Interacting Weyl semimetals on a lattice

Vieri Mastropietro
Università degli Studi di Milano, Via Cesare Saldini 50, Milano, Italy

We perform an exact Renormalization Group analysis of an interacting Weyl semimetal on a lattice. The existence of a Weyl semimetallic phase, and the absence of quantum instabilities is rigorously proved for interactions not too strong, even arbitrarily close to the boundary of the semimetallic regime, where the Fermi points are merging and the Fermi velocity is vanishing. Relativistic behavior emerges at low energies and the optical conductivity remains equal to the free value, with renormalized velocities replacing by bare ones and up to subdominant corrections.

PACS numbers: 73.22.Pr, 71.10.-w,05.10.Cc

I. INTRODUCTION

Weyl semimetals are three dimensional fermionic system whose Fermi "surface" consists of disconnected points. Their properties, somewhat intermediate between a metal and an insulator, are quite different with respect to systems with extended surfaces. The possibility of Weyl semimetallic behavior has been theoretically predicted to occur in several systems [1]-[23] and recent experiments in $Bi_{1-x}SB_x$ [24] or in $Bi_2Ir_2O_7$ [25] found indications of it. In Weyl semimetals the dispersion relation close to the Fermi points is approximately linear and the elementary excitations nearby can be effectively described in terms of massless Dirac (or Weyl) particles in $D = 3+1$. This fact suggests that phenomena typical of high energy physics (quarks and leptons in the standard model are Weyl particles) can have a low energy analogue in these systems. The emerging description in terms of Dirac particles [26],[27] in Weyl semimetals is common to other materials, among which is graphene, in which the excitations are $2+1$ massless Dirac particles; however the different dimensionality produces important differences. In addition to the point-like Fermi surface, other important features of Weyl semimetals are the Fermi arcs and the fact that the Weyl points approach to each other at the boundary of the semimetallic phase and finally merge together; such non trivial features are captured by a simplified lattice model proposed in [28].

It is of course important to understand the effect of the interaction in Weyl semimetals. Coulomb interaction has been analyzed in [29],[30],[31] finding logarithmic corrections to the conductivity and other physical quantities; in [32] the case of *strong* Hubbard interactions has been instead considered finding evidence of instabilities. In such studies the interaction has been considered in an effective relativistic model (or a regularization of it as in [31]) directly in terms of Dirac particles. There are however good reasons to go beyond such approximation and consider a more realistic lattice description of an interacting Weyl semimetal. First, the relativistic effective description of Weyl semimetals misses important phenomena as the merging of Fermi points or the presence of insulating phases at the boundary of the semimetallic phase. Moreover, it does not capture effects like the movement of Weyl points due to the interaction [33]. Finally, we recall that, in the case of graphene, certain quantities computed within the effective relativistic description are regularization dependent, see [34]-[43]; in particular the universality of the optical conductivity in graphene with short range interactions can be rigorously explained only taking into account the irrelevant terms due to the honeycomb lattice [44], [45]. It is therefeore likely that also in the conductivity of Weyl semimetals lattice effects and non linear bands must be taken into account.

In this paper we consider the lattice model introduced in [28] and we add to it a Hubbard like interaction. The physical properties are analyzed by using *exact* Renormalization Group methods (similar to the one already adopted for graphene [44], [45]); such methods allow to fully take into account the lattice and do not require any approximation. In §II the model is precisely defined and in §III we derive a set of lattice *Ward Identities* relating several physical quantities. In §IV we consider the system for values of the parameters well inside the semimetallic phase (where the Fermi velocity is not too small and the Weyl points are far enough). Contrary to what happens for extended Fermi surfaces, we can exclude the presence of quantum instabilities for interactions not too strong, even at a non perturbative level (convergence of the series expansion is proved); Weyl semimetallic behavior persists even in presence of interaction. In particular, the interaction does not change the qualitative behavior of the 2-point function, and simply changes by a finite amount (depending on the lattice and microscopic detail) the Fermi velocities and the wave function renormalization (effects missed in an effective relativistic description). The optical conductivity remains equal to the free value, with the renormalized velocities replacing the bare ones and up to subdominant corrections; this is a consequence of Ward Identities, providing an universal relation between the current and velocity renormalizations. Finally in §V we extend our analysis to the boundary of the semimetallic phase, where the Fermi velocity v_3 becomes arbitrarily small and the Weyl points are very close. In a description in terms of Weyl particles the parameter expansion is U/v_3; therefore one may suspect that even a weak interaction could produce some quantum instability close to the boundary of the semimetallic phase. However, this is not what happens; no quantum instabilities are found even at the boundary of the semimetallic phase, as a consequence of a crossover phenomenon between regions with different scaling properties. Relativistic behavior still emerges, but in a range of energies smaller and smaller as we are closer to the boundary of the semimetallic phase.

II. THE MODEL

Let us consider the tight binding model defined in [28], defined on a three dimensional lattice, with nearest and next to nearest neighbor hopping and with a magnetic flux density, such that the flux over each surface of the unit cell is vanishing. As it was shown in [28], the flux, whose effect is to decorate the hopping with phase factors [46], can be properly chosen so that a semimetallic phase is present. There are of course other physical and probably more realistic mechanism to produce semimetals, see [1]-[23]; this one is however simple enough to be accessible to an analytic analysis and at the same time it retains several non trivial features present in more complicated systems. It is therefore a good theoretical model where to investigate the effect of the interaction.

One introduces a sublattice $\Lambda_A = \Lambda$ with side L given by the points $\vec{x} = (\vec{n}\vec{\delta})$, with $\vec{\delta}_1 = (1,0,0)$, $\vec{\delta}_2 = (0,1,0)$, $\vec{\delta}_3 = (0,0,1)$, and a sublattice Λ_B whose points are $\vec{x}+\vec{\delta}_+$ with $\vec{\delta}_+ = \frac{\vec{\delta}_1+\vec{d}_2}{2}$ and $\vec{\delta}_- = \frac{\vec{\delta}_1-\vec{d}_2}{2}$. The planar nearest-neighbor hopping between A and B sublattice is t, the planar next-to-nearest-neighbor hopping between A and A, or B and B is t' while the vertical hopping is $t_\perp$. Introducing fermionic creation and annihilation operators $(a^\pm_{\vec{x}}, b^\pm_{\vec{x}+\vec{\delta}_+})$, the hopping (or non-interacting) Hamiltonian is $H_1 + H_2 + H_3$ where H_1 describes the hopping between the A and B sublattice

$$H_1 = \frac{1}{2}\sum_{\vec{x}\in\Lambda}\{[-it(a^+_{\vec{x}} b^-_{\vec{x}+\vec{\delta}_+} + b^+_{\vec{x}+\vec{\delta}_+} a^-_{\vec{x}+2\vec{\delta}_+}) + H.c.] + [t(a^+_{\vec{x}} b^-_{\vec{x}-\vec{\delta}_-} - b^+_{\vec{x}-\vec{\delta}_-} a^-_{\vec{x}-2\vec{\delta}_-}) + H.c.]\} \quad (1)$$

while H_2 contains AA or BB hopping

$$H_2 = \frac{1}{2}\sum_{\vec{x}\in\Lambda}\{[t_\perp(a^+_{\vec{x}} a^-_{\vec{x}+\vec{\delta}_3} - b^+_{\vec{x}+\vec{\delta}_+} b^-_{\vec{x}+\vec{\delta}_++\vec{\delta}_3}) + H.c.] - t' \sum_{i=1,2}[(a^+_{\vec{x}} a^-_{\vec{x}+\vec{\delta}_i} - b^+_{\vec{x}+\vec{\delta}_+} b^-_{\vec{x}+\vec{\delta}_++\vec{\delta}_i}) + H.c.]\} \quad (2)$$

and H_3 takes into account the on site energy difference between the two sublattices

$$H_3 = \frac{\mu}{2}\sum_{\vec{x}\in\Lambda}(a^+_{\vec{x}} a^-_{\vec{x}} - b^+_{\vec{x}+\vec{\delta}_+} b^-_{\vec{x}+\vec{\delta}_+}) \quad (3)$$

The hopping parameters $t, t_\perp, t'$ are assumed $O(1)$ and positive. The dispersion relation is given by $|\mathcal{E}(\vec{k})|$ where

$$\mathcal{E}(\vec{k}) = t\sin(\vec{k}\vec{\delta}_+)\sigma_1 + t\sin(\vec{k}\vec{\delta}_-)\sigma_2 + \sigma_3(\mu + t_\perp \cos k_3 - \frac{1}{2}t'(\cos k_1 + \cos k_2)) \quad (4)$$

The functions $\sin(\vec{k}\vec{\delta}_+)$ and $\sin(\vec{k}\vec{\delta}_-)$ vanish in correspondence of two points $(k_x, k_y) = (0,0)$ and $(k_x, k_y) = (\pi, \pi)$; we assume from now on $\mu + t' > t_\perp$ so that the coefficient multiplying σ_3 is always non zero for $(k_x, k_y) = (\pi, \pi)$. Therefore the dispersion relation vanishes only in correspondence of $(k_x, k_y) = (0,0)$ and $\mu - t' + t_\perp \cos k_3 = 0$. In the region in the space of parameters such that $\frac{|\mu - t'|}{t_\perp} < 1$ one has semimetallic behavior [28]; the dispersion relation vanishes in correspondence of the two *Weyl points* $\pm\vec{p}_F$, with $\vec{p}_F = (0, 0, \cos^{-1}(\frac{t'-\mu}{t_\perp}))$, and close to such points it becomes approximately *linear*. The boundary of the semimetallic region is $\frac{t'-\mu}{t_\perp} = \pm 1$; moving close to the boundary the Weyl points move closer to each other and the quadratic corrections to the linear dispersion relation become more relevant. At $\frac{t'-\mu}{t_\perp} = \pm 1$ the two Weyl points are merging and beyond two insulating phases are reached. Such features, and others like the presence of Fermi arcs [28], are present also in more realistic realization of lattice semimetals.

We want now to investigate if the semimetallic behavior survives in presence of an interaction. We assume that the electrons interact through a short range interaction, so that the Hamiltonian becomes

$$H = H_1 + H_2 + H_3 + UV \quad (5)$$

where

$$V = \sum_{\vec{x},\vec{y}} v(\vec{x}-\vec{y})[a^+_{\vec{x}} a^-_{\vec{x}} + b^+_{\vec{x}+\vec{\delta}_+} b^-_{\vec{x}+\vec{\delta}_+}][a^+_{\vec{y}} a^-_{\vec{y}} + b^+_{\vec{y}+\vec{\delta}_+} b^-_{\vec{y}+\vec{\delta}_+}] \quad (6)$$

and $v(\vec{x})$ is a short-range interaction.

The *currents* are defined as usual via the *Peierls substitution*, see App. A, by modifying the hopping parameter along the bond : using the notation $\int d\mathbf{k} = \frac{1}{\beta L^3}\sum_{\mathbf{k}}$ one obtains, if $\widehat{\psi}^\pm_{\vec{k}} = (\widehat{a}^\pm_{\vec{k}}, \widehat{b}^\pm_{\vec{k}})$

$$\widehat{j}_{+;\vec{p}} = e\int d\vec{k}\widehat{\psi}^+_{\vec{k}+\vec{p}}[w_{a,+}(\vec{k},\vec{p})\sigma_1 + w_{b,+}(\vec{k},\vec{p})\sigma_3]\widehat{\psi}^-_{\vec{k}}$$
$$\widehat{j}_{-;\vec{p}} = e\int d\vec{k}\widehat{\psi}^+_{\vec{k}+\vec{p}}[w_{a,-}(\vec{k},\vec{p})\sigma_2 + w_{b,-}(\vec{k},\vec{p})\sigma_3]\widehat{\psi}^-_{\vec{k}}$$
$$\widehat{j}_{3,\vec{p}} = e\int d\vec{k}\psi^+_{\vec{k}} w_3(\vec{k},\vec{p})\sigma_3\widehat{\psi}^-_{\vec{k}} \quad (7)$$

where

$$w_{a,\pm}(\vec{k},\vec{p}) = \frac{i}{2}t\eta_\pm(\vec{p})(e^{i(\vec{k}+\vec{p})\delta_\pm} + e^{-i\vec{k}\delta_\pm})$$
$$w_{b,\pm}(\vec{k},\vec{p}) = i\frac{t'}{2}\sum_{i=1,2}\eta_i(\vec{p})[e^{i(\vec{k}+\vec{p})\vec{\delta}_i} - e^{-i\vec{k}\vec{\delta}_i}]$$
$$w_3(\vec{k},\vec{p}) = -i\frac{t_\perp}{2}\eta_3(\vec{p})(e^{i(\vec{k}+\vec{p})\vec{\delta}_3} - e^{-i\vec{k}\vec{\delta}_3}) \quad (8)$$

with $\eta_i(\vec{p}) = \frac{1-e^{-i\vec{p}\vec{\delta}}}{i\vec{p}\vec{\delta}} = 1 + O(\vec{p})$. We have defined $a^+_{\vec{x}} = \frac{1}{L^3}\sum_{\vec{k}} e^{i\vec{k}\vec{x}}\widehat{a}^+_{\vec{k}}$ and $b^+_{\vec{x}+\vec{\delta}_+} = \frac{1}{L^3}\sum_{\vec{k}} e^{i\vec{k}(\vec{x}+\vec{\delta}_+)}\widehat{b}^+_{\vec{k}}$, where $\vec{k} = \frac{2\pi}{L}\vec{n}$. The diamagnetic current is defined as $j_{\vec{p}} = \frac{\partial^2 H(\vec{A})}{\partial \widehat{A}_{\vec{p}}\partial \widehat{A}_{\vec{p}}}|_0$ and the density operator is defined as $\rho_{\vec{x}} = a^+_{\vec{x}}a^-_{\vec{x}} + b^+_{\vec{x}+\vec{\delta}_+}b^-_{\vec{x}+\vec{\delta}_+}$. Moreover $\sigma_0 = I$ and σ_i, $i = 1, 2, 3$ are Pauli matrices.

If $O_{\mathbf{x}} = e^{x_0 H} O_{\vec{x}_i} e^{-x_0 H}$, with $\mathbf{x} = (x_0, \vec{x})$, we denote by

$$\langle O^{(1)}_{\mathbf{x}_1} \cdots O^{(n)}_{\mathbf{x}_n}\rangle_\beta = \lim_{L\to\infty} \Xi^{-1}\mathrm{Tr}\{\mathrm{e}^{-\beta \mathrm{H}}\mathbf{T}(\mathrm{O}^{(1)}_{\mathbf{x}_1}\cdots \mathrm{O}^{(n)}_{\mathbf{x}_n})\} \tag{9}$$

where $\Xi = \mathrm{Tr}\{\mathrm{e}^{-\beta\mathrm{H}}\}$ and $\mathbf{T}$ is the operator of fermionic time ordering; moreover we denote by $\langle O^{(1)}_{\mathbf{x}_1}; \cdots ; O^{(n)}_{\mathbf{x}_n}\rangle_\beta$ the corresponding *truncated expectations* and by $\langle O^{(1)}_{\mathbf{x}_1}; \cdots ; O^{(n)}_{\mathbf{x}_n}\rangle$ their zero temperature limit.

We will particularly interested in the *two-point Schwinger function* $\langle \psi^-_{\mathbf{x}}\psi^+_{\mathbf{y}}\rangle$ and in the conductivity, which is defined via Kubo formula. Denoting by $\langle \widehat{j}_{i;\mathbf{p}}\widehat{j}_{i;-\mathbf{p}}\rangle$, $\mathbf{p} = (\omega, \vec{p})$, the Fourier transform of $\langle \widehat{j}_{i;\mathbf{x}}\widehat{j}_{i;\mathbf{y}}\rangle$, the Kubo formula for conductivity is

$$\sigma_{ii}(i\omega) = -\frac{1}{\omega}\left[\langle \widehat{j}_{i;\omega,0}; \widehat{j}_{i;-\omega,0}\rangle_\beta + \Delta_i\right] \tag{10}$$

where Δ_i is the diamagnetic contribution.

In the *non interacting case* $U = 0$ the properties of the system can be easily computed. The 2-point Schwinger function $\langle \psi^-_{\mathbf{x}}\psi^+_{\mathbf{y}}\rangle|_{U=0} \equiv g(\mathbf{x}-\mathbf{y})$ is given by

$$g(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{k}\mathbf{x}} \begin{pmatrix} -ik_0 + t_\perp(\cos k_3 - \cos k_F) + E(\vec{k}) & t(\sin k_+ - i\sin k_-) \\ t(\sin k_+ + i\sin k_-) & -ik_0 - t_\perp(\cos k_3 - \cos k_F) - E(\vec{k}) \end{pmatrix}^{-1} \tag{11}$$

with $E(\vec{k}) = t'(\cos k_+ \cos k_- - 1)$. For momenta respectively close $\mathbf{p}_F$ or $-\mathbf{p}_F$, $\mathbf{p}_F = (0, \vec{p}_F)$, the 2point function has the following form, if $\mathbf{k} = \mathbf{k}' \pm \mathbf{p}_F$, $\mathbf{p}_F = (0, \vec{p}_F)$

$$\widehat{g}_\pm(\mathbf{k}') \sim \begin{pmatrix} -ik_0 \pm v_{3,0}k'_3 & v_{\pm,0}(k_+ - ik_-) \\ v_{\pm 0}(k_+ + ik_-) & -ik_0 \mp v_{3,0}k'_3 \end{pmatrix}^{-1} \tag{12}$$

with

$$v_{\pm,0} = t \qquad v_{3,0} = t_\perp \sin p_F \tag{13}$$

The two 2×2 matrices $\widehat{g}(\mathbf{k}' + \mathbf{p}_F)$ and $\widehat{g}(\mathbf{k}' - \mathbf{p}_F)$ can be combined in a 4×4 matrix coinciding with the propagator of a massless Dirac (or Weyl) article, with anisotropic light velocity. Close to the boundary of the semimetallic phase this relativistic behavior is present only in a region $O(\sqrt{\varepsilon})$ around the Fermi momentum, where $\varepsilon = 1 - \frac{|t'-\mu|}{t_\perp}$; indeed for ε small the the distance between the Fermi points and v_3 is $O(\sqrt{\varepsilon})$, and therefore the linear part of the dispersion relation $v_3 k'_3$ is dominating over the quadratic correction only in that region.

In the *collisionsless* regime $\omega >> \beta^{-1}$ information on the optical conductivity can be obtained by Kubo formula, as collisions with thermally excited carriers can be neglected; for $\beta^{-1} << \omega << v_{3,0}$ (10) is approximately equal to the conductivity of Weyl fermions up to a rescaling factor ; indeed if $l = +, -, 3$

$$\sigma_{ll}(i\omega)|_{U=0} \sim e^2 \frac{v^2_{l,0}}{(v_{\pm,0})^2 v^0_{3,0}} \sigma_{ii,weyl}(i\omega) \tag{14}$$

where $\sigma_{ii,weyl}(i\omega)$ is the conductivity of Weyl fermions with light velocity $c = 1$. By analytic continuation $i\omega \to \omega + \imath\varepsilon$ one can verify that the real part of $\sigma_{ll,weyl}(\omega)$ vanishes as $O(\omega)$ [47] while the imaginary part vanishes as $O(\omega \log|\omega|)$ [31]. Therefore, the optical conductivity in the free case has a rather different behavior with respect to graphene, in which the optical conductivity in a wide range of frequencies is essentially constant and *universal*, that is independent from the microscopic parameters like the hopping or the Fermi velocity [48]. In the present case one has instead a linear dependence from the frequencies, with a *non-universal* prefactor depending from the velocities.

III. WARD IDENTITIES

The physical observables can be expressed as usual in terms of Grassmann integrals. We denote by $\psi^\pm_{\mathbf{x}} = (a^\pm_{\mathbf{x}}, b_{\mathbf{x}+\mathbf{d}_+})$, $\mathbf{x} = (x_0, x)$, $\mathbf{d} = (0, \vec{\delta})$ a set of Grassmann variables; with abuse of notation, we denote them by the same symbol as the fermionic fields. As we expect that the location of the Weyl points will be in general modified by the presence of the interaction, we find convenient to fix them to their non interacting value by replacing μ with $t_\perp cos p_F + \nu$, where ν is a counterterm to be suitably chosen as function of U; a non vanishing ν means that the location of the Weyl points is shifted. We introduce the *generating functional*, $\mathbf{A} = (A_0, \vec{A})$

$$e^{\mathcal{W}(\mathbf{A},\phi^+,\phi^-)} = \int P(d\psi) e^{\mathcal{V}(\psi) + \mathcal{B}(A) + (\psi,\phi)} \tag{15}$$

where $P(d\psi)$ is the fermionic integration with propagator (11) and $\mathcal{V}$ is the interaction

$$\mathcal{V} = \nu N + UV \tag{16}$$

where, if $\int d\mathbf{x} = \int dx_0 \sum_{\vec{x}}$

$$N = \int d\mathbf{x} \psi^+_{\mathbf{x}} \sigma_3 \psi^-_{\mathbf{x}} \tag{17}$$
$$V = \int d\mathbf{x} d\mathbf{y} v(\mathbf{x}-\mathbf{y}) (\psi^+_{\mathbf{x}} \sigma_0 \psi^-_{\mathbf{x}})(\psi^+_{\mathbf{y}} \sigma_0 \psi^-_{\mathbf{y}})$$

if $v(\mathbf{x}) = \delta(x_0) v(x)$. Moreover $(\psi, \phi) = \int d\mathbf{x} [\psi^+_{\mathbf{x}} \sigma_0 \phi^-_{\mathbf{x}} + \psi^-_{\mathbf{x}} \sigma_0 \phi^+_{\mathbf{x}}]$ and

$$\mathcal{B}(\mathbf{A}, \psi) = \int d\mathbf{x} A_0(\mathbf{x}) \psi^+_{\mathbf{x}} \sigma_0 \partial_0 \psi^-_{\mathbf{x}} + \mathcal{B}_1(A, \psi) + \mathcal{B}_2(A, \psi) \tag{18}$$

where the explicit expression of $\mathcal{B}_1(A,\psi)$ and $B_2(A,\psi)$ is obtained by $H_1(A)$ and $H_2(A)$ (see (56) below) by replacing $\sum_{\vec{x}}$ with $\int dx_0 \sum_{\vec{x}}$, the Fermi operators $a_{\vec{x}}, b_{\vec{x}}$ with the Grassmann variables $a_{\mathbf{x}}, b_{\mathbf{x}}$ and $U_{\mathbf{x},\mathbf{x}+\mathbf{d}}(A) = e^{ie \int_0^1 \vec{\delta} \cdot \vec{A}(\mathbf{x}+s\mathbf{d}) ds}$. As in lattice gauge theory, by the change of variables $a^\pm_{\mathbf{x}} \to e^{\mp ie\alpha_{\mathbf{x}}} a^\pm_{\mathbf{x}}$, $b^\pm_{\mathbf{x}+\mathbf{d}_+} \to e^{\mp ie\alpha_{\mathbf{x}+\mathbf{d}_+}} b^\pm_{\mathbf{x}+\mathbf{d}_+}$ and using the relation $U_{\mathbf{x},\mathbf{x}+\mathbf{d}}(A) = e^{ie\alpha_{\mathbf{x}+\mathbf{d}} - ie\alpha_{\mathbf{x}}} U(\mathbf{A})$ one obtains

$$W(\mathbf{A} + \partial\alpha, \phi^+ e^{ie\alpha}, \phi^- e^{-ie\alpha}) = W(\mathbf{A}, \phi^+, \phi^-) \tag{19}$$

where the fact that the Jacobian of the transformation is equal to 1 has be exploited; due to the presence of the lattice, no anomalies are present. From (19) we get the following identity

$$\partial_\alpha W(\mathbf{A} + \partial\alpha, \phi^+ e^{ie\alpha}, \phi^- e^{-ie\alpha}) = 0 \tag{20}$$

from which by differentiating with respect to the external fields $\mathbf{A}, \phi$ an infinite number of *Ward Identities* connecting correlation functions is obtained.

In particular, if $\mathbf{p} = (\omega, \vec{p})$

$$-i\omega < \widehat{\rho}_{\mathbf{p}}; \widehat{\psi}_{\mathbf{k}} \widehat{\psi}_{\mathbf{k}+\mathbf{p}} > + \sum_{j=\pm,3} p_j < \widehat{j}_{j,\mathbf{p}}; \widehat{\psi}_{\mathbf{k}} \widehat{\psi}_{\mathbf{k}+\mathbf{p}} >= 0 \tag{21}$$

where the currents j_+, j_-, j_3 are given by (7).

Similarly we can derive equation for the current-current correlation

$$-i\omega \langle \widehat{\rho}_{\mathbf{p}}; \widehat{\rho}_{-\mathbf{p}} \rangle + \sum_{l=\pm,3} p_l \langle \widehat{j}_{l,\mathbf{p}}; \widehat{\rho}_{-\mathbf{p}} \rangle = 0 \tag{22}$$
$$-i\omega \langle \widehat{\rho}_{\mathbf{p}}; \widehat{j}_{i,-\mathbf{p}} \rangle + \sum_{l=\pm,3} p_l \langle \widehat{j}_{l,\mathbf{p}}; \widehat{j}_{i,-\mathbf{p}} \rangle + p_i \Delta_i = 0$$

where we have used that

$$\frac{\partial^2 \mathcal{W}(A)}{\partial A_{i,\mathbf{p}} \partial A_{l,-\mathbf{p}}}|_0 = \langle \widehat{j}_{i,\mathbf{p}}; \widehat{j}_{l,\mathbf{p}} \rangle \quad i \neq j$$
$$\frac{\partial^2 \mathcal{W}(A)}{\partial A_{i,\mathbf{p}} \partial A_{i,-\mathbf{p}}}|_0 = \langle \widehat{j}_{i,\mathbf{p}}; \widehat{j}_{i,-\mathbf{p}} \rangle + \Delta_i \tag{23}$$

From (22) we get the following equality

$$\frac{\omega^2}{p_i^2} \langle \widehat{\rho}_{\bar{\mathbf{p}}}; \widehat{\rho}_{-\bar{\mathbf{p}}} \rangle = \langle \widehat{j}_{i;\bar{\mathbf{p}}}; \widehat{j}_{i;-\bar{\mathbf{p}}} \rangle + \Delta_i \tag{24}$$

where $\bar{\mathbf{p}}$ is obtained from $\mathbf{p}$ setting $p_l = 0$, $l \neq (0, i)$. From the above equation we get, differentiating with respect to ω

$$\frac{2\omega}{p_i^2} \langle \widehat{\rho}_{\bar{\mathbf{p}}}; \widehat{\rho}_{-\bar{\mathbf{p}}} \rangle + \frac{\omega^2}{p_i^2} \partial_\omega \langle \widehat{\rho}_{\bar{\mathbf{p}}}; \widehat{\rho}_{-\bar{\mathbf{p}}} \rangle = \partial_\omega \langle \widehat{j}_{i;\bar{\mathbf{p}}}; \widehat{j}_{i;-\bar{\mathbf{p}}} \rangle \tag{25}$$

from which we get

$$\langle \widehat{j}_{i;\bar{\mathbf{p}}}; \widehat{j}_{i;-\bar{\mathbf{p}}} \rangle|_{\omega=0} + \Delta_i = 0 \qquad \partial_\omega \langle \widehat{j}_{i;\bar{\mathbf{p}}}; \widehat{j}_{i;-\bar{\mathbf{p}}} \rangle|_{\omega=0} = 0 \tag{26}$$

From (26) we see that the properties of the conductivity (10) depend crucially on the continuity and differentiability of the Fourier transform of the current-current correlations. Indeed if $\langle \widehat{j}_{i,\mathbf{p}}; \widehat{j}_{i,-\mathbf{p}} \rangle$ is continuous in $\mathbf{p}$ then from the first of (26) we get

$$\sigma_{ll}(i\omega) = -\frac{1}{\omega} \Big[\langle \widehat{j}_{l;\omega,0}; \widehat{j}_{l;-\omega,0} \rangle - \langle \widehat{j}_{i;0,0}; \widehat{j}_{l;0,0} \rangle \Big] \tag{27}$$

and if the derivative is continuous then is vanishing by the second of (26). In the non interacting case $\langle j_{i;\mathbf{x}}; \widehat{j}_{i;\mathbf{x}} \rangle$ decays as $O(|\mathbf{x}-\mathbf{y}|^{-6})$ at large distance; therefore the Fourier transform is continuous and with continuous first derivative, hence $\sigma^0_{ii}(\omega)$ vanishes as $\omega \to 0$. However the second derivative is not continuous, and this explains the logarithmic behavior to the free conductivity.

The above Ward Identities provide relations between physical observables in presence of interactions and will play a crucial role for the analysis of the interaction effects in the following sections.

IV. THE EFFECT OF THE INTERACTION WELL INSIDE THE SEMIMETALLIC PHASE

Renormalization Group analysis

We consider first the behavior well inside the semimetallic phase, that is for values of the parameters $\mu, t, t', t_\perp$ such that $\mu + t' > 2t_\perp$ and $\frac{\pi}{4} \leq p_F \leq \frac{3\pi}{4}$ where $\cos p_F = \frac{\mu - t'}{t_\perp}$; this condition ensures that the Fermi velocity $v_F = \sin p_F$ is not too small, that is $v_F \geq \frac{t_\perp}{\sqrt{2}}$.

In order to investigate the effect of the electron-electron interaction on the physical behavior we perform a Renormalization Group analysis of the generating functional (45). The starting point is the decomposition of the frequency-energy space in circular sectors of radius and width $O(2^h)$; more technically, one introduces a decomposition of the unity $1 = \sum_{h=\infty}^{\infty} f_h(\mathbf{k})$ where $f_h(\mathbf{k})$ is non vanishing in $a2^{h-1} \leq \sqrt{k_0^2 + |\mathcal{E}(\vec{k})|^2} \leq a2^{h+1}$ and a is a suitable constant. We can then write the propagator as $\widehat{g}(\mathbf{k}) = \sum_{-\infty}^{\infty} \widehat{g}^{(h)}(\mathbf{k})$ with $\widehat{g}^{(h)}(\mathbf{k}) = f_h(\mathbf{k})\widehat{g}(\mathbf{k})$; this implies a similar decomposition of the Grassman variables $\psi_{\mathbf{k}} = \sum_{h=-\infty}^{\infty} \psi_{\mathbf{k}}^{(h)}$.

The first step of the Renormalization Group analysis of the generating function (15) is the integration of the positive *ultraviolet scales*; due to the presence of the lattice, making the dispersion relation $|\mathcal{E}(\vec{k})|$ bounded, the ultraviolet integration poses no problems and the results of the integration of $\psi^{(>0)}$ is an expression similar to (15) with $\psi^{(\leq 0)}$ replacing ψ and $\mathcal{V}(\psi) + \mathcal{B}(\psi, A)$ replaced by $\mathcal{V}^{(0)}(\psi^{(\leq 0)}) + \mathcal{B}^{(0)}(\psi^{(\leq 0)}, A)$, sum of monomials of any order in $\psi^{(\leq 0)}$ and A.

The propagator $\widehat{g}^{(\leq 0)}(\mathbf{k})$ is non vanishing in a range of frequency and energies verifying $\sqrt{k_0^2 + |\mathcal{E}(\vec{k})|^2} \leq 2a$. We can choose the constant a so that such region corresponds to two *disconnected* regions in $\mathbf{k}$ space, centered around the Fermi points and labeled by $\varepsilon = \pm$. This means that the Grassmann fields can be conveniently written as sum of two independent fields

$$\psi_{\mathbf{x}}^{(\leq 0)} = \sum_{\varepsilon=\pm} e^{i\varepsilon \mathbf{p}_F \mathbf{x}} \psi_{\varepsilon,\mathbf{x}}^{(\leq 0)} \tag{28}$$

with $\psi_{\varepsilon,\mathbf{x}}^{(\leq 0)}$ with propagator $g_\varepsilon^{(\leq 0)}(\mathbf{x}) = \int d\mathbf{k}' e^{i\mathbf{k}'\mathbf{x}} \vartheta(\varepsilon k_3' + p_F)\widehat{g}^{(\leq 0)}(\mathbf{k}' + \varepsilon \mathbf{p}_F)$. As we will see, $\widehat{g}_\varepsilon^{(h)}(\mathbf{k})$, $h \leq 0$, is very close to the propagator of the upper ($\varepsilon = +$) or lower ($\varepsilon = -$) components of Weyl fermions, up to corrections smaller and smaller as h increase; therefore, the emerging description in terms of Weyl fermions is a natural byproduct of the Renormalization Group analysis.

Let us consider the integration of the *infrared* scales. After the integration of $\psi^{(>0)}, \psi_\varepsilon^{(0)}, .., \psi_\varepsilon^{(h+1)}$ one gets $e^{\mathcal{W}} =$

$$\int \prod_\varepsilon P(d\psi_\varepsilon^{(\leq h)}) e^{\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}) + \mathcal{B}^{(h)}(\mathbf{A}, \sqrt{Z_h}\psi^{(\leq h)}, \phi)} \tag{29}$$

where $P(d\psi_\varepsilon^{(\leq h)})$ has propagator given by

$$g^{(\leq h)}(\mathbf{x}) = \int d\mathbf{k}' e^{i\mathbf{k}'\mathbf{x}} \frac{\chi_h(\mathbf{k})}{Z_h} \begin{pmatrix} -ik_0 + v_{3,h}(\cos(k_3' + \varepsilon \vec{p}_F) - \cos k_F) + E(\vec{k}) & v_{\pm,h}(\sin k_+ - i \sin k_-) \\ v_{\pm,h}(\sin k_+ + i \sin k_-) & -ik_0 - v_{3,h}(\cos(k_3' + \vec{p}_F) - \cos k_F) - E(\vec{k}) \end{pmatrix}^{-1} \tag{30}$$

with $\chi_h(\mathbf{k}') = \sum_{h=-\infty}^{h} \vartheta(\varepsilon k_3' + p_F) f_h(\mathbf{k}' + \varepsilon \mathbf{p}_F)$ is non vanishing in a region $O(2^h)$ around $\varepsilon \mathbf{p}_F$. Moreover $\mathcal{V}^{(h)}$, the *effective potential*, is sum over monomials of $e^{i\sigma_l p p_F \mathbf{x}_l} \psi_{\varepsilon,\mathbf{x}}^{\sigma_l(\leq h)}$ multiplied by kernels $W_{n,0}^{(h)}$; similarly the *effective source* (at $\phi = 0$ for definiteness) is sum of monomials with n fields $e^{i\sigma_l p p_F \mathbf{x}_l} \psi_{\varepsilon,\mathbf{x}}^{\sigma_l(\leq h)}$ and m A-fields multiplied by kernels $W_{n,m}^{(h)}$. Note that the Fermi velocities and the wave function renormalization are modified by interaction and depend on the energy scale.

At large distances the single scale propagator decays faster than any power; that is for any N

$$g^{(h)}(\mathbf{x}) \sim \frac{2^{3h}}{1 + [2^h|\mathbf{x}|]^N} \tag{31}$$

As consequence of (45), the *scaling dimension* of the monomials in the effective potential and in the effective source with n ψ-fields and m A-field is

$$D_1 = 4 - \frac{3}{2}n - m \tag{32}$$

and the terms with negative scaling dimension are called *irrelevant*. As in the case of graphene with short range interaction (in which $D = 3 - n - m$), the terms with four or more fermionic fields are irrelevant; this is in sharp contrast to what happens for Dirac fermions in $1 + 1$ dimension ($D = 2 - n/2 - m$) in which the quartic terms are marginal.

The only non irrelevant terms are the ones with $(n, m) = 2, 0$ ($D_1 = 1$ *relevant*) or $(n, m) = (2, 1)$ ($D_1 = 0$ *marginal*). Before integrating the field $\psi^{(h)}$ one has renormalize the relevant and marginal terms. This consists in rewriting $W_{2,0}^{(h)}(\mathbf{k})$ as its Taylor expansion around the Fermi point $\mathbf{k} = \varepsilon \mathbf{p}_F$ up to the first order ($D_1 = 1$ is its dimension) plus a rest; one them moves the first order terms in the free integration, where they produce a renormalization of the wave function and the velocities , while the zero-th order terms contribute to the running coupling constant ν_h expressing the possible shift of the Fermi points. The fact that the terms generated by the integration of higher energy fields are of the same kind of the terms present originally is a consequence of symmetries, see App. B. Similarly we rewrite $W_{2,1}^{(h)}(\mathbf{k}, \mathbf{p})$ as its Taylor expansion around $\varepsilon \mathbf{p}_F$ up to zero-th order ($D_1 = 0$ is its dimension) and a rest, and the zeroth-term contribute to the renormalization of the currents $Z_{\mu,h}$, $\mu = 0, \pm, 3$. We can therefore write (29) as

$$\int \prod_\varepsilon \widetilde{P}(d\psi_\varepsilon^{(\leq h)}) e^{\widetilde{V}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)}) + \widetilde{B}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)}, A, \phi)}$$

where $\widetilde{P}(d\psi_\varepsilon^{(\le h)})$ has a propagator similar to $g^{(\le h)}$ (30) with $Z_h, v_{\pm,h}, v_{3,h}$ replaced by $Z_{h-1}, v_{\pm,h-1}, v_{3,h-1}$. Moreover $\widetilde{V}^{(h)}(\psi) =$

$$2^j \nu_h \sum_{\varepsilon=\pm} \int d\mathbf{k}' \widehat{\psi}^+_{\mathbf{k}'+\varepsilon\mathbf{p}_F,\varepsilon} \sigma_3 \widehat{\psi}^-_{\mathbf{k}'+\varepsilon\mathbf{p}_F,\varepsilon} + \text{Irr.Terms} \quad (33)$$

and Irr.Terms are terms with negative scaling dimension, for instance terms like $\psi^+\partial^2\psi^-$ or monomials with $n > 2$ fermionic fields. Similarly the effective source is given by $B^{(h)}(\mathbf{A}, \sqrt{Z_{h-1}}\psi^{(\le h)}, 0) =$

$$\sum_{\varepsilon=\pm} \int d\mathbf{k}' d\mathbf{p} \widehat{\psi}^{(\le h)}_{\mathbf{k}'+\varepsilon\mathbf{p}_F,\varepsilon} [Z_{0,h} A_0(\mathbf{p})\sigma_0 + Z_{+,h} A_+(\mathbf{p})\sigma_1$$
$$+ Z_{-,h} A_-(\mathbf{p})\sigma_2 + \varepsilon Z_{3,h}\sigma_3] \widehat{\psi}^{(\le h)}_{\mathbf{k}'+\mathbf{p}+\varepsilon\mathbf{p}_F,\varepsilon} + \text{Irr.Terms}$$

The relation between the renormalized parameters at scale $h-1$ and h is the following

$$\nu_{h-1} = \frac{Z_h}{Z_{h-1}}(\gamma\nu_h + \gamma^{-h}\widehat{W}_2^{(h)}(\varepsilon\mathbf{p}_F))$$
$$\frac{Z_{h-1}}{Z_h} = 1 + \partial_0 \widehat{W}_2^h(\varepsilon\mathbf{p}_F) \quad (34)$$
$$v_{\alpha,h-1} = \frac{Z_h}{Z_{h-1}}(v_{\alpha,h} + \partial_\alpha \widehat{W}_{2,0}^{(h)}(\varepsilon\mathbf{p}_F)) \quad \alpha = \pm, 3$$
$$\frac{Z_{\mu,h-1}}{Z_{\mu,h-1}} = \frac{Z_{\mu,h}}{Z_{h-1}}[1 + \widehat{W}_{2,1}^{(h)}(\varepsilon\mathbf{p}_F)] \quad \mu = 0, \pm, 3 \quad (35)$$

We can write $\chi_h(\mathbf{k}') = \chi_{h-1}(\mathbf{k}') + f_h(\mathbf{k}')$ and $\widetilde{P}(d\psi_\varepsilon^{(\le h)}) = P(d\psi_\varepsilon^{(\le h-1)})P(d\psi_\varepsilon^{(h)})$ where $P(d\psi_\varepsilon^{(\le h-1)})$ and $P(d\psi_\varepsilon^{(h)})$ have propagator given by $g^{(\le h-1)}$ and $g^{(h)}$, similar to (44) with respectively χ_{h-1} and f_h replacing χ_h. The integration of the single scale propagator can be performed, and one obtains an expression similar to (29) with h replaced by $h-1$; the procedure can be then iterated.

It can be shown that, for U not too large, for $0 < \vartheta < 1$

$$\frac{1}{L^3\beta} \int d\underline{\mathbf{x}} |W_{n,m}^{(h)}| \le C|U| 2^{(4-\frac{3}{2}n-m)h} 2^{\vartheta h} \quad (36)$$

Note that in addition to the factor $2^{(4-\frac{3}{2}n-m)h}$ corresponding to the scaling dimension there is a dimensional gain $2^{\vartheta h}$ due to the irrelevance of the effective electron-electron interaction: every contribution in perturbation theory involving an effective scattering in the infrared is suppressed thanks to the irrelevance of the kernels with four or more legs.

The extra factor $2^{\vartheta h}$ in (36) has a crucial role in the study of the flow of the effective velocities and renormalizations; for instance by using (36) in (34) we get $v_{\pm,h-1} = v_{\pm,h} + O(U2^{\vartheta k})$ so that by iteration $v_{\pm,h-1} = v_{\pm,0} + \sum_k O(U2^{\vartheta k}) = v_0 + O(U)$. Therefore

$$Z_h \to_{h\to-\infty} Z = 1 + O(U^2) \quad (37)$$
$$v_{3,h} \to_{h\to-\infty} v_3 = t_\perp \sin(p_F) + a_3 U + O(U^2)$$
$$v_{\pm,h-1} \to_{h\to-\infty} v_\pm = t + a_\pm U + O(U^2)$$
$$Z_{0,h} \to_{h\to-\infty} Z_0 = 1 + O(U^2)$$
$$Z_{\pm,h} \to Z_\pm = t + b_\pm U + O(U^2)$$
$$Z_{3,h} \to_{h\to-\infty} Z_3 = t_\perp \sin p_F + b_3 U + O(U^2)$$

where the explicit form of the coefficients is in App. C. Finally note that ν_h is a relevant coupling so that it would increase at each RG iteration; however we can choose the counterterm ν so that ν_h is $O(U2^{\vartheta h})$ for any h.

Absence of quantum instabilities

As a consequence of the previous analysis we get that the the interacting 2-point function is given by, in the limit $L, \beta \to \infty$ $< \widehat{\psi}^-_{\mathbf{k}'+\varepsilon\mathbf{p}_F} \widehat{\psi}^+_{\mathbf{k}'+\varepsilon\mathbf{p}_F} >=$

$$\frac{1}{Z}\begin{pmatrix} -ik_0 \pm v_3 k_3' & v_{\pm,}(k_+ - ik_-) \\ v_\pm(k_+ + ik_-) & -ik_0 \mp v_3 k_3' \end{pmatrix}^{-1} (1 + R(\mathbf{k}')) \quad (38)$$

with $|R(\mathbf{k}')| \le CU|\mathbf{k}|^\vartheta$. Moreover, for $\mu = 0, \pm, 3$ and $|\mathbf{p}| << |\mathbf{k} - \varepsilon\mathbf{p}_F| << 1$

$$< \widehat{j}_{\mu,\mathbf{p}}; \widehat{\psi}^+_{\mathbf{k}} \widehat{\psi}^-_{\mathbf{k}+\mathbf{p}} >= \quad (39)$$
$$e Z_\mu \langle \widehat{\psi}^-_{\mathbf{k}} \widehat{\psi}^+_{\mathbf{k}} \rangle \widetilde{\sigma}_\mu \langle \widehat{\psi}^-_{\mathbf{k}+\mathbf{p}} \widehat{\psi}^+_{\mathbf{k}+\mathbf{p}} \rangle \ (1 + O(|\mathbf{k} - \varepsilon\mathbf{p}_F|^\vartheta))$$

where Z_μ is given by (37) and $\widetilde{\sigma}_+ = \sigma_1$, $\widetilde{\sigma}_- = \sigma_2$, while $\widetilde{\sigma}_0 = \sigma_0$ and $\widetilde{\sigma}_3 = \sigma_3$.

The interaction does not qualitatively change the asymptotic behavior of the 2-point Schwinger function close to the Fermi points for couplings not too large; its effect is essentially to change by a *finite amount* the wave function renormalization, the Fermi velocities and the location of the Fermi points. Similarly, it changes the current renormalization in the vertex functions.

The presence of quantum instabilities in the ground state is then excluded at weak coupling. This is peculiar feature of interacting Weyl semimetals (or for graphene with weak short range interaction, see [44]) due to point-like Fermi surfaces; in case of Fermi liquids in $D = 2, 3$ with extended Fermi surface the short range interactions is *marginal* and quantum instabilities at $T = 0$ are generically present.

We stress that the persistence of the Weyl semimetallic behavior in presence of interaction is proved at a *non perturbative* level. Usually the physical quantities of interacting fermionic systems are written in terms of expansion in Feynman diagrams, which are not convergent or at least whose convergence is not known. Without knowing convergence, some non-perturbative phenomenon cannot be excluded. However in a small number of case the methods of constructive Quantum Field Theory are able to establish convergence. In particular, this is true for QED_4 for massless fermions and massive photons, [49], and one can check that the proof of convergence there can be easily adapted to the present case. The possibility of a non perturbative construction of interacting Weyl semimetals is a property shared with graphene which makes them radically different with respect to Fermi liquids with extended (symmetric) Fermi surfaces, for which a non-perturbative understanding of the ground state properties is still far to be reached.

Consequences of the Ward Identities

The above Renormalization Group analysis makes clear the close relationship between the model (15) and a relativistic Quantum Field Theory like QED_4 with massless fermions and massive photons, or a Nambu-Jona Lasinio model in $d = 4+1$ with an ultraviolet cut-off is imposed (in Weyl semimetals the cut-off is provided by the lattice). There are however important differences related to the symmetries. Indeed any infrared regularization of a relativistic Quantum Field Theory must respect Lorentz symmetry, which implies that the velocity should not be modified by the interaction, and the renormalizations of the currents should be all equal.

In the case of Weyl semimetals described by (45) a different behavior is found: the velocities are *renormalized* by the interaction and they have a non universal value function on all the microscopic details. In addition, the current renormalizations are not constrained to be equal by Lorentz symmetry. Nevertheless, lattice symmetries still imply relations among the dressed quantities. Inserting (38) and (40) in the Ward Identity (21) one finds the following identities

$$\frac{Z_\pm}{Z} = v_\pm \qquad \frac{Z_3}{Z} = v_3 \tag{40}$$

At lowest order, the above identities simply says that $a_\perp = b_\perp$ and $a_\pm = b_\pm$, and this can be explicitly checked, see App. C. However, (40) is a non perturbative identity, implying an infinite series of identities between Feynman graphs of arbitrary order.

The optical conductivity

The current current correlation is given by, $i = \pm, 3$, $< j_{i,\mathbf{x}}; j_{i,0} >=$

$$e^2[\frac{Z_i}{Z}]^2 \sum_{\varepsilon=\pm} \int d\mathbf{k} \mathrm{Tr}(\widetilde{\sigma}_i g^{(\le 0)}_{rel,\varepsilon}(\mathbf{k}) \widetilde{\sigma}_i g^{(\le 0)}_{rel,\varepsilon}(\mathbf{k}+\mathbf{p})) + H_i(\mathbf{x})$$

where $g^{(\le 0)}_{rel,\varepsilon}(\mathbf{k}) =$

$$\chi_0(\mathbf{k}) \begin{pmatrix} -ik_0 \pm v_3 k'_3 & v_{\pm,}(k_+ - ik_-) \\ v_\pm(k_+ + ik_-) & -ik_0 \mp v_3 k'_3 \end{pmatrix}^{-1}$$

and $|H_i(\mathbf{x})|$ contains the higher order contribution or the correction to the zero-th contribution due to the non linear term

$$|H_i(\mathbf{x})| \le C|\mathbf{x}|^{-6-\vartheta} \tag{41}$$

The first term is the same as the one in a non-interacting theory with velocities $v_\pm, v_3$ instead of $v_{0,\pm}, v_{0,3}$. In addition $H_i(\mathbf{x})$ has a faster decay, as for large distances behaves as $O(|\mathbf{x}|^{-6-\vartheta})$; therefore the Fourier transform $\widehat{H}_i(\mathbf{p})$ admits a continuous second derivative; hence

$$\widehat{H}_i(\omega,\vec{0}) - \widehat{H}_i(0,\vec{0}) = \omega \partial_\omega \widehat{H}_i(0,\vec{0}) + \frac{1}{2}\omega^2 \partial^2_\omega \widehat{H}_i(0,\vec{0}) + \omega^2 R_i(\omega,\vec{0}) \tag{42}$$

with $R_i(0,\vec{0}) = 0$ and by (26) $\partial_\omega H_i(0,\vec{0}) = 0$; by using (27) we get therefore, for $\beta^{-1} << \omega << v_3$ and using (40) to relate the current renormalization to the velocities

$$\sigma_{ll}(i\omega) = \sum_{\varepsilon=\pm} v_l^2 \int d\mathbf{k} \mathrm{Tr}(\widetilde{\sigma}_i g^{(\le 0)}_{rel,\varepsilon}(\mathbf{k}) \widetilde{\sigma}_i g^{(\le 0)}_{rel,\varepsilon}(\mathbf{k}+(\omega,0))) + R(\omega)) \tag{43}$$

with $|R(\omega)| \le C|\omega|$. Therefore the optical conductivity is equal to the one of Weyl fermions with renormalized velocities, up to subdominant corrections (they are $O(\omega)$ while the conductivity of free Weyl fermions is $O(\omega \log \omega)$).

V: ANALYSIS CLOSE TO THE MERGING OF THE FERMI POINTS

Renormalization Group analysis in the first regime

The analysis of the previous section is restricted to value of parameters well inside the semi-metallic phase, where the Fermi velocity is $O(1)$; however at the boundary of the semimetallic regime the Fermi velocity v_3 can be arbitrarily small. An effective description in terms of Weyl fermions would lead to a series expansion in terms of U/v_3, hence small v_3 apparently corresponds to a strong coupling regime even for a weak electronic coupling U. Therefore one may suspect that even a weak interaction could produce some quantum instability close to the boundary of the semimetallic phase. As we will see, that is not what happens; a weak interaction cannot produce instabilities even close to the boundary of the semimetallic phase. This is a phenomenon that cannot be understood, even qualitatively, using a linearized model of Weyl fermions, and the full lattice description is necessary.

Let us introduce the parameters ε defined as $\varepsilon = 1-\frac{t'-\mu}{t_\perp}$, and we consider ε small (the case $\frac{t'-\mu}{t_\perp}\sim -1$ is done similarly); when $\varepsilon=0$ the Weyl points coalesces and the Fermi velocity vanishes, that is $v_{3,0}=O(\sqrt{\varepsilon})$ and $p_F=O(\sqrt{\varepsilon})$. As before, we decompose frequency-energy space in circular sectors of radius and width $O(2^h)$, that is each field at scale h lives in the region $a2^{h-1}\le\sqrt{k_0^2+|\mathcal{E}(\vec k)|^2}\le a2^{h+1}$ with a small enough. The renormalization Group analysis naturally identifies two different regimes, separated by an energy scale $a2^{\bar h}\sim\varepsilon$. For scales greater than $\bar h$ the dispersion relation in the third direction is essentially quadratic and $\cos k-1+\varepsilon\sim\frac{k^2}{2t_\perp}$; the behavior in this region is essentially the same as at the boundary of the semimetallic regime, and the Weyl points have no role. It is only at smaller scales, that is $h\le\bar h$, that the dispersion relation in the third direction becomes linear around the Weyl points, and the fermions acquire an extra label corresponding to their closeness to $\mathbf{p}_F$ or $-\mathbf{p}_F$.

After the integration of the ultraviolet scales $h>0$ we start the integration of the scales $h\ge\bar h$; we integrate of the fields $\psi^{(>0)},\psi^{(0)},\psi^{(-1)},..,\psi^{(h)}$, $h\ge\bar h$ and we get

$$\int\widetilde P(d\psi^{(\le h)})e^{\widetilde V^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})+\widetilde B^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)},A,\phi)}$$

where $\widetilde P(d\psi^{(\le h)})$ has propagator given by

$$g^{(\le h)}(\mathbf{x})=\int d\mathbf{k}e^{i\mathbf{kx}}\frac{\chi_h(\mathbf{k})}{Z_h}\begin{pmatrix}-ik_0+v_{3,h}(\cos(k_3)-\cos p_F)+E(\vec k) & v_{\pm,h}(\sin k_+-i\sin k_-)\\ v_{\pm,h}(\sin k_++i\sin k_-) & -ik_0-v_{3,h}(\cos(k_3)-\cos p_F)-E(\vec k)\end{pmatrix}^{-1}\quad(44)$$

Note that $\cos p_F=1-\varepsilon$ and, choosing (say) $a2^{\bar h}=10\varepsilon$ one has $\cos k_3-1+\varepsilon\le a2^h$ so that k_3 is smaller than $O(2^{\frac{h}{2}})$; therefore the single scale propagator behaves asymptotically as

$$g^{(h)}(\mathbf{x})\sim\frac{2^{\frac{5h}{2}}}{1+[2^h(|x_0|+|x_+|+|x_|)+2^{\frac{h}{2}}|x_3|]^N}\quad(45)$$

and correspondingly the scaling dimension is

$$D_2=\frac{7}{2}-\frac{5}{4}n-m\quad(46)$$

As before, the non irrelevant terms are only the ones with $(n,m)=(2,0)$ $(D_2=1)$ and $(n,m)=(2,0)$ $(D_2=0)$. Before integrating the field $\psi^{(h)}$ one has renormalize the relevant and marginal terms. This consists in rewriting $\widehat W^{(h)}_{2,0}(\mathbf{k})$ as its Taylor expansion around $\mathbf{k}=0$ up to the first order ($D_2=1$ is its dimension) in k_0,k_+,k_- and up to second order in k_3 (remember that $k_3\sim 2^{h/2}$ so first order in k_3 is not sufficient to make the dimension negative) plus a rest; one moves the first and second order terms in the free integration, where they produce a renormalization of the wave function and the velocities , while the zero-th order terms contribute to the running coupling constant ν_h. Note that the Taylor expansion is now around $\mathbf{k}=0$ and not around the Fermi points; moreover $\partial_3\widehat W^{(h)}_{2,0}(0)=0$ by parity. Similarly we rewrite $\widehat W^{(h)}_{2,1}(\mathbf{k},\mathbf{p})$ as its Taylor expansion around $\mathbf{k}=0$ up to zero-th order ($D_2=0$ is its dimension) and a rest, and the zeroth-term contribute to the renormalization of the currents $Z_{\mu,h}$, $\mu=0,\pm$; note that $Z_{3,h}=0$ by symmetry so that the current in the third direction is $\sim A_3\psi^+\partial\psi$, hence irrelevant according to dimensional arguments. The relation between the effective renormalizations at scale h and $h-1$ is

$$\begin{aligned}\nu_{h-1}&=\frac{Z_h}{Z_{h-1}}(\gamma\nu_h+\gamma^{-h}\widehat W^{(h)}_2(0)\\ \frac{Z_{h-1}}{Z_h}&=1+\partial_0\widehat W^h_2(0)\\ v_{\pm,h-1}&=\frac{Z_h}{Z_{h-1}}(v_{\pm,h}+\partial_\pm\widehat W^{(h)}_{2,0}(0))\quad\alpha=\pm\\ v_{3,h-1}&=\frac{Z_h}{Z_{h-1}}(v_{\alpha,h}+\partial_3^2\widehat W^{(h)}_{2,0}(0))\end{aligned}\quad(47)$$

$$\frac{Z_{\mu,h-1}}{Z_{\mu,h-1}}=\frac{Z_{\mu,h}}{Z_{h-1}}[1+\widehat W^{(h)}_{2,1}(0,0)]\quad\mu=0,\pm\quad(48)$$

As before, we can choose $\nu=O(U)$ so that ν_h is bounded uniformly in h; moreover a bound like (36) holds, with $2^{(4-\frac{3}{2}n-m)h}2^{\vartheta h}$ in the r.h.s. replaced by $2^{(\frac{7}{2}-\frac{5}{4}n-m)h}2^{\vartheta h}$; again the presence of the extra factor $2^{\vartheta h}$ implies (as in (37)) that the limiting $Z_{\bar h},v_{\pm,\bar h},v_{3,\bar h},Z_{\pm,\bar h}$ remain close $O(U)$ to their initial value.

Renormalization Group analysis in the second regime

We have now to discuss the integration of the scales $\le\bar h$; after a finite number of integrations again the region $\sqrt{k_0^2+|\mathcal{E}(\vec k)|^2}\le 2^{\bar h}a$ corresponds to two *disconnected* regions in moments space $\mathbf{k}$, centered around the Fermi points, so that the Grassmann fields can be conveniently written as sum of two independent fields

$$\psi^{(\leq h)}_{\mathbf{x}} = \sum_{\varepsilon=\pm} e^{i\varepsilon \mathbf{p}_F \mathbf{x}} \psi^{(\leq h)}_{\varepsilon,\mathbf{x}} \tag{49}$$

In this case $v_3 k' \sim \gamma^h$; therefore the asymptotic behavior of the singe scale propagator is

$$g^{(h)}(\mathbf{x}) \sim \frac{1}{v_3} \frac{2^{3h}}{1+[2^h|\bar{\mathbf{x}}|]^N} \tag{50}$$

where $\bar{\mathbf{x}}$ is equal to $\mathbf{x}$ with x_3 replaced by $v_3^{-1} x_3$. The Renormalization group analysis proceed essential as in §IV; the starting point is a functional integral with integration $P(d\psi^{(\leq \bar{h})})$ and effective potential given by $\mathcal{V}^{(\bar{h})}$, sum of monomials of any degree in $\psi^{(\leq \bar{h})}$. The main difference, with resepct to the case treated in §IV, is in the $\frac{1}{v_3}$ factors in (50) producing "small divisors" for small ε which could destroy convergence. Indeed each term of the renormalized expansion contributing to $W^{(h)}_{n,m}$ obtained contracting m_i vertices with i-fields has a factor v_3^{-P} where

$$P = \sum_i \frac{i m_i}{2} - \bar{n} + 1 - \frac{n}{2} \tag{51}$$

and $\bar{n} = \sum_i m_i$. For large $\bar{n}$ one as $P = O(\bar{n})$ and positive; as $v_3 = O(\sqrt{\varepsilon})$ this factor seems to destroy the convergence of the renormalized expansion for ε small enough (unless U is not chosen vanishing with ε). However the scaling dimension of the second regime D_2 is different from the dimension of the first regime, and the difference is

$$D_2 - D_1 = -\frac{1}{2} + \frac{n}{4} \tag{52}$$

Therefore each term in $\mathcal{V}^{\bar{h}}$ has an extra factor $\gamma^{\bar{h}(D_2-D_1)}$ and therefore one has an extra ε^Q where

$$Q = -\frac{\bar{n}}{2} + \sum_i \frac{i m_i}{4} \tag{53}$$

and as $v_3 = O(\sqrt{\varepsilon})$ one has that the total extra factor (with respect to the analysis in §IV) at each order of the renormalized expansion is

$$\varepsilon^{-\frac{P}{2}} \varepsilon^Q = \varepsilon^{\frac{n}{4}-1} \tag{54}$$

Therefore the small denominators due to the vanishing of the Fermi velocity v_3 are exactly compensated by the extra factors due to the difference in scaling dimensionles in the two regimes; convergence is achieved for couplings U small but independent from ε.

In conclusions, even near the merging of the Fermi points there are no quantum instabilities for U not too large: Weyl semimetallic behavior persists up to the boundary in presence of interaction. The two point function behaves as (38), provided that $|\bar{\mathbf{k}}| << 1$, where $\bar{\mathbf{k}}$ is equal to $\mathbf{k}$ with k_3 replaced by $v_3 k_3$ (that is $|R(\mathbf{k}')| \leq CU|\bar{\mathbf{k}}'|^\vartheta$). In other words, relativistic behavior is found at extremely small momenta $k'_3 << O(\sqrt{\varepsilon})$. Similarly the optical conductivity is given by (43) for extremely small frequencies, that is $\omega << O(\sqrt{\varepsilon})$.

VI.CONCLUSIONS

We have considered for the first time a true interacting lattice model for Weyl semimetals exhibiting, in addition to the semimetallic phase, the merging of Weyl points at the onset of the insulating phase. The Renormalization group analysis reveals that there are generically two regimes, one in which the dispersion relation is essentially quadratic and another in which asymptotic massless Dirac fermion behavior emerges; this second regime begins at lower and lower energy scales as we approach the boundary of the semimetallic behavior. The analysis establishes the absence of quantum phase transitions for coupling small enough, even arbitrarily close to the boundary where the Fermi velocity is arbitrarily small, a region not accessible to a effective relativistic description. The optical conductivity remains equal to the free value, with the renormalized velocities replacing by bare one and up to subdominant corrections.

APPENDIX A: LATTICE CURRENTS

The *current* is defined as usual via the *Peierls substitution*, by modifying the hopping parameter along the bond $(\vec{x}, \vec{x}+\vec{\delta})$ as $U_{\vec{x},\vec{x}+\vec{\delta}}(\vec{A}) = e^{ie\int_0^1 \vec{\delta}\cdot\vec{A}(\vec{x}+s\vec{\delta})ds}$, where e is the electric charge and $\vec{A} = (A_1, A_2, A_3)$ is the vector electromagnetic field; the modified Hamiltonian is

$$H(\vec{A}) = H_1(\vec{A}) + H_2(\vec{A}) + H_3 + UV \tag{55}$$

where

$$H_1(A) = \frac{1}{2}\sum_{\vec x\in\Lambda}\{[-it(a^+_{\vec x}U_{\vec x,\vec x+\vec\delta_+}b^-_{\vec x+\vec\delta_+}+b^+_{\vec x+\vec\delta_+}U_{\vec x+\vec\delta_+,\vec x+2\vec\delta_+}b^-_{\vec x+2\vec\delta_+})+it(b^+_{\vec x+\vec\delta_+}U_{\vec x+\vec\delta_+,\vec x}a^-_{\vec x}+a^+_{\vec x+2\vec\delta_+}U_{\vec x+2\vec\delta_+,\vec x+\vec\delta_+}b^-_{\vec x+\vec\delta_+})] \quad (56)$$
$$+[t(a^+_{\vec x}U_{\vec x,\vec x-\vec\delta_-}b^-_{\vec x-\vec\delta_-}-b^+_{\vec x-\vec\delta_-}U_{\vec x-\vec\delta_-,\vec x-2\vec\delta_-}a_{\vec x-2\vec\delta_-})+(b^+_{\vec x-\vec\delta_-}U_{\vec x-\vec\delta_-,\vec x}a^-_{\vec x}-a^+_{\vec x-2\vec\delta_-}U_{\vec x-2\vec\delta_-,\vec x-\vec\delta_-}b_{\vec x-\vec\delta_-})]$$
$$H_2(A) = \frac{1}{2}\sum_{\vec x\in\Lambda}\{[t_\perp(a^+_{\vec x}U_{\vec x,\vec x+\vec\delta_3}a^-_{\vec x+\vec\delta_3}-b^+_{\vec x+\vec\delta_+}U_{\vec x+\vec\delta_+,\vec x+\vec\delta_++\vec\delta_3}b_{\vec x+\vec\delta_++\vec\delta_3})+a^+_{\vec x+\vec\delta_3}U_{\vec x+\vec\delta_3,\vec x}a^-_{\vec x}-$$
$$b^+_{\vec x+\vec\delta_++\vec\delta_3}U_{\vec x+\vec\delta_++\vec\delta_3,\vec x+\vec\delta_+}b_{\vec x+\vec\delta_+})]$$
$$-t'\sum_{i=1,2}[(a^+_{\vec x i}U_{\vec x,\vec x+\vec\delta_i}a^-_{\vec x+\vec\delta_i}-b^+_{\vec x+\vec\delta_++\vec\delta_i}U_{\vec x+\vec\delta_+,\vec\delta_j,\vec x+\vec\delta_+}b^-_{\vec x+\vec\delta_++\vec\delta_i})+a^+_{\vec x}U_{\vec x,\vec x+\vec\delta_i}a^-_{\vec x+\vec\delta_i}-b^+_{\vec x+\vec\delta_+}U_{\vec x+\vec\delta_+,\vec x+\vec\delta_++\vec\delta_i}b^-_{\vec x+\vec\delta_++\vec\delta_i})]\}$$

The paramagnetic lattice current is given by

$$j_\pm(\vec p) = -\frac{\partial H(\vec A)}{\partial A_{\pm,\vec p}}|_0, \quad j_3(\vec p) = -\frac{\partial H(\vec A)}{\partial A_{3,\vec p}}|_0 \qquad (57)$$

if $A_\pm = \frac{A_1\pm A_2}{2}$.

APPENDIX B: SYMMETRY PROPERTIES

In the effective action there are no bilinear terms $\psi^+_\varepsilon\psi^-_{-\varepsilon}$; in the case $m=0$ this follows from conservation of momentum and when $m=1$ this follows from the fact that we assume $\mathbf{p}$ small. Note that the propagator verifies the following symmetry properties, calling $\mathbf{k}^* = (k_0,-k_1,-k_2,k_3)$

$$\widehat g_{1,1}(\mathbf{k}) = \widehat g_{1,1}(\mathbf{k}^*) \quad \widehat g_{2,2}(\mathbf{k}) = \widehat g_{2,2}(\mathbf{k}^*)$$
$$\widehat g_{1,2}(\mathbf{k}) = -\widehat g_{1,2}(\mathbf{k}^*) \quad \widehat g_{2,1}(\mathbf{k}) = -\widehat g_{2,1}(\mathbf{k}^*) \qquad (58)$$

Moreover the kernels of the currents verify

$$w_{a,\pm}(\vec k,0) = w_{a,\pm}(\vec k^*,0) \quad w_{b,\pm}(\vec k,0) = -w_{b,\pm}(\vec k^*,0)$$
$$w_3(\vec k,0) = w_3(\vec k^*,0) \qquad (59)$$

By using the above symmetry properties it is easy to check that

1. The non diagonal terms $\widehat W^{(h)}_{2,0}(\varepsilon\mathbf{p_F})$ are vanishing by (58) as they contain an odd number of non diagonal propagators, by (58).

2. The non diagonal terms contributing to $\partial_0\widehat W^{(h)}_{2,0}(\varepsilon\mathbf{p}_F)$ or $\partial_3\widehat W^{(h)}_{2,0}(\varepsilon\mathbf{p}_F)$ are vanishing as they contain an odd number of non diagonal contributions; similarly diagonal terms contributing to $\partial_1\widehat W^{(h)}_{2,0}(\varepsilon\mathbf{p}_F)$ or $\partial_2\widehat W^{(h)}_{2,0}(\varepsilon\mathbf{p}_F)$.

3. The diagonal contributions to $\widehat W_{\pm;2,1}(\varepsilon\mathbf{p}_F,0)$ are vanishing; indeed the terms containing $w_{b,\pm}$ contains an even number of non diagonal propagators, hence they are vanishing; the terms containing $w_{a,\pm}$ contains instead an odd number of non diagonal propagators.

4. The non diagonal contributions to $\widehat W^{(h)}_{3;2,1}(\varepsilon\mathbf{p}_F,0)$ are vanishing as they contain an odd number of non diagonal propagators.

APPENDIX C: LOWEST ORDER COMPUTATIONS

The explicit value of the coefficients in (37) is

$$a_3\sigma_3 = \int d\mathbf{k}\widehat v(\mathbf{k})\partial_3\widehat g(\mathbf{k})$$
$$a_+\sigma_1 = \int d\mathbf{k}\widehat v(\mathbf{k})\partial_+\widehat g(\mathbf{k}) \qquad (60)$$
$$b_3\sigma_3 = \int d\mathbf{k}\widehat v(\mathbf{k})w_3(\mathbf{k},0)\widehat g(\mathbf{k})\sigma_3\widehat g(\mathbf{k})$$
$$b_+\sigma_1 = \int d\mathbf{k}\widehat v(\mathbf{k})[w_{a,+}(\vec k,0)\widehat g(\mathbf{k})\sigma_1\widehat g(\mathbf{k}) +$$
$$w_{b,+}(\vec k,0)\widehat g(\mathbf{k})\sigma_3\widehat g(\mathbf{k})]$$

In agreement with (38) $a_\perp = b_\perp$, $a_\pm = b_\pm$ as they can be easily checked from the relations, $\mathbf{p}_3 = (0,\vec p\vec\delta_3)$, $\mathbf{p}_\pm = (0,\vec p\vec\delta_\pm)$

$$g^{-1}(\mathbf{k}) - g^{-1}(\mathbf{k}+\mathbf{p}_3) = p_3w_3(\vec k,0)\sigma_3 + O(\mathbf{p}^2)$$
$$g^{-1}(\mathbf{k}) - g^{-1}(\mathbf{k}+\mathbf{p}_\pm) = p_\pm[w_{a,\pm}(\vec k,0)\sigma_1$$
$$+w_{b,\pm}(\vec k,0)\sigma_3] + O(\mathbf{p}^2) \qquad (61)$$

[1] X. Wan, A. M. Turner, A. Vishwanath, and S. Y. Savrasov, Phys. Rev. B 83, 205101 (2011).
[2] S. Murakami, New J. Phys. 9, 356 (2007).
[3] B. Singh, A. Sharma, H. Lin, M. Z. Hasan, R. Prasad, and A. Bansil, Phys. Rev. B 86, 115208 (2012).
[4] A. A. Burkov and L. Balents, Phys. Rev. Lett. 107, 127205 (2011).
[5] A. A. Burkov, M. D. Hook, and L. Balents, Phys. Rev. B 84, 235126 (2011).

[6] A. A. Zyuzin, S. Wu, and A. A. Burkov, Phys. Rev. B 85, 165110 (2012).
[7] G. B. Halasz and L. Balents, Phys. Rev. B 85, 035103 (2012).
[8] G. Y. Cho, arXiv:1110.1939 (unpublished).
[9] C.-X. Liu, P. Ye, and X.-L. Qi, Phys. Rev. B 87, 235306 (2013).
[10] D. Bulmash, C.-X. Liu, and X.-L. Qi, arXiv:1309.6327 (unpublished).
[11] W. Witczak-Krempa and Y. B. Kim, Phys. Rev. B 85, 045124 (2012).
[12] A. Go, W. Witczak-Krempa, G. S. Jeon, K. Park, and Y. B. Kim, Phys. Rev. Lett. 109, 066401 (2012).
[13] G. Chen and M. Hermele, Phys. Rev. B 86, 235129 (2012).
[14] G. Xu, H. Weng, Z. Wang, X. Dai, and Z. Fang, Phys. Rev. Lett. 107, 186806 (2011). 26 H. Zhang, J. Wang, G. Xu, Y. Xu, and S.-C. Zhang, arXiv:1308.0349 (unpublished).
[15] E. V. Gorbar, V. A. Miransky, and I. A. Shovkovy, Phys. Rev. B 88, 165105 (2013).
[16] S. M. Young, S. Zaheer, J. C. Y. Teo, C. L. Kane, E. J. Mele, and A. M. Rappe, Phys. Rev. Lett. 108, 140405 (2012).
[17] Z. Wang, Y. Sun, X.-Q. Chen, C. Franchini, G. Xu, H. Weng, X. Dai, and Z. Fang, Phys. Rev. B 85, 195320 (2012).
[18] Z. Wang, H. Weng, Q. Wu, X. Dai, and Z. Fang, Phys. Rev. B 88, 125427 (2013).
[19] J. A. Steinberg, S. M. Young, S. Zaheer, C. L. Kane, E. J. Mele, and A. W. Rappe, arXiv:1309.5967 (unpublished).
[20] M. Neupane, S.-Y. Xu, R. Sankar, N. Alidoust, G. Bian, C. Liu, I. Belopolski, T.-R. Chang, H.-T. Jeng, H. Lin, A. Bansil, F. Chou, and M. Z. Hasan, arXiv:1309.7892
[21] S. Borisenko, Q. Gibson, D. Evtushinsky, V. Zabolotnyy, B. Buechner, and R. J. Cava, arXiv:1309.7978 (unpublished). 34 Z. K. Liu, B. Zhou, Z. J. Wang, H. M. Weng, D. Prabhakaran, S.-K. Mo, Y. Zhang, Z. X. Shen, Z. Fang, X. Dai, Z. Hussain, and Y. L. Chen, arXiv:1310.0391 (unpublished).
[22] M. Orlita, D. M. Basko, M. S. Zholudev, F. Teppe, W. Knap, V. I. Gavrilenko, N. N. Mikhailov, S. A. Dvoretskii, P. Neugebauer, C. Faugeras, A.-L. Barra, G. Martinez, G. and M. Potemski, arXiv:1310.0969 (unpublished). 36 H.-J. Kim, K.-S. Kim, J. F. Wang, M. Sasaki, N. Satoh, A. Ohnishi, M. Kitaura, M. Yang, and L. Li, arXiv:1307.6990 (unpublished)
[23] J.-H. Chu, S. C. Riggs, M. Shapiro, J. Liu, C. R. Serero, D. Yi, M. Melissa, S. J. Suresha, C. Frontera, A. Vishwanath, X. Marti, I. R. Fisher, and R. Ramesh, arXiv:1309.4750 (unpublished).
[24] H.-J. Kim, K.-S. Kim, J. F. Wang, M. Sasaki, N. Satoh, A. Ohnishi, M. Kitaura, M. Yang, and L. Li, arXiv:1307.6990 (unpublished).
[25] J.-H. Chu, S. C. Riggs, M. Shapiro, J. Liu, C. R. Serero, D. Yi, M. Melissa, S. J. Suresha, C. Frontera, A. Vishwanath, X. Marti, I. R. Fisher, and R. Ramesh, arXiv:1309.4750 (unpublished).
[26] G. Volovik, Lect. Notes. Phys. 718, 31 (2007)
[27] O. Vafek and A. Vishwanath, arXiv:1306.2272 (unpublished).
[28] Delplace, P, Li J, Carepntier D EPL 97 67004(6) (2011)
[29] P. Hosur, S. A. Parameswaran, and A. Vishwanath, Phys. Rev. Lett. 108, 046602 (2012).
[30] Y.-S. Jho and K.-S. Kim, Phys. Rev. B 87, 205133 (2013).
[31] B. Rosenstein and M. Lewkowicz, Phys. Rev. B **88**, 045108 (2013)
[32] J. Maciejko, R. Nandkishore arXiv:1311.7133 (unpublished).
[33] B. Dora, I. F. Herbut, R. Moessner Phys. Rev. B 88, 075126 (2013)
[34] E. G. Mishchenko, Europhys. Lett. 83, 17005 (2008);
[35] I. F. Herbut, V. Juričić and O. Vafek, Phys. Rev. Lett 100, 046403 (2008)
[36] D. E. Sheehy and J. Schmalian, Phys. Rev. B 80, 193411 (2009)
[37] V. Juričić, O. Vafek and I. F. Herbut, Phys. Rev. B 82, 235402 (2010)
[38] I. Sodemann and M. M. Fogler, Phys. Rev. B 86, 115408 (2012)
[39] A.Giuliani, V.Mastropietro, M.Porta Annals of Phys. 327, 461-511 (2012)
[40] A.Giuliani, V.Mastropietro, Phys. Rev. B 85, 045420 (2012)
[41] B. Rosenstein, M. Lewkowicz, T. Maniv, Phys. Rev. Lett. 110, 066602 (2013)
[42] H.Herbut, V.Mastropietro Phys. Rev. B, 87, 205445 (2013)
[43] F. Guinea, M. I. Katsnelson arXiv:1307.6221
[44] A. Giuliani, V.Mastropietro Phys. Rev. B 79, 201403(R) (2009);Comm. Math. Phys. 293, 301-346 (2010)
[45] A. Giuliani, V. Mastropietro and M. Porta, Phys. Rev. B 83, 195401 (2011); Comm. Math. Phys. 311, 317 (2012).
[46] D.Haldane Phys. Rev Lett. 61, 182015 (1988)
[47] P. Hosur, S.A. Parameswaran, and A. Vishwanath, Phys. Rev. Lett. 108, 046602 (2012).
[48] T. Stauber, N. Peres and A. Geim, Phys. Rev. B 78, 085432 (2008).
[49] V.Mastropietro, J. Math. Phys. 48, 102023 (2006)