

Aspects of Lagrange's Mechanics and their legacy

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Abstract

From the “vibrating string” and “Kepler’s equation” theories to relativistic quantum fields, (divergent) series resummations, perturbation theory, KAM theory.

Key words: *Vibrating string, Quantum fields, Perturbation theory, Divergent series resummation, KAM*⁰

1 Law of continuity for the vibrating string

In 1759 an important open problem was to establish the correctness of Euler’s solution of the wave equation for a string starting from an initial configuration with a given shape $u_0(x) = \varphi(x)$ and no initial velocity:

$$\begin{aligned} \partial_t^2 u &= c^2 \partial_x^2 u, & x \in [0, a], & u(a) = u(b) = 0 \\ u(x, t) &= \varphi(x - ct) + \varphi(x + ct) \end{aligned} \tag{1.1}$$

According to D’Alembert’s arguments it should have been necessary that $\varphi(x)$ be at least a *smooth* function (infinitely differentiable in present notations or, in D’Alembert’s language, subject to the *loi de continuité*) of $x \in R$, *periodic with period* $2a$ and odd around 0 and a . In his notations φ had to satisfy conditions at $0, a$ allowing its continuation to a $2a$ -periodic function of class $\mathcal{C}_\infty(R)$, odd around 0 and a (the arguments really would only require class $\mathcal{C}_2(R)$),

Euler claimed that Eq.(1.1) would be a solution $u(x, t) = \varphi(x - ct) + \varphi(x + ct)$ if $\varphi(x)$ was smooth in $[0, a]$ and simply defined outside the interval

⁰giovanni.gallavotti@roma1.infn.it extended version of a talk at the “*Il segno di Lagrange nella Matematica contemporanea*”, Torino, Politecnico, 10 April 2013.

$[0, a]$ by just continuing it as a periodic function of period $2a$ odd around 0 and a . He insisted that the smoothness in $[0, a]$ of $\varphi(x)$ is sufficient: this means that as time t becomes > 0 the form of the string appears (in general) to keep points with discontinuous curvature: *i.e.* points in which the second derivative is not defined. This led D'Alembert to think that when the continuation of φ was not smooth the solutions did not make sense.

Neither was able to produce a rigorous argument. Other theories were due to Taylor who (starting from an initial configuration in which the string, at rest, is given a shape $u_0(x) = \varphi(x)$) had proposed that a general solution of the string motion is a sum $u(x) = \sum_n \alpha_n \sin(\frac{2\pi}{2a}nx) \cos(\frac{2\pi}{2a}nct)$: a view supported by D. Bernoulli (according to Lagrange). This is criticized by Euler who objects that such expressions only represent D'Alembert's solutions (which is true *if the series converges* in class $\mathcal{C}_2(R)$).

All solutions had been obtained by arguments relying on unproved properties based on intuition or experience and this is the reason identified by Lagrange as the source of the controversies.

Lagrange's idea is that the problem must first be (in modern locution) *regularized*: this means imagining the string to consist of an indefinite number m of small particles aligned (in the rest state) and elastically interacting with the nearest neighbors possibly moving orthogonally to the rest line and with extremes $y_0 = y_m = 0$ fixed.

In this way the problem becomes a clearly posed mechanical problem and the equations of motion are readily derived: the only question is therefore finding the properties of their solutions and of the limit in which the mass of the particles tends to zero while their number and the strength of the elastic force tend to infinity so that a continuum string motion emerges, [1, T.I, p.71]:¹

Il resulte de tout cet exposé que l'Analyse que nous avons proposée dans le Chapitre précédent est peut-être, la seule qui puisse jeter sur ces matières obscures une lumière suffisante à éclaircir les doutes qu'on forme de part et d'autre.^{t1}

The analysis is today well known: he notices that the problem is (in modern language) the diagonalization of a $(m - 1) \times (m - 1)$ tridiagonal symmetric matrix. The solution is perhaps the first example of the diagonalization procedure of a large matrix. The result is the representation of the general motion of the chain with general initial data for positions *and velocities*, [1, T.I, p.97]:

¹English translation of quotations in the endnotes correspondingly labeled **t***

.. je ne crois pas qu'on ait jamais donné pour cela une formule générale, telle que nous venons de la trouver.^{t2} see however [2].

Lagrange's very remarkable three memories, the first two constitute a veritable monograph, consist in showing (in full detail and rigor) that the general motion has, for suitable ω_h , the form (translated in modern language)

$$y^{(\delta)}(\xi, t) = \sum_{h=1}^{m-1} \left\{ \tilde{A}_h \sqrt{\frac{2}{m}} \sin \frac{\pi h}{a} \xi \cdot \cos \omega_h t + \tilde{B}_h \sqrt{\frac{2}{m}} \sin \frac{\pi h}{a} \xi \cdot \sin \omega_h t \right\}, \quad (1.2)$$

where, denoting by δ (called dx by Lagrange) the mesh of the discretized positions so that the number of small masses, located at points $\xi = i\delta$, is $\frac{a}{\delta}$ and $\tilde{A}_h, \tilde{B}_h, \omega_h$ are derived from the initial profiles of positions $Z(\xi)$ and velocities $U(\xi)$, [1, T.I., p.163]: via the expressions

$$\begin{aligned} \sqrt{\frac{2}{m}} \tilde{A}_h &= \frac{2}{m} \sum_{i=1}^{m-1} \left(\sin \frac{\pi h}{a} \xi \right) Z(\xi) \xrightarrow{\delta \rightarrow 0} \frac{2}{a} \int_0^a Z(x) \left(\sin \frac{\pi h}{a} x \right) dx \\ \sqrt{\frac{2}{m}} \tilde{B}_h &= \frac{2}{\omega_h m} \sum_{i=1}^{m-1} \left(\sin \frac{\pi h}{a} \xi \right) U(\xi) \xrightarrow{\delta \rightarrow 0} \frac{2}{\bar{\omega}_h a} \int_0^a U(x) \left(\sin \frac{\pi h}{a} x \right) dx, \quad (1.3) \\ \omega_h &= c \sqrt{2 \frac{1 - \cos(\frac{\pi h \delta}{a})}{\delta^2}} \xrightarrow{\delta \rightarrow 0} \bar{\omega}_h = c \frac{\pi h}{a} \end{aligned}$$

where c is $\sqrt{\frac{\tau}{\mu}}$, with τ the tension and μ the density of the string. The formulae Eq.(1.3) do not require smoothness assumptions on the data Z, U other than, for instance, continuity as stated in answer to D'Alembert's critique, [1, T.I, p.324]:

Mais je le prie de faire attention que, dans ma solution, la détermination de la figure de la corde à chaque instant dépend uniquement des quantités Z et U , lesquelles n'entrent point dans l'opération dont il s'agit. Je conviens que la formule à laquelle j'applique la méthode de M. Bernoulli est assujettie à la loi de continuité; mais il ne me paraît pas s'ensuire que les quantités Z et U , qui constituent le coefficient de cette formule, le soient aussi, comme M. d'Alembert le prétend.^{t3}

Hence, setting $\xi = x$, the general solution:

$$u(x, t) = \sum_{h=0}^{\infty} \sin \frac{\pi h}{a} x \left\{ \left(\frac{2}{a} \int_0^a Z(x') \sin \frac{\pi h}{a} x' dx' \right) \cos \bar{\omega}(h)t \right. \\ \left. + \left(\frac{2}{a} \int_0^a U(x') \sin \frac{\pi h}{a} x' dx' \right) \frac{\sin \bar{\omega}(h)t}{\bar{\omega}(h)} \right\} \quad (1.4)$$

is found: and it is far more general than the previous solutions because it permits initial data with $U \neq 0$, *i.e.* with initial data in which the string is already in motion. In the case $U = 0$ it can be written $\varphi(x + ct) + \varphi(x - ct)$ with φ odd around 0 and a and $2a$ -periodic aside from the obvious convergence and exchange of sum and limits problems.

Check of Eq.(1.3) is based on trigonometric properties: basically on “Cote’s formula”, [1, T.I, p.75], $|a^m - b^m| = \prod_{h=0}^{m-1} (a^2 - 2ab \cos \frac{2\pi}{m} h + b^2)^{\frac{1}{2}}$ (derivable from $a^m - b^m = \prod_{h=0}^{m-1} (ae^{i\frac{2\pi}{m}h} - b)$) which is used instead of the modern $\sum_{h=0}^{m-1} e^{i\frac{2\pi}{m}h(p-q)} = m\delta_{p,q}$.

The continuum limit Eq.(1.4) is identified with Euler’s result: it is remarkably found together with interesting considerations and attempted justifications of resummations of divergent series like, [1, T.I, p.111],

$$\cos x + \cos 2x + \cos 3x + \dots = -\frac{1}{2} \quad (1.5)$$

for $x \neq 0$, immediately criticized by D’Alembert, [1, T.I, p.322].²

²Lagrange uses this formula to infer that for $t = 0$ the string shape is described by the function Z and the speed configuration by U . The argument amounts to proving $\frac{1}{a} \sum_{h=1}^{\infty} \sin \frac{\pi}{a} hx \sin \frac{\pi}{a} hy = \delta(x-y)$: looking at Lagrange’s theory it appears, as he correctly exposes in [1, T.I, p.111], that what is really used is the truncated version of the above relations and the Eq.(1.5) only plays the role of an intermediate illustrative proposition: *il ne sera pas hors de propos de démontrer encore la même proposition d’une autre manière*, [1, T.I, p.109] (“it will not be out of place to prove again the same proposition by another method”). The analysis essentially repeats in different form the just proved completeness of the trigonometric sines basis in the finite m case; it was derived at a time when even the notion of continuous functions (not to mention of distributions) was not formalized, is masterful although not formal by our standards; and is used also to infer that for $t > 0$ the solution coincides with Euler’s: for this purpose the bold statement that $\sin m(\frac{\pi}{a}(x \pm ct)) = 0$ if $m = \infty$ is made and better justified in the answer to D’Alembert quoted above after Eq.(1.3), [1, T.I, p.324], although D’Alembert’s questions are not really answered also because of the needed further exchange of limits intervening to replace $\omega_h t$ with $hct\frac{\pi}{a}$.

To D'Alembert's objections Lagrange later gave also new arguments, apparently "far less" convincing:

Or je demande si, toutes les fois que dans une formule algébrique il se trouvera par exemple une série géométrique infinie, telle que $1 + x + x^2 + x^3 + x^4 + \dots$, on ne sera pas en droit d'y substituer $\frac{1}{1-x}$ quoique cette quantité ne soit réellement égale à la somme de la série proposée qu'en supposant le dernier terme x^∞ nul. Il me semble qu'on ne saurait contester l'exactitude d'une telle substitution sans renverser les principes les plus communs de l'analyse^{t4}

[1, T.I, p.323], where the comment refers to the case $x \neq 1$ and illustrates the conceptions on divergent series common already at the time.^{3 t5} And, about a further similar criticism by D'Alembert, applying to the alternating series obtained setting $x = \frac{\pi}{4}$:

Je réponds qu'avec un pareil raisonnement on soutiendrait aussi que $\frac{1}{1+x}$ n'est point l'expression générale de la somme de la suite infinie $1 - x + x^2 - x^3 + \dots$ parce que, en faisant $x = 1$, on a $1 - 1 + 1 - 1 + \dots$ ce qui est, ou 0, ou 1, selon que le nombre des termes qu'on prend est pair ou impair, tandis que la valeur de $\frac{1}{1+x}$ est $\frac{1}{2}$. Or je ne crois pas qu'aucun Géomètre voulût admettre cette conclusion^{t6}

[1, T.I, p.323].

The analysis might appear not rigorous in today sense: not really because of the statement Eq.(1.5); nor because of $\sin(\frac{\pi}{g}m(x \pm ct)) = 0$ as consequence of $m = \infty$, [1, T.I, p.102], see footnote²: which may make the eyebrows frown, as of course D'Alembert's did, and for which Lagrange almost "apologized":

"Je conviens que je ne me suis pas exprimé assez exactement .. t7",

[1, T.I, p.322], while not making it more plausible (it seems to be an early version of what is today the "Lebesgue theorem" for Fourier series).

Also several other objections by D'Alembert and D. Bernoulli do not appear to have been answered very convincingly, by the present standards, in the memory in defense of the theory in [1, T.I, p.319-332]. The short note is still of great interest as it shows that Lagrange struggles and gets very close to the modern notion of "weak solution" of a PDE, see also [1, T.I,

³The case $x = 1$ was discussed in the original paper, [1, T.I, p.111]:

Mais, dira-t-on, comment peut-il se faire que la somme de la suite infinie $\cos x + \cos 2x + \cos 3x + \dots$ soit toujours égale à $-\frac{1}{2}$ puisque, dans le cas de $x = 0$, elle devient nécessairement égale à une suite d'autant d'unités? Je réponds que ...

p.177], and to a formalization of a theory of resummation of divergent series. It is unlikely that D'Alembert was ever convinced by Euler or Lagrange: but Lagrange was strongly supported by Euler,[1, T. XIV, p.111], writing:

je vous avoue qu'elles ne me paraissent pas assez fortes pour renverser votre solution. Ce grand génie me semble un peu trop enclin à détruire tout ce qui n'est pas construit par lui-même.

...

Après cette remarque, je vous accorde aisément, monsieur, que pour que le mouvement de la corde soit conforme à la loi de continuité, il faut que, dans la figure initiale, les $\frac{d^2y}{dx^2}$, $\frac{d^4y}{dx^4}$, $\frac{d^6y}{dx^6}$, soient égales à 0 aux deux extrémités; mais, quoique ces conditions n'aient pas lieu, je crois pouvoir soutenir que notre solution donnera néanmoins le véritable mouvement de la corde; car

...^{t8}

In conclusion today a mathematician or a physicist will probably consider it not completely rigorous only because although it applies to twice $C_2([0, a])$ initial data Z, U , for the positions and speeds of the string elements, nevertheless a proof of the exchanges of limits (and their existence) needed to “pass to the continuum limit” is not even mentioned. As it would be expected since the condition is not met at $t > 0$ as the discontinuity in the second derivatives (if present at $t = 0$) migrates from the boundary to points moving with the wave in the interior of $[0, a]$ (which is probably what really worried D'Alembert).

The simple (sufficient) condition, implying the existence of the continuum limit, that initial data vanishing at the extremes should also have two continuous derivatives became clear only later when Fourier established the theory of trigonometric series.⁴

Its need did not occur to Lagrange, in his 23-d year of age, nor it did occur to Euler himself: the easy proof is now in all textbooks, *e.g.* [3, Ch.4.5] for the D'Alembert's case, and it is rightly considered that, *de facto*, Lagrange gave a key argument that later led to Fourier's solution of the controversies on the proof of completeness of the basis $\sin \frac{\pi}{a}nx$ for the functions on $[0, a]$ vanishing at 0 and a , *i.e.* proved, for instance, the convergence of their Fourier's series when twice continuously differentiable. For a detailed discussion of the vibrating string controversy with the points of view of D'Alembert, Euler, Bernoulli see [2].

Having determined completely the motion of a discrete chain with arbi-

⁴Thus obtaining a rigorous extension beyond D'Alembert's “continuity condition” which required two continuous derivatives everywhere, vanishing second derivative at the extremes and a string initially deformed but at rest.

trary initial conditions was enough to justify the “extension” to the continuum limit, non rigorous by standards that were just beginning to emerge, as D’Alembert’s objections witness.

The theory of the vibrating strings, in Lagrange, was motivated and became part of a long and detailed study of the propagation of sound in two exhaustive Memoires (followed by a short one to answer criticism): in which sound propagation, *i.e.* the wave equation, is analyzed in the one dimensional case and for three dimensional spherical waves (whose equation is taken from Euler and which he reduces to the one dimensional theory via the remarkable change of variables $u(x) = x^{-2} \int^x z(x) x dx$).

In the second Memory the law of continuity problem is also examined from a new viewpoint: namely to study the motion of the *internal points* of a string with extremes fixed. It is proved that the points move following the solution found with the discretization method (in spite of the travel of the continuity break through $[0, a]$). This is interesting also because the method presented will be recognized to be a precursor of the modern notion of *weak solution* of a PDE, [1, T.I, p.177]:

Les transformations dont je fais usage dans cette occasion sont celles qu’on appelle intégrations par parties, et qui se démontrent ordinairement par les principes du calcul différentiel; mais il n’est pas difficile de voir qu’elles ont leur fondement dans le calcul général des sommes et des différences; d’où il suit qu’on n’a point à craindre d’introduire par là dans notre calcul aucune loi de continuité entre les différentes valeurs de z .^{t9}

The vibrating string theory and the continuum as a limit of a microscopically discrete reality was developed by Lagrange at a time when the atomistic conceptions were being established. The approach he adopted is a key legacy and the basis of methods currently employed in the most diverse fields, see for instance [4]: a further example will be discussed in the next section.

2 QFT: quantum elastic string

The key idea in the vibrating string has been that it is a continuous system which should be regarded as, and behaves as, a limit case of a system of infinitely many adjacent particles whose motion should be described via the ordinary equations *without* requiring new principles.

Interestingly this problem has reappeared essentially for the same reasons in recent times. The theory of elementary particles requires at the

same time quantum mechanics and (at least) special relativity: it became soon clear that it could be appropriately formulated as a theory of quantized fields which met immediately impressive successes in the description of electromagnetic interactions of photons and electrons and of weak interactions. Particles were naturally represented by particular states of a field which could describe waves as well, [5, Sec.I].

The simplest example is a “scalar field”, $\varphi(x)$, in space time dimension 2 corresponding classically to the Lagrangian

$$\mathcal{L} = \frac{\mu}{2} \int_{\alpha}^{\beta} \left(\dot{\varphi}(x)^2 - c^2 \left(\frac{d\varphi}{dx}(x) \right)^2 - \left(\frac{m_0 c^2}{\hbar} \right)^2 \varphi(x)^2 - I(\varphi(x)) \right) dx \quad (2.1)$$

where $I(\varphi)$ is some function of φ . If $I = 0$ this is a vibrating string with density μ , tension $\tau = \mu c^2$ and an elastic pinning force $\mu \left(\frac{m_0 c^2}{\hbar} \right)^2$.

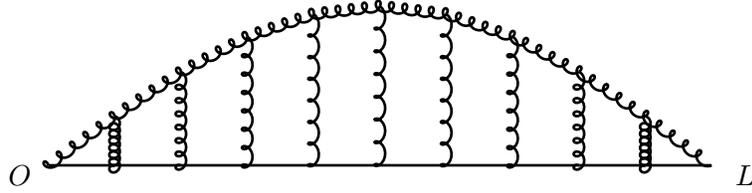


Fig.1

Figure 4.1: chain of oscillators elastically bound by nearest neighbors and to centers aligned on an axis orthogonal to their vibrations and Dirichlet’s boundary condition.

The nonlinearity of the associated wave equation produces the result that when two or more wave packets collide they emerge out of the collision quite modified and do not just go through each other as in the case of the linear string, so that their interaction is nontrivial.

The model is naturally generalized to space-time dimension $D + 1$ if x is imagined a point \mathbf{x} of a cubic lattice in a D -dimensional cube and $\varphi(\mathbf{x})$ describes the deformation of an elastic film ($D = 2$) or body ($D = 3$).

Naively the quantum states will be, by the “natural extension of the usual quantization rules”, *i.e.* functions $F(\varphi)$ of the profile φ describing the configurational shape of the elastic deformations. The Hamiltonian operator acts on the wave function F as

$$\begin{aligned} (\mathcal{H}F)(\varphi) = \int_0^L \left(-\frac{\hbar^2}{2\mu} \frac{\delta^2 F}{\delta \varphi(\mathbf{x})^2}(\varphi) + \frac{\mu}{2} \left(c^2 \left(\frac{\partial \varphi}{\partial \mathbf{x}}(\mathbf{x}) \right)^2 + \right. \right. \\ \left. \left. + \left(\frac{m_0 c^2}{\hbar} \right)^2 \varphi(\mathbf{x})^2 + I(\varphi(\mathbf{x})) \right) F(\varphi) \right) d^D \mathbf{x} \end{aligned} \quad (2.2)$$

where $\frac{\delta}{\delta \varphi(\mathbf{x})}$ is the functional derivative operator (a notion also due to Lagrange and to his calculus of variations) and it should be defined in the space

$L_2("d\varphi")$, where the scalar product ought to be $(F, G) = \int \overline{F(\varphi)} G(\varphi) "d\varphi"$ and $"d\varphi" = \prod_{\mathbf{x} \in [0, L]} d\varphi(\mathbf{x})$.

Even though by now the mathematical meaning that one should try to attach to expressions like the above, as “infinite dimensional elliptic operators” and “functional integrals”, is quite well understood, particularly when $I \equiv 0$, formulae like the above are still quite shocking for conservative mathematicians, even more so because they turn out to be very useful and deep.

One possible way to give meaning to (2.2) is to go back to first principles and recall the classical interpretation of the vibrating string as a system of finitely many oscillators, following Lagrange’s brilliant theory of the discretized wave equation and of the related Fourier series summarized in Sec.1.

Suppose, for simplicity, that the string has periodic boundary conditions instead of the Dirichlet’s conditions studied by Lagrange; replace it with a lattice Z_δ with mesh $\delta > 0$ and such that L/δ is an integer. In every point $n\delta$ of Z_δ

- (1) locate an oscillator with mass $\mu\delta$, described by a coordinate $\varphi_{n\delta}$ giving the elongation of the oscillator over its equilibrium position,
- (2) subject to elastic pinning force with potential energy $\frac{1}{2}\mu\delta \left(\frac{m_0 c^2}{\hbar}\right)^2 \varphi_{n\delta}^2$,
- (3) to a nonlinear pinning force with potential energy $\frac{1}{2}\mu\delta I(\varphi_{n\delta})$
- (4) finally to a linear elastic tension, coupling nearest neighbors at positions $n\delta, (n+1)\delta$, with potential energy $\frac{1}{2}\mu\delta^{-1} c^2 (\varphi_{n\delta} - \varphi_{(n+1)\delta})^2$.

Therefore the Lagrangian of the classical system, in the more general case of space dimension $D \geq 1$ ($D = 1$ is the vibrating string, $D = 2$ is the vibrating film, *etc.*, *i.e.* space-time dimension $d = D + 1$), is

$$\mathcal{L} = \frac{\mu}{2} \delta^D \sum_{n\delta \in \Lambda_0} \left(\dot{\varphi}_{n\delta}^2 - c^2 \sum_{j=1}^D \frac{(\varphi_{n\delta + e_j \delta} - \varphi_{n\delta})^2}{\delta^2} - \left(\frac{m_0 c^2}{\hbar}\right)^2 \varphi_{n\delta}^2 - I(\varphi_{n\delta}) \right) \quad (2.3)$$

where e_j is a unit vector oriented as the directions of the lattice; if $n\delta + e_j \delta$ is not in Λ_0 but $n\delta$ is in Λ_0 then the j -th coordinate equals L and $n\delta + e_j \delta$ has to be interpreted as the point whose j -th coordinate is replaced by 1; *i.e.* is interpreted with periodic boundary conditions with coordinates identified modulo L .

It should be remarked that for $I = 0$ and $m_0 = 0$ and Dirichlet boundary conditions Eq.(2.3) is, for $D = 1$, the Lagrangian of the discretized vibrating string introduced by Lagrange in his theory of sound.

Of course there is no conceptual problem in quantizing the system: it will

correspond to the familiarly elliptic operator on $L_2(\prod_{n\varepsilon} d\varphi_{n\varepsilon}) = L_2(R^{\frac{L}{\delta}})$:

$$\mathcal{H}_\delta = -\frac{\hbar^2}{2\mu\delta} \sum_{n\delta \in \Lambda_o} \frac{\partial^2}{\partial \varphi_{n\delta}^2} + \frac{\mu\delta}{2} \sum_{n\delta \in \Lambda_o} \left(c^2 \frac{(\varphi_{n\delta+\varepsilon\delta} - \varphi_{n\delta})^2}{\delta^2} + \left(\frac{m_0 c^2}{\hbar} \right)^2 \varphi_{n\delta}^2 + I(\varphi_{n\delta}) \right) \quad (2.4)$$

with $\mathcal{C}_0^\infty(R^{\frac{L}{\delta}})$ as domain (of essential self-adjointness) provided $I(\varphi)$ is assumed bounded below, as it should always be: *e.g.* in the case of the so called $\lambda\varphi^4$ theory with $I(\varphi) = \int_0^L (\lambda\varphi(x)^4 + \mu\varphi(x)^2 + \nu)dx$ with $\lambda > 0$.

At this point it could be claimed, with Lagrange [1, T.I, p.55],

Ces équations, comme il est aisé de le voir, sont en même nombre que les particules dont on cherche les mouvements; c'est pourquoi, le problème étant déjà absolument déterminé par leur moyen, on est obligé de s'en tenir là, de sorte que toute condition étrangère ne peut pas manquer de rendre la solution insuffisante et même fautive.^{t10}

This can be appreciated by recalling that in developing the theory, *even in the simplest case of the $\lambda\varphi^4$ field, just mentioned*, difficulties, which generated many discussions, arise.

In particular the attempts to study the properties of the operator \mathcal{H} via expansions in λ, μ, ν immediately lead to nonsensical results (infinities or indeterminate expressions). At the beginning of QFT the results were corrected by adding “*counterterms*” amounting, in the case of the string *i.e.* of space-time of dimension 2, to make μ, ν infinite in the continuum limit (*i.e.* as $\delta \rightarrow 0$), suitably fast.

The subtraction prescriptions, known as “*renormalization*” were not really arbitrary: the remark was that *all results* were given by many integrals which were divergent but in which a divergent part could be naturally isolated by bounding first the integration domains and by determining μ, ν as functions of δ and of the parameters defining the domains boundaries so that the infinities disappear when the boundaries of the domains of integration are removed.

Reassuringly the choice of the divergent “*counterterms*” μ, ν turned out to be essentially independent of the particular result that was being computed.

Even though the *renormalization* procedure was unambiguous, at least it was claimed to be such in the physically more interesting model of quantum electrodynamics as Feynman states in the abstract of [6]:

A complete unambiguous and presumably consistent, method is therefore available for the calculation of all processes involving electrons and photons

it was not clear (at least not to many) that some new rule, *i.e.* a new physical assumption, was not implicitly introduced in the process. For instance Dyson:

Finally, it must be said that the proof of the finiteness and unambiguity of $U(\infty)$ given in this paper makes no pretence of being complete and rigorous. It is most desirable that these general arguments should as soon as possible be supplemented by an explicit calculation of at least one fourth-order radiative effect, to make sure that no unforeseen difficulties arise in that order, [7, p.1754],

or similarly, later, Heitler.⁵

This is analogous to the controversy on the vibrating string solved by Lagrange with his theory of sound via

- (1) discretization of the string,
- (2) solution of its motion and
- (3) removal of the regularization, *i.e.* taking the particles mass to 0, the tension to ∞ at proper rates as $\delta \rightarrow 0$ so that all results remained well defined and converging to limits.

In the 1960's reducibility of renormalization theory to the rigorous study of the properties of the operator in Eq.(2.4) was started and at least in the above cases of the quantum string and of the quantum film (*i.e.* the cases of Eq.(2.4) with space-time dimension 2 and 3, respectively) it was fully understood in a few (very few) cases via the work of Nelson, Glimm-Jaffe, Wilson who showed clearly that no infinities really arise if the problem is correctly studied, [5]: *i.e.* first taking seriously the discretized Hamiltonian, then computing physically relevant quantities and finally passing to the continuum limit, just as Lagrange did in his theory of sound.

This has been a major success of Physics and Analysis. The problem remains open in the case of $d = D + 1 = 4$ space-time dimensions and the

⁵ *On the one hand we can state that the present theory cannot be final. We have found a number of divergent quantities, although all of them are unobservable in principle. These are ...*

Furthermore, we found in some cases that even observable effects are described by ambiguous mathematical expressions...

The ambiguities can always be settled by applying a certain amount of 'wishful mathematics', namely by using additional conditions for the evaluation of such ambiguous integrals... On the other hand, these difficulties do not prevent us from giving a theoretical answer to every legitimate question concerning observable effects. These answers are, whenever they can be tested, always in excellent agreement with the facts..., [8, p.354].

reason is precisely that the nature of the regularization becomes essential in higher dimension: it seems that the naive discretization on a square lattice described here implies assumptions on the Physics at very small scale so that it might be impossible to perform the continuum limit $\delta \rightarrow 0$ unless the regularization of the formal functional derivatives in the Hamiltonian is chosen conveniently. It is conjectured that a naive discretization like the above Eq.(2.4) cannot lead to a nontrivial result if $d = 4$ and $I(\varphi)$ is a fourth order polynomial.

Therefore a challenge remains to explain why the renormalized quantum electrodynamics in 4-space time dimensions, only defined as a formal power expansion in the elementary electric charge, gives amazingly precise results perfectly agreeing with experiments, although the known regularizations are all conjectured to yield trivial result if studied under full mathematical rigor. In recent times the regularization choice appears to have been related to very new conceptions of the nature at very small space-time scale (QCD and string theory): even in this respect Lagrange's method of attacking problems avoiding the introduction of new principles or of understanding their necessity is still fertile.

3 Perturbations in Field Theory

The problems of QFT are often studied by perturbation theory and a key role is played by *i.e.* scale invariant theories. For instance consider the elliptic operator \mathcal{H}_δ in Eq.(2.4) for a quantum string, written as $H_\delta^0 + I_\delta$ separating the term with $I(\varphi_{n\delta}) = \lambda\varphi_{n\delta}^4 + \mu\varphi_{n\delta}^2 + \nu$, *i.e.* with the interaction.

Perturbation theory as developed by Lagrange might seem at first sight quite far from its use in QFT: yet it is quite close as it will be discussed in this section after giving some details on the form in which it arises in renormalization theory and how it appears as an implicit functions problem.

Restricting the discussion to the scalar φ^4 -systems, Eq.(2.4), the breakthrough has been Wilson's theory which in particular shows that the values of physical observables can be can be constructed, for the φ^4 system in dimensions $d = 2, 3$, as power series in a sequence of parameters $\Lambda_k \stackrel{def}{=} (\lambda_k, \mu_k, \nu_k)$, $k = 0, 1, 2, \dots$, called *running coupling constants*, which are related by

$$\Lambda_k = M\Lambda_{k+1} + B(\{\Lambda_r\}_{r=k+1}^\infty), \quad k = 0, 1, \dots \quad (3.1)$$

where M is a diagonal matrix with elements m_1, m_2, m_3 ; B, M define, via Eq.(3.1) two operators \mathcal{B}, \mathcal{M} on the sequences $\{\Lambda_r\}_{r=0}^\infty$.

The physical meaning of the running constants is that they control the physical phenomena that occur on length scale $2^{-k}\ell_0$, where ℓ_0 is a natural length scale associated with the system, *e.g.* $\ell_0 = \frac{\hbar}{m_0c^2}$.

The values of observable quantities are power series in the running constants which, order by order, are *well defined and no infinities appear provided* the sequence $\{\Lambda_r\}_{r=0}^\infty$ consists of uniformly bounded elements.

Hence the physical observables values can be expressed formally in terms of a well defined power series in few parameters, the Λ_k 's, which are in turn functions of three independent parameters (the Λ_0 , for instance, often called the “physical” or “dressed couplings”) whose values define the theory.

However the values of physical observables are *very singular functions* of the Λ_r and their expansion in powers of the physical parameters (*i.e.* Λ_0 , for instance) would be meaningless. In other words the infinities appearing in the heuristic theory are due to singularities of the Λ_r 's as functions of a single one among them: in other words they are due to overexpanding the solution.

This leaves however open the problems:

- (1) existence of a bounded sequence of “running couplings” Λ_k
- (2) convergence of the series for the observables values

The second question has a negative answer because it is immediate that $\lambda_k < 0$ cannot be expected to be allowed: however in the φ^4 -systems they can be shown to be asymptotic series in space-time dimensions 2, 3.

The first question has an answer in space-time dimensions 2, 3 (ultimately due to $m_1 < 1$ and $m_2, m_3 > 1$) while in 4 dimensions space-time one more expansion parameter α_k is needed and $m_1, m_4 = 1$ but it seems impossible to have a bounded sequence with $\lambda_k > 0$ (the *triviality conjecture* proposes actual impossibility, but it is a delicate and open problem).

A further relation with Lagrange's work comes from Eq.(3.1) considered as an equation for $\Lambda \stackrel{def}{=} \{\Lambda_k\}_{k=0}^\infty$ of the form $\Lambda - \mathcal{M}(\Lambda) - \mathcal{B}(\Lambda) = 0$. In the development of perturbation theory (and in cases more general than the present) the request of a bounded solution of the latter implicit function problem arises in a form that has strong similarity with the solution algorithm proposed in Lagrange's theory of “litteral equations” (see below).

In [1, T.III, p.25] the following formula is derived

$$\alpha = x - \varphi(x), \quad \longleftrightarrow \quad \psi(x) = \psi(\alpha) + \sum_{k=1}^{\infty} \frac{1}{k!} \partial_\alpha^{k-1} (\varphi(\alpha)^k \partial_\alpha \psi(\alpha)) \quad (3.2)$$

for all ψ . In particular for $\psi(\alpha) \equiv \alpha$

$$x(\alpha) = \alpha + \sum_{k=1}^{\infty} \frac{1}{k!} \partial_{\alpha}^{k-1}(\varphi(\alpha)^k), \quad x(0) = \sum_{k=1}^{\infty} \frac{1}{k!} \partial_{\alpha}^{k-1}(\varphi(\alpha)^k) \Big|_{\alpha=0} \quad (3.3)$$

are expressions for the inverse function and, respectively, for a solution of $x = \varphi(x)$ (for the formula to work the series convergence is necessary).

In general $x(0)$ in Eq.(3.3) does not always select among the roots the closest to α as pointed out in [9], where a related statement by Lagrange is criticized and a discussion is given of the properties of the root obtained by replacing φ by $t\varphi$ and studying which among the roots is associated with the sum of the series if the latter converges up to $t = 1$, theorem 3, p.18.

The formulae can be generalized to α, x in R^n , $n \geq 1$ and are used in the derivation and on the detailed analysis of Eq.(3.1) via the remark that the fixed point equation of Lagrange is equivalent to the following *tree expansion* of the $k - th$ term in Eq.(3.2) (when $\psi(x) = x$):

$$\frac{1}{k!} \partial_{\alpha}^{k-1}(\varphi(\alpha)^k) = \sum_{\theta} \text{Val}(\theta) \quad (3.4)$$

where θ is a tree graph, *i.e.* it is a “decorated” tree, see Fig.2, with

- (1) k branches λ of equal length oriented towards the “root” r ,
- (2) each node as well as the root (not considered a node) carries a label $j_v \in \{1, \dots, n\}$
- (3) at each node v enter k_v branches $\lambda_1 \equiv vv_1, \dots, \lambda_{k_v} \equiv vv_{k_v}$ where v_1, \dots, v_{k_v} are the k_v nodes preceding v (it is $\sum_{v < r} k_v = k$). and
- (4) the node v symbolizes the tensor $\partial_{x_{j_{v_1}}, x_{j_{v_2}}, \dots, x_{j_{v_{k_v}}}}^{k_v} \varphi_{j_v}(x)$

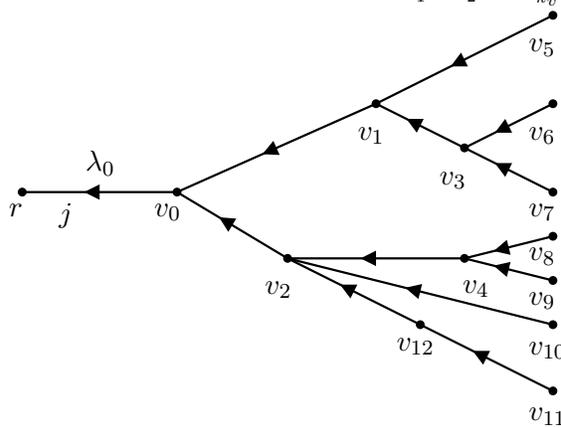


Fig.2

Fig.2: A decorated tree. Labels $j_v, v < r$ not marked and intended as contracted.

- (5) Two such decorated trees are considered equivalent if superposable by pivoting the branches around the nodes without permitting overlapping.
- (6) The *value*, $\text{Val}(\theta)$ is defined as

$$\text{Val}(\theta) = \prod_{v \in \theta} \left(\frac{1}{k_v!} \partial_{j_{v_1}, \dots, j_{v_{k_v}}}^{k_v} \varphi_{j_v}(x) \right) \quad (3.5)$$

where sum over node labels, *except the root label* $j = j_r$, is understood.

It can be said that also the determination and theory of the functions B follows a path that can be traced to Lagrange's theory of "litteral equations": the formula is also a graphical reformulation of a combinatorial version of Eq.(3.3), [10, 11], and can be used to derive several combinatorial formulae (like Cayley's count of trees, [12, (5.11)], [13, p.324], [11] and as recently pointed out more, [11, p.4]).

The analysis in the φ^4 -Field Theory context of the just described method of getting to the running couplings expansion can be found in [5]: the analogy seems manifest and it would be interesting to find a closer relation.

Lagrange made use of Eq.(3.4) in his works on Celestial Mechanics, relying on his study of Kepler's equation giving eccentric anomaly ξ in terms of average anomaly ℓ :

$$\ell = \xi + e \sin \xi, \quad \xi = \ell + \sum_{k=1}^{\infty} \frac{e^k}{k!} \partial_{\ell}^{k-1} (\sin^k \ell) \quad (3.6)$$

The coefficients can be readily evaluated if $\sin^k \xi$ is expressed in terms of angles multiples of ℓ ; and also the true anomaly u can be computed in powers of the eccentricity via Eq.(3.2) with $\varphi(\ell) = e \sin \ell$ and $\psi(\ell) = \frac{\sqrt{1-e^2}}{1+e \cos \ell}$, because $\frac{du}{d\xi} = \frac{(1-e^2)^{\frac{1}{2}}}{1+e \cos \xi}$, [1, T.III, p.114].

Several applications of perturbation theory have been considered by Lagrange often leading, as a byproduct, to key discoveries like the determination of the five Lagrangian points, [1, T.VI, p.280], found while attempting to find approximate solutions to the three body problem, or the many works on the secular variations of the planetary nodes, the determination of the orbit of comets, the Moon librations.

4 Overview on Mechanics

Lagrange's contribution to the formulation and application of the principles of Mechanics is so well known and modern that it is continuously used today.

The 1788 *Mécanique Analytique*, whose second edition is in [1, T.XI] and [1, T.XII], is entirely based on a consistent reduction to first principles of every problem considered.

This means constantly returning to applying the least action principle in the form of the combination the principle of virtual works with the principle of D'Alembert, [14, p.51], which says:

Décomposés les Mouvements $a, b, c, \&tc$ imprimés à chaque Corps, chacun en deux autres $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma' \&tc$ qui soient tels, que si l'on n'eût imprimé aux Corps que les Mouvements $\alpha, \beta, \gamma, \&tc$ ils eussent pû conserver ces Mouvements sans se nuire réciproquement; et que si on ne leur eût imprimé que les Mouvements $\alpha', \beta', \gamma', \&tc$ le système fut demeuré en repos; il est clair que $\alpha, \beta, \gamma, \&tc$ seront les Mouvements que ces Corps prendront en vertu de leur action. Ce Q.F. trouver.^{t116}

Lagrange's addition is clearly explicitly stated in the second work on the Moon librations where he says, about his method:

... Elle n'est autre chose que le principe de Dynamique de M. d'Alembert, réduit en formule au moyen du principe de l'équilibre appelé communément loi des vitesses virtuelles. Mais la combinaison de ces deux principes est un pas qui n'avait pas été fait, et c'est peut-être le seul degré de perfection qui, après la découverte de M. d'Alembert, manquait encore à la Théorie de la Dynamique.^{t12}, [1, T.V, p.11].⁷

The equivalent action principle is used in the form

Dans le mouvement d'un système quelconque de corps animés par des forces mutuelles d'attraction, ou tendantes à des centres fixes, et proportionnelles à des fonctions quelconques des distances, les courbes décrites par les différents corps, et leurs vitesses, sont nécessairement telles que la somme des produits de chaque masse par l'intégrale de la vitesse multipliée par l'élément de la courbe est un maximum ou un minimum, pourvu que l'on regarde les premiers et les derniers points de chaque courbe comme donnés, en sorte que les variations des coordonnées répondantes à ces points soient nulles.

⁶The last words, "What was to be found", are there because the principle is "derived" as an answer to a problem posed in the previous page. Here $a, b, c, \&tc$ are the impressed forces \mathbf{f}_i , and $\alpha, \beta, \gamma, \&tc$ are $m_i \mathbf{a}_i$ while $\alpha', \beta', \gamma', \&tc$ are the constraint reactions $-\mathbf{R}_i$: i.e. $\mathbf{f}_i = m_i \mathbf{a}_i - \mathbf{R}_i$; in modern form, [15, §3.18, Vol.2], the $\mathbf{f}_i - m_i \mathbf{a}_i$ may be called the "lost forces" and the principle is *During the motion of a system of point masses, however constrained and subject to forces, at each instant the lost forces are equilibrated by virtue of the constraints.*

⁷i.e. $\sum_i (\mathbf{f}_i - m_i \mathbf{a}_i) \cdot \delta \mathbf{x}_i = 0$.

C'est le théorème dont nous avons parlé à la fin de la première Section, sous le nom de Principe de la moindre action^{t13}, [1, T.XI, p. 318].

He arrives at the *Mécanique Analytique*, which “brought Rational Mechanics to a state of perfection”, [16, p.221], as the completion of a long series of remarkable applications. Namely

(1) the theory of sound (*i.e.* the vibrating string) and the theory of the librations of the Moon, 1763, [1, T.VI, p.8] (revisited in [1, T.V, p.5], 1781), which is perhaps the first example of development and systematic application of perturbation theory based on analytical mechanics, *i.e.* on the action principle and the Euler-Lagrange equations.

(2) the theory of rigid body, which he undertakes curiously insisting in avoiding starting from the proper axes of rotation, [1, T.III, p.579]:

... ce qui exige la résolution d'une équation cubique. Cependant, à considérer le Problème en lui-même, il semble qu'on devrait pouvoir le résoudre directement et indépendamment des propriétés des axes de rotation, propriétés dont la démonstration est assez difficile, et qui devraient d'ailleurs être plutôt des conséquences de la solution même que les fondements de cette solution^{t14},

he rederives the theory in an original way, certainly very involved for today eyes (used to Euler's equations as beginning point [17, p. 402], [18]), obtaining, as a byproduct, the theorem about the reality of the eigenvalues of a 3×3 symmetric matrix, [1, T.III, p. 605], and (later, in the *Mécanique*) the motion of the “Lagrange's top”, [1, T.XII, p. 253]

Ce cas est celui où l'axe des coordonnées c , c'est--dire la droite qui passe par le point de suspension et par le centre de gravité, est un axe naturel de rotation, et où les moments d'inertie autour des deux autres axes sont égaux (art. 32), ce qui a lieu en général dans tous les solides de révolution, lorsque le point fixe est pris dans l'axe de révolution. La solution de ce cas est facile, d'après les trois intégrales qu'on vient de trouver^{t15},

a remarkable integrability property that Poisson rediscovered (in fact he does not quote Lagrange, see the footnote by J. Bertrand at the *loc.cit.*).

(3) the theory of the secular variations of the planetary elements and several other celestial questions (the fifth volume of the collected paper is entirely dedicated to celestial mechanics, [1, T.V]) opening the way to Laplace's celestial mechanics and slightly preceding the completion and publication of the *Mécanique*. This was the first time analytical mechanics was used to attempt *long time* orbital predictions.

(4) the contributions are not only directed towards the principles but concentrate on concrete problems like the integration by separation of variables, *i.e.* by quadratures, of several dynamical systems: *e.g.* the two centers of gravitational attraction in the three dimensional case, [1, T.II, p.67] or the mentioned symmetric top.

(5) The *Mécanique Analytique* also reflects the attitude of Lagrange regarding matter as constituted by particles: *i.e.* his consistently kept atomistic view, see for instance [1, T.XI, p.189]:

Quoique nous ignorions la constitution intérieure des fluides, nous ne pouvons douter que les particules qui les composent ne soient matérielles, et que par cette raison les lois générales de l'équilibre ne leur conviennent comme aux corps solides^{t16},

his conception of Mechanics as based on a variational principle was adopted universally by the physicists that developed further the atomistic theories: in particular Clausius and Boltzmann make constant use of the action principle using also Lagrange's notations, [19, p. 25], [20], including the commutation rule $\delta d = d\delta$ (simply reflecting the equality of the second derivatives with respect to different arguments) become universal after Lagrange introduced it in the calculus of variations, [1, T.I, p.337].

In this respect it is worth mentioning that by introducing the variation δ of the entire extremal curves rather than relying on Euler's local variations (changing the curves only in infinitesimally close points) answered in a simple way Euler's question, [1, T. I, p.336], on why his analysis led to replacing pdV by $-Vdp$ in his calculations: although simple it was a major change in point of view leading to the modern formulation of the calculus of variations.

(6) Very often the "Lagrange multipliers", that he introduced, [1, T.I, p.247], (extin various cases calculus of variations problems, are incorporated in the variational equation of Euler-Lagrange, see also [1, T.XI, p.340].

(7) It is important to comment also on the style of all his papers which continues to inform the *Mécanique Analytique*: at a time when quoting other works was not very usual, all papers of Lagrange start with a dense summary of the previous works on the subject from which he draws the foundations of his contributions and gives an important help to the historians of science.

(8) The terminology used by Lagrange has become the adopted terminology in most cases: at times, however, the reader might be confused by the use of terms which no longer have the same meaning and sorting things out requires very careful reading: an example is in [1, T.XI, p.244] where the footnote added by J. Bertrand provides an essential help to the reader who has not read enough in detail the earlier pages and tomes.

(9) In the *Mécanique Analytique* the theory of the small oscillations in systems of several degrees of freedom is treated as a perturbation theory problem and applied to the rigid body motions and to celestial mechanics: it was employed extensively in the 1800-ths, starting with Laplace, until Poincaré made clear the need of substantial improvements on the method by pointing out the existence of motions that could not be reduced to simple “quasi periodic” ones. New ideas came much later with the work of Siegel, Kolmogorov, Arnold, Moser, Eliasson: Lagrange’s imprint on the problem however remains not only through the secular perturbations treatment but also through another of his contributions, namely the theory of the implicit functions, Eq.(3.2), as outlined in the next section.

5 KAM

The problem of stability of quasi periodic motions is closely related to Lagrange’s inversion formula Eq.(3.2) and to its graphical version (3.4). It can be formulated as an implicit functions problem for a function $\mathbf{h}(\boldsymbol{\alpha})$ defined on the torus T^ℓ satisfying, in the simplest non trivial cases, the *Hamilton–Jacobi* equation

$$\mathcal{K}\mathbf{h}(\boldsymbol{\alpha}) \stackrel{def}{=} \mathbf{h}(\boldsymbol{\alpha}) + (\boldsymbol{\omega} \cdot \boldsymbol{\partial})^{-2} \left(\varepsilon \boldsymbol{\partial} f(\boldsymbol{\alpha} + \mathbf{h}(\boldsymbol{\alpha})) \right) = \mathbf{0} \quad (5.1)$$

where $\boldsymbol{\omega} \in R^\ell$ is a vector with Diophantine property: $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| \geq C_0^{-1} |\boldsymbol{\nu}|^{-\tau}$ for some $C_0, \tau > 0$ and $\boldsymbol{\partial} f$ is the gradient of an analytic even (for simplicity) function on T^ℓ , $(\boldsymbol{\omega} \cdot \boldsymbol{\partial})^{-2}$ is the linear (pseudo)differential operator on the functions analytic and odd on T^ℓ applied to the function of $\boldsymbol{\alpha} \rightarrow \varepsilon \boldsymbol{\partial} f(\boldsymbol{\alpha} + \mathbf{h}(\boldsymbol{\alpha}))$ and $\mathbf{h}(\boldsymbol{\alpha})$ is to be determined, odd in $\boldsymbol{\alpha} \in T^\ell$.

This leads to consider an infinite dimensional version of Lagrange’s inversion: it can be solved in exactly the same way writing $\mathbf{A} = \mathbf{h} + \mathcal{K}\mathbf{h}$ and solving for \mathbf{h} when $\mathbf{A} = \mathbf{0}$, as in Eq.(3.2). This is more easily done if the Eq.(5.1) is considered in Fourier’s transform. Writing $\mathbf{h}(\boldsymbol{\alpha}) = \sum_{k=1}^{\infty} \varepsilon^k \mathbf{h}^{[k]}(\boldsymbol{\alpha})$ and denoting $\mathbf{h}_{\boldsymbol{\nu},j}^{(k)}$ the Fourier transform of $\mathbf{h}_j^{(k)}$, $\boldsymbol{\nu} \in Z^\ell$ an expression for j -th component $h_{\boldsymbol{\nu},j}^{[k]}$ is given via trees with k root-oriented branches as:

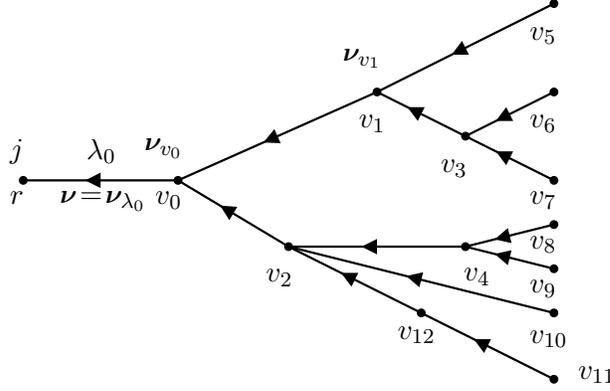


Fig.3

Fig.3: A tree θ con $k_{v_0} = 2, k_{v_1} = 2, \dots$ and $k = 13$, with a few decorations

In this case on each node there is an extra label ν_v (marked in Fig.3 only on v_0 and v_1) and on each line there is an extra label $\nu(\lambda) = \sum_{w \leq v} \nu_w$; also in this case the labels $j_v, j_{v_1}, \dots, j_{v_{k_v}}$ associated with the nodes have to be contracted when appearing twice (*i.e.* unless $v = r$ as $j_r = j$ appears only once). The trees will be identified if reducible to each other by pivoting as in the simple scalar case of Sec.3. After some algebra it appears that

$$\mathbf{h}_{\nu}^{[k]} = \sum_{\theta} \text{Val}(\theta), \quad \text{Val}(\theta) = \left(\prod_v \frac{f_{\nu_v}}{k_v!} \right) \left(\prod_{\lambda} \frac{\nu_v \cdot \nu_{v'}}{(\omega \cdot \nu(\lambda))^2} \right) \quad (5.2)$$

where the sum is over all trees with k branches.

Eq.(5.2) was developed in the context of celestial mechanics by Lindstedt and Newcomb. It turns the proof of the KAM theorem into a simple algebraic check in which the main difficulty of the *small divisors*, which appears because a naive estimate of $\mathbf{h}_{\nu}^{[k]}$ has size $O(k!^{\tau})$ (making the formula illusory, because apparently divergent), can be solved by checking that the values of the trees which are too large have competing signs and almost cancel between themselves leading to an estimate $|\mathbf{h}_{\nu}^{[k]}| \leq c^k e^{-\kappa|\nu|}$ for suitable $c, \kappa > 0$.

The tree representation is particularly apt to exhibit the cancellations that occur: the consequent proof of the KAM theorem, [21], is not the classical one and it is often considered too complicated. In this respect a comment of Lagrange is relevant:

D'ailleurs mes recherches n'ont rien de commun avec le leurs que le problème qui en fait l'object; et c'est toujours contribuer à l'avancement des Mathématiques que de montrer comment on peut résoudre les même questions et parvenir au même resultats par des voies très-différentes; les méthodes se

prêtent par ce moyen un jour mutuel et en acquièrent souvent un plus grand degré d'évidence et de généralité.^{t17}

[1, T. VI, p.280].

Going back to Kepler's problem the work of Carlini and of Levi-Civita (independent, later) made clear that Lagrange's series, Eq.(3.6), can be resummed into a power series in the parameter $\eta = \frac{e \exp \sqrt{1-e^2}}{1+\sqrt{1-e^2}}$ with radius of convergence 1, [22, Appendice, p.44],[23], thus redetermining the D'Alembert's radius of convergence $r^* = 0.6627434\dots$ of the power series in e (called "Laplace's limit")¹ as the closest point to 0 of the curve $z = e(\eta)$, $|\eta| = 1$, *i.e.* $e = ir^*$ or $r^* \frac{\exp(\sqrt{1+r^{*2}})}{1+\sqrt{1+r^{*2}}} = 1$. Furthermore for e real and $e < 1$ it is $\eta < 1$ and an expansion for the eccentric anomaly is obtained by a power series in $\eta(e)$ convergent for all eccentricities $e < 1$.

Recently the same formula has been used to study resonant quasi periodic motions in integrable systems subject to a perturbing potential εV with ε small: the resulting tree expansion allowed the study of the series in cases in which it is likely to be not convergent: in spite of this it has been shown that a resummation is possible and leads to a representation of the invariant torus which is analytic in ε in a region of the form, [13],

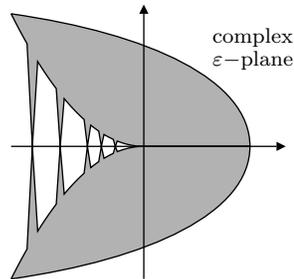


Fig.4

where the contact points of the holomorphy region and the negative real axis have a Lebesgue density point at 0; and in other cases to a fractional power series representation, [28], or also to a Borel summability in a region with $\varepsilon = 0$ on its boundary, [29].

The resummation leading to the above result is a typical resummation that appears in the eighteenth century analysis and is precisely the one that is used against the objections of D'Alembert to the vibrating string solution (*e.g.* $\sum_{k=0}^{\infty} x^k = (1-x)^{-1}$ for $x > 1$): see the quotation following Eq.(5.1)

¹This is not the only resummation of Lagrange's series: the most famous is perhaps Carlini's resummation in terms of the Bessel functions $J_n(z)$, namely $\xi = \ell + \sum_{n=1}^{\infty} \frac{2}{n} J_n(n\ell) \sin n\ell$, found at the same time by Bessel, [24], [22], [25], [26], [27].

above: the modernity of Lagrange's defying viewpoint will not escape the readers' attention.

6 Comments

(1) The work of Lagrange has raised a large number of comments and deep critical analysis, here I mention a few: [30], [31], [32], [33].

(2) Although born and raised in Torino there are very few notes written in Italian: all of them seem to be in mail exchanges; a remarkable one has been inserted in the Tome VII of the collected works, [1, T.VII, p. 583]. It is a very deferent letter in which he explains a somewhat unusual algorithm to evaluate derivatives and integrals. Consider the series

$$[xy]^m \stackrel{def}{=} \sum_{k=0}^{\infty} \binom{m}{k} [x]^{m-k} [y]^k \quad (6.1)$$

(which the Lagrange writes without the brackets, added here for clarity). The positive powers are interpreted as derivatives (or more properly as infinitesimal increments with respect to the variation of an *unspecified* variable) and the negative as integrals provided at least x or y is an infinitesimal increment. Thus for $m > 0$ and integer the expression is a finite sum and yields the Leibnitz differentiation rule $[xy]^m = \sum_{k=0}^{\infty} \binom{m}{k} (dx)^{m-k} (dy)^k$. If $m = -1 - k < 0$ then $[dx]^{-1-k}$ is interpreted as $k + 1$ iteration of indefinite integration over an infinitesimal interval dx ; this means $dx^{-1} = x$, $dx^{-2} = \frac{x^2}{2dx}$, \dots , $[dx]^{-1-k} = \frac{x^{k+1}}{(k+1)! dx^k}$.

For $m = -1$ Lagrange gives the example $\int y dx$: from Eq. (6.1)

$$\begin{aligned} [dx y]^{-1} &= \sum_{k=0}^{\infty} (-1)^k [dx]^{-1-k} d^k y \\ \int y dx &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{(k+1)! dx^k} d^k y = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{(k+1)!} \frac{d^k y}{dx^k} \end{aligned} \quad (6.2)$$

(a relation that the 18 years old Lagrange attributes to "Giovanni Bernoulli", 1694). A further example is worked out:

$$[dxdy]^{-2} = \sum_{k=0}^{\infty} \binom{-2}{k} [dx]^{-2-k} d^{k+1} y = \sum_{k=0}^{\infty} (-1)^k \frac{k+1}{(k+2)!} x^{k+2} \frac{d^{k+1} y}{dx^{k+1}} \quad (6.3)$$

which therefore yields the indefinite integral $\int \int dy dx$ which is shown to be identical to $\int y dx$ simply by differentiating the *r.h.s.* of (6.3) and checking

its identity with the *r.h.s.* of (6.2) (Lagrange suggests, equivalently, to differentiate both equations twice).

The letter is signed *Luigi De La Grange*, and addressed to “Illustrissimo Signor Da Fagnano”, well known mathematician, who would soon help Lagrange to get his first paper published.

(3) Consideration of friction is not frequent in the works of Lagrange, it is mentioned in a remark on the vibrating strings theory, [1, T.I, p.109, 241], and the analysis on the tautochrone curves in T.II,III.

It is also considered in the astronomical problems to examine the consequences, on the variations of the planetary elements, of a small rarefied medium filling the solar system (if any) in the Tomes VI and VII. Or in the influence of friction on the oscillations of a pendulum in the *Mécanique Analytique* (Tome XII) and in fluid mechanics problems.

Although his attention to applications has been constant (for instance studying the best shape to give to a column to strengthen it) friction enters only very marginally in the remarkable theory of the anchor escapement, [1, T.IV, p.341]: this is surprising because it is an essential feature controlling the precision of the clocks and the very possibility of building them, [5, Ch.1,Sec.2.17].

(4) Among other applications discussed by Lagrange are problems in Optics, again referring to the variational properties of light paths, and in Probability theory.

(5) Infinitesimals, in the sense of Leibnitz, are pervasive in his work (a nice example is the 1754 letter to G.C. Fagnano quoted above which shows that Lagrange learnt very early their use and their formidable power of easing the task of long algebraic steps (at the time there were already some objections to their use): this makes reading his papers easy and pleasant; at the time there were several objections to their use which eventually lead to the rigorous refoundation of analysis.

However Lagrange himself seems to have realized that something ought to be done in systematizing the foundations of analysis: the tellingly long title of his lecture notes, [1, T. IX], *Théorie des Fonctions analytiques, contenant les principes du calcul différentiel dégagé de toutes considérations d'infiniment petits ou d'évanouissants, de limites ou de fluxions et réduits à l'analyse algébrique de quantités finies*, and the very first page of the subsequent *Leçons sur le calcul des fonctions*^{t18}, [1, T. X],

On connaît les difficultés qu'offre la supposition des infiniment petits, sur laquelle Leibnitz a fondé le Calcul différentiel. Pour les éviter, Euler regarde les différentielles comme nulles, ce qui réduit leur rap- rapport à l'expression

zéro divisé par zéro, laquelle ne présente aucune idée.^{t19}

are a clear sign of his hidden qualms on the matter.

The fact that they developed at a late stage, while he wholeheartedly adopted the Leibnitz methods in his youth, shows that the problem of mathematical rigor had grown, even for the great scientists, to a point that it was necessary to work more on it. Nevertheless his last words on the subject are probably to be found in the preface to the second edition of the *Mécanique Analytique* where he (reassuringly) admits

On a conservé la notation ordinaire du Calcul différentiel, parce qu'elle répond au système des infiniment petits, adopté dans ce Traité. Lorsqu'on a bien conçu l'esprit de ce système, et qu'on s'est convaincu de l'exactitude de ses résultats par la méthode géométrique des premières et dernières raisons, ou par la méthode analytique des fonctions dérivées, on peut employer les infiniment petits comme un instrument sûr et commode pour abrégé et simplifier les démonstrations. C'est ainsi qu'on abrège les démonstrations des Anciens par la méthode des indivisibles. ^{t20}

(6) An informative history of his life can be found in the “Notices sur la vie et les ouvrages” in the preface by M. Delambre of the Tome I of the collected works, [1] and in the commemoration by P. Cossali at the University of Padova, [34]. If abstraction is made of the bombastic rethoric celebrating, in the first (quite a) few pages, Eugene Napoleon and, in the last (quite a) few pages, Napoleon I himself the remaining about 130 pages of Cossali's *Elogio* contain a useful and detailed summary and evaluation of all the works of Lagrange (exposed, as well, in a circumvolved rethorical style). (7) A recent analysis of Lagrange's contributions to Celestial Mechanics in perspective with the historical development up to contemporary works can be found in [35].

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End Notes: translation of the quotations

^{t1}From the above considerations follows that the analysis that we have proposed in the previous chapter is perhaps the only one which could thorow on such obscure subjects a light sufficient to clarify the doubts arising from various sources

^{t2}I do not believe that it for this never has been given a genaral formula, like the one that we just gave

^{t3}But I ask him to please pay attention that, in my solution, the determination of the string shape at each instant depends uniquely on the quantities Z and U , which do not enter, by any means, in the considered operation. I agree that the formula to which I apply Mr. Bernoulli's method is subject to the continuity law; however it does not seem to me that the quantities Z and U , constituting the coefficients of this formula, are also subject to it, as D'Alembert pretends.

^{t4}Now I ask whether every time that in an algebraic formula it eill for instance occur an infinite geometric series, such as $1 + x + x^2 + x^3 + x^4 + \dots$, one had not the right to replace it by $\frac{1}{1-x}$ although this quantity is not really equal to the sum of the proposed series unless one supposed that the last term x^∞ vanishes. It seems to me that we could not context the correctness of such a substitution without overthrowing the most common principles of analysis.

^{t5}But, one would say, how can it be that the sum of the infinite sequence $\cos x + \cos 2x + \cos 3x + \dots$ is always equal to $-\frac{1}{2}$ since, in the case $x = 0$, it becomes necessarily equal to a sequence of as many unities? I answer that ...

^{t6}I answer that by a similar argument one would also maintain that $\frac{1}{1+x}$ is not the general expression of the sum of the infinite sequence $1 - x + x^2 - x^3 + \dots$ because, setting $x = 1$, one gets $1 - 1 + 1 - 1 + \dots$ which is eithe 0 or 1 depending on whether the number of terms considered is even or odd, while the value of $\frac{1}{1+x}$ is $\frac{1}{2}$. Now I do not believe that any Geometer would be willing to admit such conclusion

^{t7}I admit that I did not express myself sufficiently exactly

^{t8}I confess the that they do not look to me strong enough to infirm your solution. The great genius seems to me too incline to destroy what he himself does not construct.

...

After this remark, I easily concede, Sir, that in order that the string motion be conform to the continuity law, it is necessary that the initial shape the derivatives

$\frac{d^2y}{dx^2}, \frac{d^4y}{dx^4}, \frac{d^6y}{dx^6}$ be equal to 0 at the extremes but, whether or not such conditions take place, I believe that I could maintain that our solution will nevertheless give the true motion of the string because ...

^{t9}The transformations that I use here are those called integration by parts, and which are normally proved by the principles of differential calculus; but it is not difficult to see that their foundations lie in the general calculus of sums and differences; hence it follows that one does not at all have to fear that for this any continuity law between the different z -values is introduced in our calculation

^{t10}Such equations, as is easy to see, equal un number that of the particues of which we study the motions; therefore the probem being already fully determined via them, one is obliged to follow course, so that any foreing condition cannot fail to make the solution not sufficient and possibly even wrong.

^{t11}Decompose each of the motions a, b, c & tc impressed to each body in two others $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ & tc such that if we impressed on the bodies only the motions $\alpha, \beta, \gamma,$ & tc that they could have kept such motions without influencing each other; and if we had not impressed any motion other than the motions $\alpha', \beta', \gamma',$ & tc the system would have remained motionless; it is clear that $\alpha, \beta, \gamma,$ & tc will be the motions that the bodies will undergo becuse of their actions. What was to be found.

^{t12}.. That is just the dynamical principle of M. d'Alembert, transformed into a formula via the law of virtual velocities. But the combination of such two principles is a step which had never been done, and thisis erhaps the only degree of perfection which, after M. d'Alembert's discovery, was still missing in the Theory of Dynamics

^{t13}For motions of an arbitrary system driven by forces mutually attractive, or directed towards fixed centers and proportional to arbitrary functions of the distances, the curves described by the various bodys, as well as their velocities, are necessarily such that the velocity times the line element of the curve is maximum or a minimum, provided the first and last point of each curve are regarded as given, so that the variations of the corresponding coordinates vanish.

This is the theorem about which we spoke at the end of the first section, under the name of Principle of least action,

^{t14}... this demands the resolution of a cubic equation. However, thinking about the essence of the problem, it seems that it should be possible to solve it directly and independently of the rotation axes properties, whose demonstration is rather difficult and which, on the other hand, should rather be consequences of the solution itself than of its the foundation.

^{t15}This is the case in which the coordinates axis c , *i.e.* the line passing through the suspension point and the baricenter is a natural rotation axis, and in which the inertia moments around the other two axes are equal (art. 32), this happens in general for ll solids of rotation when the fixed point is on the revolution axis. The solution of this case ie easy because of the three integrals just found.

^{t16}Although we ignore the internal constitution of fluids, we cannot doubt that the particles that compose them are material, and for this reason the the general equilibrium laws pertain to them as they do to solid bodies

^{t17}However my researches have nothing else in common with theirs besides the problem of which they are the object: and it is always a contribution to the progress of Mathematiques to show how the same questions can be solved and the same results obtained through very different ways; the methods provide in this was mutual support and often a better degree of evidence and generality.

^{t18}Analytic functions theory, containing the principles of differential calculus freed of all considerations of infinitesimal or evanescent quantities, of limits or fluxions and reduced the the algebraic analysis of finite quantities

^{t19}We know the difficulties arising when assuming the infinitesimals, on which Leibnitz built differential calculus. To avoid them Euler condiders the differentials as vanishing, which reduces their ratios to the expression zero divided by zero, which does not suggest any idea ..

^{t20}We kept the ordinary differential Calculus notation, because it agrees with the syste of infinitesimals, adopted n this Traité. When the spirit of this systme has been well understood, and one is convinced of the exactness of the results through the geometrical method of the first and last ratios, or through the analytical method of the derivative functions, one can use the infinitesimals as a sure and convenient instrument to abridge and simplify the proofs. It is in this way that the classical proofs are abridged via the method of the indivisibles.