Ward Identities and chiral anomalies for coupled fermionic chains

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Abstract

Coupled fermionic chains are usually described in terms of an effective model in the bonding and anti-bonding variables with linear dispersion relation. We will derive for the first time exact chiral Ward Identities (WI) for this model valid in the limit of removed ultraviolet cut-off and at finite volume, proving the existence of chiral anomalies verifying the Adler-Bardeen non-renormalization property. Such WI are expected to play a crucial role in the understanding of the thermodynamic properties of the system. Our results are non-perturbative and obtained analyzing Grassmann functional integrals through the methods of Constructive Quantum Field Theory.

1 The model

Coupled fermionic chains have been extensively analyzed in the last years, since the suggestion [1]-[3] that their properties are related to the physics of high T_c superconductors; however, very few rigorous results are known. The Hamiltonian of two spinless interacting fermionic chain with an hopping term is

$$H = H^{(1)} + H^{(2)} + t_\perp \sum_x [a^+_{x,1} a^-_{x,2} + a^+_{x,2} a^-_{x,1}] \tag{1}$$

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where $a^\pm_{x,1}, a^\pm_{x,2}$ are fermionic creation or annihilation operators, $x = 1, 2, .., L$ and

$$H^{(i)} = -\frac{1}{2}\sum_{x=1}^{L-1}[t(a^+_{x+1,i}a^-_{x,i}+a^+_{x,i}a^-_{x+1,i})-\mu a^+_{x+1,i}a^-_{x,i}]-\lambda\sum_{x,y=1}^{L-1}v(x-y)a^+_{x,i}a^-_{x,i}a^+_{y,i}a^-_{y,i} \qquad (2)$$

with $i = 1, 2$ and where $|v(x)| \leq e^{-\kappa|x|}$.

When $\lambda = 0$ the Hamiltonian can be easily diagonalized in terms of the bonding and antibonding operators

$$\psi^\pm_{x,a} = \frac{1}{\sqrt{2}}(a^\pm_{1,x} + a^\pm_{2,x}), \qquad \psi^\pm_{x,b} = \frac{1}{\sqrt{2}}(a^\pm_{1,x} - a^\pm_{2,x}) \qquad (3)$$

and $\psi^\pm_{x,i}$ are *free* fermions with Fermi points k^i_F, $i = a, b$ respectively equal to $\cos^{-1}(\mu + t_\perp)$ and $\cos^{-1}(\mu - t_\perp)$. On the other hand if $t_\perp = 0$ the system reduces to two uncoupled one dimensional Fermi system, and in such a case a behavior radically different with respect to the non interacting case is found; the correlations decay for large distances with anomalous exponents, and such behavior is called in the literature *Luttinger liquid behavior* [4]. The dramatic difference between the two limiting behavior at $\lambda = 0$ and $t_\perp = 0$ suggests that a naive perturbation theory in $t_\perp$ or λ cannot be valid, and that the physical properties in the general case will result from a complicate interplay of such two limiting regimes.

The difficulty in analyzing the two chains model (1) has suggested the introduction of a low energy effective field theory in terms of bonding and antibonding chiral fermions with linear dispersion relations [5, 6], "linearizing" the dispersion relation close to the Fermi points. This procedure for the spinless uncoupled chain leads to the Luttinger, which was solved exactly in [7] using bosonization, showing that it can be mapped in a *non interacting* boson systems. In the case of the two chain model, however, the corresponding effective model is *not* solvable; the analogue of the bosonization procedure followed in [7] lead to a systems of *interacting* bosons with a rather complicate non linear term in the Hamiltonian. This is the reason why the thermodynamic properties are still controversial, despite an intense investigation along the years, see *e.g.* [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18].

The effective model can be conveniently expressed in terms of Grassmann variables. Given $L > 0$ we consider the set $\mathcal{D}$ of space-time momenta $\mathbf{k} = (k, k_0)$, with $k = \frac{2\pi}{L}(n+\frac{1}{2})$ and $k_0 = \frac{2\pi}{L}(n_0+\frac{1}{2})$; with each $\mathbf{k} \in \mathcal{D}$ we associate eight *Grassmann variables* (sometimes also called fields) $\widehat{\psi}^+_{\mathbf{k},\omega,j}, \widehat{\psi}^-_{\mathbf{k},\omega,j}$ with $\omega = \pm$ a *quasi-particle* index and $j = \pm$ a *band* index. We also define

Grassmanian fields as

$$\psi^{\pm}_{\mathbf{x},\omega,j} = \frac{1}{L^2}\sum_{\mathbf{k}\in\mathcal{D}} e^{\pm i\mathbf{k}\mathbf{x}}\widehat{\psi}^{\pm}_{\mathbf{k},\omega,j} \tag{4}$$

where $\mathbf{x} = (x, x_0) \in \Lambda$, Λ being a bidimensional square torus of size L^2. The *Generating Function* of the effective model is the following Grassmann integral

$$e^{W_{N,L}(\eta,J)} = \int P(d\psi)\ \exp\left\{-V(\psi) + \sum_{\omega,j}\int d\mathbf{x}\ J_{\mathbf{x},\omega,s}\psi^{+}_{\mathbf{x},\omega,j}\psi^{-}_{\mathbf{x},\omega,j}\right.$$
$$\left. + \sum_{\omega,j}\int d\mathbf{x}\ \left[\psi^{+}_{\mathbf{x},\omega,j}\eta^{-}_{\mathbf{x},\omega,j} + \eta^{+}_{\mathbf{x},\omega,j}\psi^{-}_{\mathbf{x},\omega,j}\right]\right\} \tag{5}$$

where η, J are (respectively commuting and anticommuting) *external fields*, $P_(d\psi^{[l,N]})$ is the *Gaussian Grassmann measure* with propagator $\delta_{\omega,\omega'}\delta_{j,j'}g_{\mathrm{D},\omega}(\mathbf{x}-\mathbf{y})$ with

$$g_{\omega,j}(\mathbf{x}-\mathbf{y}) = \frac{1}{L^2}\sum_{\mathbf{k}\in\mathcal{D}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{\chi^{\varepsilon}_N(\mathbf{k})}{-ik_0 + \omega v_j k} \tag{6}$$

where $\chi^{\varepsilon}_N(\mathbf{k})$ is a cut-off function nonvanishing for all $\mathbf{k}$, depending on a small positive parameter ε, and reducing as $\varepsilon \to 0$, to a compact support function $\chi_N(|\mathbf{k}|) = \bar{\chi}(2^{-N}|\widetilde{\mathbf{k}}|)$, $\widetilde{\mathbf{k}} = (k_0, v_j k)$ with $\bar{\chi}(t) = 1$ for $0 \le t \le 2^N$ and vanishing for $t \ge 2^{N+1}$. The interaction is given by

$$V(\psi) = \sum_{j,\omega}\int d\mathbf{x}d\mathbf{y}v(\mathbf{x}-\mathbf{y})\Big\{g_0\psi^{+}_{\mathbf{x},\omega,j}\psi^{-}_{\mathbf{x},\omega,j}\psi^{+}_{\mathbf{y},-\omega,j}\psi^{-}_{\mathbf{y},-\omega,j} +$$
$$g_f\psi^{+}_{\mathbf{x},\omega,j}\psi^{-}_{\mathbf{x},\omega,j}\psi^{+}_{\mathbf{y},-\omega,-j}\psi^{-}_{\mathbf{y},-\omega,-j} + g_u\psi^{+}_{\mathbf{x},\omega,j}\psi^{-}_{\mathbf{x},\omega,-j}\psi^{+}_{\mathbf{y},-\omega,j}\psi^{-}_{\mathbf{y},-\omega,-j} +$$
$$g_{bs}\psi^{+}_{\mathbf{x},\omega,j}\psi^{-}_{\mathbf{x},\omega,-j}\psi^{+}_{\mathbf{y},-\omega,-j}\psi^{-}_{\mathbf{y},-\omega,j} + g_4\psi^{+}_{\mathbf{x},\omega,j}\psi^{-}_{\mathbf{x},\omega,j}\psi^{+}_{\mathbf{y},\omega,-j}\psi^{-}_{\mathbf{y},\omega,-j} +$$
$$\widetilde{g}_4\psi^{+}_{\mathbf{x},\omega,j}\psi^{-}_{\mathbf{x},\omega,j}\psi^{+}_{\mathbf{y},\omega,j}\psi^{-}_{\mathbf{y},\omega,j}\Big\},$$

with $|v(\mathbf{x}-\mathbf{x}')| \le e^{-\kappa|\mathbf{x}-\mathbf{x}'|}$, $\widehat{v}(0) = 1$. The *Schwinger functions* are determined by the functional derivatives of the generating function; for instance

$$< \psi^{-}_{\mathbf{x},j,\alpha}\psi^{+}_{\mathbf{y},j,\alpha} >_{N,L} = \frac{\partial^2 W_{N,L}(J,\xi)}{\partial\eta^{+}_{\mathbf{x},j,\alpha}\partial\eta^{-}_{\mathbf{y},j,\alpha}}|_{J=\xi=0} \tag{7}$$

$$< \rho_{\mathbf{z},j',\alpha'};\psi^{-}_{\mathbf{x},j,\alpha}\psi^{+}_{\mathbf{y},j,\alpha} >_{N,L} = \frac{\partial^3 W_{N,L}(J,\xi)}{\partial J_{\mathbf{z},j',\alpha'}\partial\eta^{+}_{\mathbf{x},j,\alpha}\partial\eta^{-}_{\mathbf{y},j,\alpha}}|_{J=\xi=0} \tag{8}$$

The model (**??**) is a regularization of the effective model for the two chain considered in literature [5, 6] Heuristically the model (5) is obtained from the model (1) performing the change of variables (3) and linearizing the dispersion relation around k_F^i; in the interaction one keeps only the terms corresponding to a sum of momenta , measured from the respective Fermi points, which are non vanishing for $t_\perp = 0$. In particular g_{bs} corresponds to an interaction between two fermions whose sum of momenta, measured from the respective Fermi points, given by $\pm 2(k_F^a - k_F^b) = O(t^\perp)$ and such process is essential for $t^\perp \to 0$ when the system reduces to the uncoupled chains limit. In order to have a well defined functional integral there is an infrared cut-off (the volume) and an ultraviolet cut-off 2^N; moreover, the non local nature of the interaction has also the role of an ultraviolet cut-off.

2 Ward Identities and chiral anomalies

In the case of the single chain model, that is (1) with $t_\perp = 0$, Ward Identities and Renormalization Group methods have allowed a complete rigorous understanding of the low temperature properties, see [20, 21]. Therefore, one expects that the role of Ward Identites will be crucial also for the understanding of the model (5). However such a derivation is not trivial at all. The formal WI found by a local chiral phase transformation in a functional integral are known to be violated even in the case of the single chain model. Usually, see for instance [29], the WI for the Luttinger model are derived in the Hamiltonian formalism using the fact that the Hamiltonian is quadratic in the boson fields. The fact that the commutators are anomalous implies the presence of an extra term in the WI which, as stressed in [22], is the analogue in $(1+1)$ dimensions of the chiral anomaly in $(3+1)$ dimensional gauge theories found in [24] and [23].

However the coupled chains model is not solvable since the hamiltonian is not quadratic in the boson fermionic operators and so one cannot derive the WI directly in the Hamiltonian approach. One has therefore to rely on functional integrals methods which are more involved but of more general applicability. In recent times, developments in the analysis of functional Grassmann integrals obtained in the context of Constructive Renormalization has allowed to rigorously establish at a non perturbative level the properties of the anomalies in $(1+1)$ dimensional quantum field theory (QFT) models, like the Thirring [26] or the QED_{1+1} [27] (see also the related perturbative analysis in [28] in higher dimensions). We will apply such methods to the derivation of Ward Identities in the model (5).

We perform in the functional integral (5) the *local* change of variables

$$\psi^{\pm}_{\mathbf{x},\omega,j} \to e^{\pm i\lambda_{\omega,\mathbf{x}}}\psi^{\pm}_{\mathbf{x},\omega,j} \tag{9}$$

and performing derivatives with respect to the external fields one obtains, calling $D_{j,\omega}(\mathbf{k}) = -ik_0 + \omega v_j k$

$$\sum_j D_{j,\omega}(\mathbf{q}) < \widehat{\rho}_{j,\omega,\mathbf{q}}; \widehat{\psi}^{-}_{j',\omega',\mathbf{k}}\widehat{\psi}^{+}_{j',\omega',\mathbf{k}+\mathbf{q}} >_{N,L}= \tag{10}$$

$$\delta_{\omega,\omega'}[< \widehat{\psi}^{-}_{j',\omega',\mathbf{k}+\mathbf{q}}\widehat{\psi}^{+}_{j',\omega',\mathbf{k}+\mathbf{q}} >_{N,L} - < \widehat{\psi}^{-}_{j',\omega',\mathbf{k}}\widehat{\psi}^{+}_{j',\omega',\mathbf{k}} >_{N,L}] + \widehat{\Delta}_{N,L}(\mathbf{k},\mathbf{k}+\mathbf{q})$$

where

$$\widehat{\Delta}(\mathbf{k},\mathbf{k}+\mathbf{q}) = \frac{2\pi}{L^2}\sum_{\mathbf{k}'} C_{\omega,j}(\mathbf{k}') < \widehat{\psi}^{+}_{\mathbf{k}',j,\omega}\widehat{\psi}^{+}_{\mathbf{k}'+\mathbf{q},j,\omega}; \widehat{\psi}^{-}_{\mathbf{k},j',\omega'}\widehat{\psi}^{+}_{\mathbf{k}+\mathbf{q},j',\omega'} > \tag{11}$$

and

$$C_{\alpha,j}(\mathbf{k},\mathbf{q}) = [(\chi^{\varepsilon}_N(\mathbf{k}+\mathbf{q}))^{-1} - 1]D_{\alpha,j}(\mathbf{k}+\mathbf{q}) - [(\chi^{\varepsilon}_N(\mathbf{k})^{-1} - 1]D_{\alpha,j}(\mathbf{k}). \tag{12}$$

Note the last term in (1) which would be formally vanishing if we replace the momentum cut-off $\chi^{\varepsilon}_N(|\mathbf{k}|)$ with 1. Our main result is the following Theorem.

Theorem. *Given the generating function (5), there exists ε_L such that, for $|g_0|, |g_f|, |g_u|, |g_{bs}|, |g_4|, |\widetilde{g}_4| \leq \varepsilon_L$ the limits*

$$\lim_{N\to\infty} < \widehat{\rho}_{j,\omega,\mathbf{q}}; \widehat{\psi}^{-}_{j',\omega',\mathbf{k}}\widehat{\psi}^{+}_{j',\omega',\mathbf{k}+\mathbf{q}} >_{N,L}=< \widehat{\rho}_{j,\omega,\mathbf{q}}; \widehat{\psi}^{-}_{j',\omega',\mathbf{k}}\widehat{\psi}^{+}_{j',\omega',\mathbf{k}+\mathbf{q}} >_{L}$$

$$\lim_{N\to\infty} < \widehat{\psi}^{-}_{j',\omega',\mathbf{k}}\widehat{\psi}^{+}_{j',\omega',\mathbf{k}} >_{N,L}=< \widehat{\psi}^{-}_{j',\omega',\mathbf{k}}\widehat{\psi}^{+}_{j',\omega',\mathbf{k}} >_{N} \tag{13}$$

exist and verify the Ward Identity (1) with

$$\lim_{N\to\infty} \widehat{\Delta}_{L,N}(\mathbf{k},\mathbf{k}+\mathbf{q}) = \sum_j \{\frac{g_0}{4\pi v_j} D_{-\omega,j}(\mathbf{q}) < \widehat{\rho}_{j,-\omega,\mathbf{q}}; \widehat{\psi}^{-}_{j',\omega',\mathbf{k}}\widehat{\psi}^{+}_{j',\omega',\mathbf{k}+\mathbf{q}} >_L +$$
$$\frac{g_f}{4\pi v_j} D_{-\omega,j}(\mathbf{q}) < \widehat{\rho}_{-j,-\omega,\mathbf{q}}; \widehat{\psi}^{-}_{j',\omega',\mathbf{k}}\widehat{\psi}^{+}_{j',\omega',\mathbf{k}+\mathbf{q}} >_L +$$
$$\frac{g_4}{4\pi v_j} D_{-\omega,j}(\mathbf{q}) < \widehat{\rho}_{-j,\omega,\mathbf{q}}; \widehat{\psi}^{-}_{j',\omega',\mathbf{k}}\widehat{\psi}^{+}_{j',\omega',\mathbf{k}+\mathbf{q}} >_L$$
$$+\frac{\widetilde{g}_4}{4\pi v_j} D_{-\omega}(j) < \widehat{\rho}_{j,\omega,\mathbf{q}}; \widehat{\psi}^{-}_{j',\omega',\mathbf{k}}\widehat{\psi}^{+}_{j',\omega',\mathbf{k}+\mathbf{q}} >_L\} \tag{14}$$

Remarks

1. The proof of the existence of the Schwinger functions of the model (5) when the ultraviolet cut-off is removed $N \to \infty$ is very similar to the proof of the ultraviolet limit in the Yukawa model [30] or in QED_{1+1} [27]. Note that we are instead unable to remove the infrared cut-off, that is to prove the existence of the $L \to \infty$ limit.

2. In the limit $N \to \infty$ the correction term to the Ward Identities is non vanishing; the fact that the naive Ward Identities are are violated even if the ultraviolet cut-off is removed is a phenomenon called *quantum anomaly*. Note that he coefficient multiplying the Schwinger function is *linear* in the couplings, that is all the possible radiative corrections cancel out. Such a property is the *non perturbative* analogue of the *anomaly non renormalization* in QFT, established by Adler and Bardeen for $(3+1)$ dimensional Quantum Electrodynamics in [23] and extended to QFT models in $(1+1)$ dimensions in [22].

3. If we set $g_u = g_{bs} = 0$ and we consider the Hamiltonian analogue of the model (5) (defined with a cut-off only on the spatial part of the momenta), one gets a system which can be solved by bosonization. One could derive for such system the Ward Identities starting from the anomalous commutation relation getting WI with an anomalous terms given by (2) (with $D_{-\omega,j}(\mathbf{q})$ replaced by $\alpha v_j q$ as the cut-off acts only on the spatial components); see App. 2. However in the general case $g_u \neq 0, g_{bs} \neq 0$ no one has been able to derive WI (1),(2) starting using the Hamiltonian formalism, and non perturbative functional integrals methods are necessary.

4. The WI 1,(2) are derived using the methods introduced in [27]. The main novelty with respect to such paper is that in the model (5) one cannot take the infrared limit $L \to \infty$ (contrary to the model considered in [27]); nevertheless we can prove the validity of exact WI with anomalies *even at L finite*. Note that the main role of Ward Identities is to provide relations between correlations. A finite ultraviolet cut-off produces the correction term $\Delta_{N,L}$ which is not a combination of correlations; therefore (1) at finite N cannot be used for obtaining relations between correlations. However in the limit $N \to \infty$ Δ_L is expressed in terms of correlations *even at L finite*; we have then obtained an exact relations between correlations valid for any L, and which could be used for the understanding of the thermodynamic limit.

5. Another important difference with respect to the case of QED_{1+1} is that some interaction do no contribute to the anomlay; if one looks to

(2) one recognized that there are no anomaies in g_u, g_{bs} appear, even if of course the correlations depend from such couplings.

3 Renormalization Group analysis

The proof of the Theorem above follows from an extension of the proof of Theorem 1 of [27], and we will explain below how such a proof has to be modified in order to get our result. The starting point is the proof of the existence of the limits (13), which can be done using a multiscale analysis. We write the cut-off function as

$$\chi_N(\mathbf{k}) = \sum_{k=-\infty}^{N} f_k(\mathbf{k}), \tag{15}$$

where $f_k(\mathbf{k}) = \bar{\chi}(2^{-k}|\widetilde{\mathbf{k}}|) - \bar{\chi}(2^{-k+1}|\widetilde{\mathbf{k}}|)$ is a smooth function of $\sqrt{k_0^2 + v_j^2 k^2}$ with supports 2^{k-1} and 2^{k+1}. Note that since $|k_0|$ and $|k|$ are greater than $\frac{\pi}{L}$

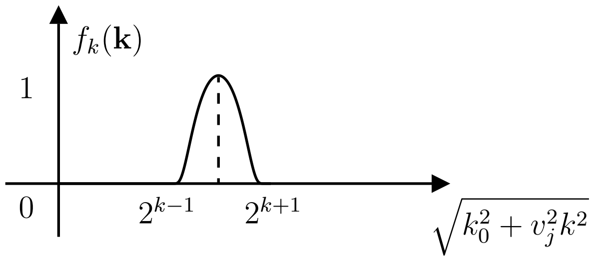

Figure 1: The cut-off function $f_j(\mathbf{k})$ in (15).

there exists a scale h_L such that $f_j(\mathbf{k}) = 0$ for $j \leq h_L$; we can then write

$$g_{j,\alpha}(\mathbf{x}-\mathbf{y}) = \sum_{k=h_L}^{N} (\frac{2\pi}{L})^2 \sum_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \frac{f_k(\mathbf{k})}{-ik_0 + \alpha v_j k} = \sum_{k=h_L}^{N} g^{(k)}_{j,\alpha}(\mathbf{x}-\mathbf{y}) \tag{16}$$

and, for $0 \leq k \leq N$ and for a constant C,

$$|g^{(k)}|_{L_1} \leq C 2^{-k}, \qquad |g^{(k)}|_{L_\infty} \leq C 2^k \tag{17}$$

Note also that $g^{(k)}_{j,\alpha}(\mathbf{x}-\mathbf{y})$ admits a Gram representation:

$$g^{(h)}(\mathbf{x}-\mathbf{y}) = \int d\mathbf{z}\, A^*_h(\mathbf{x}-\mathbf{z}) \cdot B_h(\mathbf{y}-\mathbf{z}, \tag{18}$$

with

$$
\begin{aligned}
A_h(\mathbf{x}) &= \frac{1}{L^2}\sum_{\mathbf{k}\in\mathcal{D}}\sqrt{f_k(\mathbf{k})}\frac{e^{i\mathbf{k}\mathbf{x}}}{k_0^2+v_j^2k^2}\;, \\
B_h(\mathbf{x}) &= \frac{1}{L^2}\sum_{\mathbf{k}\in\mathcal{D}}\sqrt{f_k(\mathbf{k})}e^{i\mathbf{k}\mathbf{x}}(ik_0+v_jk)
\end{aligned}
\tag{19}
$$

and

$$
||A_h||^2=\int d\mathbf{z}|A_h(\mathbf{z})|^2\leq C2^{-2h}\;,\qquad ||B_h||^2\leq C2^{4h}
\tag{20}
$$

for a suitable constant C.

The decomposition of the propagator (16) allows us to make a decomposition of the fermionic measure $P(d\psi)=\prod_{k=h_L}^N P(d\psi^{(k)})$, where $P(d\psi^{(k)})$ is the fermionic measure with propagator $g^{(k)}_{j,\alpha}(\mathbf{x})$ and the corresponding decomposition of the field $\psi_{j,\alpha}=\sum_{k=h_L}^N\psi^{(k)}_{j,\alpha}$. Indeed

$$
\begin{aligned}
&\int P(d\psi)e^{\mathcal{V}}=\int P(d\psi^{(\leq N-1)})\int P(d\psi^{(\leq N)})e^{\mathcal{V}}= \\
&e^{-L^2E_{N-1}+S_{N-1}(J)}\int P(d\psi^{(\leq N-1)})e^{\mathcal{V}^{(N-1)}}
\end{aligned}
\tag{21}
$$

where

$$
-L^2E_{N-1}+S_{N-1}(J)+\mathcal{V}^{(N-1)}\equiv\sum_{n=0}^\infty\frac{1}{n!}\mathcal{E}^T_n(\mathcal{V};n)
\tag{22}
$$

and $\mathcal{E}^T_n$ are the fermionic truncated expectations.

We can integrate iteratively the fields $\psi^{(N)},..,\psi^{(h)}$, $h\geq 0$ so obtaining

$$
e^{\mathcal{W}_{N,L}(0,J)}=e^{-L^2E_h+S_h(J)}\int P(d\psi^{(\leq h)})e^{\mathcal{V}^{(h)}(\psi^{(\leq h)},J)}
\tag{23}
$$

where

$$
\begin{aligned}
&\mathcal{V}^{(h)}(\psi)= \\
&\sum_{m=1}^\infty\sum_{n=0}^\infty\sum_{\underline{\omega}',\underline{j}',\underline{t}'}\int d\underline{\mathbf{z}}\int d\underline{\mathbf{x}}\int d\underline{\mathbf{y}}W^{(n;2m)(h)}(\underline{\mathbf{z}};\underline{\mathbf{x}},\underline{\mathbf{y}})[\prod_{i=1}^n J_{\mathbf{z}_i}][\prod_{i=1}^m\psi^+_{\mathbf{x}_i,\omega'_i,j_i}\psi^-_{\mathbf{y}_i,\omega'_i,j_i}]
\end{aligned}
\tag{24}
$$

We write the r.h.s. of (23) as

$$
e^{-L^2E_h+S_h(J)}\int P_Z(d\psi^{[j,h]})e^{-\mathcal{L}V^{(h)}(\sqrt{Z}\psi^{[j,h]},J)-\mathcal{R}V^{(h)}(\sqrt{Z}\psi^{[j,h]},J)}
\tag{25}
$$

where $\mathcal{R} = 1 - \mathcal{L}$ and $\mathcal{L}$ is a linear operation acting on the kernels so that

$$\mathcal{L}W^{(n;2m)(h)}(\underline{\mathbf{z}};\underline{\mathbf{x}},\underline{\mathbf{y}}) := \begin{cases} W^{(n;2m)(h)}(\underline{\mathbf{z}};\underline{\mathbf{x}},\underline{\mathbf{y}}) & \text{if } n+m \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

Therefore we can write

$$e^{-L^2 E_h + S_h(J)} \int P(d\psi^{[(\le h-1)}) \int P(d\psi^{(h)}) e^{-\mathcal{L}V^{(h)}(\sqrt{Z}\psi^{[j,h]},J) - \mathcal{R}V^{(h)}(\sqrt{Z}\psi^{[j,h]},J)} =$$
$$e^{-L^2 E_{h-1} + S_{h-1}(J)} \int P(d\psi^{[(\le h-1)}) \int P(d\psi^{[(h)}) e^{-V^{(h-1)}(\psi^{[j,h]},J)} \tag{27}$$

and the procedure can be iterated. The outcome of this integration procedure is that the kernels of the effective potentials are expressed in terms of an expansion in terms of running coupling functions $W^{(1;2)(k)}, W^{(0;2)(k)}, W^{(0;4)(k)}$, with $k > h$. We define the following norm

$$||W^{(h)}_{n,2m}|| = \frac{1}{L^2} \int d\underline{\mathbf{x}} |W^{(h)} n, 2m(\mathbf{x})| \tag{28}$$

It can be proved, see Lemma 1 of [25], that if

$$\sup_{h>k} ||2^{-k} W^{(0;2)(k)} + ||W^{(0;4)(k)}|| \le \varepsilon_0, \quad \sup_{h>k} ||W^{(2;1)(k)}|| le 2 \tag{29}$$

with ε_0 independent from h, L, N, then for a suitable constant C

$$||W^{(h)}_{n,2m}|| \le C 2^{-h(n+m-2)} \tag{30}$$

The above claim follows from the representation of $W^{(n;m)(k)}$ in terms of Gallavotti-Nicolo' trees (see [25]) and by the *Brydges-Battle-Federbush* formula (e.g., see [30]), if $s > 1$ and $\widetilde{\psi}^{(h)}(P) = \prod_{i \in P} \psi^{\varepsilon_i}_{\mathbf{x},\omega_i,j_i}$

$$\mathcal{E}^T_h(\widetilde{\psi}^{(h)}(P_1), \ldots, \widetilde{\psi}^{(h)}(P_s)) = \sum_T \prod_{l \in T} \left[g^{(h)}(\mathbf{x}_l - \mathbf{y}_l) \right] \int dP_T(\mathbf{t}) \det G^{h,T}(\mathbf{t}) \tag{31}$$

where T is a set of lines forming an *anchored tree graph* between the clusters of points $\mathbf{x}^{(i)} \cup \mathbf{y}^{(i)}$, that is T is a set of lines, which becomes a tree graph if one identifies all the points in the same cluster. Moreover $\mathbf{t} = \{t_{ii'} \in [0,1], 1 \le i, i' \le s\}$, $dP_T(\mathbf{t})$ is a probability measure with support on a set of $\mathbf{t}$ such that $t_{ii'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i \in \mathbb{R}^s$ of unit norm. Finally $G^{h,T}(\mathbf{t})$ is a $(k-s+1) \times (k-s+1)$ matrix, whose elements are given by

$$G^{h,T}_{ij,i'j'} = t_{ii'} \delta_{s_{ij}, s_{i'j'}} \, g^{(h)}(\mathbf{x}_{ij} - \mathbf{y}_{i'j'}) \tag{32}$$

with $(f^-_{ij}, f^+_{i'j'})$ not belonging to T and $s_{ij}, s_{i'j'}$ the corresponding spin variables. In the following we shall use (**??**) even for $s = 1$, when T is empty, by interpreting the r.h.s. as equal to 1, if $|P_1| = 0$, otherwise as equal to $\det G^h = \mathcal{E}^T_h(\widetilde{\psi}^{(h)}(P_1))$. The determinants appearing in (36) are bounded using the Gram-Hadamard inequality using (18).

In order to prove (29) one proceeds by Induction as in Lemma 2 in [27]. The core of the analysis consists in improving the scaling dimension "extracting a loop line" and showing that all resulting terms have negative scaling dimensions. Consider for instance the kernels with $n = 2$ and $m = 0$. They can be decomposed, for $k \geq 0$, as shown in Fig. 2.

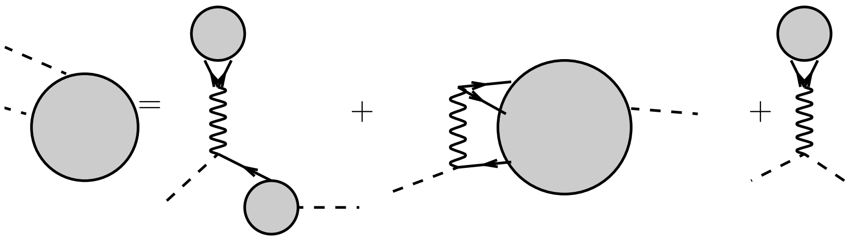

Figure 2: Graphical representation of the decomposition of the kernel $W^{(h)}_{2,0}$; ; the blobs represent $W^{(h)}_{n,m}$, the paired wiggly lines represent v, the full lines $g^{(h,N)}$ and the dotted lines are the external fields

Let us consider the second class of terms in the r.h.s. of Fig. 2. Its naive bound is

$$|\int d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 v(\mathbf{x}_1 - \mathbf{x}_2) g^{[h,N]}(\mathbf{x}_1 - \mathbf{x}_3) W^{(h)}_{2,1}(\mathbf{x}_2; \mathbf{x}_3, 0)| \leq$$
$$|g^{[h,N]}|_{L_\infty} |v|_{L_1} \int d\mathbf{x}_2 d\mathbf{x}_3 |W^{(h)}_{2,1}(\mathbf{x}_2; \mathbf{x}_3, 0)| \leq C 2^N, \tag{33}$$

which is diverging as $N \to \infty$; one can read in the exponent of the r.h.s. the scaling dimension which is $+1$. However one can improve such estimate noting that the wiggly line is not necessary to ensure the connectivity of the diagram, and one can instead integrate over the fermionic propagator getting the bound

$$|\int d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 v(\mathbf{x}_1 - \mathbf{x}_2) g^{[h,N]}(\mathbf{x}_1 - \mathbf{x}_3) W^{(h)}_{2,1}(\mathbf{x}_2; \mathbf{x}_3, 0)| \leq$$
$$|g^{[h,N]}|_{L_1} |v|_{L_\infty} \int d\mathbf{x}_2 d\mathbf{x}_3 |W^{(h)}_{2,1}(\mathbf{x}_2; \mathbf{x}_3, 0)| \leq C 2^{-h}, \tag{34}$$

that is the effective dimension is -1 instead of the previous $+1$. That is, such a contribution now behaves as an irrelevant term. This argument cannot be repeated for the first and third terms in the r.h.s. of Fig. 4, since in those cases the wiggly line is essential to ensure the connectivity of the diagrams. However such terms are vanishing by parity consideration.

In the same way, we can consider $W^{(h)}_{0,2}$ which can be decomposed as in the r.h.s. Fig. 5. The second term in Fig. 5 can again be bounded for $|\lambda| = \max(|g_0|, |g_u|, |g_4|, |\widetilde{g}_4|)$ with C_i constants

$$C|v|_{L^\infty}|W^{(k)}_{2,2}|_{L^1} \sum_{k\leq i'\leq j\leq i\leq N} |g^{(j)}|_{L^1}|g^{(i)}|_{L^1}|g^{(i')}|_{L^\infty} \leq$$
$$C_1|\lambda|^2 2^{-2k} \sum_{k\leq i\leq N} (i-k)2^{-i+k} \leq C_2|\lambda|^2 2^{-2k}. \qquad (35)$$

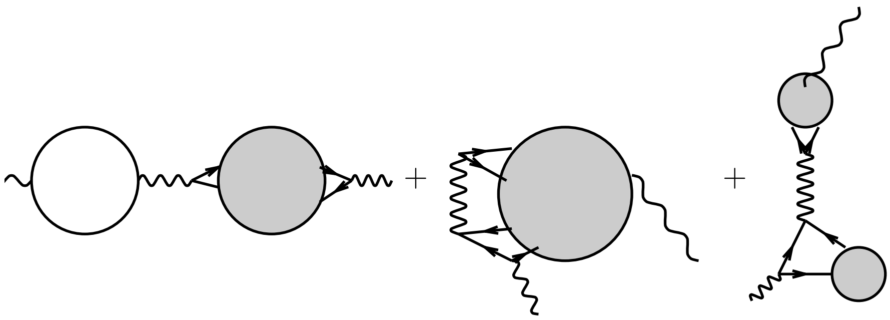

Figure 3: Decomposition of $W^{(k)}_{2,0}$: the blobs represent $W^{(k)}_{n,m}$, the paired wiggly lines represent v, the paired line $g^{(k,N)}$

The main difference with the analysis in [27] is in the first term in Fig. 5, which becomes disconnected if we cut the internal wiggly line. There are now several possible contributions to such bubble; however alll such terms are vanishing at zero external momentum, or by the fact that the propagator is diagonal in ω and j, or by parity as, if $C_{k,N}(\mathbf{k}) = \sum_{i=k}^N f_i(\mathbf{k})$

$$\frac{1}{L^2}\sum_{\mathbf{k}} \frac{C_{k,N}(\mathbf{k})}{(-ik_0+v_j k)^2} = \frac{1}{L^2}\sum_{\mathbf{k}} C_{k,N}(\mathbf{k}) \frac{k_0^2 - v_j^2k^2 + 2iv_jk_0k_j}{(k_0^2+v_j^2k^2)^2} = 0. \qquad (36)$$

In this way the bound in the first term has an extra 2^{-k}. A similar improvement can be repeated for the other terms to show that the scaling dimension is always negative.

Another important diference with respect to [27] is in the integration of the scale ≤ 0; in [27] one exploits that the Beta function is vanishing in the infrared region, what is not true in the present case. However as we keep L

finite we can integrate in a single step all the scale between 0 and h_L, and we call the corresponding kernl simply $W^{(h_L)}_{n,2m}$. Using that the propagator has still a Gram rapresentation

$$g^{(h_L,0)}(\mathbf{x}-\mathbf{y}) = \int d\mathbf{z}\, A^*(\mathbf{x}-\mathbf{z})\cdot B(\mathbf{y}-\mathbf{z}), \tag{37}$$

with

$$||A||^2 \leq CL^2 \quad ||B|^|2 \leq C \tag{38}$$

it is an immediate consequence of (36) and (37) that for $|\bar{g}| \leq CL$

$$||W^{(h_L)}_{n,2m}|| \leq C(|\bar{g}|L) \tag{39}$$

In order to prove (23) we can consider the difference with respect to the l.h.s. and r.h.s. of (24) at a finite Λ, which can be expressed by the appropriate functional derivative of the functional integral

$$\widetilde{W}^R(\xi,\chi) = \int \mathcal{D}[\psi^+\psi] e^{-S_f - S_{int} + B(\psi,\xi_{+,j},\xi_{-,j},0) + \widetilde{B}(\chi,\psi)},$$

with

$$\widetilde{B}(\chi,\psi) = \sum_j \int d\mathbf{k} d\mathbf{q} \chi_{\alpha,\mathbf{q}} \{ \widehat{\psi}^+_{\alpha,j,\mathbf{k}+\mathbf{q}} C_{\alpha,j}(\mathbf{k},\mathbf{q}) \widehat{\psi}^-_{\alpha,j,\mathbf{k}} -$$
$$[\frac{g_0}{4\pi v_j} \sum_j D_{-\alpha,j}(\mathbf{q})\widehat{v}(\mathbf{q})\rho_{-\alpha,j} + \frac{g_f}{4\pi v_j} D_{-\alpha,j}(\mathbf{q})\widehat{v}(\mathbf{q})\rho_{-\alpha,-j} +$$
$$\frac{g_4}{4\pi v_j} D_{-\alpha,j}(\mathbf{q})\widehat{v}(\mathbf{q})\rho_{\alpha,-j} + \frac{\widetilde{g}_4}{4\pi v_j} D_{-\alpha,j}(\mathbf{q})\widehat{v}(\mathbf{q})\rho_{\alpha,j}]\}; \tag{40}$$

This functional integral is similar to (**??**), with the source term $\sum_{j,\alpha} J_{j,\alpha,\mathbf{x}} \psi^+_{j,\alpha,\mathbf{x}} \psi_{j,\alpha,\mathbf{x}}$ replaced by $\widetilde{B}(\chi,\psi)$. The proof that the functional derivative of $\widetilde{W}^R(\xi,\chi)$ with respect to the external current is vanishing for $\Lambda \to \infty$ is similar to what is done in Lemma 3 of [27] or the Lemma 3.2 of [26] for the Thirring model. The main difference in the two chains model is the much more involved structure of the anomalies due to the presence of several kinds of interactions. After the integration of the fields $\psi^{(N)}, \psi^{(N-1)}, .., \psi^{(k)}$ the effective potential is the sum of monomials with n ψ fields, and m χ fields with kernels $\widetilde{W}^{(k)}_{n,m}$.

After the integration of the fields $\psi^{(N)}, \psi^{(N-1)}, .., \psi^{(k)}$ the effective potential is the sum of monomials with n ψ fields, and m χ fields with kernels $\widetilde{W}^{(k)}_{n,m}$. In Fig.6 we consider a decomposition of the contribution to $\widetilde{W}^{(k)}_{1,2}$ in-

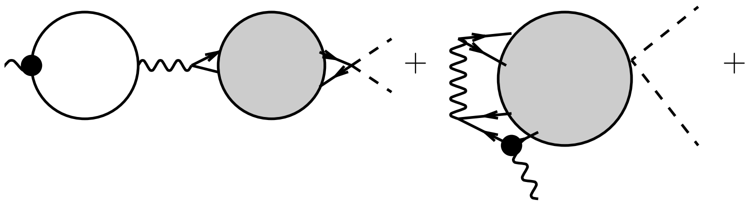

Figure 4: Contributions to $\widetilde{W}^{(k)}_{1,2}$; the black dot represents the first term in the r.h.s. of (40)

volving the first term in (40). In this analysis a crucial role is played by the function $C_{\alpha,j}(\mathbf{k},\mathbf{q})$ defined in (12). In particular, we note that for $k \neq N$

$$[\frac{1}{C(\mathbf{k})} - 1] f^{(k)}_{\mathbf{k}} = 0, \tag{41}$$

since $C(\mathbf{k};\Lambda) = 1$, for $\sqrt{k_0^2 + v_j^2 k^2} \leq 2^N$, and as a result we have that

$$C_{\alpha,j}(\mathbf{k},\mathbf{q}) \psi_{\alpha,j,\mathbf{k}} g^{(k_1)}_{\alpha,j}(\mathbf{k}) g^{(k_2)}_{\alpha,j}(\mathbf{k}+\mathbf{q}) \tag{42}$$

vanishes unless either k_1 or k_2 equals the ultraviolet scale N. Therefore, at least one of the fields in (42) must have the scale N. Therefore, the second term in Fig. 6 can be bounded by

$$\begin{aligned} &C|\lambda||v|_{L^\infty}|\widetilde{W}^{(k)}_{2,2}|_{L^1} \sum_{k \leq i' \leq i \leq N} |g^{(N)}|_{L^1}|g^{(i)}|_{L^1}|g^{(i')}|_{L^\infty} \leq \\ &\text{Const.}\lambda^2 2^{-2k}(N-k)2^{-N+k} \leq \text{Const.}\lambda^2 2^{-2k} 2^{-(N-k)/2}, \end{aligned} \tag{43}$$

leading to the vanishing of this contribution for $N \to \infty$.

It now remains to consider the contribution from the first term in Fig. 6. This is the main novelty with respect to [27]. One can verify that there is no contribution of this kinfinvolving g_u and g_{bs}; on the contrary, the contribution froomthe other coupling are compenstated by (40) using that the contribution from the bubble is

$$\frac{1}{L^2} \sum_{\mathbf{k}} \frac{C_{\alpha,j}(\mathbf{k},\mathbf{q})}{D_{-\alpha}(\mathbf{q})} g_\alpha(\mathbf{k}) g_\alpha(\mathbf{k}+\mathbf{q}) = \frac{1}{4\pi v_j}. \tag{44}$$

Therefore the contributions from the first term in Fig.6 compensate exactly the contributions from the second and third lines of (40). Note that there is no bubble graph in the first term in Fig.4 involving either a g_{bs} or a g_u interaction and this explains why they do not contribute directly to the anomaly.

4 Conclusions

We have considered a low energy model for the two chains model described in terms of bonding and antibonding fields with linear dispersion relations. Contrary to the single chain problem, the model is not exactly solvable and the associated WI cannot be derived by bosonization methods. By analyzing the functional integrals using Constructive Quantum Field Theory methods we have derived for the first time exact WI which display the presence of chiral anomalies. Such anomalies do not depend on the umklapp and backscattering interactions and verify the Adler-Bardeen non renormalization property. Such a WI provide non trivial relations between correlation functions and are an important ingredient for the full Renormalization Group analysis of the TCCM (see [**?**]) and for tackling the problem of the renormalization of the interacting Fermi surface and possibly prove the confinement conjecture.

5 Appendix A: comparison with bosonization

In the case $g_{bs} = g_u = 0$ the Hamiltonian version of the two chain model can be solved by bosonization; in such case the regularization is a momentum cut-off $C(\mathbf{k};\Lambda) = \vartheta(\Lambda - |k)$, one assumes $v(p)$ deending only on the space momenta and the fermionic density operators are boson fields verifying the following commutations rules

$$[\rho_{\alpha,j}(p), \rho_{\alpha',j'}(-p')] = \delta_{\alpha,\alpha'}\delta_{j,j'}\frac{pL}{2\pi}. \tag{45}$$

Therefore as for $g_{bs} = g_u = 0$ the Hamiltonian is bilinear in the boson fields one gets

$$\begin{aligned}\frac{\partial \rho_{\alpha,j}(p)}{\partial t} &= [H, \rho_{\alpha,j}(p)] = \alpha p \rho_{\alpha,j}(p) + \alpha\frac{pL}{2\pi} g_0 v(p)\rho_{-\alpha,j}(p) + \\ \alpha\frac{pL}{2\pi} g_f v(p)\rho_{-\alpha,-j}(p) &+ \alpha\frac{pL}{2\pi} g_4 v(p)\rho_{\alpha,-j}(p) + \alpha\frac{pL}{2\pi} g_4 v(p)\rho_{\alpha,j}(p),\end{aligned} \tag{46}$$

from which, summing over j, one deduces a WI in agreement with (23), found with a completely different method, with momentum cut-off.

On the other hand when g_{bs}, g_u are not vanishing the Hamiltonian cannot be rewritten as a quadratic expression in the boson fields; the model is not solvable and the above steps cannot be repeated so that apparently fermionic functional integrals methods are more effective to get the WI.

6 Appendix B: Two loop analysis

It is useful to compare the above non perturbative derivation with a perturbative analysis up to two-loop order assuming the cut-off only for spatial momenta. For convenience, we rewrite the WI equation (**??**),(**??**) in the case $v_j = v_F$ in term of the amputated correlations

$$\sum_j (iq_0 - \alpha' v_F q)\Lambda^{j,j'}_{\alpha,\alpha'}(\mathbf{k}+\mathbf{q},\mathbf{k}) - \sum_j \alpha' v_F q[(\bar g_0 + \bar g_f)\Gamma^{j,j'}_{\alpha,-\alpha'}(\mathbf{k}+\mathbf{q},\mathbf{k}) + (\bar g_4 + \bar{\widetilde g}_4)\Gamma^{j,j'}_{\alpha,\alpha'}(\mathbf{k}+\mathbf{q},\mathbf{k})] = \delta_{\alpha',\alpha}[\Sigma^j_\alpha(\mathbf{k}) - \Sigma^j_\alpha(\mathbf{k}+\mathbf{q})], \tag{47}$$

where as usual Γ and Σ are the vertex part and the self-energy, $\bar g_i = g_i/2\pi v_F$ and

$$\Gamma^{j,j'}_{\alpha,\alpha'}(\mathbf{p}+\mathbf{q},\mathbf{p}) = \delta_{\alpha,\alpha'} + \Lambda^{j,j'}_{\alpha,\alpha'}(\mathbf{p}+\mathbf{q},\mathbf{p}). \tag{48}$$

Let us choose $\alpha = +$ and $j = b$. In this way, in 1-loop order equation becomes

$$(iq_0 - v_F q)[\Lambda^{b,b\,(1)}_{+,+}(\mathbf{k}+\mathbf{q},\mathbf{k}) + \Lambda^{b,a\,(1)}_{+,+}(\mathbf{k}+\mathbf{q},\mathbf{k})] - v_F q(\bar g_4 + \bar{\widetilde g}_4) = \Sigma^{b\,(1)}_+(\mathbf{k}) - \Sigma^{b\,(1)}_+(\mathbf{k}+\mathbf{q}), \tag{49}$$

for $\alpha' = +$, and

$$(iq_0 + v_F q)[\Lambda^{b,b\,(1)}_{+,-}(\mathbf{k}+\mathbf{q},\mathbf{k}) + \Lambda^{b,a\,(1)}_{+,-}(\mathbf{k}+\mathbf{q},\mathbf{k})] + v_F q(\bar g_0 + \bar g_f) = 0, \tag{50}$$

for $\alpha' = -$. In our notation, $\Lambda^{j,j'\,(1)}_{\alpha\alpha'}$ represents the corresponding 1-loop vertex function. Notice that in 1-loop order, the anomalous vertex contributions in Eq. (50), which are not matched by self energy diagrams, are originated in the diagrams (a) and (b) in Fig. 7 below. They are cancelled exactly by the g_4 and $\widetilde g_4$ contributions in this order. In contrast, the vertex diagram (c) in Fig. 7 is matched exactly by the corresponding self-energy difference in 1-loop order.

Similarly, Eq. (51) is represented diagrammatically in Fig. 8. Since there are no self-energy counterparts in that equation, the vertex contributions are cancelled entirely by the $\bar g_0$ and $\bar{\widetilde g}_f$ terms which are produced by the anomaly.

In 2-loop order, Eq. (48) reduces to the two new equations:

$$(iq_0 - v_F q)\sum_{j=b,a}\Lambda^{b,j\,(2)}_{+,+}(\mathbf{k}+\mathbf{q},\mathbf{k}) - v_F q\sum_{j=b,a}[(\bar g_0 + \bar g_f)\Lambda^{b,j\,(1)}_{+,-}(\mathbf{k}+\mathbf{q},\mathbf{k}) + (\bar g_4 + \bar{\widetilde g}_4)\Lambda^{b,j\,(1)}_{+,+}(\mathbf{k}+\mathbf{q},\mathbf{k})] = \Sigma^{b\,(2)}_+(\mathbf{k}) - \Sigma^{b\,(2)}_+(\mathbf{k}+\mathbf{q})], \tag{51}$$

Figure 5: Equation (50) in diagrammatic form.

Figure 6: Equation (51) in diagrammatic form.

and

$$
\begin{aligned}
(iq_0 \quad &+ \quad v_F q)\sum_{j=b,a}\Lambda^{b,j\,(2)}_{+,-}(\mathbf{k}+\mathbf{q},\mathbf{k}) + v_F q \sum_{j=b,a}[(\bar{g}_0+\bar{g}_f)\Lambda^{b,j\,(1)}_{+,+}(\mathbf{k}+\mathbf{q},\mathbf{k}) + \\
&+ \quad (\bar{g}_4+\bar{\bar{g}}_4)\Lambda^{b,j\,(1)}_{+,-}(\mathbf{k}+\mathbf{q},\mathbf{k})] = 0, \qquad (52)
\end{aligned}
$$

The number of diagrams associated with these two equations is now considerably larger. The anomalous contributions in Eq. (52) are of the same kind as before and are produced by the vertex diagrams displayed in Fig. 9, which are again exactly cancelled by the corresponding $\bar{g}_4, \bar{\bar{g}}_4$ and $\bar{g}_0, \bar{g}_f$ contributions.

Figure 7: 2-loop anomalous vertex diagrams for Equation (52).

We display the full Eq. (52) in our Fig. 10. The remaining non-anomalous 2-loop vertex diagrams in Figure 10 either cancel each other out, such as diagrams (2), (3), (4), and diagrams (14) and (15) or they are matched by their corresponding self-energy differences (like diagrams (7),(8),(9) with the

Figure 8: Equation (52) in diagrammatic form.

second contribution to the self-energy difference, (10), (11), (12) with the first contribution, (13) with the third and so on).

The same goes for the 2-loop WI equation for $\alpha' = -$. The corresponding anomalous vertex diagrams are shown in Fig. 11.

These diagrams are cancelled by the corresponding $\bar{g}_0, \bar{g}_f$ and $\bar{g}_4, \bar{\widetilde{g}}_4$ contributions. The whole set of diagrams is shown in Fig. 12 below. Notice that vertex diagrams (2) and (3), (9) and (10), (11) and (12) as well as diagrams (4), (5) and (6) cancel each other out exactly.

One clearly sees that all anomalous diagrams are associated with the g_0, g_f and $g_4, \widetilde{g}_4$ couplings which are originated in 1-loop order. In other words there

Figure 9: Anomalous 2-loop vertex diagrams for $\alpha = -$.

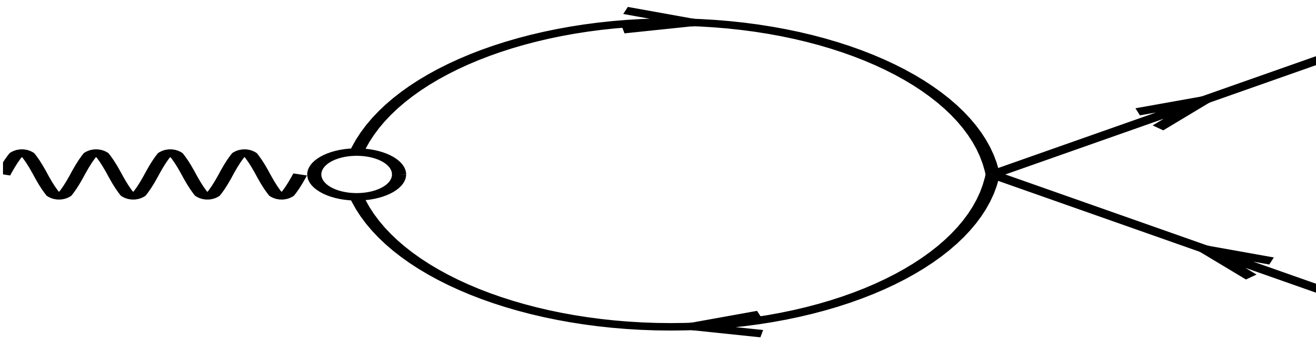

Figure 10: 2-loop WI equation for $\alpha-$.

are no new classes of anomalous vertex diagrams in higher loops so that the anomaly is linear and the g_u, g_{bs} couplings contribute in many ways to both the vertex and the self-energy but not to the anomaly, as predicted by the Theorem in §4.

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