n\}) \prod_{\tilde{\gamma}\in\tilde{\Gamma}} z(\tilde{\gamma}) \;, \end{aligned} \tag{4.5}$$

where

$$z(\tilde{\gamma}) = \sum_{\substack{\mathcal{S}\text{ connected:}\\ \tilde{\gamma}(\mathcal{S})=\tilde{\gamma}}} \Big[\prod_{S\in\mathcal{S}} \tanh(\tfrac{1}{2}\beta\lambda v_S)\Big]\Big[\prod_{b\in \mathrm{bl}(\mathcal{S})} \tilde{\sigma}_b\Big] \tag{4.6}$$

and:

- b indicates a nearest neighbor bond and $\tilde{\sigma}_b$ is the bond spin, i.e. the product of the two spins at the vertices of b;
- $\mathcal{S} = \{S_1,\ldots,S_m\}$ is a set of strings, where each string S_i is the union of the bonds in a finite lattice path that can be either horizontal, or vertical, or "corner-like", as in Figure 1; moreover, $v_S := v(\mathbf{x}-\mathbf{y})$, where $\mathbf{x}$ and $\mathbf{y}$ are the first and last points connected by the path S on $\Lambda_{\ell,L}$. We say that a set of strings $\mathcal{S} = \{S_1,\ldots,S_m\}$ is connected if, given $1 \leq i_0 < j_0 \leq m$, we can find a sequence $(S_{i_0}, S_{i_1},\ldots,S_{i_p} \equiv S_{j_0})$ such that $S_{i_l}\cap S_{i_{l+1}} \neq \emptyset$. From a graphical point of view, every connected component $\mathcal{S}$ corresponds in a non-unique way to a *polymer* $\tilde{\gamma}(\mathcal{S})$, i.e., a connected set of bonds. It is helpful to color the bonds in $\tilde{\gamma}(\mathcal{S})$ black or gray, depending on whether the given bond belongs to an odd or even number of strings in $\mathcal{S}$, and denote the set of bonds thus colored black by $\mathrm{bl}(\mathcal{S})$. See [14, Fig.4].
- the sum $\sum_{\{\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n\}}$ in the r.h.s. runs over sets of polymers $\{\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n\}$, such that each polymer is contained in $\Lambda_{\ell,L}$. Moreover, the function $\varphi(\{\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n\})$ implements the hard core condition, that is φ is equal to 1 if none of the polymers overlap, and 0 otherwise (here two polymers overlap if and only if they have at least one bond in common); the term with $n = 0$ should be interpreted as 1.

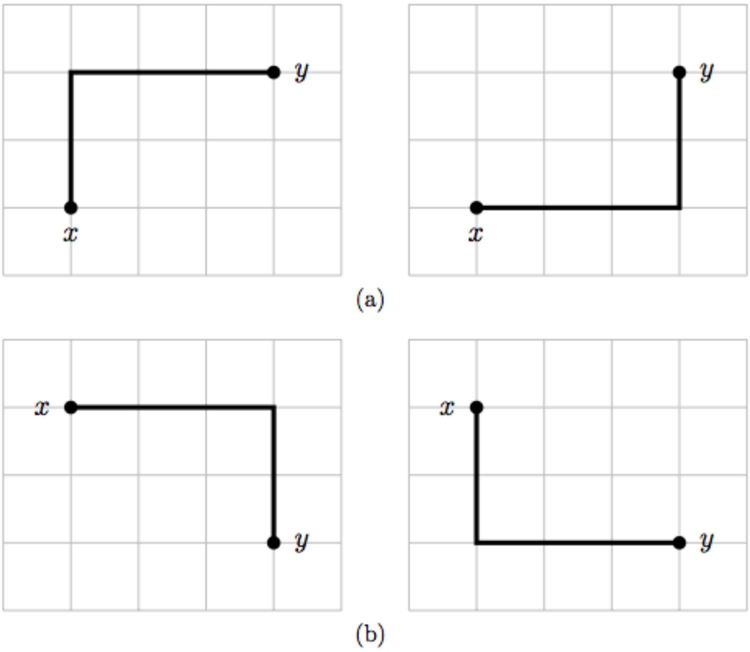

Figure 1: The four possible types of “corner like” strings

Now, in order to obtain the Grassmann representation Eq.(3.1) one can proceed as explained in [14, Section II.A]: the idea is simply to rewrite Eq.(4.5) as

$$\mathcal{Z}(\Lambda_{\ell,L}) = \Big[\prod_{\{\mathbf{x},\mathbf{y}\}} \cosh^2\big(\tfrac{1}{2}\beta\lambda v(\mathbf{x}-\mathbf{y})\big)\Big] \cdot \sum_{n\geq 0} \sum_{\{\tilde\gamma_1,\dots,\tilde\gamma_n\}\subseteq\Lambda_{\ell,L}} \varphi(\{\tilde\gamma_1,\dots,\tilde\gamma_n\}) \cdot$$
$$\cdot \sum_{\substack{\mathcal{S}_1,\dots,\mathcal{S}_n \text{ connected:} \\ \tilde\gamma(\mathcal{S}_i)=\tilde\gamma_i}} \Big[\prod_{i=1}^{n}\prod_{S\in\mathcal{S}_i} \tanh(\tfrac{1}{2}\beta\lambda v_S)\Big] \sum_{\underline{\sigma}} \prod_{b\in \mathrm{bl}(\underline{\mathcal{S}})} \tilde\sigma_b \, e^{\beta J \sum_b \tilde\sigma_b} \,, \tag{4.7}$$

where $\underline{\mathcal{S}} := (\mathcal{S}_1,\dots,\mathcal{S}_n)$, $\mathrm{bl}(\underline{\mathcal{S}}) = \cup_{i=1}^n \mathrm{bl}(\mathcal{S}_i)$ and

$$\sum_{\underline{\sigma}} \prod_{b\in \mathrm{bl}(\underline{\mathcal{S}})} \tilde\sigma_b \, e^{\beta J \sum_b \tilde\sigma_b} = \Big[\prod_{b\in \mathrm{bl}(\underline{\mathcal{S}})} \frac{1}{\beta}\frac{\partial}{\partial J_b}\Big] \mathcal{Z}^0(\{J_b\};\Lambda_{\ell,L})\big|_{J_b\equiv J} \tag{4.8}$$

and then to re-express $\mathcal{Z}^0(\{J_b\};\Lambda_{\ell,L})$ as a sum of Grassmann integrals, via

Eqs.(2.10) and (2.2). This leads to Eq.(3.1), with

$$\mathcal{Z}_{\boldsymbol{\alpha}}(\Lambda_{\ell,L}) = \Big[\prod_{\{\mathbf{x},\mathbf{y}\}} \cosh^2\big(\tfrac{1}{2}\beta\lambda v(\mathbf{x}-\mathbf{y})\big)\Big] \cdot \sum_{n\geq 0} \sum_{\{\tilde\gamma_1,\ldots,\tilde\gamma_n\}\subseteq\Lambda_{\ell,L}} \varphi(\{\tilde\gamma_1,\ldots,\tilde\gamma_n\}) \cdot$$

$$\cdot \sum_{\substack{\mathcal{S}_1,\ldots,\mathcal{S}_n \text{ connected:} \\ \tilde\gamma(\mathcal{S}_i)=\tilde\gamma_i}} \Big[\prod_{i=1}^{n}\prod_{S\in\mathcal{S}_i} \tanh(\tfrac{1}{2}\beta\lambda v_S)\Big]\Big[\prod_{b\in \mathrm{bl}(\underline{\mathcal{S}})} \frac{1}{\beta}\frac{\partial}{\partial J_b}\Big] \mathcal{Z}^0_{\boldsymbol{\alpha}}(\{J_b\};\Lambda_{\ell,L})\big|_{J_b\equiv J} \ , \quad (4.9)$$

which is equivalent to [14, Eq.(2.11)] that, if further manipulated, implies the representation Eq.(3.2). On the other hand, if we plug Eqs.(2.4)–(2.7) into Eq.(4.9), we immediately get

$$\begin{aligned}
\mathcal{Z}_{++}(\Lambda_{\ell,L}) &= \mathcal{Z}_{e-e}(\Lambda_{\ell,L}) - \mathcal{Z}_{e-o}(\Lambda_{\ell,L}) - \mathcal{Z}_{o-e}(\Lambda_{\ell,L}) - \mathcal{Z}_{o-o}(\Lambda_{\ell,L}) \ , \\
\mathcal{Z}_{+-}(\Lambda_{\ell,L}) &= \mathcal{Z}_{e-e}(\Lambda_{\ell,L}) + \mathcal{Z}_{e-o}(\Lambda_{\ell,L}) - \mathcal{Z}_{o-e}(\Lambda_{\ell,L}) + \mathcal{Z}_{o-o}(\Lambda_{\ell,L}) \ , \\
\mathcal{Z}_{-+}(\Lambda_{\ell,L}) &= \mathcal{Z}_{e-e}(\Lambda_{\ell,L}) - \mathcal{Z}_{e-o}(\Lambda_{\ell,L}) + \mathcal{Z}_{o-e}(\Lambda_{\ell,L}) + \mathcal{Z}_{o-o}(\Lambda_{\ell,L}) \ , \\
\mathcal{Z}_{--}(\Lambda_{\ell,L}) &= \mathcal{Z}_{e-e}(\Lambda_{\ell,L}) + \mathcal{Z}_{e-o}(\Lambda_{\ell,L}) + \mathcal{Z}_{o-e}(\Lambda_{\ell,L}) - \mathcal{Z}_{o-o}(\Lambda_{\ell,L}) \ ,
\end{aligned} \quad (4.10)$$

with

$$\mathcal{Z}_{e-e}(\Lambda_{\ell,L}) = \Big[\prod_{\{\mathbf{x},\mathbf{y}\}} \cosh^2\big(\tfrac{1}{2}\beta\lambda v(\mathbf{x}-\mathbf{y})\big)\Big] \cdot \sum_{n\geq 0} \sum_{\{\tilde\gamma_1,\ldots,\tilde\gamma_n\}\subseteq\Lambda_{\ell,L}} \varphi(\{\tilde\gamma_1,\ldots,\tilde\gamma_n\}) \cdot$$

$$\cdot \sum_{\substack{\mathcal{S}_1,\ldots,\mathcal{S}_n \text{ connected:} \\ \tilde\gamma(\mathcal{S}_i)=\tilde\gamma_i}} \Big[\prod_{i=1}^{n}\prod_{S\in\mathcal{S}_i} \tanh(\tfrac{1}{2}\beta\lambda v_S)\Big]\Big[\prod_{b\in \mathrm{bl}(\underline{\mathcal{S}})} \frac{1}{\beta}\frac{\partial}{\partial J_b}\Big] \mathcal{Z}^0_{e-e}(\{J_b\};\Lambda_{\ell,L})\big|_{J_b\equiv J} \ , \quad (4.11)$$

and analogously for the three other partition functions, with the label $e-e$ replaced at both sides by $e-o$, $o-e$, $o-o$, respectively. If we use the definition Eq.(2.8), we can rewrite the last factor in Eq.(4.11) as:

$$\begin{aligned}
&\Big[\prod_{b\in \mathrm{bl}(\underline{\mathcal{S}})} \frac{1}{\beta}\frac{\partial}{\partial J_b}\Big] \mathcal{Z}^0_{e-e}(\{J_b\};\Lambda_{\ell,L})\big|_{J_b\equiv J} = \\
&= 2^{\ell L} t^{|\mathrm{bl}(\underline{\mathcal{S}})|}\Big[\prod_{b\in\mathcal{B}_{\ell,L}} \cosh(\beta J_b)\Big] \sum\nolimits^{(e-e)}_{\Gamma\subseteq\Lambda_{\ell,L}} \prod_{\gamma\in\Gamma}\prod_{b\in\gamma} t_b(\underline{\mathcal{S}}) \ ,
\end{aligned} \quad (4.12)$$

where $t_b(\underline{\mathcal{S}})$ is equal either to t, if $b \notin \mathrm{Bl}(\underline{\mathcal{S}}) = \cup_{\mathcal{S}_i\in\underline{\mathcal{S}}}\mathrm{bl}(\mathcal{S}_i)$, or to $1/t$, if $b \in \mathrm{Bl}(\underline{\mathcal{S}})$. An equivalent way of rewriting Eq.(4.12) is

$$\Big[\prod_{b\in \mathrm{bl}(\underline{\mathcal{S}})} \frac{1}{\beta}\frac{\partial}{\partial J_b}\Big] \mathcal{Z}^0_{e-e}(\{J_b\};\Lambda_{\ell,L})\big|_{J_b\equiv J} = t^{|\mathrm{bl}(\underline{\mathcal{S}})|}\mathcal{Z}^0_{e-e}(\{\bar J_b(\underline{\mathcal{S}})\};\Lambda_{\ell,L}) \ , \quad (4.13)$$

where $\bar{J}_b(\underline{\mathcal{S}})$ is equal either to J, if $b \notin \mathrm{Bl}(\underline{\mathcal{S}}) = \cup_{\mathcal{S}_i \in \underline{\mathcal{S}}} \mathrm{bl}(\mathcal{S}_i)$, or to $\beta^{-1}\mathrm{arctanh}(1/t)$, if $b \in \mathrm{Bl}(\underline{\mathcal{S}})$. Plugging this back into Eq.(4.11) gives

$$\mathcal{Z}_{e-e}(\Lambda_{\ell,L}) = \Big[\prod_{\{\mathbf{x},\mathbf{y}\}} \cosh^2\big(\tfrac{1}{2}\beta\lambda v(\mathbf{x}-\mathbf{y})\big)\Big] \cdot \sum_{n\geq 0} \sum_{\{\tilde\gamma_1,\ldots,\tilde\gamma_n\}\subseteq\Lambda_{\ell,L}} \varphi(\{\tilde\gamma_1,\ldots,\tilde\gamma_n\}) \cdot$$
$$\cdot \sum_{\substack{\mathcal{S}_1,\ldots,\mathcal{S}_n \text{ connected:} \\ \tilde\gamma(\mathcal{S}_i)=\tilde\gamma_i}} \Big[\prod_{i=1}^n \prod_{S\in\mathcal{S}_i} \tanh(\tfrac{1}{2}\beta\lambda v_S)\Big] t^{|\mathrm{bl}(\underline{\mathcal{S}})|} \mathcal{Z}^0_{e-e}(\{\bar{J}_b(\underline{\mathcal{S}})\};\Lambda_{\ell,L}) \;, \tag{4.14}$$

and analogous formulas are valid for the three other partition functions, with the label $e-e$ replaced at both sides by $e-o$, $o-e$, $o-o$, respectively.

Now, the key observation is that if $\lambda v_S \geq 0$ for all S, then all the factors appearing in Eq.(4.14) (as well as in its analogues with $e-e$ replaced by $e-o$, $o-e$, $o-o$) are non-negative. Therefore, if we insert Eq.(4.14) and its analogues with the label $e-e$ replaced by $e-o$, $o-e$, $o-o$, into Eq.(4.10), and if we use the known bounds on the partition functions at $\lambda=0$ with bond-dependent couplings, proved at the beginning of this section, namely

$$\mathcal{Z}^0(\{\bar{J}_b(\underline{\mathcal{S}})\};\Lambda_{\ell,L}) \leq \mathcal{Z}^0_{--}(\{\bar{J}_b(\underline{\mathcal{S}})\};\Lambda_{\ell,L}) + \mathcal{Z}^0_{-+}(\{\bar{J}_b(\underline{\mathcal{S}})\};\Lambda_{\ell,L}) + \mathcal{Z}^0_{+-}(\{\bar{J}_b(\underline{\mathcal{S}})\};\Lambda_{\ell,L})$$
$$\mathcal{Z}^0_{--}(\{\bar{J}_b(\underline{\mathcal{S}})\};\Lambda_{\ell,L}) + \mathcal{Z}^0_{-+}(\{\bar{J}_b(\underline{\mathcal{S}})\};\Lambda_{\ell,L}) + \mathcal{Z}^0_{+-}(\{\bar{J}_b(\underline{\mathcal{S}})\};\Lambda_{\ell,L}) \leq 3\mathcal{Z}^0(\{\bar{J}_b(\underline{\mathcal{S}})\};\Lambda_{\ell,L})$$
$$\mathcal{Z}^0_{-+}(\{\bar{J}_b(\underline{\mathcal{S}})\};\Lambda_{\ell,L}) + \mathcal{Z}^0_{+-}(\{\bar{J}_b(\underline{\mathcal{S}})\};\Lambda_{\ell,L}) \geq 0$$

then we finally obtain the desired estimates Eqs.(3.11)-(3.12). This concludes the proof of Lemma 1 and, therefore, of Theorem 1.1.

A Computation of the Grassmann partition functions in the nearest neighbor Ising model

In this appendix we prove Eqs.(2.14)-(2.15). We start from Eq.(2.14), with $\boldsymbol{\alpha} \neq (+,+)$. We go to Fourier space, by defining the unitary transformation

$$\hat\Phi_{\mathbf{k}} = |\Lambda_{\ell,L}|^{-1/2} \sum_{\mathbf{x}} e^{i\mathbf{k}\mathbf{x}} \Phi_{\mathbf{x}} \;, \qquad \Phi_{\mathbf{x}} = \begin{pmatrix} \overline{H}_{\mathbf{x}} \\ H_{\mathbf{x}} \\ \overline{V}_{\mathbf{x}} \\ V_{\mathbf{x}} \end{pmatrix} \;. \tag{A.1}$$

Note that the momenta in $\mathcal{D}_{\boldsymbol{\alpha}}$ can be grouped into pairs, namely:

$$\mathcal{D}_{\boldsymbol{\alpha}} = \cup_{\mathbf{k}\in\mathcal{D}^+_{\boldsymbol{\alpha}}} \{\mathbf{k},-\mathbf{k}\} \;, \qquad \mathcal{D}^+_{\boldsymbol{\alpha}} := \{\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}} : k_1 > 0 \text{ or } k_1 = 0 \text{ and } k_2 > 0\} \;. \tag{A.2}$$

In terms of these definitions, we can rewrite the quadratic action $S_t(\Phi)$ in block diagonal form as $S_t(\Phi) = -\sum_{\mathbf{k}\in\mathcal{D}^+_{\boldsymbol{\alpha}}} \hat{\Phi}^T_{-\mathbf{k}} M_{\mathbf{k}} \hat{\Phi}_{\mathbf{k}}$, where

$$M_{\mathbf{k}} := \begin{pmatrix} 0 & -(1+te^{-ik_1}) & 1 & 1 \\ 1+te^{ik_1} & 0 & -1 & 1 \\ -1 & 1 & 0 & -(1+te^{-ik_2}) \\ -1 & -1 & 1+te^{ik_2} & 0 \end{pmatrix} \tag{A.3}$$

Correspondingly, if $\boldsymbol{\alpha} \neq (+,+)$ and $d\hat{\Phi}_{\mathbf{k}} = d\hat{\overline{H}}_{\mathbf{k}} d\hat{H}_{\mathbf{k}} d\hat{\overline{V}}_{\mathbf{k}} d\hat{V}_{\mathbf{k}}$,

$$\int \mathcal{D}\Phi e^{S_t(\Phi)} = \prod_{\mathbf{k}\in\mathcal{D}^+_{\boldsymbol{\alpha}}} \int d\hat{\Phi}_{-\mathbf{k}} d\hat{\Phi}_{\mathbf{k}} e^{-\hat{\Phi}^T_{-\mathbf{k}} M_{\mathbf{k}} \hat{\Phi}_{\mathbf{k}}} = \prod_{\mathbf{k}\in\mathcal{D}^+_{\boldsymbol{\alpha}}} \det M_{\mathbf{k}} \ . \tag{A.4}$$

A computation shows that

$$\det M_{\mathbf{k}} = (1+t^2)^2 - 2t(1-t^2)(\cos k_1 + \cos k_2) \ , \tag{A.5}$$

which proves Eq.(2.14). Let us now turn to the case $\boldsymbol{\alpha} = (+,+)$. Note that now the modes $\mathbf{k} = \mathbf{0}$ and $\mathbf{k} = (\pi,\pi)$ are unpaired, namely:

$$\mathcal{D}_{+,+} = \{\mathbf{0}, (\pi,\pi)\} \bigcup_{\substack{\mathbf{k}\in\mathcal{D}^+_{+,+}: \\ \mathbf{k}\neq\mathbf{0},(\pi,\pi)}} \{\mathbf{k}, -\mathbf{k}\} \ . \tag{A.6}$$

Correspondingly, the Grassmann action can be rewritten in block diagonal form as

$$S_t(\Phi) = -\sum_{\substack{\mathbf{k}\in\mathcal{D}^+_{+,+}: \\ \mathbf{k}\neq\mathbf{0},(\pi,\pi)}} \hat{\Phi}^T_{-\mathbf{k}} M_{\mathbf{k}} \hat{\Phi}_{\mathbf{k}} - \frac{1}{2}\hat{\Phi}_{\mathbf{0}} M_{\mathbf{0}} \hat{\Phi}_{\mathbf{0}} - \frac{1}{2}\hat{\Phi}_{(\pi,\pi)} M_{(\pi,\pi)} \hat{\Phi}_{(\pi,\pi)} \tag{A.7}$$

so that, if $\boldsymbol{\alpha} = (+,+)$,

$$\begin{aligned} \int \mathcal{D}\Phi e^{S_t(\Phi)} &= \Big[\prod_{\substack{\mathbf{k}\in\mathcal{D}^+_{+,+}: \\ \mathbf{k}\neq\mathbf{0},(\pi,\pi)}} \int d\hat{\Phi}_{-\mathbf{k}} d\hat{\Phi}_{\mathbf{k}} e^{-\hat{\Phi}^T_{-\mathbf{k}} M_{\mathbf{k}} \hat{\Phi}_{\mathbf{k}}} \Big] \cdot \\ &\quad \cdot \int d\hat{\Phi}_{\mathbf{0}} e^{-\frac{1}{2}\hat{\Phi}^T_{\mathbf{0}} M_{\mathbf{0}} \hat{\Phi}_{\mathbf{0}}} \int d\hat{\Phi}_{(\pi,\pi)} e^{-\frac{1}{2}\hat{\Phi}^T_{(\pi,\pi)} M_{(\pi,\pi)} \hat{\Phi}_{(\pi,\pi)}} \\ &= \Big[\prod_{\substack{\mathbf{k}\in\mathcal{D}^+_{+,+}: \\ \mathbf{k}\neq\mathbf{0},(\pi,\pi)}} \det M_{\mathbf{k}} \Big] \cdot \mathrm{Pf} M_{\mathbf{0}} \cdot \mathrm{Pf} M_{(\pi,\pi)} \ . \end{aligned} \tag{A.8}$$

A computation shows that $\mathrm{Pf} M_{\mathbf{0}} = (1+t)^2 - 2$ and $\mathrm{Pf} M_{(\pi,\pi)} = (1-t)^2 - 2$. Combining these two identities with Eq.(A.5), we get Eq.(2.15).

B The contribution to the pressure from the ratio of the Grassmann partition functions

B.1 The non-interacting case

In this section we prove the vanishing of the limit in the second line of Eq.(2.19),

$$\lim_{\ell\to\infty}\lim_{L\to\infty}\frac{\ell}{L}\log\Big[\frac{1}{2}\big(1+\frac{\mathcal{Z}^0_{-+}(\Lambda_{\ell,L})}{\mathcal{Z}^0_{--}(\Lambda_{\ell,L})}+\frac{\mathcal{Z}^0_{+-}(\Lambda_{\ell,L})}{\mathcal{Z}^0_{--}(\Lambda_{\ell,L})}\big)\Big|_{\beta=\beta_c}\Big]=0\,. \tag{B.1}$$

To this purpose, we rewrite the partition functions in Eq.(2.17) as

$$\begin{aligned}
\mathcal{Z}^0_{--}(\Lambda_{\ell,L})\Big|_{\beta=\beta_c} &= (\sqrt{2})^{\ell L}\prod_{r=0}^{\ell-1}\Big[\prod_{n=0}^{L-1}(2a_{2r+1}-z_{2n+1}-z^{-1}_{2n+1})\Big]^{1/2}\,,\\
\mathcal{Z}^0_{-+}(\Lambda_{\ell,L})\Big|_{\beta=\beta_c} &= (\sqrt{2})^{\ell L}\prod_{r=0}^{\ell-1}\Big[\prod_{n=0}^{L-1}(2a_{2r+1}-z_{2n}-z^{-1}_{2n})\Big]^{1/2}\,,\\
\mathcal{Z}^0_{+-}(\Lambda_{\ell,L})\Big|_{\beta=\beta_c} &= (\sqrt{2})^{\ell L}\prod_{r=0}^{\ell-1}\Big[\prod_{n=0}^{L-1}(2a_{2r}-z_{2n+1}-z^{-1}_{2n+1})\Big]^{1/2}\,,
\end{aligned} \tag{B.2}$$

where $a_p := 2-\cos(\frac{\pi}{\ell}p)$ and $z_p := e^{i\frac{\pi}{L}p}$. The expressions in square brackets in the r.h.s of these equations can be further rewritten and put in the form used by [10]. Consider first the expression in square brackets appearing in the definition of $\mathcal{Z}^0_{--}(\Lambda_{\ell,L})\Big|_{\beta=\beta_c}$. Note that z_{2n+1} are the L roots of -1 and, therefore, $\prod_{n=0}^{L-1}(z-z_{2n+1})=z^L+1$, $\forall z\in\mathbb{C}$. In particular, $\prod_{n=0}^{L-1}z_{2n+1}=(-1)^L$, so that

$$\begin{aligned}
&\prod_{n=0}^{L-1}(2a_{2r+1}-z_{2n+1}-z^{-1}_{2n+1})=\prod_{n=0}^{L-1}(z^2_{2n+1}-2a_{2r+1}z_{2n+1}+1)=\\
&=\prod_{n=0}^{L-1}(z_{2n+1}-a^+_{2r+1})(z_{2n+1}-a^-_{2r+1})=\big[(a^+_{2r+1})^L+1\big]\cdot\big[(a^+_{2r+1})^L+1\big]\,,
\end{aligned}$$

where $a^\pm_p=a_p\pm\sqrt{a_p^2-1}$. Choosing $\gamma_p\ge 0$ in such a way that $\cosh\gamma_p=a_p$, we can further simplify this into

$$\prod_{n=0}^{L-1}(2a_{2r+1}-z_{2n+1}-z^{-1}_{2n+1})=(e^{L\gamma_{2r+1}}+1)(e^{-L\gamma_{2r+1}}+1)=4\cosh^2\frac{L\gamma_{2r+1}}{2}\,. \tag{B.3}$$

Plugging this back into the definition of $\mathcal{Z}^0_{--}(\Lambda_{\ell,L})\Big|_{\beta=\beta_c}$ gives

$$\mathcal{Z}^0_{--}(\Lambda_{\ell,L})\Big|_{\beta=\beta_c} = (\sqrt{2})^{\ell L}\prod_{r=0}^{\ell-1} 2\cosh\frac{L\gamma_{2r+1}}{2}\ ,$$

which is the same as [10, Eq.(2.3)]. Actually, by exchanging the roles of ℓ and L we can rewrite $\mathcal{Z}^0_{--}(\Lambda_{\ell,L})\Big|_{\beta=\beta_c}$ in two equivalent ways:

$$\mathcal{Z}^0_{--}(\Lambda_{\ell,L})\Big|_{\beta=\beta_c} = (\sqrt{2})^{\ell L}\prod_{r=0}^{\ell-1} 2\cosh\frac{L\gamma_{2r+1}}{2} = (\sqrt{2})^{\ell L}\prod_{n=0}^{L-1} 2\cosh\frac{\ell\tilde{\gamma}_{2n+1}}{2}\ , \tag{B.4}$$

where $\tilde{\gamma}_p \geq 0$ is defined by the condition that $\cosh\tilde{\gamma}_p = 2-\cos(\frac{\pi}{L}p)$. Proceeding exactly in the same way for $\mathcal{Z}^0_{-+}(\Lambda_{\ell,L})\Big|_{\beta=\beta_c}$, $\mathcal{Z}^0_{+-}(\Lambda_{\ell,L})\Big|_{\beta=\beta_c}$ gives:

$$\mathcal{Z}^0_{-+}(\Lambda_{\ell,L})\Big|_{\beta=\beta_c} = (\sqrt{2})^{\ell L}\prod_{r=0}^{\ell-1} 2\sinh\frac{L\gamma_{2r+1}}{2} = (\sqrt{2})^{\ell L}\prod_{n=0}^{L-1} 2\cosh\frac{\ell\tilde{\gamma}_{2n}}{2}\ ,$$
$$\mathcal{Z}^0_{+-}(\Lambda_{\ell,L})\Big|_{\beta=\beta_c} = (\sqrt{2})^{\ell L}\prod_{r=0}^{\ell-1} 2\cosh\frac{L\gamma_{2r}}{2} = (\sqrt{2})^{\ell L}\prod_{n=0}^{L-1} 2\sinh\frac{\ell\tilde{\gamma}_{2n+1}}{2}\ .$$

By using these formulas we rewrite the l.h.s. of Eq.(B.1) as

$$\lim_{\ell\to\infty}\lim_{L\to\infty}\frac{\ell}{L}\log\Big[\frac{1}{2}\big(1+\prod_{r=0}^{\ell-1}\tanh\frac{L\gamma_{2r+1}}{2}+\prod_{n=0}^{L-1}\tanh\frac{\ell\tilde{\gamma}_{2n+1}}{2}\big)\Big]\ . \tag{B.5}$$

Since $0 \leq \tanh x \leq 1$ for $x \geq 0$, it is apparent that the argument of the logarithm in this equation is positive and smaller than 3/2, which implies that the limit in Eq.(B.5) is zero.

B.2 The interacting case

In order to bound the ratios $\frac{|\mathcal{Z}_{-+}(\Lambda_{\ell,L})|}{\mathcal{Z}_{--}(\Lambda_{\ell,L})}$ and $\frac{|\mathcal{Z}_{+-}(\Lambda_{\ell,L})|}{\mathcal{Z}_{--}(\Lambda_{\ell,L})}$ appearing in Eq.(3.15), we compute $\mathcal{Z}_{-+}(\Lambda_{\ell,L})$ and $\mathcal{Z}_{+-}(\Lambda_{\ell,L})$ by a renormalization group construction analogous to the one used to analyze $\mathcal{Z}_{--}(\Lambda_{\ell,L})$. Everything is the same, with a few obvious changes induced by the different boundary conditions. The important fact that makes the renormalization group construction of $\mathcal{Z}_{-+}(\Lambda_{\ell,L})$ and $\mathcal{Z}_{+-}(\Lambda_{\ell,L})$ possible, is that h^* is finite for both: in the first case $h^* = \lfloor\log_2(\pi/\ell)\rfloor$, as for $\mathcal{Z}_{--}(\Lambda_{\ell,L})$, while in the second $h^* = \lfloor\log_2(\pi/L)\rfloor$.

Of course, the bulk contributions to the free energy are the same for all these partition functions, so that

$$\frac{|\mathcal{Z}_{-+}(\Lambda_{\ell,L})|}{\mathcal{Z}_{--}(\Lambda_{\ell,L})} \leq \Big[\frac{\mathcal{Z}^0_{-+}(\Lambda_{\ell,L})}{\mathcal{Z}^0_{--}(\Lambda_{\ell,L})}\Big]\Big|_{\beta=\beta_c} \cdot e^{\ell L R_1(\lambda)} , \tag{B.6}$$

$$\frac{|\mathcal{Z}_{+-}(\Lambda_{\ell,L})|}{\mathcal{Z}_{--}(\Lambda_{\ell,L})} \leq \Big[\frac{\mathcal{Z}^0_{+-}(\Lambda_{\ell,L})}{\mathcal{Z}^0_{--}(\Lambda_{\ell,L})}\Big]\Big|_{\beta=\beta_c} \cdot e^{\ell L R_2(\lambda)} , \tag{B.7}$$

where $\mathcal{Z}^0_{--}(\Lambda_{\ell,L}),\big|_{\beta=\beta_c}, \mathcal{Z}^0_{-+}(\Lambda_{\ell,L})\big|_{\beta=\beta_c}, \mathcal{Z}^0_{+-}(\Lambda_{\ell,L})\big|_{\beta=\beta_c}$ are the non interacting partition functions at criticality defined in Eqs.(2.16)-(2.17), and R_1, R_2, according to be renormalization group analysis, are bounded by $c|\lambda|\ell^{-2-\theta'}$ for some $c > 0$ and $0 < \theta' < 1$. As proved in Appendix B, the ratios $\Big[\frac{\mathcal{Z}^0_{-+}(\Lambda_{\ell,L})}{\mathcal{Z}^0_{--}(\Lambda_{\ell,L})}\Big]\Big|_{\beta=\beta_c}$ and $\Big[\frac{\mathcal{Z}^0_{+-}(\Lambda_{\ell,L})}{\mathcal{Z}^0_{--}(\Lambda_{\ell,L})}\Big]\Big|_{\beta=\beta_c}$ are positive and smaller than 1 and, therefore,

$$\frac{|\mathcal{Z}_{-+}(\Lambda_{\ell,L})|}{\mathcal{Z}_{--}(\Lambda_{\ell,L})} + \frac{|\mathcal{Z}_{+-}(\Lambda_{\ell,L})|}{\mathcal{Z}_{--}(\Lambda_{\ell,L})} \leq e^{C|\lambda|\ell^{-1-\theta'}L} , \tag{B.8}$$

for a suitable constant $C > 0$. Plugging this back into the l.h.s. of Eq.(3.15) gives

$$\lim_{\ell\to\infty}\lim_{L\to\infty}\frac{\ell}{L}\log\Big(1+\frac{|\mathcal{Z}_{-+}(\Lambda_{\ell,L})|}{\mathcal{Z}_{--}(\Lambda_{\ell,L})}+\frac{|\mathcal{Z}_{+-}(\Lambda_{\ell,L})|}{\mathcal{Z}_{--}(\Lambda_{\ell,L})}\Big) \leq \lim_{\ell\to\infty}\lim_{L\to\infty}\frac{\ell}{L}C|\lambda|\frac{L}{\ell^{1+\theta'}} , \tag{B.9}$$

which proves Eq.(3.15).

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