Realization of stripes and slabs in two and three dimensions

Alessandro Giuliani,[1] Elliott H. Lieb,[2,3] and Robert Seiringer[4]

[1]*Dipartimento di Matematica e Fisica, Università di Roma Tre, L.go S. L. Murialdo 1, 00146 Roma, Italy*
[2]*Department of Mathematics, Princeton University, Washington Road, Princeton, NJ 08544, USA*
[3]*Department of Physics, Princeton University, P. O. Box 708, Princeton, NJ 08542, USA*
[4]*Institute of Science and Technology Austria (IST Austria), Am Campus 1, A - 3400 Klosterneuburg, Austria*

We consider Ising models in two and three dimensions with nearest neighbor ferromagnetic interactions and long range, power law decaying, antiferromagnetic interactions. If the strength of the ferromagnetic coupling J is larger than a critical value J_c, then the ground state is homogeneous and ferromagnetic. As the critical value is approached from smaller values of J, it is believed that the ground state consists of a periodic array of stripes ($d = 2$) or slabs ($d = 3$), all of the same size and alternating magnetization. Here we prove rigorously that the ground state energy per site converges to that of the optimal periodic striped/slabbed state, in the limit that J tends to the ferromagnetic transition point. While this theorem does not prove rigorously that the ground state is precisely striped/slabbed, it does prove that in any suitably large box the ground state is striped/slabbed with high probability.

PACS numbers: 05.50.+q, 75.10.Hk, 75.70.Kw, 89.75.Kd

The spontaneous emergence of periodic states in translation invariant systems is still an incompletely understood phenomenon. Even less well understood is the phenomenon of translation symmetry breaking in only one direction, that is to say formation of striped patterns in two dimensions or slabbed patterns in three dimensions, which we collectively refer to as stripes. Particularly interesting is the formation of wide stripes, by which we mean that stripes have a width much larger than the microscopic length scales. Stripes of this kind are expected to display a sort of universal phenomenology, which is in fact observed in a variety of different systems, ranging from magnetic films [1–5], to manganites [6], to high-temperature superconductors [7–11], MOSFETs [12, 13], polymer suspensions [14, 15], twinned martensites [16, 17], Coulomb glasses [18], and many others [19–26].

While there exist some rigorous examples of symmetry breaking in two dimensions into doubly-periodic crystalline structures [27–30], we are aware of only one rigorous proof of formation of periodic arrays of wide stripes in isotropic two-dimensional systems: this is a system of inplane spins with four possible orientations interacting via a short range exchange plus the actual three-dimensional dipolar interaction [31]. It would be nice to find more examples of this kind. A simple and very popular model used to understand stripe formation in the classical setting is a d-dimensional Ising model with the following Hamiltonian:

$$H = -J \sum_{\langle \mathbf{x},\mathbf{y}\rangle} (\sigma_{\mathbf{x}}\sigma_{\mathbf{y}} - 1) + \sum_{\{\mathbf{x},\mathbf{y}\}} \frac{(\sigma_{\mathbf{x}}\sigma_{\mathbf{y}} - 1)}{|\mathbf{x}-\mathbf{y}|^p} \qquad (1)$$

where $J > 0$ is the relative strength of the attractive exchange interaction, the first sum ranges over nearest neighbor pairs in $\mathbb{Z}^d$, $d = 2, 3$, and the second over pairs of distinct sites in $\mathbb{Z}^d$. Depending on the specific value of the exponent p, the second term in the Hamiltonian can describe a Coulomb ($p = 1$), a dipolar ($p = 3$), or a more general repulsive interaction. Note that the Hamiltonian is normalized so that the homogeneous ferromagnetic state has zero energy. The question is to determine the ground state of the system, as the parameters J and p are varied. In some limiting cases, it is easy to identify the minimal energy states: e.g., if J is sufficiently small, the ground state is the Néel antiferromagnet, as one can prove by using reflection positivity [32]. If $p > d + 1$, there exists a critical value $J_c(p)$, of the form

$$J_c(p) = \sum_{y_1=1}^{\infty} \sum_{y_2,\dots,y_d=-\infty}^{\infty} \frac{y_1}{(y_1^2 + \cdots + y_d^2)^{p/2}} ,$$

such that the homogeneous ferromagnetic state is the ground state for $J \geq J_c$, and it is not the ground state for $J < J_c$ [33]. Note that $J = J_c(p)$ is the value of the ferromagnetic strength at which the surface tension of an infinite, isolated, straight domain wall vanishes. The expected region where wide stripes should occur is $p \leq d+1$ if $J \gg 1$, and $p > d+1$ if $J \lesssim J_c$. This is the region that we call "universal", in the sense that the structures displayed by the ground state in this regime are large compared to the lattice spacing and, therefore, their shape is expected to be independent of the microscopic details of the Hamiltonian. See Fig.1.

In this article, we report a recent advance in the understanding of the ground state phase diagram of model (1) in the universal regime, for $p > 2d$. Our new estimates are the sharpest rigorous bounds available on the ground state of the class of models under consideration here. To the best of our knowledge, these are the first results of this kind in three dimensions. Before we state them, let us introduce a few more definitions. Let $e_{\rm s}(h)$ be the

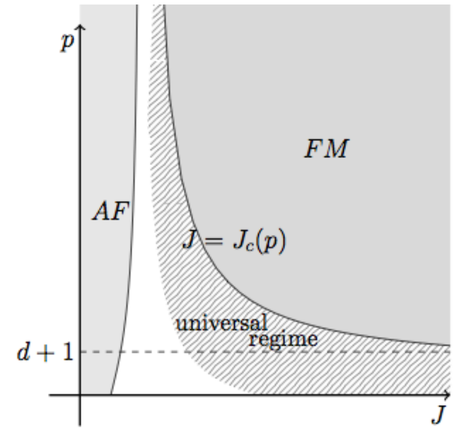

FIG. 1. Ground state phase diagram as a function of the antiferromagnetic decay exponent p and of the ferromagnetic strength J. In the leftmost region the ground state is the Néel antiferromagnetic state, while in the rightmost region it is the homogeneous ferromagnetic state. These two phases are rigorously known, while the rest of the phase diagram remains to be understood. The light-gray shaded region is the "universal regime", where the ferromagnetic islands (droplets) have typical size much larger than the lattice spacing. The conjecture is that the ground state is periodic and striped in the whole universal regime. A partial proof of this fact is given in this paper.

energy per site in the thermodynamic limit of periodic striped configurations consisting of stripes all of width h. We denote by $h(J)$ the optimal stripe width, which can be obtained by minimizing $e_s(h)$ over $h \in \mathbb{N}$. For $p > d+1$, $h(J)$ turns out to be of the order $(J_c-J)^{-\frac{1}{p-d-1}}$ as $J \nearrow J_c$. Let us denote by $e_S(J) \equiv e_s(h(J))$ the optimal striped energy per site and by $e_0(J)$ the actual ground state energy per site in the thermodynamic limit. Note that $e_0(J) = 0$ for $J \geq J_c$. Our main results can be summarized as follows.

Theorem. *Let us consider model (1) with $d = 2, 3$ and $p > 2d$. As $J \to J_c$ from below,*

$$\lim_{J \to J_c} \frac{e_0(J)}{e_S(J)} = 1 \ .$$

A few remarks are in order. The theorem says that asymptotically, as we approach the ferromagnetic transition line $J = J_c(p)$, the actual ground state energy approaches the optimal striped energy, which is a very strong indication of the conjectured periodic striped structure of the ground state. The proof comes with explicit error bounds on the difference $e_0(J)/e_S(J) - 1$, namely

$$1 \geq \frac{e_0(J)}{e_S(J)} \geq 1 + O\big((J_c - J)^{\frac{p-2d}{(d-1)(p-d-1)}}\big) \ . \qquad (2)$$

More precisely, the proof shows that the density of corners in the minimizing configuration is much smaller than $(J_c-J)^{d/(d-1)}$, i.e., the average mutual distance between corners is much larger than the typical stripe width $h(J)$. By corners here we mean the points ($d = 2$) or edges ($d = 3$) where domain walls bend by 90^o. The notion of corner and corner energy was introduced in [33] and understood there to play an important role for the case $p > 2d$: if widely separated from each other, the corners give a finite, positive contribution to the energy and, therefore, can be thought of as the elementary excitation of the system, at least in some approximate sense. Our new estimate (2) implies that the ground state configuration, if restricted to a suitable large window of side $\ell \gg h(J)$, with high probability has no corners, i.e., with high probability it is exactly striped. Similarly, we can show that with high probability these stripes have width all very close to $h(J)$.

The proof of the theorem is based on refined lower bounds on the ground state energy. The details of the proof are lengthy and will be given elsewhere [34]. Here we explain the main strategy behind the proof. These ideas may prove useful for subsequent developments in this subject. The key steps are the following.

1. Representation of the energy in terms of droplets: these are simply the maximal connected regions of negative spins, whose boundaries are the standard low-temperature contours of the nearest neighbor Ising model. The energy can be written as a sum of droplet self-energies, plus a long-range antiferromagnetic repulsion among different droplets.

2. Localization of the droplet energy functional into boxes Q of proper size, to be optimized over. By localization we mean that we bound from below the original energy of a generic droplet configuration in terms of a sum of independent local energies, each depending only on the restriction of the droplet configuration to the given box Q. Of course, the non-trivial aspect of this localization bound is due to the long range nature of the antiferromagnetic potential. The important fact is that our lower bound is sharp for striped configurations, up to unimportant boundary corrections. On top of that, we show that the localized energy of any droplet with one or more corners is positive, irrespective of any details of the configuration: therefore, for the purpose of a lower bound, corners can be eliminated in every box.

3. Minimization of the corner-free configurations by the method of block reflection positivity, introduced in [35] and further developed in [31, 36–38]. Once the corners have been eliminated, we are left with purely striped configurations, whose energy can be further bounded from below by iterative re-

flections across the straight domain walls. After repeated reflections, we end up with periodic striped configurations, and the proof is complete.

We believe that the ground state displays striped order also for values $p \leq 2d$. However, our proof only works for $p > 2d$, the reason being two-fold: (i) the energy of an isolated corner ($d = 2$) or of a trihedral vertex ($d = 3$) becomes infinite at smaller values of p and, therefore, there is no obvious way of identifying the local excitations of the system; (ii) the optimal striped energy per site is of the same order as that of other putative ordered ground states, such as checkerboard or columnar states, and, therefore, it is difficult to exclude the emergence of other ordered structures on the basis of local energy estimates.

In *conclusion*, we considered Ising models in two and three dimensions with nearest neighbor ferromagnetic and power law decaying antiferromagnetic interactions. We presented new rigorous bounds on the ground state energy and, in particular, we showed that the actual ground state energy per site tends to the one of the optimal periodic striped configuration, as we approach the ferromagnetic transition line. Moreover, we proved that the minimizing spin configurations are striped in a suitable sense; namely, if restricted to finite windows of proper size (much larger than the optimal stripe width), they all look precisely striped with very high, explicitly estimated, probability. These are the most refined rigorous bounds on the ground state energy of the considered model, and the first of this kind in three dimensions. Our new methods, which the proof of the theorem is based on, combine for the first time the ideas of energy localization into boxes and of block reflection positivity, in the context of isotropic systems with competing interactions in two and three dimensions. We expect them to be crucial for further developments in the subject and, in particular, for a proof of exact, macroscopic, stripe ordering in two and three dimensions.

Acknowledgments. The research leading to these results has received funding from the European Research Council under the EU's Seventh Framework Programme ERC Starting Grant CoMBoS (grant agreement n^o 239694; A.G. and R.S.), the U.S. National Science Foundation (grant PHY 0965859; E.H.L.), the Simons Foundation (grant # 230207; E.H.L.) and the NSERC (R.S.). The work is part of a project started in collaboration with Joel Lebowitz, whom we thank for many useful discussions and for his constant encouragement.

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