HEIGHT FLUCTUATIONS IN INTERACTING DIMERS

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Abstract. We consider a non-integrable model for interacting dimers on the two-dimensional square lattice. Configurations are perfect matchings of $\mathbb{Z}^2$, i.e. subsets of edges such that each vertex is covered exactly once ("close-packing" condition). Dimer configurations are in bijection with discrete height functions, defined on faces $\boldsymbol{\xi}$ of $\mathbb{Z}^2$. The non-interacting model is "integrable" and solvable via Kasteleyn theory; it is known that all the moments of the height difference $h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}$ converge to those of the Gaussian Free Field, asymptotically as $|\boldsymbol{\xi} - \boldsymbol{\eta}| \to \infty$. We prove that the same holds for small non-zero interactions, as was conjectured in the theoretical physics literature. Remarkably, dimer-dimer correlation functions are instead not universal and decay with a critical exponent that depends on the interaction strength. Our proof is based on an exact representation of the model in terms of lattice interacting fermions, which are studied by constructive field theory methods. In the lattice language, the height difference $h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}$ takes the form of a non-local fermionic operator, consisting of a sum of fermionic monomials along an *arbitrary* path connecting $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. As in the non-interacting case, this path-independence plays a crucial role in the proof.

1. Introduction and main results

Two-dimensional dimer models were studied extensively in the 1960s for their equivalence with various statistical physics models such as the Ising model. At close packing, dimer models are critical (correlations decay polynomially with distance) and, as was later discovered, enjoy conformal invariance properties [42]. Their early study culminated in the *exact solution* of non-interacting dimers by Kasteleyn, Temperley and Fisher [26, 40, 61] and the related computation of the correlations [27]. However, even in the presence of a solution, a number of properties used in the physical literature were left for decades without a mathematical justification. In particular, the height field (see Section 1.1) was believed to be effectively described in terms of a continuum Gaussian field theory. The difficulty in substantiating mathematically such belief is due to the *ultraviolet divergences* that arise in the continuum limit. They produce ambiguities in the final formulas for the moments of the height function, which require ad hoc regularizations, see e.g. [4, 20, 62] for an analogous discussion in the context of the critical Ising model. It is fair to say that not only a mathematical proof, but even a solid, convincing, non-rigorous argument, proving the correctness of the scaling limit for the height function, was missing until very recent. The progress came from the mathematical community: in the last 15 years, radically new

ideas and methods have been introduced [41, 42, 43, 44, 45], which provided a firm basis for the continuum field picture in the *non-interacting* dimer model. These works take advantage of the underlying discrete holomorphicity properties of the model, which arise from its integrability, and can be used to prove the emergence of conformal symmetry in the scaling limit [42, 43]. Similar ideas also appeared and developed in the context of percolation and of the Ising model [59, 60]. However, these methods fail as soon as integrability is lost, and the very natural question of whether the Gaussian Free Field description survives for the interacting case requires radically new ideas. It was proposed in [23] to apply the methods of constructive Renormalization Group (RG) theory to interacting dimers, and in this way the large-distance asymptotics of the dimer-dimer correlations were derived, as well as certain universality relations between critical exponents. In this paper we extend the approach of [23] to the computation of all the moments of the height function, and we succeed in proving their convergence to those of the massless Gaussian Free Field. The control of the height fluctuations, as compared to that of the dimer correlations, poses new non trivial problems, due to the non-local nature of the height function, as opposed to the local nature of single-dimer observables.

Constructive RG methods have proven, along the decades, an invaluable tool to control rigorously (in some cases) non-integrable critical models and their universality properties, see references below. On the other hand, these methods seem to be very little known in the probability/combinatorics/discrete complex analysis communities, despite the fact that they are interested in very similar mathematical questions for the Ising model, percolation, etc. One of the aims of the present work is to make these methods accessible to a wider audience. For this reason, we make an effort to present the main ideas and steps in a pedagogical way (within reasonable limits: for the technical details of some constructive RG estimates we refer the reader to the relevant literature), which (partly) explains the length of the article.

1.1. **The model.** To be definite, we study the model of interacting classical dimers proposed in [2] and [55]. The original motivation of [2, 55] was to study the classical limit of quantum dimers [56] (see [51] for a review): in the classical limit, the interaction is short-ranged and tends to align neighboring dimers. Originally motivated by the physics of high T_c superconductors [3], quantum dimer models have been used in theoretical physics to describe new and unusual forms of collective behavior in a broader variety of settings, ranging from Resonant Valence Bond theory [52] to gauge theories [19]. The understanding of interacting classical dimers appears to be a necessary prerequisite for a full comprehension of the quantum model.

We consider a periodic box Λ of side L (with L even), whose sites are labelled as follows: $\Lambda = \{\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2 : \ x_i = -L/2 + 1, \cdots, L/2\}$. "Periodic", as usual, means that if $\hat{e}_i$ are the two unit coordinate vectors, then $(L/2, x_2)+\hat{e}_1$ should be identified with $(-L/2+1, x_2)$, and $(x_1, L/2)+\hat{e}_2$ with $(x_1, -L/2+1)$. The partition function of interest is

$$Z_\Lambda(\lambda, m) = \sum_{M\in\mathcal{M}_\Lambda} \Big[\prod_{b\in M} t_b^{(m)}\Big] e^{\lambda W_\Lambda(M)} \equiv \sum_{M\in\mathcal{M}_\Lambda} \mu_{\Lambda;\lambda,m}(M) : \qquad (1.1)$$

- $\mathcal{M}_\Lambda$ is the set of dimer coverings (or perfect matchings) of Λ. We recall that a dimer covering is a subset of edges such that each vertex of Λ is contained in exactly one edge in M. We choose L even, otherwise $\mathcal{M}_\Lambda$ would be empty.
- $m > 0$ is the amplitude of a periodic modulation of the horizontal bond weights, playing the role of an "infrared regularization" (see later), to be eventually removed after performing the thermodynamic limit, by sending $m \to 0$. The modulation is defined as follows: $t^{(m)}_{(\mathbf{x},\mathbf{x}+\hat{e}_j)} = 1 + \delta_{j,1} m(-1)^{x_1}$. Note that $\lim_{m\to 0} t_b^{(m)} = 1$.
- $W_\Lambda(M) = \sum_{P\subset\Lambda} N_P(M)$, where P is a plaquette (face of $\mathbb{Z}^2$) and $N_P(M) = 1$ if the plaquette P is occupied by two parallel dimers in M, and $N_P(M) = 0$ otherwise.

If one sets $\lambda = m = 0$, one recovers the usual integrable, translation invariant, dimer model studied e.g. in [40, 42, 44].

Since Λ is bipartite we can paint white and black the sites of the two sublattices; with no loss of generality we can assume that the coordinates of the white sites are either (even, even) or (odd, odd). The expectation w.r.t. the measure corresponding to the partition function $Z_\Lambda(\lambda, m)$ will be denoted $\langle\cdot\rangle_{\Lambda;\lambda,m}$: if $O(M)$ is a function of the dimer configuration, we define

$$\langle O\rangle_{\Lambda;\lambda,m} = \frac{1}{Z_\Lambda(\lambda,m)} \sum_M \mu_{\Lambda;\lambda,m}(M) O(M). \tag{1.2}$$

Truncated expectations are denoted by a semicolon: e.g., $\langle O; O'\rangle_{\Lambda;\lambda,m} := \langle OO'\rangle_{\Lambda;\lambda,m} - \langle O\rangle_{\Lambda;\lambda,m}\langle O'\rangle_{\Lambda;\lambda,m}$. The massless infinite volume measure is defined via the following weak limit:

$$\langle\cdot\rangle_\lambda := \lim_{m\to 0} \lim_{\Lambda\nearrow\mathbb{Z}^2} \langle\cdot\rangle_{\Lambda;\lambda,m}\ . \tag{1.3}$$

The name "massless" refers to the fact that $\langle\cdot\rangle_\lambda$ exhibits algebraic decay of correlations, irrespective of the value of λ. For example, if $\mathbb{1}_b(M)$ denotes the dimer occupancy, i.e., the observable that is equal to 1 if b is occupied by a dimer in M, and 0 otherwise, then, letting $b = (\mathbf{x}, \mathbf{x}+\hat{e}_1)$ and $b' = (\mathbf{y}, \mathbf{y}+\hat{e}_1)$:

$$\begin{aligned}\langle \mathbb{1}_b; \mathbb{1}_{b'}\rangle_\lambda \;&=\; (-1)^{\mathbf{x}-\mathbf{y}} c \frac{(x_1-y_1)^2 - (x_2-y_2)^2}{|\mathbf{x}-\mathbf{y}|^4} + \\ &\quad + (-1)^{x_1-y_1} c_- \frac{1}{|\mathbf{x}-\mathbf{y}|^{2\kappa_-}} + R(\mathbf{x}-\mathbf{y})\ ,\end{aligned} \tag{1.4}$$

where, assuming λ sufficiently small, c, c_- and κ_- are analytic functions of λ such that $c|_{\lambda=0} = c_-|_{\lambda=0} = 1/(2\pi^2)$ and $\kappa_-|_{\lambda=0} = 1$, while $R(\mathbf{x})$ is subdominant at large distances, that is $|R(\mathbf{x})| \le C_\theta(|\mathbf{x}|+1)^{-3+\theta}$, $\forall\theta \in (0,1)$. This formula appears in [23] together with a sketchy derivation, see Section 6.2 below for a detailed proof. If, instead of sending $m \to 0$ in (1.3), we keep $m > 0$ fixed in the thermodynamic limit, then the truncated correlations decay exponentially to zero at large distances, with rate proportional to m itself. In this sense, m plays the role of a mass (infrared regularization).

Given a dimer covering M and two faces of Λ centered at $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, one defines the *height* difference between $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ as

$$h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}} = \sum_{b \in \mathcal{C}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}} \big(\mathbb{1}_b(M) - \langle \mathbb{1}_b \rangle_{\Lambda;\lambda,m}\big)\sigma_b \tag{1.5}$$

where $C_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ is a nearest-neighbor path on the dual lattice of Λ (i.e. a path on faces of Λ), the sum runs over the edges crossed by the path and $\sigma_b = +1/-1$ depending on whether the oriented path $C_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ from $\boldsymbol{\xi}$ to $\boldsymbol{\eta}$ crosses b with the white site on the right/left. See figure 1.

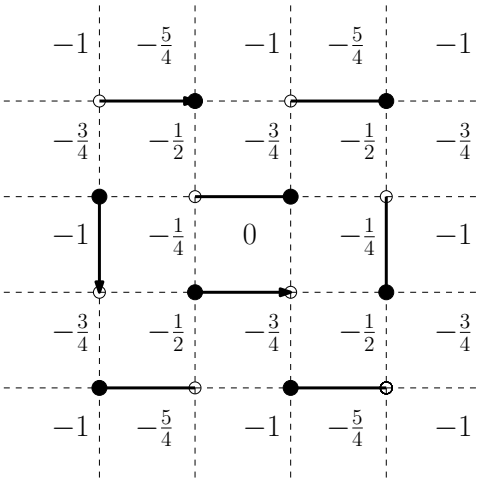

FIGURE 1. A dimer configuration for $L = 4$ and the height function computed according to (1.5). In this picture we assume that $\langle \mathbb{1}_b \rangle = 1/4$ for every b, which is the case on the torus for $m = 0$. Moreover, we conventionally set the value of the height in the central plaquette equal to 0.

We have centered the height function to have gradients with zero average; remark that, for $m = 0$, $\langle \mathbb{1}_b \rangle_{\Lambda;\lambda,m} = 1/4$ by symmetry. A priori, the definition (1.5) depends on the choice of the path. The remarkable fact is that it is actually independent of it, provided the path "does not wind around the torus": more precisely, the right side of (1.5) computed along two different paths is the same, provided the loop obtained by taking the union of the two paths does not wind around the torus[1] [44]. We shall say that two such paths are equivalent. In particular, if $||\boldsymbol{\xi} - \boldsymbol{\eta}||_\infty < L/2$, then all the shortest lattice paths are equivalent, and we uniquely define the height difference between $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ as the right side of (1.5), computed along any path equivalent to one of the shortest lattice paths. In this way, given two faces with fixed (i.e., L-independent) coordinates $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, their height difference is

[1] In general, if a path wraps n_1 times horizontally and n_2 times vertically over the torus, the right side of (1.5) picks up an additive term $n_1 T_1(M) + n_2 T_2(M)$, for suitable constants, called *periods*. In this sense, the height on the torus is additively multi-valued. The example in Fig.1 is special, in that $T_1(M) = T_2(M) = 0$ for the configuration M depicted there; however, it is easy to exhibit other configurations for which these periods are non zero.

uniquely defined, for sufficiently large L. If we arbitrarily assign height zero to the "central" plaquette (the one centered at $(1/2, 1/2)$), then the height profile is uniquely determined everywhere, asymptotically as $L \to \infty$. In conclusion, each plaquette is associated with a value of the height function, and one can view each plaquette as the basis of a block which extends out of the page by an amount given by the height function. From this perspective, dimer covering may be viewed as a two-dimensional representation of the surface of a three-dimensional crystal.

Let us mention that the bijection between discrete interfaces and perfect matchings of planar bipartite graphs is a general fact: see for instance Figure 2 for the (visually more obvious) case of the honeycomb lattice.

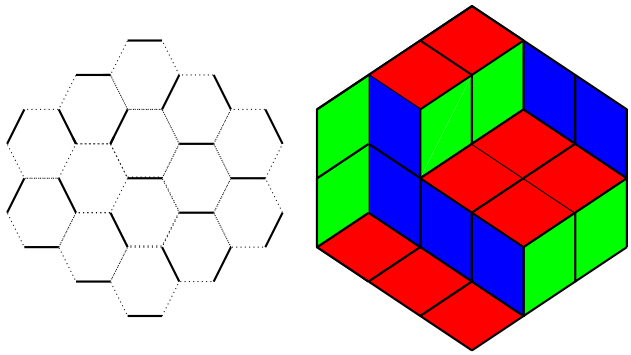

FIGURE 2. A dimer covering of a domain of the honeycomb lattice and the corresponding discrete height function. The correspondence is established by drawing a segment along the main diagonal of each lozenge in the figure on the right: these segments are the same as the dimers in the figure on the left, and they make apparent the fact that dimer configurations are in one-to-one correspondence with lozenge tilings of planar domains. The vertices of the lozenges correspond to the centers of the hexagonal cells in the figure on the left.

1.2. **Correlations and expected behavior.** Among the physically interesting correlations are the *dimer correlations* $\langle \mathbb{1}_{b_1}; \cdots ; \mathbb{1}_{b_n} \rangle_\lambda$, already mentioned above in the $n = 2$ case, the *height fluctuations* $\langle (h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}})^n \rangle_\lambda$ and the so-called *electric correlator*

$$\langle e^{i\alpha(h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}})} \rangle_\lambda. \tag{1.6}$$

For $\lambda = 0$ (non-interacting dimers) the partition function was exactly computed in [26, 40, 61], where it was shown that it can be expressed in terms of the Pfaffian of the Kasteleyn matrix (see below); such a Pfaffian can be rewritten exactly as a *gaussian* Grassmann integral, so that the case $\lambda = 0$ is also called free fermion point (see e.g. [34, 54] for a definition and an illustration of the basic properties of Grassmann integrals). The *dimer* correlations are easily computable from their Grassmann representation (the dimer occupancy $\mathbb{1}_b$ becomes a *local* quadratic monomial in the language of Grassmann variables), by using the fermionic Wick theorem, see for instance [25]: one finds that if $m = 0$, the dimer correlations decay as a power law

modulated by an oscillating factor; in particular, the decay of the two-point dimer correlation is proportional to the inverse distance squared, as it can be seen from (1.4) at $\lambda = 0$.

The computation of the height or electric correlations is a completely different matter: the height and electric observables take the form of *non local* expressions in the Grassmann variables (as can be guessed from (1.5)) and their computation is much harder. Indeed, the proof of Gaussian Free Field-like behavior for the height function (for $\lambda = 0$) is very recent [43, 45]. Also, closed expressions for the electric correlator are only known in certain special directions (along a row or column, or along a main diagonal). Indeed the dimer model with $\lambda = 0$ strongly resembles the two-dimensional Ising model at the critical temperature, which admits a similar fermionic representation in terms of gaussian Grassmann integrals [30, 57]. The dimer correlations are the analogues of the Ising energy density correlations (i.e. the correlation between $\sigma_x\sigma_{x'}$ and $\sigma_y\sigma_{y'}$, if (x, x') and (y, y') are two lattice bonds) and the electric correlator at $\alpha = \pi$ is the analogue of the (square of the) spin-spin correlation at criticality. The analogy is not just formal, but also quantitative: it was recently shown in [22] that there is an exact identity, valid at the lattice level and at finite volume, between the energy correlations of the critical Ising model and the dimer correlations, as well as between the (square of the) two-point spin correlation of critical Ising and the electric correlator at $\alpha = \pi$. These identities play the role of *lattice bosonization* identities, see [22], and imply in particular that the critical exponents of the corresponding Ising and dimer observables are the same.

If $\lambda \neq 0$ the model is not solvable anymore. The Grassmann representation, reviewed below, shows that the interacting model can be expressed exactly in terms of a non-gaussian Grassmann integral. That is, the interacting dimer model is equivalent to a model of interacting lattice fermions in two dimensions [23]. The critical exponents of the dimer observables are in general expected to change, as apparent from (1.4) where a non-trivial critical exponent $2\kappa_-$ appears. Nevertheless, a heuristic mapping of the theory into a sine-gordon model [2] predicts that the height function, at least for small λ, still behaves in the continuum limit as a massless Gaussian free field. In order to make this conjecture precise and to keep the discussion simple, let us pretend for a moment that the model is defined not on the torus but on the square $\{-L/2+1, \dots, L/2\}^2$ with open boundary conditions (i.e. there are no bonds connecting $(x_1, L/2)$ with $(x_1, -L/2+1)$ or $(L/2, x_2)$ with $(-L/2+1, x_2)$; correspondingly, the height difference is defined uniquely, rather than being additively multi-valued). If space is rescaled by $1/L$ so that the box Λ becomes the unit square D, then it is expected that

$$h_u - h_v \sim \sqrt{\frac{K}{\pi}}(\varphi_u - \varphi_v),\ u \in D \tag{1.7}$$

where (formally) φ is the field in D with quadratic Hamiltonian $\frac{1}{2}\int_D du|\partial\varphi|^2$, while $K = K(\lambda)$ is an analytic function of λ such that $K(0) = 1$. Mathematically, φ is known as the massless Gaussian Free Field: it is the unique centered Gaussian process in D whose covariance is the Green's function of the Laplacian with Dirichlet boundary conditions on D [58]. In particular, since the Green function behaves like $-1/(2\pi)\log|u-v|$ at short distances,

for the original un-rescaled lattice this means that one conjectures

$$\langle (h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}})^2 \rangle_\lambda \sim (K/\pi^2) \log |\boldsymbol{\xi} - \boldsymbol{\eta}| \tag{1.8}$$

for $|\boldsymbol{\xi} - \boldsymbol{\eta}|$ large.

This remarkable correspondence, if proven, makes the computation of the height correlations straightforward. More in general, from (1.7) one can (heuristically at least) compute the exponent κ_- of the two-point dimer correlation in terms of the amplitude K [39, 55], see (1.11)-(1.12) below. In addition, the correspondence (1.7) is the basis of the construction of the quantum dimer model, see e.g. [51].

As already noticed, the identification (1.7) in the scaling limit has been rigorously proved in the *non interacting* case only [42, 43]. In the presence of interactions, (1.7) was until now a phenomenological assumption, not derived from the microscopic Hamiltonian but confirmed by numerical simulations, see [2] and [55]. From simulations, K appears to be a non trivial function of λ, which suggest that the model should be in the same universality class of the Ashkin-Teller model, see [2, 55].

Our Theorem 1 is the first rigorous confirmation of (1.7) in the interacting case $\lambda \neq 0$. Let us mention that, in the same spirit, convergence of Ginzburg-Landau type $\nabla\phi$ interface models to a Gaussian Free Field was obtained for instance in [35, 50, 53].

1.3. **Results and perspectives.** In the last years methods based on constructive Renormalization Group (RG) have been applied to various classical and quantum statistical mechanics models, starting from [47]. In contrast with field theoretic RG, they can be applied in the presence of a lattice, they allow for a mathematically rigorous control of the effects of momentum cutoffs, of the irrelevant terms, and of the convergence of perturbation theory. These methods have already been successfully applied to the computation of the critical exponents associated with several different observables that, once re-expressed in the language of Grassmann variables, are local or quasi-local operators: examples include the energy and crossover observables in the eight vertex [7, 47] and anisotropic Ashkin-Teller models [37], energy density correlations in non-integrable Ising models [38], the correlations of S^z (the z-component of the spin) in the XXZ model [12] and, more recently, the already mentioned dimer correlation of the interacting dimer model [23].

In this paper, we combine this approach with the methods used in [45] to compute the height variance in the integrable dimer model, thus applying for the first time constructive RG methods to the study of a *non-local* observable such as the height. Our main result is the following:

Theorem 1. *There exist:*

1. *a positive constant λ_0 and an analytic function $K(\lambda)$ on $|\lambda| < \lambda_0$ satisfying $K(0) = 1$,*
2. *positive constants C_n, with $n \geq 2$, and a bounded function $R(\boldsymbol{\xi})$ satisfying $|R(\boldsymbol{\xi})| \leq C_2$, $\forall \boldsymbol{\xi} \neq \mathbf{0}$,*

such that the following is true: if $\boldsymbol{\xi} \neq \boldsymbol{\eta}$, then

$$\langle (h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}})^2 \rangle_\lambda = \frac{K(\lambda)}{\pi^2} \log |\boldsymbol{\xi} - \boldsymbol{\eta}| + R(\boldsymbol{\xi} - \boldsymbol{\eta}) \, . \tag{1.9}$$

Moreover, if $n > 2$, the n-th cumulant of $(h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}})$ is bounded uniformly in $|\boldsymbol{\xi} - \boldsymbol{\eta}|$ as

$$|\langle \underbrace{h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}; \cdots ; h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}}_{n \text{ times}} \rangle_\lambda| \le C_n \, . \tag{1.10}$$

In the non-interacting case $\lambda = m = 0$, the result is a refinement of previously known estimates: in fact, in that case (1.9) is proven in [41, 45] (in a much more general setting of bipartite planar graphs), see also [46] for the height moments of order $n \ge 2$. Neither in [45] nor in [46] there is a sharp control of the error terms: for instance, for the variance the error term in[45] is $o(\log |\boldsymbol{\xi} - \boldsymbol{\eta}|)$ instead of $O(1)$.

Let us also mention that the logarithmic growth of the height variance (without sharp control of the constant in front of the log) for some discrete $(2+1)$-dimensional interface models (Solid-on-Solid and discrete Gaussian model) was obtained in [29]. Moreover, an asymptotic computation of the height variance in the six vertex model was recently presented in [24].

Remark 1. *Theorem 1 can be straightforwardly extended to the case where the nearest neighbor interaction is replaced by a finite range interaction that respects the symmetries of the lattice. Another possible generalization (in the spirit of* [45]*) that we did not work out in detail but we believe would not entail new conceptual difficulties, is to work on different planar bipartite lattices, like the honeycomb lattice.*

The proof of Theorem 1 is constructive and, therefore, we have explicit estimates on the convergence radius λ_0, as well as on the constants C_n. However, since we do not expect them to be optimal, we do not spell them explicitly here (e.g., our estimates on C_n grow proportionally to $(n!)^\beta$ with n, for some $\beta > 1$). The proof is based on precise asymptotics on multipoint dimer correlations, which requires the identification of a number of remarkable cancellations in the (renormalized, convergent) expansion for the correlations, which follow from hidden Ward Identities [12] (i.e., asymptotic identities among correlation functions). The name "hidden" refers to the fact that these identities are not exact in the model at hand, while they are so in a relativistic reference model, which displays the same large-distance behavior as the interacting dimer model but on the other hand has more symmetries.

Note that in the above theorem no continuum limit is performed. Therefore, the $n \ge 3$ cumulants are not exactly vanishing, but are finite, while the 2-point function is log-divergent as $|\boldsymbol{\xi} - \boldsymbol{\eta}| \to \infty$.

Let us also remark that our result is not just a corollary of the dimensional estimates on the dimer correlations, which can be inferred from (the methods of) [23]. In fact, a naive substitution of these estimates into the expression of the n-th cumulant of $(h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}})$ obtained by plugging (1.5) into the left sides of (1.9)-(1.10) leads to very poor bounds, growing faster than a logarithm to the power $n/2$ at large distances. A key fact that we need to implement is the path-independence of the right side of (1.5), which is a (weak) instance of the underlying discrete holomorphicity of the model, and relies crucially on the presence of the oscillatory factor σ_b: therefore, these

oscillatory factors produce remarkable cancellations in the perturbation series, which we keep track of within our constructive multiscale computation of the height correlations.

As mentioned above, the amplitude $K(\lambda)$ is expected to be related to the critical exponent of the electric correlator, in the sense that our theorem suggests that, at least for α small,

$$\langle e^{i\alpha(h_{\boldsymbol{\xi}}-h_{\boldsymbol{\eta}})}\rangle_\lambda \sim |\boldsymbol{\xi}-\boldsymbol{\eta}|^{-K\alpha^2/(2\pi^2)}\,, \tag{1.11}$$

asymptotically at large distances, since formally the electric correlator is equal to

$$\langle e^{i\alpha(h_{\boldsymbol{\xi}}-h_{\boldsymbol{\eta}})}\rangle_\lambda = \exp\Big\{\sum_{n\ge 2}\frac{(i\alpha)^n}{n!}\langle \underbrace{h_{\boldsymbol{\xi}}-h_{\boldsymbol{\eta}};\cdots;h_{\boldsymbol{\xi}}-h_{\boldsymbol{\eta}}}_{n\ times}\rangle_\lambda\Big\}\,. \tag{1.12}$$

Of course, such an identity makes sense provided one can control the convergence of this infinite sum, which we cannot, due to our poor bounds on C_n. It is an important open problem to rigorously compute the electric correlator at $\lambda \neq 0$, even just for α small. We hope to come back to this issue in a future publication, possibly by combining the methods of constructive RG with the (strong) discrete holomorphicity used at $\lambda = 0$ in [21] to compute the electric correlator.

Finally let us mention that, for $\lambda = 0$, height correlations in finite domains exhibit conformal covariance properties in the scaling limit where the lattice spacing tends to zero; this was proven for instance by Kenyon [42, 43] for some suitably chosen boundary conditions. It would be extremely interesting to prove that conformal invariance survives for $\lambda \neq 0$, where integrability is lost. While we believe that constructive RG is again the right approach to attack this problem, new difficulties will need to be overcome with respect to the present work, notably due to the loss of translation invariance arising from non-periodic boundary conditions.

1.4. **Organization of the paper.** In Section 2 we derive the Grassmann representation for the partition function and for the multipoint dimer correlations. In particular, we show that the model can be expressed in the form of a non-gaussian Grassmann integral, the non-gaussian part of the action being proportional to the interaction strength λ in (1.1). In Section 3 we discuss the long-distance properties of the covariance (propagator) of the reference gaussian Grassmann integral, to which the model reduces at $\lambda = 0$. We introduce convenient coordinates (i.e., Grassmann variables), whose covariance in the massless limit have a well-defined, non-oscillating, power law decay at large distance: they are suitable linear combinations of the original Grassmann variables and, due to the formal analogy of their action with that of Majorana and Dirac fermions, are called Majorana and Dirac fields. In Section 4 we discuss the Grassmann representation of the height function, including its perturbative expansion in λ. We review some of its properties at $\lambda = 0$ and explain in the Grassmann language how to asymptotically compute and estimate all the moments of the height fluctuations, thus proving our main theorem in the case $\lambda = 0$. Section 5 is devoted to the construction of the theory at $\lambda \neq 0$. We first explain why naive perturbation theory fails to provide a convergent expansion for the pressure

and the correlations. Next, we review the constructive RG computation of non-gaussian Grassmann integrals. Finally, in Section 6 we explain how to compute the asymptotic behavior of multipoint dimer correlations, as well as of the height functions, and we prove Theorem 1.

2. Grassmann representation

In this section we explain how to derive a representation of the interacting partition function $Z_\Lambda(\lambda, m)$ and dimer correlation functions in terms of non-gaussian Grassmann integrals. This representation is exact and valid as an algebraic identity for every finite lattice Λ. For the reader who is not used to Grassmann variables, we refer for instance to [34, Section 4] for some of their basic properties. The key points to keep in mind are the following: Grassmann variables anti-commute, in particular $\psi^2_{\mathbf{x}} = 0$. Gaussian Grassmann integrals are just an alternative way of writing determinants (or Pfaffians); non-gaussian Grassmann integrals are just an alternative, compact, way of writing certain series of determinants (or of Pfaffians); the rewriting of $Z_\Lambda(\lambda, m)$ in terms of a non-gaussian Grassmann integral is very convenient for its subsequent computation via the methods of constructive field theory, which makes the analogy with the rigorous multiscale analysis of perturbed gaussian measures as apparent as possible.

2.1. Partition function.

We rewrite the partition function (1.1) as

$$\begin{aligned} Z_\Lambda(\lambda, m) &= \sum_{M\in\mathcal{M}_\Lambda} \Big[\prod_{b\in M} t_b^{(m)}\Big] \prod_{P\subset\Lambda} (1+\alpha N_P(M)) \qquad (2.1)\\ &= \sum_{M\in\mathcal{M}_\Lambda} \Big[\prod_{b\in M} t_b^{(m)}\Big] \prod_{\langle b,b'\rangle\subset\Lambda} (1+\alpha \mathbb{1}_b(M)\mathbb{1}_{b'}(M)) \end{aligned}$$

where $\alpha = e^\lambda - 1$ and the product $\prod_{\langle b,b'\rangle\subset\Lambda}$ runs over pairs of neighboring parallel bonds b, b' (i.e., such that the union of the four vertices of b and b' are the four vertices of a plaquette in Λ). In the second identity we used the fact that, if P is the plaquette with sites $\mathbf{x}, \mathbf{x}+\hat e_1, \mathbf{x}+\hat e_2, \mathbf{x}+\hat e_1+\hat e_2$, then

$$N_P = \mathbb{1}_{(\mathbf{x},\mathbf{x}+\hat e_1)}\mathbb{1}_{(\mathbf{x}+\hat e_2,\mathbf{x}+\hat e_1+\hat e_2)} + \mathbb{1}_{(\mathbf{x},\mathbf{x}+\hat e_2)}\mathbb{1}_{(\mathbf{x}+\hat e_1,\mathbf{x}+\hat e_1+\hat e_2)}$$

and

$$\mathbb{1}_{(\mathbf{x},\mathbf{x}+\hat e_1)}\mathbb{1}_{(\mathbf{x}+\hat e_2,\mathbf{x}+\hat e_1+\hat e_2)}\mathbb{1}_{(\mathbf{x},\mathbf{x}+\hat e_2)}\mathbb{1}_{(\mathbf{x}+\hat e_1,\mathbf{x}+\hat e_1+\hat e_2)} = 0$$

as an observable over dimer configurations; therefore,

$$1+\alpha N_P = (1+\alpha\mathbb{1}_{(\mathbf{x},\mathbf{x}+\hat e_1)}\mathbb{1}_{(\mathbf{x}+\hat e_2,\mathbf{x}+\hat e_1+\hat e_2)})(1+\alpha\mathbb{1}_{(\mathbf{x},\mathbf{x}+\hat e_2)}\mathbb{1}_{(\mathbf{x}+\hat e_1,\mathbf{x}+\hat e_1+\hat e_2)})\,. \qquad (2.2)$$

By expanding the product in the r.h.s. of (2.1), we can rewrite $Z_\Lambda(\lambda, m)$ as a sum of terms, each of which is proportional to

$$\sum_{M\in\mathcal{M}_\Lambda} \Big[\prod_{b\in M} t_b^{(m)}\Big] \mathbb{1}_{b_1}(M)\cdots\mathbb{1}_{b_k}(M)$$

for a suitable k-ple of bonds $(b_1, \dots, b_k)$, $b_i \in \Lambda$. The key fact is that each of these sums can be computed by using Kasteleyn's solution. This can be seen as follows. Kasteleyn's solution [26, 40, 61] gives an explicit formula for the dimer partition function with bond-dependent activities $\mathbf{t} = \{t_b\}_{b\subset\Lambda}$.

Introduce the Kasteleyn matrix $K_{\mathbf{t}}$, which is a $|\Lambda| \times |\Lambda|$ antisymmetric matrix indexed by vertices in Λ, such that its elements $(K_{\mathbf{t}})_{\mathbf{x},\mathbf{y}}$ are non-zero if and only if $\mathbf{x}$ and $\mathbf{y}$ are nearest neighbors; in this case $(K_{\mathbf{t}})_{\mathbf{x},\mathbf{x}+\hat{e}_1} = -(K_{\mathbf{t}})_{\mathbf{x}+\hat{e}_1,\mathbf{x}} = t_{(\mathbf{x},\mathbf{x}+\hat{e}_1)}$, and $(K_{\mathbf{t}})_{\mathbf{x},\mathbf{x}+\hat{e}_2} = -(K_{\mathbf{t}})_{\mathbf{x}+\hat{e}_2,\mathbf{x}} = it_{(\mathbf{x},\mathbf{x}+\hat{e}_2)}$. Also, for $\theta, \tau \in \{0,1\}$ let $K_{\mathbf{t}}^{(\theta\tau)}$ be the antisymmetric matrix obtained from $K_{\mathbf{t}}$ by multiplying the matrix elements $(K_{\mathbf{t}})_{\mathbf{x},\mathbf{x}+\hat{e}_1} = -(K_{\mathbf{t}})_{\mathbf{x}+\hat{e}_1,\mathbf{x}}$ by $(-1)^\theta$ if $\mathbf{x}$ belongs to the rightmost column of Λ and $(K_{\mathbf{t}})_{\mathbf{x},\mathbf{x}+\hat{e}_2} = -(K_{\mathbf{t}})_{\mathbf{x}+\hat{e}_2,\mathbf{x}}$ by $(-1)^\tau$ if $\mathbf{x}$ is in the top row of Λ. Of course, $K_{\mathbf{t}}^{(00)} = K_{\mathbf{t}}$. Then one has (cf. [44] and [45, Sect. 3.1.2])

$$Z_\Lambda(\mathbf{t}) := \sum_{M \in \mathcal{M}_\Lambda} \prod_{b \in M} t_b = \frac{1}{2}(-\mathrm{Pf}K_{\mathbf{t}}^{(00)} + \mathrm{Pf}K_{\mathbf{t}}^{(01)} + \mathrm{Pf}K_{\mathbf{t}}^{(10)} + \mathrm{Pf}K_{\mathbf{t}}^{(11)}) \tag{2.3}$$

$$:= \frac{1}{2} \sum_{\theta,\tau=0,1} C_{\theta,\tau}\, \mathrm{Pf}K_{\mathbf{t}}^{(\theta\tau)}.$$

Here, $\mathrm{Pf}(A)$ indicates the Pfaffian of A. [We recall that the Pfaffian of a $2n \times 2n$ antisymmetric matrix A is defined as

$$\mathrm{Pf}A := \frac{1}{2^n n!} \sum_\pi (-1)^\pi A_{\pi(1),\pi(2)} \cdots A_{\pi(2n-1),\pi(2n)}; \tag{2.4}$$

π is a permutation of $(1,\dots,2n)$, $(-1)^\pi$ is its signature. One of the properties of the Pfaffian is that $(\mathrm{Pf}A)^2 = \det A$.] Since the ordering of the labels $(1,\dots,2n)$ matters in the definition of Pfaffian (changing the ordering, the sign of the Pfaffian can change), in (2.3) we use the convention that the sites $\mathbf{x} \in \Lambda$ that label the elements of $K_{\mathbf{t}}$ are ordered from left to right on every row, starting from the bottom and going upwards to the top row. Using (2.3) we immediately obtain:

$$\sum_{M \in \mathcal{M}_\Lambda} \Big[\prod_{b \in M} t_b^{(m)} \Big] \mathbb{1}_{b_1} \cdots \mathbb{1}_{b_k} = \frac{1}{2} \sum_{\theta,\tau=0,1} C_{\theta,\tau}\; t_{b_1} \partial_{t_{b_1}} \cdots t_{b_k} \partial_{t_{b_k}} \mathrm{Pf}K_{\mathbf{t}}^{(\theta\tau)} \Big|_{\mathbf{t}=\mathbf{t}^{(\mathbf{m})}} \tag{2.5}$$

where $\mathbf{t} = \mathbf{t}^{(\mathbf{m})}$ means $t_b = t_b^{(m)}$ for every b. The right side of (2.5) is itself a sum over Pfaffians, and can be conveniently represented in terms of gaussian Grassmann integrals. In fact, given any $2n \times 2n$ antisymmetric matrix A,

$$\mathrm{Pf}A = \int \big[\prod_{i=1}^{2n} d\psi_i \big]\, e^{-\frac{1}{2}(\psi, A\psi)}\,, \tag{2.6}$$

where the Grassmann integration is normalized in such a way that

$$\int \big[\prod_{i=1}^{2n} d\psi_i \big]\, \psi_{2n} \cdots \psi_1 = 1.$$

For later purposes, it is also useful to recall that the averages of Grassmann monomials with respect to the Grassmann gaussian integration can be computed in terms of the fermionic Wick rule:

$$\langle \psi_{k_1} \cdots \psi_{k_m} \rangle_A := \frac{1}{\mathrm{Pf}A} \int \big[\prod_{i=1}^{2n} d\psi_i \big]\, \psi_{k_1} \cdots \psi_{k_m} e^{-\frac{1}{2}(\psi, A\psi)} = \mathrm{Pf}G\,, \tag{2.7}$$

where, if m is even, G is the $m \times m$ matrix with entries

$$G_{ij} = \langle \psi_{k_i} \psi_{k_j} \rangle_A = [A^{-1}]_{k_i,k_j} \tag{2.8}$$

(if m is odd, the r.h.s. of (2.7) should be interpreted as 0).

Specializing these formulas to the case $A = K_{\mathbf{t}}^{(\theta\tau)}$ we find:

$$\mathrm{Pf} K_{\mathbf{t}}^{(\theta\tau)} = \int_{(\theta\tau)} \big[\prod_{\mathbf{x}\in\Lambda} d\psi_{\mathbf{x}} \big] e^{S_{\mathbf{t}}(\psi)} \,, \tag{2.9}$$

$$S_{\mathbf{t}}(\psi) = -\frac{1}{2} \sum_{\mathbf{x},\mathbf{y}\in\Lambda} \psi_{\mathbf{x}} (K_{\mathbf{t}}^{(\theta\tau)})_{\mathbf{x},\mathbf{y}} \psi_{\mathbf{y}} \tag{2.10}$$

$$= -\sum_{\mathbf{x}\in\Lambda} \big[t_{(\mathbf{x},\mathbf{x}+\hat{e}_1)} E_{(\mathbf{x},\mathbf{x}+\hat{e}_1)} + t_{(\mathbf{x},\mathbf{x}+\hat{e}_2)} E_{(\mathbf{x},\mathbf{x}+\hat{e}_2)} \big] \tag{2.11}$$

where $E_{(\mathbf{x},\mathbf{x}+\hat{e}_1)} = \psi_{\mathbf{x}} \psi_{\mathbf{x}+\hat{e}_1}$ while $E_{(\mathbf{x},\mathbf{x}+\hat{e}_2)} = i\psi_{\mathbf{x}} \psi_{\mathbf{x}+\hat{e}_2}$ and the index $(\theta\tau)$ under the integral means that we have to identify $\psi_{(L/2+1,y)} \equiv \psi_{(-L/2+1,y)} (-1)^\theta$ and similarly $\psi_{(x,L/2+1)} \equiv \psi_{(x,-L/2+1)} (-1)^\tau$. The choice $\theta = 0$ (resp. $\theta = 1$) means periodic (resp. antiperiodic) boundary conditions for the Grassmann field in the horizontal direction, and similarly τ determines periodic/antiperiodic boundary conditions in the vertical direction.

Inserting (2.11) into (2.5) we find that, *if the bonds $b_1, \dots, b_k$ are all different* [here we say that two bonds are different if they are not identical; their geometrical supports may overlap], then

$$\sum_{M\in\mathcal{M}_\Lambda} \Big[\prod_{b\in M} t_b^{(m)} \Big] \mathbb{1}_{b_1} \cdots \mathbb{1}_{b_k} = \sum_{\theta\tau} \frac{C_{\theta,\tau}}{2} \int_{(\theta\tau)} \prod_{\mathbf{x}\in\Lambda} d\psi_{\mathbf{x}} \, (-1)^k E_{b_1}^{(m)} \cdots E_{b_k}^{(m)} \, e^{S(\psi)} \,, \tag{2.12}$$

where $S(\psi) = S_{\mathbf{t}^{(\mathbf{m})}}(\psi)$, see (2.11), $E_b^{(m)} = t_b^{(m)} E_b$ and the r.h.s. can be computed via (2.7).

We now go back to (2.1). By using (2.12), we would like to re-express it in terms of a Grassmann integral. To this purpose, we rewrite

$$\prod_{\langle b,b'\rangle\subseteq\Lambda} (1 + \alpha \mathbb{1}_b \mathbb{1}_{b'}) = \sum_{n\geq 0} \sum_{\{\gamma_1,\dots,\gamma_n\}\subset\Lambda}^{*} \zeta(\gamma_1) \cdots \zeta(\gamma_n) \tag{2.13}$$

where γ_i are "contours", each consisting of a sequence of 2 or more adjacent parallel bonds and the constrained sum $\sum^*_{\{\gamma_1,\dots,\gamma_n\}}$ runs over unordered *compatible* n-ples of contours (here we say that $\{\gamma_1, \dots, \gamma_n\}$ is compatible if $\gamma_i \cap \gamma_j = \emptyset$, $\forall i \neq j$, where $\gamma_i \cap \gamma_j = \emptyset$ means that the bonds in γ_i are all different from those in γ_j; note that the geometric supports of two compatible contours may overlap). Moreover, if $b_1, \dots, b_k$ are adjacent parallel bonds (with $k \geq 2$) and $\gamma = \{b_1, \dots, b_k\}$, then

$$\zeta(\gamma) = \zeta(\{b_1, \dots, b_k\}) = \alpha^{k-1} \mathbb{1}_{b_1} \cdots \mathbb{1}_{b_k} \,. \tag{2.14}$$

Finally, the term with $n = 0$ in the right side of (2.13) should be interpreted as 1. By inserting (2.13) into (2.1) we find:

$$Z_\Lambda(\lambda, m) = \sum_{n\geq 0} \sum_{\{\gamma_1,\dots,\gamma_n\}\subseteq\Lambda}^{*} \sum_{M\in\mathcal{M}_\Lambda} \Big[\prod_{b\in M} t_b^{(m)} \Big] \zeta(\gamma_1) \cdots \zeta(\gamma_n) \,. \tag{2.15}$$

Note that each term $\zeta(\gamma_1)\cdots\zeta(\gamma_n)$ in the r.h.s. of (2.15) is proportional to a product of operators $\mathbb{1}_b$ over *different* bonds: actually, having a representation involving only products of $\mathbb{1}_b$ over different bonds was the very purpose of grouping the bonds into contours and of rewriting the product in the l.h.s. of (2.13) as a sum over compatible collections of contours. Therefore, we can evaluate the sum $\sum_{M\in\mathcal{M}_\Lambda}\Big[\prod_{b\in M} t_b^{(m)}\Big]\zeta(\gamma_1)\cdots\zeta(\gamma_n)$ by using (2.12):

$$Z_\Lambda(\lambda,m)=\frac{1}{2}\sum_{\theta,\tau}C_{\theta,\tau}\sum_{n\geq 0}\sum_{\{\gamma_1,\dots,\gamma_n\}\subset\Lambda}^{*}\int_{(\theta\tau)}\big[\prod_{\mathbf{x}\in\Lambda}d\psi_{\mathbf{x}}\big]\xi(\gamma_1)\cdots\xi(\gamma_n)\,e^{S(\psi)}\,, \tag{2.16}$$

where, if $b_1,\dots,b_k$ are adjacent parallel bonds (with $k\geq 2$) and $\gamma=\{b_1,\dots,b_k\}$, then

$$\xi(\gamma)=\xi(\{b_1,\dots,b_k\})=(-1)^k\alpha^{k-1}E_{b_1}^{(m)}\cdots E_{b_k}^{(m)}\,. \tag{2.17}$$

Finally, note that by the Grassmann anti-commutation rules, it holds that

$$\sum_{n\geq 0}\sum_{\{\gamma_1,\dots,\gamma_n\}\subset\Lambda}^{*}\xi(\gamma_1)\cdots\xi(\gamma_n)=e^{\sum_{\gamma\subset\Lambda}\xi(\gamma)}\,, \tag{2.18}$$

(in the expansion of the exponential, terms containing incompatible contours vanish since $E_b^2=0$) so that (2.16) simplifies into

$$Z_\Lambda(\lambda,m)=\frac{1}{2}\sum_{\theta,\tau}C_{\theta,\tau}\int_{(\theta\tau)}\big[\prod_{\mathbf{x}\in\Lambda}d\psi_{\mathbf{x}}\big]e^{S(\psi)+V_\Lambda(\psi)}\,, \tag{2.19}$$

with

$$V_\Lambda(\psi)=\sum_{\gamma\subset\Lambda}\xi(\gamma)\,, \tag{2.20}$$

which is our final expression. It is worth noting that V_Λ can be written as

$$V_\Lambda(\psi)=\alpha\sum_{\mathbf{x}\in\Lambda}(2+2m(-1)^{x_1}+m^2)\psi_{\mathbf{x}}\psi_{\mathbf{x}+\hat{e}_1}\psi_{\mathbf{x}+\hat{e}_2}\psi_{\mathbf{x}+\hat{e}_1+\hat{e}_2}+W_{\geq 6}(\psi)\,. \tag{2.21}$$

where $W_{\geq 6}(\psi)$ is a sum over Grassmann monomials of order larger or equal than 6, whose kernels decay exponentially in space (with rate $\kappa=-\log|\alpha|$, which is positive for λ small).

2.2. **The free propagator.** We introduce the Grassmann gaussian "measure"[2]

$$P_\Lambda^{(\theta\tau)}(d\psi):=\frac{1}{\mathrm{Pf}K_{\mathbf{t}(\mathbf{m})}^{(\theta\tau)}}\,e^{S(\psi)}\prod_{\mathbf{x}\in\Lambda}d\psi_{\mathbf{x}},$$

where it is understood that boundary conditions on the ψ field are (θ,τ). Averages w.r.t. $P_\Lambda^{(\theta\tau)}$ of products of Grassmann fields are given by (2.7),

[2]We emphasize that $P_\Lambda^{(\theta\tau)}(d\psi)$ is not a measure in the usual probabilistic sense: it is however a linear application that associates with a polynomial function f of the ψ variables a number, the "average" $\int_{(\theta\tau)}P_\Lambda^{(\theta\tau)}(d\psi)f(\psi)\in\mathbb{C}$, and it is normalized so that $\int_{(\theta\tau)}P_\Lambda^{(\theta\tau)}(d\psi)1=1$.

with *propagator* given by (see Appendix A)

$$g_\Lambda^{(\theta\tau)}(\mathbf{x},\mathbf{y}) = \int_{(\theta\tau)} P_\Lambda^{(\theta\tau)}(d\psi)\psi_{\mathbf{x}}\psi_{\mathbf{y}} = \big[\big(K^{(\theta\tau)}_{\mathbf{t}(\mathbf{m})}\big)^{-1}\big]_{\mathbf{x},\mathbf{y}} = \tag{2.22}$$
$$= \frac{1}{L^2}\sum_{\mathbf{k}\in\mathcal{D}_\Lambda^{(\theta\tau)}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{N(\mathbf{k},m,y_1)}{2D(\mathbf{k},m)}\ ,$$

where

$$N(\mathbf{k},m,y_1) = i\sin k_1 + \sin k_2 + m(-1)^{y_1}\cos k_1,$$
$$D(\mathbf{k},m) = m^2 + (1-m^2)(\sin k_1)^2 + (\sin k_2)^2$$

and

$$\mathcal{D}_\Lambda^{(\theta\tau)} = \{(2\pi/L)(\mathbf{n}+(\theta,\tau)/2),\ \mathbf{n}\in\Lambda\}\ . \tag{2.23}$$

Note that $g_\Lambda^{(\theta\tau)}(\mathbf{x},\mathbf{y})$ is zero whenever $\mathbf{x}$ and $\mathbf{y}$ have the same parity (this can be seen by observing that the ratio in (2.22) changes sign if $\mathbf{k}$ is changed to $\mathbf{k}+(\pi,\pi)$, while $e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}$ remains unchanged if $\mathbf{x}$ has the same parity as $\mathbf{y}$). The propagator is not translation invariant, but is invariant under translations in $2\mathbb{Z}\times\mathbb{Z}$ (because of the horizontal periodic modulation of the bond weights). Of course, when $m=0$ full translation invariance is recovered.

In the following we will need to evaluate the propagator for fixed $\mathbf{x},\mathbf{y}\in\mathbb{Z}^2$, as $\Lambda\nearrow\mathbb{Z}^2$. In this limit, the propagator takes a particularly simple form, independent of $(\theta\tau)$:

$$g(\mathbf{x},\mathbf{y}) = \lim_{\Lambda\nearrow\mathbb{Z}^2} g_\Lambda^{(\theta\tau)}(\mathbf{x},\mathbf{y}) = \int_{\mathbb{T}^2}\frac{d\mathbf{k}}{(2\pi)^2}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{N(\mathbf{k},m,y_1)}{2D(\mathbf{k},m)}, \tag{2.24}$$

where the torus $\mathbb{T}^2=\mathbb{R}^2/2\pi\mathbb{Z}^2$ is also called the *Brillouin zone*. In analogy with its finite volume counterpart, $g(\mathbf{x},\mathbf{y})$ is zero whenever $\mathbf{x}$ and $\mathbf{y}$ have the same parity. We will see in Appendix A.2 that the finite-volume corrections to $g_\Lambda^{(\theta\tau)}$ are exponentially small in L, if $m>0$.

2.3. **Infrared regularization.** At this point the role of the regularization parameter $m>0$ should be apparent. If $m=0$ then the integrand in $g(\mathbf{x},\mathbf{y})$ has poles whenever $\sin k_1 = \sin k_2 = 0$. As we will see in next section, the propagator then decays slowly at large distances (like $1/|\mathbf{x}-\mathbf{y}|$), signaling that the system is critical (or massless). When instead $m>0$ the integrand is analytic on the Brillouin zone and therefore $g(\mathbf{x},\mathbf{y})$ (that is its Fourier transform) decays exponentially fast and the system is off-critical (or massive). The exponential decay however kicks in only when $|\mathbf{x}-\mathbf{y}|\gtrapprox 1/m$, and for $m\to 0$ the critical decay is recovered. In the language of [45], one says that the non-interacting ($\lambda=0$) system is in the "liquid phase" when $m=0$ and in the "gaseous phase" when $m>0$.

2.4. **Dimer correlation functions.** Besides the partition function $Z_\Lambda(\lambda,m)$, we are interested in computing *truncated* multipoint dimer correlations (cumulants) of the form

$$\langle \mathbb{1}_{b_1};\cdots;\mathbb{1}_{b_k}\rangle_{\Lambda;\lambda,m} = \frac{\partial^k}{\partial A_{b_1}\cdots\partial A_{b_k}}\log\mathcal{Z}_\Lambda(\lambda,m,\mathbf{A})\Big|_{\mathbf{A}=\mathbf{0}}\ , \tag{2.25}$$

with

$$\mathcal{Z}_\Lambda(\lambda, m, \mathbf{A}) := \sum_{M\in\mathcal{M}_\Lambda} \Big[\prod_{b\in M} t_b^{(m)}\Big] e^{\lambda W_\Lambda(M)+\sum_{b\subset\Lambda} A_b \mathbb{1}_b} \tag{2.26}$$

and $b_1, \dots, b_k$ a k-ple of bonds all different among each other. Moreover, $\mathbf{A} = \{A_b\}_{b\subset\Lambda}$. The modified partition function $\mathcal{Z}_\Lambda(\lambda, m, \mathbf{A})$ can be expressed in the form of a Grassmann integral, by proceeding in the same way that we followed for $Z_\Lambda(\lambda, m)$. The result is (details are left to the reader):

$$\mathcal{Z}_\Lambda(\lambda, m, \mathbf{A}) = \frac{1}{2}\sum_{\theta,\tau=0,1} C_{\theta\tau} \int_{(\theta\tau)} \big[\prod_{\mathbf{x}\in\Lambda} d\psi_{\mathbf{x}}\big] e^{S(\psi)+V_\Lambda(\psi)+\mathcal{B}_\Lambda(\psi,\mathbf{J})}\ , \tag{2.27}$$

where $\mathbf{J} = \mathbf{J}(\mathbf{A}) = \{J_b(A_b)\}_{b\subset\Lambda}$ with $J_b = e^{A_b} - 1$, and

$$\mathcal{B}_\Lambda(\psi, \mathbf{J}) = \sum_{k\geq 1} \sum_{\gamma=\{b_1,\dots,b_k\}\subset\Lambda} \sum_{\emptyset\neq R\subseteq\gamma} \tilde{\xi}(\gamma; R), \tag{2.28}$$

$$\tilde{\xi}(\gamma; R) = (-1)^k \alpha^{k-1} \prod_{b\in\gamma} E_b^{(m)} \prod_{b\in R} J_b. \tag{2.29}$$

Here, as above, $b_1, \dots, b_k$ are adjacent parallel bonds. Note that once the truncated correlations are known, the standard correlations can be reconstructed via the inversion formula:

$$\langle \mathbb{1}_{b_1} \cdots \mathbb{1}_{b_k} \rangle_{\Lambda;\lambda,m} = \tag{2.30}$$
$$= \sum_{\{\underline{i}^{(1)},\dots,\underline{i}^{(s)}\}\in\mathcal{P}[1,\dots,k]} \langle \mathbb{1}_{b_{i_1^{(1)}}}; \cdots; \mathbb{1}_{b_{i_{k_1}^{(1)}}} \rangle_{\Lambda;\lambda,m} \cdots \langle \mathbb{1}_{b_{i_1^{(s)}}}; \cdots; \mathbb{1}_{b_{i_{k_s}^{(s)}}} \rangle_{\Lambda;\lambda,m}\ ,$$

where $\underline{i}^{(j)} = \{i_1^{(j)}, \dots, i_{k_j}^{(j)}\} \subseteq \{1, \dots, k\}$, with $k_j \geq 1$, is a non-empty set of indices, and $\mathcal{P}[1, \dots, k]$ is the set of partitions of $\{1, \dots, k\}$. In (2.30), the single-bond average $\langle \mathbb{1}_b \rangle_{\Lambda;\Lambda'}$, $b \subset \Lambda$, is given by

$$\langle \mathbb{1}_b \rangle_{\Lambda;\lambda,m} = \frac{\partial}{\partial A_b} \log \mathcal{Z}_\Lambda(\lambda, m, \mathbf{A})\Big|_{\mathbf{A}=\mathbf{0}}\ . \tag{2.31}$$

2.5. **Reducing to a single Pfaffian.** Not only the propagator $g_\Lambda^{(\theta\tau)}$, but also $\mathrm{Pf}K_\Lambda^{(\theta\tau)}$, the normalization of the measure $P_\Lambda^{(\theta\tau)}$, loses dependence on (θ, τ) in the limit $\Lambda \nearrow \mathbb{Z}^2$. More precisely, while each Pfaffian grows exponentially in L^2, for $m > 0$ one has (cf. Appendix A.2)

$$\lim_{\Lambda\nearrow\mathbb{Z}^2} \frac{\mathrm{Pf}K_\Lambda^{(11)}}{\mathrm{Pf}K_\Lambda^{(\theta\tau)}} = 1 \tag{2.32}$$

and the limit is reached exponentially fast in L. This is a consequence of the fact that at very large distances the propagator decays exponentially (actually this is the main technical reason why we introduced the infrared regularization $m > 0$).

The observation (2.32) implies important simplifications in the thermodynamic limit. Suppose that we want to compute the average of a dimer

observable, say $\mathbb{1}_{b_1} \cdots \mathbb{1}_{b_k}$, for the non-interacting system ($\lambda = 0$). From (2.12) we get

$$\langle \mathbb{1}_{b_1} \cdots \mathbb{1}_{b_k} \rangle_{\Lambda;0,m} = \frac{\sum_{\theta\tau} \frac{C_{\theta,\tau}}{2} \left[\mathrm{Pf} K_\Lambda^{(\theta\tau)}\right] \int_{(\theta\tau)} P_\Lambda^{(\theta\tau)}(d\psi)\, (-1)^k E_{b_1}^{(m)} \cdots E_{b_k}^{(m)}}{\sum_{\theta\tau} \frac{C_{\theta,\tau}}{2} \left[\mathrm{Pf} K_\Lambda^{(\theta\tau)}\right]}.$$

We have seen above that the free propagator, and therefore the integrals in the numerator, become independent of $(\theta\tau)$ when $\Lambda \nearrow \mathbb{Z}^2$. Together with (2.32), this implies that

$$\lim_{\Lambda\nearrow\mathbb{Z}^2} \langle \mathbb{1}_{b_1} \cdots \mathbb{1}_{b_k} \rangle_{\Lambda;0,m} = \lim_{\Lambda\nearrow\mathbb{Z}^2} \int_{(11)} P_\Lambda^{(11)}(d\psi)\, (-1)^k E_{b_1}^{(m)} \cdots E_{b_k}^{(m)} \quad (2.33)$$

$$= \int P(d\psi)\, (-1)^k E_{b_1}^{(m)} \cdots E_{b_k}^{(m)}, \quad (2.34)$$

with $P(d\psi)$ the gaussian Grassmann measure with propagator $g(\mathbf{x},\mathbf{y})$. That is, it is sufficient to consider (11) boundary conditions in the Grassmann integrations (these are more convenient than (00) conditions since even for $m = 0$ the denominator in (2.22) is never singular for $\mathbf{k} \in \mathcal{D}_\Lambda^{(11)}$).

An analogous fact holds also for the interacting model ($\lambda \neq 0$), as a consequence of the fact that the interacting propagator also decays exponentially as long as $m > 0$ (the model remains off-critical even in the presence of interactions, see Remark 4 below). More precisely, for $m > 0$ the following holds: given $n \geq 1$ distinct $\mathbf{x}_1, \ldots, \mathbf{x}_n$,

$$\lim_{\Lambda\nearrow\mathbb{Z}^2} \frac{\int_{(\theta\tau)} P_\Lambda^{(\theta\tau)} e^{V_\Lambda(\psi)} \psi_{\mathbf{x}_1} \ldots \psi_{\mathbf{x}_n}}{\int_{(11)} P_\Lambda^{(11)} e^{V_\Lambda(\psi)} \psi_{\mathbf{x}_1} \ldots \psi_{\mathbf{x}_n}} = 1 \quad (2.35)$$

and actually the limit is reached exponentially fast in L. The proof is a corollary of the multiscale construction described in Section 5 below, and goes along the same lines as [47, Appendix G].

As a consequence of (2.35) and (2.32), using (2.27), we see that

$$\lim_{\Lambda\nearrow\mathbb{Z}^2} \frac{\partial^n}{\partial A_{b_1} \ldots \partial A_{b_n}} \log \mathcal{Z}_\Lambda(\lambda, m, \mathbf{A})\Big|_{\mathbf{A}=\mathbf{0}} = \quad (2.36)$$

$$= \lim_{\Lambda\nearrow\mathbb{Z}^2} \frac{\partial^n}{\partial A_{b_1} \ldots \partial A_{b_n}} \log \mathcal{Z}_\Lambda^{(11)}(\lambda, m, \mathbf{A})\Big|_{\mathbf{A}=\mathbf{0}} \quad (2.37)$$

with

$$\mathcal{Z}_\Lambda^{(11)}(\lambda, m, \mathbf{A}) = \int_{(11)} P_\Lambda^{(11)}(d\psi) e^{V_\Lambda(\psi) + \mathcal{B}_\Lambda(\psi, \mathbf{J})}. \quad (2.38)$$

3. Majorana and Dirac fermions

In this section we discuss in some detail the large-distance behavior of the non-interacting propagator $g(\mathbf{x},\mathbf{y})$ introduced above. The same estimates (with different notations and in greater generality) are obtained also in [45]. The fall-off properties of g play a key role in the computation of the dimer correlations, as well as of the height fluctuations, to be discussed in the next sections. As we will see, it is convenient to split $\psi_{\mathbf{x}}$ as the sum of oscillating functions times four new Grassmann variables $\psi_{\mathbf{x},\gamma}$, $\gamma = 1, \ldots, 4$,

each of which has a propagator with well-defined limiting behavior for large distances,

$$\langle \psi_{\mathbf{x},\gamma}\psi_{\mathbf{y},\gamma}\rangle \simeq \frac{1}{4\pi}\frac{1}{(x_1-y_1)+i\ (-1)^{\gamma+1}(x_2-y_2)}\ , \tag{3.1}$$

when $|\mathbf{x}-\mathbf{y}|$ is large (but $|\mathbf{x}-\mathbf{y}| \lessapprox 1/m$). For $m=0$, the four fields ψ_γ are independent (i.e. their propagator is diagonal in the γ index) and are the lattice analogues of "real", massless, Majorana fermions, see [30, Section 2.3.1]. These lattice Majoranas can be also combined in pairs, to form two "complex", massless, Dirac fields, $\psi^{\pm}_{\omega}$, $\omega=\pm 1$. Besides the terminology, which is borrowed from high energy physics, the transformations from the original Grassmann field, to the Majorana, and then the Dirac fields, are just restatements of a couple of simple, and convenient, algebraic manipulations of the propagator, which are discussed in the following.

Consider (2.24). Of course, the large distance asymptotics of $g(\mathbf{x},\mathbf{y})$ is dominated by the contributions from the momenta close to the singularity points where the denominator is small (for m small), which are $\mathbf{p}_1=(0,0)$, $\mathbf{p}_2=(\pi,0)$, $\mathbf{p}_3=(\pi,\pi)$, $\mathbf{p}_4=(0,\pi)$. Therefore, $g(\mathbf{x},\mathbf{y})$ can be naturally written as the superposition of four terms:

$$g(\mathbf{x},\mathbf{y}) = \sum_{\gamma=1}^{4}\int_{\mathbb{T}^2}\frac{d\mathbf{k}}{(2\pi)^2}\chi_\gamma(\mathbf{k})e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{N(\mathbf{k},m,y_1)}{2D(\mathbf{k},m)}\ , \tag{3.2}$$

where $\chi_\gamma(\mathbf{k})$ are suitable smooth (say, C^∞) functions over the torus, centered at $\mathbf{p}_\gamma$, and defining a partition of the identity: $\sum_{\gamma=1}^4\chi_\gamma(\mathbf{k})=1$. More specifically, we assume that the functions $\chi_\gamma(\mathbf{k})$ satisfy the following: first of all,

$$\chi_\gamma(\mathbf{k}) = \bar\chi(\mathbf{k}-\mathbf{p}_\gamma)\ , \tag{3.3}$$

for a nonnegative compactly supported smooth function $\bar\chi(\mathbf{k})$, centered at the origin and even in $\mathbf{k}$. We also require that the support of $\bar\chi(\mathbf{k})$ does not include $(0,\pi),(\pi,0),(\pi,\pi)$. For definiteness, one should think of $\bar\chi(\mathbf{k})$ as a suitably smoothed version of $\mathbf{1}_{\|\mathbf{k}\|_\infty\le\pi/2}$. Finally, we assume that $\bar\chi$ is in the Gevrey class of order 2 (say). We recall that a $C^\infty(\mathbb{R}^d)$ function f is said to be in the Gevrey class of order s if on every compact $K\subset\mathbb{R}^d$ there are two constants $A=A(K,f)$ and $\mu=\mu(K,f)$ so that for any non-negative integers $n_1,\dots,n_d$

$$\|\partial_1^{n_1}\cdots\partial_d^{n_d}f\| \le A\mu^{n_1}\cdots\mu^{n_d}(n_1!\cdots n_d!)^s. \tag{3.4}$$

For $s=1$ we have a common characterization of real-analytic functions; the class of order 2 includes the function $e^{-1/x_1}\mathbf{1}_{x_1\ge 0}$. A useful feature of Gevrey functions, which we will use in the following, is the fact that the Fourier transform $\tilde f(\mathbf{x})$ of a compactly supported Gevrey function $f(\mathbf{k})$ of order s decays at large distances like a stretched exponential $e^{-(\text{const.})|\mathbf{x}|^{1/s}}$. An example of a function $\bar\chi(\mathbf{k})$ is the following. Given $\varepsilon>0$, let $f_\varepsilon(k) := e^{-(1-k^2/\varepsilon^2)^{-1}}\mathbf{1}_{|k|\le\varepsilon}$ and $F_\varepsilon(k)=\int_{-\infty}^k f_\varepsilon(t)dt/\int_{-\infty}^{+\infty}f_\varepsilon(t)dt$. Note that F_ε is a smoothed version of the Heaviside step function. It is Gevrey of order 2, and such that $F_\varepsilon(k)+F_\varepsilon(-k)=1$. For $\varepsilon<\pi/2$ we let $\theta_\varepsilon(k)=F_\varepsilon(k+\pi/2)F_\varepsilon(-k+\pi/2)$. We also define $\tilde\chi(k):=\sum_{n\in\mathbb{Z}}\theta_\varepsilon(k+2\pi n)$, which we can naturally think as a function on the circle $\mathbb{T}:=\mathbb{R}\backslash 2\pi\mathbb{Z}$. It is straightforward

to check that $\bar\chi(\mathbf{k}) := \tilde\chi(k_1)\tilde\chi(k_2)$ satisfies all the properties required above. For future reference, we also introduce $\chi(\mathbf{k}) := \sum_{\mathbf{n}\in\mathbb{Z}^2}\theta_\varepsilon(|\mathbf{k}+2\pi\mathbf{n}|)$, which is (as a function on the torus) a rotationally invariant version of $\bar\chi(\mathbf{k})$.

The decomposition (3.2) with $\chi_\gamma(\mathbf{k})$ as in (3.3) induces the following decomposition on the Grassmann fields:

$$\psi_{\mathbf{x}} = e^{i\mathbf{p}_1\mathbf{x}}\psi_{\mathbf{x},1} - ie^{i\mathbf{p}_2\mathbf{x}}\psi_{\mathbf{x},2} + ie^{i\mathbf{p}_3\mathbf{x}}\psi_{\mathbf{x},3} + e^{i\mathbf{p}_4\mathbf{x}}\psi_{\mathbf{x},4}\ , \tag{3.5}$$

with $\psi_{\mathbf{x},\gamma}$ Grassmann variables with propagator

$$\langle\psi_{\mathbf{x},\gamma}\psi_{\mathbf{y},\gamma'}\rangle = \int P(d\psi)\psi_{\mathbf{x},\gamma}\psi_{\mathbf{y},\gamma'} = \begin{pmatrix} G(\mathbf{x}-\mathbf{y}) & 0\\ 0 & G(\mathbf{x}-\mathbf{y})\end{pmatrix}_{\gamma,\gamma'} \tag{3.6}$$

where $G(\mathbf{x}) = \{G(\mathbf{x})_{\omega\omega'}\}, \omega,\omega' = \pm 1$ is the 2×2 matrix

$$G(\mathbf{x}) = \int_{\mathbb{T}^2}\frac{d\mathbf{k}}{(2\pi)^2}\bar\chi(\mathbf{k})\frac{e^{-i\mathbf{k}\mathbf{x}}}{2D(\mathbf{k},m)}\begin{pmatrix} i\sin k_1+\sin k_2 & im\cos k_1\\ -im\cos k_1 & i\sin k_1-\sin k_2\end{pmatrix}. \tag{3.7}$$

[The reader should simply check that with this definition the field $\psi_{\mathbf{x}}$ has the correct propagator $g(\mathbf{x},\mathbf{y})$ as in (3.2). Keep in mind that for x integer one has $\exp(i\pi x) = \exp(-i\pi x)$.] Note the symmetry properties

$$G_{++}(\mathbf{x}) = G_{--}(\mathbf{x})^*,\quad G_{+-}(\mathbf{x}) = G_{-+}(\mathbf{x})^*,\quad G_{\omega\omega'}(\mathbf{x}) = -\omega\omega' G_{\omega\omega'}(-\mathbf{x}) \tag{3.8}$$

The large-distance behavior of $G(\mathbf{x})$ is given as follows (cf. Appendix A.1):

Proposition 1. *If* $\mathbf{x}\neq\mathbf{0}$,

$$G(\mathbf{x}) = \mathfrak{g}(\mathbf{x}) + R(\mathbf{x})\ , \tag{3.9}$$

with

$$\mathfrak{g}(\mathbf{x}) = \int_{\mathbb{R}^2}\frac{d\mathbf{k}}{(2\pi)^2}\frac{e^{-i\mathbf{k}\mathbf{x}}}{2\bar D(\mathbf{k},m)}\begin{pmatrix} ik_1+k_2 & im\\ -im & ik_1-k_2\end{pmatrix}, \tag{3.10}$$

$$\bar D(\mathbf{k},m) = m^2 + (1-m^2)k_1^2 + k_2^2 \tag{3.11}$$

and R a remainder such that

$$|R_{\omega\omega'}(\mathbf{x})| \le \frac{C}{|\mathbf{x}|^2}\ , \tag{3.12}$$

with C independent of the infrared regularization parameter m.

When $m=0$, $\mathfrak{g}$ is the diagonal matrix $\mathfrak{g}^{(0)}$ with diagonal elements

$$\mathfrak{g}^{(0)}_{\omega\omega}(\mathbf{x}) = \frac{1}{4\pi}\frac{1}{x_1+i\omega x_2}. \tag{3.13}$$

Remark 2. *Using* (3.9)*, we see that $\langle\psi_{\mathbf{x},\gamma}\psi_{\mathbf{y},\gamma}\rangle$ behaves asymptotically as the propagator of a real Majorana field of mass m, in the sense of* [30, Section 2.3.1]*: therefore, the fields $\psi_{\mathbf{x},\gamma}$ are referred to as lattice Majorana fields.*

From the above discussion we see that the propagator G decays as the inverse of the distance, without any oscillating factor. The discrete derivatives of G decay as the inverse distance squared, while the same is not true for g, due to oscillatory factors in (3.5).

3.1. **Rewriting the partition function in terms of Majorana or Dirac fields.** The decomposition of the field ψ in terms of four Majorana fields ψ_γ can be done analogously in finite volume and for the boundary conditions $(\theta\tau)$. In this case, one should simply interpret, e.g. in (3.7), integrals as sums for $\mathbf{k} \in \mathcal{D}_\Lambda^{(\theta\tau)}$, the estimates in Proposition 1 still hold and the Grassmann integration w.r.t. ψ_γ will be denoted $P_\Lambda^{(\theta\tau)}(d\psi_\gamma)$.

The partition function (and, similarly, the generating function $\mathcal{Z}_\Lambda(\lambda, m, \mathbf{A})$ for dimer correlations) can be rewritten in terms of the Majorana fields: going back to (2.19) and (2.27) we get for instance

$$Z_\Lambda(\lambda, m) = \frac{1}{2}\sum_{\theta\tau} C_{\theta\tau} \mathrm{Pf} K_\Lambda^{(\theta\tau)} \int_{(\theta\tau)} \prod_{\gamma=1,\dots,4} P_\Lambda^{(\theta\tau)}(d\psi_\gamma) \exp\Big\{V_\Lambda(\psi)\Big\} , \quad (3.14)$$

with $\psi = \{\psi_{\mathbf{x}}\}_{\mathbf{x}\in\Lambda}$ as in (3.5). We used the addition formula for normalized Grassmann gaussian integrations; namely, if $P_g(d\psi)$ is the normalized Grassmann gaussian integration with propagator g, and if $g = g_1 + g_2$, then $P_g(d\psi) = P_{g_1}(d\psi_1)P_{g_2}(d\psi_2)$, in the sense that for every polynomial F:

$$\int P_g(d\psi)F(\psi) = \int P_{g_1}(d\psi_1) \int P_{g_2}(d\psi_2)F(\psi_1 + \psi_2) . \quad (3.15)$$

Since the propagator of ψ_γ depends only on the parity of γ, it can also be convenient to group these two so-called "real" fields to form a single "complex field" $\psi_\omega^\pm$:

$$\psi^\pm_{\mathbf{x},1} := \frac{1}{\sqrt{2}}(\psi_{\mathbf{x},1} \mp i\psi_{\mathbf{x},3}); \quad \psi^\pm_{\mathbf{x},-1} := \pm\frac{i}{\sqrt{2}}(\psi_{\mathbf{x},2} \mp i\psi_{\mathbf{x},4}) . \quad (3.16)$$

which is inverted (recall (3.5)) as

$$\psi_{\mathbf{x}} = \sqrt{2} \cdot \begin{cases} \psi^-_{\mathbf{x},1} + \psi^-_{\mathbf{x},-1} \text{ if } (x_1, x_2) = (\text{even}, \text{even}) \\ \psi^+_{\mathbf{x},1} - \psi^+_{\mathbf{x},-1} \text{ if } (x_1, x_2) = (\text{even}, \text{odd}) \\ \psi^+_{\mathbf{x},1} + \psi^+_{\mathbf{x},-1} \text{ if } (x_1, x_2) = (\text{odd}, \text{even}) \\ \psi^-_{\mathbf{x},1} - \psi^-_{\mathbf{x},-1} \text{ if } (x_1, x_2) = (\text{odd}, \text{odd}) \end{cases} . \quad (3.17)$$

Here $\psi^+_{\mathbf{x},\omega}$ formally plays the role of complex conjugate of $\psi^-_{\mathbf{x},\omega}$. Using its definition we see that $\langle\psi^+_{\mathbf{x},\omega}\psi^+_{\mathbf{y},\omega'}\rangle = \langle\psi^-_{\mathbf{x},\omega}\psi^-_{\mathbf{y},\omega'}\rangle = 0$, while

$$\begin{pmatrix} \langle\psi^-_{\mathbf{x},1}\psi^+_{\mathbf{y},1}\rangle & \langle\psi^-_{\mathbf{x},1}\psi^+_{\mathbf{y},-1}\rangle \\ \langle\psi^-_{\mathbf{x},-1}\psi^+_{\mathbf{y},1}\rangle & \langle\psi^-_{\mathbf{x},-1}\psi^+_{\mathbf{y},-1}\rangle \end{pmatrix} = \begin{pmatrix} G_{++}(\mathbf{x}-\mathbf{y}) & iG_{+-}(\mathbf{x}-\mathbf{y}) \\ -iG_{-+}(\mathbf{x}-\mathbf{y}) & G_{--}(\mathbf{x}-\mathbf{y}) \end{pmatrix} . \quad (3.18)$$

The "complex" nature of the field $\psi^\pm_\omega$ justifies the name "lattice Dirac field", which is used for it. In the following, it will be sometimes convenient to work with Majorana variables and sometimes with Dirac variables.

4. The height function

In this section we explain how to re-express the height function in terms of the Grassmann fields introduced above. We are interested in computing the height fluctuations, i.e., the n-point truncated self-correlations, $n \geq 2$:

$$\lim_{m\to 0} \lim_{\Lambda\nearrow\mathbb{Z}^2} \langle \underbrace{h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}; \cdots ; h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}}_{n \text{ times}} \rangle_{\Lambda;\lambda,m} . \quad (4.1)$$

For lightness we will write here $\langle\cdot\rangle_\Lambda$ instead of $\langle\cdot\rangle_{\Lambda;\lambda,m}$. The definition (1.5) allows us to re-express (4.1) in terms of sums of multipoint dimer correlations:

$$\langle \underbrace{h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}; \cdots ; h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}}_{n \text{ times}} \rangle_\Lambda = \sum_{b_1 \in \mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}} \cdots \sum_{b_n \in \mathcal{C}^{(n)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}} \sigma_{b_1} \cdots \sigma_{b_n} \langle \mathbb{1}_{b_1}; \cdots ; \mathbb{1}_{b_n} \rangle_\Lambda \,, \tag{4.2}$$

where $\mathcal{C}^{(j)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ are paths on $(\mathbb{Z}^2)^*$ from $\boldsymbol{\xi}$ to $\boldsymbol{\eta}$, which we assume not to wind around the torus and to be independent of L. The n-point dimer correlation in the r.h.s. of (4.2) can be computed via (2.25), so that

$$\begin{aligned} &\langle \underbrace{h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}; \cdots ; h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}}_{n \text{ times}} \rangle_\Lambda = \\ &= \sum_{b_1 \in \mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}} \cdots \sum_{b_n \in \mathcal{C}^{(n)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}} \sigma_{b_1} \cdots \sigma_{b_n} \frac{\partial^n}{\partial A_{b_1} \cdots \partial A_{b_n}} \log \mathcal{Z}_\Lambda(\lambda, m, \mathbf{A})\big|_{\mathbf{A}=\mathbf{0}} \,. \end{aligned} \tag{4.3}$$

Finally, one takes the limit $\lim_{m\to 0} \lim_{\Lambda \nearrow \mathbb{Z}^2}$ of the expression thus obtained. Since the limit $\Lambda \nearrow \mathbb{Z}^2$ is taken keeping $\boldsymbol{\xi}, \boldsymbol{\eta}$ fixed, in view of (2.36) we are allowed to replace $\mathcal{Z}_\Lambda(\lambda, m, \mathbf{A})$ in the right side of (4.3) by $\mathcal{Z}^{(11)}_\Lambda(\lambda, m, \mathbf{A})$, modulo an error term that is negligible in the thermodynamic limit and that we will simply forget in the following formulas.

4.1. **The Feynman diagrams expansion of the height fluctuations.** Using the Grassmann representation discussed in Section 2, we can rewrite the right side of (4.3) in terms of expectations of Grassmann variables. Let $\mathcal{E}^T$ indicate the truncated expectation with respect to $P^{(11)}_\Lambda(d\psi)$, i.e.,

$$\mathcal{E}^T(X_1(\psi); \ldots ; X_s(\psi)) = \frac{\partial^s}{\partial\lambda_1 \cdots \partial\lambda_s} \log \int_{(11)} P^{(11)}_\Lambda(d\psi) e^{\lambda_1 X_1(\psi) + \cdots + \lambda_s X_s(\psi)} \Big|_{\lambda_i = 0} \,. \tag{4.4}$$

In particular,

$$\int_{(11)} P^{(11)}_\Lambda(d\psi) e^{X(\psi)} = \sum_{s\ge 1} \frac{1}{s!} \mathcal{E}^T(\underbrace{X(\psi); \cdots ; X(\psi)}_{s \text{ times}}) \,. \tag{4.5}$$

Therefore, recalling (2.38), we get

$$\begin{aligned} \log \mathcal{Z}^{(11)}_\Lambda(\lambda, m, \mathbf{A}) &= \\ &= \sum_{s\ge 1} \frac{1}{s!} \mathcal{E}^T(\underbrace{V_\Lambda(\psi) + \mathcal{B}_\Lambda(\psi, \mathbf{J}); \ldots ; V_\Lambda(\psi) + \mathcal{B}_\Lambda(\psi, \mathbf{J})}_{s \text{ times}}) = \\ &= E_\Lambda(\lambda, m) + \sum_{k\ge 1} \sum_{\{b_1, \ldots, b_k\} \subseteq \Lambda} \Big[\prod_{j=1}^k J_{b_j} \Big] S_{\Lambda,k}(b_1, \ldots, b_k) \,, \end{aligned} \tag{4.6}$$

where the third line is the definition of $E_\Lambda(\lambda, m)$ and of $S_{\Lambda,k}(b_1, \ldots, b_k)$, i.e., $E_\Lambda(\lambda, m)$ (resp. $S_{\Lambda,k}(b_1, \ldots, b_k) J_{b_1} \cdots J_{b_k}$) collects all the terms in the second line that are independent of $\mathbf{J}$ (resp. are proportional to $J_{b_1} \cdots J_{b_k}$ but are independent of the other J_b's). In the last line, the sum over $\{b_1, \ldots, b_k\}$ does not run just over k-ples of different bonds. Rather, $\{b_1, \ldots, b_k\} =: B$ is a *bond configuration* in which some bonds are allowed to coincide. Formally,

one such configuration is a function $b \to B(b)$ with nonnegative integer values such that $\sum_b B(b) = N(B) < +\infty$. The number $B(b)$ has the meaning of *multiplicity* of b in B. Given B, we denote by $\tilde{B}$ the set of bonds b such that $B(b) > 0$; hence $\tilde{B}$ is the *support* of B, and it consists of the bonds that are in B, each counted without taking multiplicity into account.

We emphasize that, since $\mathcal{Z}_\Lambda(\lambda, m, \mathbf{A})$ is a polynomial in λ for finite Λ and it equals 1 when $\lambda = 0, \mathbf{A} = \mathbf{0}$, the sums in the second and third lines of (4.6) are convergent for sufficiently small $\lambda, \mathbf{A}$. However, proving that the radius of convergence in λ does not shrink to zero as $\Lambda \nearrow \mathbb{Z}^2$ is a highly non-trivial task, discussed in detail in Section 5. Of course, before even attempting to prove such a uniform convergence, we need at least to understand how to compute the right side of (4.6) formally, i.e., order by order in λ. A possible way of computing the perturbation series in λ for the generating function is in terms of Feynman diagrams and Feynman rules, which we now introduce.

The goal is to prescribe rules for computing truncated expectations like those in the second line of (4.6). With reference to (4.4), assume that $X_i(\psi)$ are Grassmann monomials (which is not a restrictive assumption, since the operator $\mathcal{E}^T$ is multilinear in its arguments), i.e.

$$X_i(\psi) = c_i \psi_{\mathbf{x}_1^{(i)}} \dots \psi_{\mathbf{x}_{n_i}^{(i)}}. \tag{4.7}$$

Then, Eq.(4.4) admits a natural diagrammatical representation in terms of connected Feynman diagrams, which are obtained as follows:

(1) draw s vertices, each representing one of the monomials X_i, with a number of "legs" equal to the order n_i of the corresponding monomial; each leg is associated with a label $\mathbf{x}_j^{(i)}$, which we will think of as the point which the leg exits from (or is anchored to);
(2) contract in all possible *connected* ways the legs, by pairing them two by two and by graphically representing every such pair by a line (here a contraction, or pairing, is called connected if the s vertices are geometrically connected by the contracted lines).

In this way, each pairing is in one-to-one correspondence with its diagrammatical representation, called Feynman diagram, and (4.4) is written as the sum over such connected Feynman diagrams of their values, where the value of the graph is: the product $\prod_{i=1}^s c_i$ of the "kernels" $c_1, \dots, c_s$ of $X_1, \dots, X_s$, times the product of the propagators associated with the contracted lines, times a sign, which is equal to the sign of the permutation required for placing next to each other the contracted Grassmann fields, starting from their original ordering in $X_1(\psi) \cdots X_s(\psi)$. For instance, if $s = 2$ and $X_i(\psi) = \psi_{\mathbf{x}_{2i-1}} \psi_{\mathbf{x}_{2i}}$, $i = 1, 2$, and we contract the leg associated with $\mathbf{x}_1$ with $\mathbf{x}_3$ and $\mathbf{x}_2$ with $\mathbf{x}_4$, the value of the corresponding Feynman diagram is $(-1) g_\Lambda^{(11)}(\mathbf{x}_1, \mathbf{x}_3) g_\Lambda^{(11)}(\mathbf{x}_2, \mathbf{x}_4)$, where -1 is the signature of the permutation that transforms 1234 into 1324.

The diagrammatic interpretation of (4.4) induces a diagrammatic representation of (4.6) (and, therefore, of (4.3)). First of all, $E_\Lambda(\lambda, m)$ equals the sum of all possible connected Feynman diagrams obtained by contracting vertices of type $\xi(\gamma)$ (coming from $V_\Lambda(\psi)$, see (2.20)), where $\gamma = \{b_1, \dots, b_k\} \subseteq \Lambda$ is a collection of $k \geq 2$ parallel adjacent bonds, see (2.17); in order to

graphically represent $\xi(\gamma)$, we imagine to represent $E^{(m)}_{(\mathbf{x},\mathbf{x}+\hat{e}_j)} = (i)^{j-1}(1 + \delta_{j,1} m(-1)^{x_1})\psi_{\mathbf{x}}\psi_{\mathbf{x}+\hat{e}_j}$ as a pair of solid half-lines, each of which can be contracted with another solid half-line to form a solid line (a propagator), while the α's can be thought of as wiggly lines from b_1 to b_2, etc, to b_k, see Fig. 3.

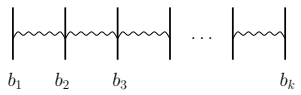

FIGURE 3. Graphical representation of a vertex of type $\xi(\{b_1, \dots, b_k\})$.

Moreover, $\big[\prod_{j=1}^k J_{b_j}\big] S_{\Lambda,k}(b_1, \dots, b_k)$ is the sum of all possible connected Feynman diagrams obtained by contracting vertices of type $\xi(\gamma)$ and of type $\tilde{\xi}(\gamma; R)$ (coming from $\mathcal{B}_\Lambda(\psi, \mathbf{J})$), with the obvious constraint that the product of the J_b factors involved produces exactly $\prod_{j=1}^k J_{b_j}$. See Fig. 4. For example, one of the diagrams contributing to $S_{\Lambda,4}(b_1, \dots, b_4)$ is shown in

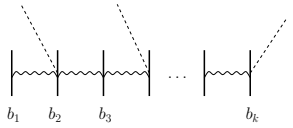

FIGURE 4. Graphical representation of a vertex of type $\tilde{\xi}(\{b_1, \dots, b_k\}; R)$, with $R = \{b_2, b_4, b_k\}$. The dotted lines represent the external fields J_{b_i}. If $|R| = 1$ the vertex is said to be of type $-J_b E^{(m)}_b$.

Fig. 5. The diagram in Fig. 5 is obtained from a contraction of the vertices depicted in Fig. 6.

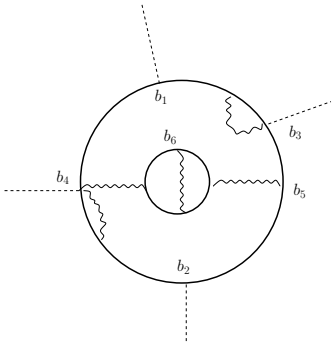

FIGURE 5. A diagram contributing to $S_{\Lambda,4}(b_1, \dots, b_4)$.

It must be stressed that there is no simple direct way to prove that the series of Feynman diagrams is convergent, uniformly in Λ. In order to prove the uniform convergence of the series in α (or, equivalently, in λ, since $\alpha = e^\lambda - 1$) we will need to modify the expansion, by implementing suitable (sophisticated, iterative) resummations, after which both $E_\Lambda(\lambda, m)$ and

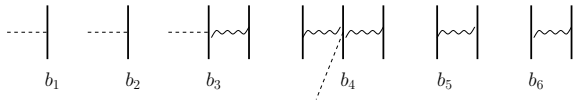

FIGURE 6. The vertex elements producing the diagram in Fig. 5, after a suitable contraction of the solid half-lines.

$S_{\Lambda,k}(b_1,\dots,b_k)$ will be cast into the form of new series in the *running coupling constants* introduced in Section 5.2. The iterative resummation procedure required for giving a well-defined meaning to the generating function has been described in detail in several specialized and review papers in the last 20 years, see [9, 10, 11, 12, 34, 49], and will be reviewed and adapted to the present case in Section 5 below.

4.2. **The height in the non-interacting case.** As a warm-up, and in order to introduce some basic ideas that will be important in Section 5, here we prove Theorem 1 in the special but important non-interacting case, $\lambda = \alpha = 0$. The strategy we use is convenient for the subsequent generalization to the interacting case. If $\lambda = 0$, then (4.3)-(4.6) lead to the following explicit representation (observe that in this case $V_\Lambda(\psi) = 0$ and $\mathcal{B}_\Lambda(\psi, \mathbf{J}) = -\sum_b J_b E_b^{(m)}$):

$$\langle \underbrace{h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}; \cdots ; h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}}_{n \text{ times}} \rangle_{\Lambda;\lambda=0,m} =$$

$$= \sum_{b_1 \in \mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}} \cdots \sum_{b_n \in \mathcal{C}^{(n)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}} \sigma_{b_1} \cdots \sigma_{b_n} \sum_{m_1=1}^{B(b'_1)} \cdots \sum_{m_s=1}^{B(b'_s)} (-1)^{m_1+\cdots+m_s} \times \tag{4.8}$$

$$\times P_{m_1}(B(b'_1)) \cdots P_{m_s}(B(b'_s)) \mathcal{E}^T(\underbrace{E^{(m)}_{b'_1}; \cdots ; E^{(m)}_{b'_1}}_{m_1 \text{ times}}; \cdots ; \underbrace{E^{(m)}_{b'_s}; \cdots ; E^{(m)}_{b'_s}}_{m_s \text{ times}})$$

where $\{b_1,\dots,b_n\} =: B$ should be thought of as a bond configuration (possibly with repetitions), $\tilde{B} = \{b'_1,\dots,b'_s\}$ as the support of B and $B(b'_i)$ as the multiplicity of b'_i, see the discussion after (4.6). Moreover,

$$P_m(N) := \frac{\partial^N}{\partial A^N} \frac{(e^A - 1)^m}{m!}\Big|_{A=0}. \tag{4.9}$$

Then, we take the limit $\lim_{m\to 0} \lim_{\Lambda \nearrow \mathbb{Z}^2}$: this simply means that in the computations of the averages $\mathcal{E}^T(\dots)$ all propagators $g^{(11)}_\Lambda(\mathbf{x},\mathbf{y})$ are replaced by $\lim_{m\to 0} g(\mathbf{x},\mathbf{y})$ and $E^{(m)}_b = t^{(m)}_b E_b$ is replaced by E_b.

Let us now discuss how to evaluate (4.8), separately for the cases $n = 2$ (the variance) and $n > 2$.

4.2.1. *The variance.* We assume for simplicity that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ have the same parity. We choose the two paths $\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}, \mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ in such a way that: (1) they are completely distinct, i.e., the bonds in $\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ are all different from those in $\mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$; (2) they are both of length comparable with $|\boldsymbol{\xi} - \boldsymbol{\eta}|$; (3) they consist of a union of straight portions, each of which is of even length. Moreover,

we assume that $\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}, \mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ are "well-separated", in the following sense. Fix $c, c' > 0$. Inside balls of radius $c|\boldsymbol{\xi}-\boldsymbol{\eta}|$ around $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, the two paths are portions of length $c|\boldsymbol{\xi}-\boldsymbol{\eta}|$ of infinite periodic paths (that is, they are portions of straight paths - apart from lattice discretization - see [46, Definition 2.1]) and have mutually different asymptotic directions, say opposite. Outside of these balls the paths stay at distance at least $c'|\boldsymbol{\xi}-\boldsymbol{\eta}|$ of each other and their length is of order $|\boldsymbol{\xi}-\boldsymbol{\eta}|$. See Fig.7. Using (4.8) for $n=2$ and the above assumptions on the paths, we can rewrite the variance as

$$\lim_{m\to 0}\lim_{\Lambda\nearrow\mathbb{Z}^2} \langle\, (h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}})^2 \,\rangle_{\Lambda;\lambda=0,m} = \sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}} \sum_{b_2\in\mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}} \sigma_{b_1}\sigma_{b_2}\mathcal{E}^T(E_{b_1}; E_{b_2}) \quad (4.10)$$

where the expectation $\mathcal{E}^T$ has propagator $\lim_{m\to 0} g(\mathbf{x},\mathbf{y})$. Let $b=(\mathbf{x},\mathbf{x}+\hat{e}_j)$ be a bond crossed say by $\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ and observe that (letting $(-1)^{\mathbf{x}} := (-1)^{x_1+x_2}$)

$$\sigma_b = \alpha_b (-1)^{\mathbf{x}} (-1)^j, \quad (4.11)$$

where α_b is $+1/-1$, depending on whether the bond b is crossed by the oriented path $\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ in the positive/negative direction (the positive direction is upwards for vertical portions of the paths, and rightwards for horizontal portions).

Next, we have to rewrite $\mathcal{E}^T(E_{b_1}; E_{b_2})$ and we start by expressing $E_{\mathbf{x},\mathbf{x}+\hat{e}_j} = i^{(j-1)}\psi_{\mathbf{x}}\psi_{\mathbf{x}+\hat{e}_j}$ in terms of Dirac variables:

(1) we replace each of the two fields by a combination of Dirac fields using (3.17);
(2) whenever $\psi^{\pm}_{\mathbf{x}+\hat{e}_j,\omega}$ appears we replace it by $\psi^{\pm}_{\mathbf{x},\omega} + \partial_j \psi^{\pm}_{\mathbf{x},\omega}$ with ∂_j the (right) discrete derivative in the j direction.

In this way we obtain (we skip lengthy but straightforward computations):

$$E_{\mathbf{x},\mathbf{x}+\hat{e}_1} = \mathcal{A}_{\mathbf{x},\mathbf{x}+\hat{e}_1} + \mathcal{R}_{\mathbf{x},\mathbf{x}+\hat{e}_1} \quad (4.12)$$

$$:= -2(-1)^{\mathbf{x}} \sum_{\omega} \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} - 2(-1)^{x_1} \sum_{\omega} \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},-\omega} + \mathcal{R}_{\mathbf{x},\mathbf{x}+\hat{e}_1} \quad (4.13)$$

$$E_{\mathbf{x},\mathbf{x}+\hat{e}_2} = \mathcal{A}_{\mathbf{x},\mathbf{x}+\hat{e}_2} + \mathcal{R}_{\mathbf{x},\mathbf{x}+\hat{e}_2} \quad (4.14)$$

$$:= -2i(-1)^{\mathbf{x}} \sum_{\omega} \omega\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} - 2i(-1)^{x_2} \sum_{\omega} \omega\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},-\omega} + \mathcal{R}_{\mathbf{x},\mathbf{x}+\hat{e}_2}, \quad (4.15)$$

where $\mathcal{R}$ is a linear combinations of terms of the type $\psi^{\varepsilon}_{\mathbf{x},\omega}\partial_j\psi^{\varepsilon'}_{\mathbf{x},\omega'}$, with $\varepsilon,\varepsilon' = \pm$. Let us consider first the "local parts" $\mathcal{A}_{\mathbf{x},\mathbf{x}+\hat{e}_i}$, i.e. let us neglect for the moment $\mathcal{R}$. When the path crosses the bond b, the change of position Δz_b in the complex plane is $i\alpha_b$ if b is horizontal and α_b is b is vertical. Therefore,

$$\sigma_b\mathcal{A}_b = -2i\Delta z_b \Big[\sum_{\omega} \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} + (-1)^{x_2} \sum_{\omega} \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},-\omega}\Big] \quad (4.16)$$

if b is horizontal, and

$$\sigma_b\mathcal{A}_b = -2i\Delta z_b \Big[\sum_{\omega} \omega\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} + (-1)^{x_1} \sum_{\omega} \omega\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},-\omega}\Big] \quad (4.17)$$

if b is vertical. At this point we can write, assuming for the moment that both b_1 and b_2 are horizontal bonds, i.e., $b_1 = (\mathbf{x}, \mathbf{x}+\hat{e}_1)$ and $b_2 = (\mathbf{y}, \mathbf{y}+\hat{e}_1)$,

$$\sigma_{b_1}\sigma_{b_2}\mathcal{E}^T(\mathcal{A}_{b_1};\mathcal{A}_{b_2}) = -8\mathrm{Re}\Big[\Delta z_{b_1}\Delta z_{b_2}\mathcal{E}^T(\psi^+_{\mathbf{x},+}\psi^-_{\mathbf{x},+};\psi^+_{\mathbf{y},+}\psi^-_{\mathbf{y},+})\Big] \tag{4.18}$$

$$-8(-1)^{x_2+y_2}\mathrm{Re}\Big[\Delta z_{b_1}\Delta z_{b_2}\mathcal{E}^T(\psi^+_{\mathbf{x},+}\psi^-_{\mathbf{x},-};\psi^+_{\mathbf{y},-}\psi^-_{\mathbf{y},+})\Big]. \tag{4.19}$$

Here we used the fact that $\Delta z_{b_1}\Delta z_{b_2}$ is real, that

$$\mathcal{E}^T(\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega};\psi^+_{\mathbf{y},\omega'}\psi^-_{\mathbf{y},-\omega'}) = 0$$

(because $\langle\psi^-_{\mathbf{x},-}\psi^+_{\mathbf{y},+}\rangle = -iG_{-+}(\mathbf{x}-\mathbf{y})$ vanishes when the mass m is zero as is the case here) and that

$$\mathcal{E}^T(\psi^+_{\mathbf{x}_1,-1}\psi^-_{\mathbf{x}_1,-1};\psi^+_{\mathbf{x}_2,-1}\psi^-_{\mathbf{x}_2,-1}) = \mathcal{E}^T(\psi^+_{\mathbf{x}_1,1}\psi^-_{\mathbf{x}_1,1};\psi^+_{\mathbf{x}_2,1}\psi^-_{\mathbf{x}_2,1})^*$$

(cf. (3.18) and the first of (3.8)). Using the Wick rule and the first and third of (3.8) we have, assuming that $b_1 = (\mathbf{x}, \mathbf{x}+\hat{e}_1)$ and $b_2 = (\mathbf{y}, \mathbf{y}+\hat{e}_1)$,

$$\sigma_{b_1}\sigma_{b_2}\mathcal{E}^T(E_{b_1};E_{b_2}) = -8\mathrm{Re}\Big[\Delta z_{b_1}\Delta z_{b_2}(G_{++}(\mathbf{x}-\mathbf{y}))^2\Big] \tag{4.20}$$

$$-8(-1)^{x_2+y_2}\Delta z_{b_1}\Delta z_{b_2}|G_{++}(\mathbf{x}-\mathbf{y})|^2. \tag{4.21}$$

In the general case $b_1 = (\mathbf{x}, \mathbf{x}+\hat{e}_{j_1}), b_2 = (\mathbf{y}, \mathbf{y}+\hat{e}_{j_2})$ one finds with similar computations

$$\sigma_{b_1}\sigma_{b_2}\mathcal{E}^T(\mathcal{A}_{b_1};\mathcal{A}_{b_2}) = -8\mathrm{Re}\Big[\Delta z_{b_1}\Delta z_{b_2}(G_{++}(\mathbf{x}-\mathbf{y}))^2\Big] + \tag{4.22}$$

$$+8\delta_{j_1=j_2}(-1)^{j_1}(-1)^{x_{3-j_1}+y_{3-j_1}}\Delta z_{b_1}\Delta z_{b_2}|G_{++}(\mathbf{x}-\mathbf{y})|^2. \tag{4.23}$$

Using Proposition 1 for $m = 0$ to express G as $\mathfrak{g}^{(0)}$ plus a fast decaying remainder, we have

$$\sigma_{b_1}\sigma_{b_2}\mathcal{E}^T(\mathcal{A}_{b_1};\mathcal{A}_{b_2}) = -\,\mathrm{Re}\,\Big[\Delta z_{b_1}\Delta z_{b_2}\frac{1}{2\pi^2}\frac{\mathbf{1}_{\mathbf{x}\neq\mathbf{y}}}{(z_{\mathbf{y}}-z_{\mathbf{x}})^2}\Big] + \tag{4.24}$$

$$+\delta_{j_1=j_2}(-1)^{j_1}(-1)^{x_{3-j_1}+y_{3-j_1}}\Delta z_{b_1}\Delta z_{b_2}\frac{1}{2\pi^2}\frac{\mathbf{1}_{\mathbf{x}\neq\mathbf{y}}}{|z_{\mathbf{y}}-z_{\mathbf{x}}|^2} + R(\mathbf{x}-\mathbf{y})\ , \tag{4.25}$$

where $z_{\mathbf{x}} = x_1 + ix_2$ and $|R(\mathbf{x}-\mathbf{y})| \le (\text{const.})(1+|\mathbf{x}-\mathbf{y}|)^{-3}$. Now we can sum over b_i in the paths $\mathcal{C}^{(i)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}, i = 1, 2$. The contribution from R is of order 1 uniformly in $\boldsymbol{\xi}, \boldsymbol{\eta}$: to see this, use the properties of the paths spelled out at the beginning of this subsection. The same holds for the second term, this time because of the oscillating factor $(-1)^{x_{3-j_1}+y_{3-j_1}}$, that in the sum has the effect of a discrete derivative in the direction $3-j_1$: in fact, recall that the paths are assumed to consist of unions of straight portions of even length; on each such portion, the sum over $\mathbf{x}$ and $\mathbf{y}$ of $(-1)^{x_{3-j_1}+y_{3-j_1}}|z_{\mathbf{x}}-z_{\mathbf{y}}|^{-2}$ is of the same order as the sum of $\partial_{x_{3-j_1}}\partial_{y_{3-j_1}}|z_{\mathbf{x}}-z_{\mathbf{y}}|^{-2}$, which decays at large distances like $|\mathbf{x}-\mathbf{y}|^{-4}$. As for the first term, it produces the Riemann approximation to the integral

$$-\frac{1}{2\pi^2}\mathrm{Re}\int_{z_{\boldsymbol{\xi}}}^{z_{\boldsymbol{\eta}}} dz\int_{z'_{\boldsymbol{\xi}}}^{z'_{\boldsymbol{\eta}}} dw\frac{1}{(z-w)^2} \tag{4.26}$$

(here $z'_{\boldsymbol{\xi}}$ and $z'_{\boldsymbol{\eta}}$ are points at a distance $O(1)$ from $z_{\boldsymbol{\xi}}$ and $z_{\boldsymbol{\eta}}$, respectively), and differs from it by a constant, independent of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. This integral is the

same found in [45] (see the second equation at p.1043); it can be explicitly evaluated and gives (see the third and fourth equation at p.1043 of [45]):

$$\frac{1}{2\pi^2}\mathrm{Re}\log\frac{(z'_{\boldsymbol{\eta}}-z_{\boldsymbol{\xi}})(z'_{\boldsymbol{\xi}}-z_{\boldsymbol{\eta}})}{(z'_{\boldsymbol{\eta}}-z_{\boldsymbol{\eta}})(z'_{\boldsymbol{\xi}}-z_{\boldsymbol{\xi}})}=\frac{1}{\pi^2}\log|z_{\boldsymbol{\xi}}-z_{\boldsymbol{\eta}}|+O(1)\ . \tag{4.27}$$

It remains to study the contribution coming from the error terms $\mathcal{R}_b$ in (4.12),(4.14), that we disregarded so far.

Remark 3. *Each remainder $\mathcal{R}_{b_i}$ is a linear combination of terms like $\psi^{\varepsilon}_{\omega}\partial\psi^{\varepsilon'}_{\omega'}$, all localized in the vicinity of $\mathbf{x}_i$, where $\mathbf{x}_i$ is such that $b_i=(\mathbf{x}_i,\mathbf{x}_i+\hat{e}_{j_i})$. Therefore, we can symbolically write the contribution to the height variance from all the terms containing at least one term $\mathcal{R}_{b_i}$ as*

$$\tilde{R}(\boldsymbol{\xi}-\boldsymbol{\eta})=\sum_{\substack{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}},\\ b_2\in\mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}}\sum_{\substack{\omega_1,\dots,\omega'_2\\ \varepsilon_1,\dots,\varepsilon'_2}}\Big[\mathcal{E}^T((\psi^{\varepsilon_1}_{\omega_1}\partial\psi^{\varepsilon'_1}_{\omega'_1})(\mathbf{x}_1);(\psi^{\varepsilon_2}_{\omega_2}\psi^{\varepsilon'_2}_{\omega'_2})(\mathbf{x}_2))+ \tag{4.28}$$

$$+\mathcal{E}^T((\psi^{\varepsilon_1}_{\omega_1}\psi^{\varepsilon'_1}_{\omega'_1})(\mathbf{x}_1);(\psi^{\varepsilon_2}_{\omega_2}\partial\psi^{\varepsilon'_2}_{\omega'_2})(\mathbf{x}_2))+\mathcal{E}^T((\psi^{\varepsilon_1}_{\omega_1}\partial\psi^{\varepsilon'_1}_{\omega'_1})(\mathbf{x}_1);(\psi^{\varepsilon_2}_{\omega_2}\partial\psi^{\varepsilon'_2}_{\omega'_2})(\mathbf{x}_2))\Big]\ .$$

Using (3.18) to express the propagator of the Dirac fields $\psi^{\pm}_{\omega}$ in terms of the propagator $G(\mathbf{x})$ of the Majorana fields ψ_γ and the decay properties of $G(\mathbf{x})$ stated in Proposition 1, we can bound the expression in square brackets by a constant times $(1+|\mathbf{x}_1-\mathbf{x}_2|)^{-3}$, so that, recalling that $b_i=(\mathbf{x}_i,\mathbf{x}_i+\hat{e}_{j_i})$,

$$|\tilde{R}(\boldsymbol{\xi}-\boldsymbol{\eta})|\le\sum_{\substack{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}},\\ b_2\in\mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}}\frac{C}{(1+|\mathbf{x}_1-\mathbf{x}_2|)^3}\le C'\ , \tag{4.29}$$

for suitable constants $C,C'>0$. Putting all together, we find

$$\Big|\langle h_{\boldsymbol{\xi}}-h_{\boldsymbol{\eta}};h_{\boldsymbol{\xi}}-h_{\boldsymbol{\eta}}\rangle_{\lambda=0}-\frac{1}{\pi^2}\log|z_{\boldsymbol{\xi}}-z_{\boldsymbol{\eta}}|\Big|\le C''\ , \tag{4.30}$$

as desired, since $|z_{\boldsymbol{\xi}}-z_{\boldsymbol{\eta}}|=|\boldsymbol{\xi}-\boldsymbol{\eta}|$.

4.2.2. *The n-point function.* Here we discuss how to evaluate (4.8) for $n>2$. Again, we assume for simplicity that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ have the same parity. As in the case of the variance, we fix $c_n,c'_n>0$, and we assume that the n paths satisfy the following: (i) inside balls of radius $c_n|\boldsymbol{\xi}-\boldsymbol{\eta}|$ around $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, the n paths are portions of length $c_n|\boldsymbol{\xi}-\boldsymbol{\eta}|$ of infinite periodic paths and have mutually different asymptotic directions, say $(\cos\theta_j,\sin\theta_j)$, with $\theta_j=2\pi j/n$, and $j=0,\dots,n-1$; (ii) outside of these balls they stay at distance at least $c'_n|\boldsymbol{\xi}-\boldsymbol{\eta}|$ of each other and their length is of order $|\boldsymbol{\xi}-\boldsymbol{\eta}|$. See Fig.7

Moreover, we require that the n paths consist of unions of straight portions of even length. Note that if $b_i=b_j$ with $i\neq j$ in (4.8), then b_i is at a distance smaller than r_n from $\boldsymbol{\xi}$, or from $\boldsymbol{\eta}$. Here and below $C_n,C'_n,\dots$, and $c_n,c'_n,\dots$ denote n-dependent constants, which might change from line to line. If we drop the index n, it means that the constants can be chosen independent of n.

We rewrite (4.8) as the contribution from the bonds $b_1,\dots,b_n$ that are all outside the balls $B_{r_n}(\boldsymbol{\xi})$ and $B_{r_n}(\boldsymbol{\eta})$ of radius r_n around $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, plus a

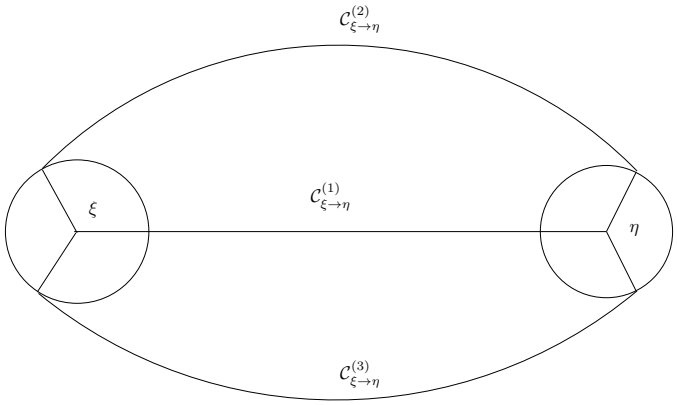

FIGURE 7. A schematic view of the paths $\mathcal{C}^{(i)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ for $n = 3$. Near $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, paths are essentially linear for a length proportional to $|\boldsymbol{\xi}-\boldsymbol{\eta}|$, with non-zero mutual angles.

rest (and the limit $\lim_{m\to 0}\lim_{\Lambda\nearrow\mathbb{Z}^2}$ has been already taken):

$$\langle \underbrace{h_{\boldsymbol{\xi}}-h_{\boldsymbol{\eta}};\cdots;h_{\boldsymbol{\xi}}-h_{\boldsymbol{\eta}}}_{n\text{ times}}\rangle_{\lambda=0} = D_n(\boldsymbol{\xi},\boldsymbol{\eta})+R_n(\boldsymbol{\xi},\boldsymbol{\eta}) \tag{4.31}$$

$$:= \sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*}\cdots\sum_{b_n\in\mathcal{C}^{(n)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*}\sigma_{b_1}\cdots\sigma_{b_n}(-1)^n\mathcal{E}^T(E_{b_1};E_{b_2};\cdots;E_{b_n})+R_n(\boldsymbol{\xi},\boldsymbol{\eta})\ ,$$

where the $*$ on the sums indicate the constraints that the b_i's are at a distance larger than r_n from $\boldsymbol{\xi}$ and from $\boldsymbol{\eta}$, and we used the fact that such constrained sums involve n-ples of bonds that are all distinct from each other. The rest $R_n(\boldsymbol{\xi},\boldsymbol{\eta})$ contains all the remaining contributions, including those where some of the bonds are coinciding.

We start by analyzing the dominant term, namely $D_n(\boldsymbol{\xi},\boldsymbol{\eta})$. With the notations of (4.12), we write

$$D_n(\boldsymbol{\xi},\boldsymbol{\eta}) = \sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*}\cdots\sum_{b_n\in\mathcal{C}^{(n)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*}(-1)^n\sigma_{b_1}\ldots\sigma_{b_n}\mathcal{E}^T(\mathcal{A}_{b_1};\ldots;\mathcal{A}_{b_n})+D'_n(\boldsymbol{\xi},\boldsymbol{\eta}) \tag{4.32}$$

where $D'_n(\boldsymbol{\xi},\boldsymbol{\eta})$ collects all the terms containing at least one remainder term $\mathcal{R}_{b_i}$ and can be symbolically written (see Remark 3) as

$$\begin{aligned} D'_n(\boldsymbol{\xi},\boldsymbol{\eta}) \;=\; & \sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*}\cdots\sum_{b_n\in\mathcal{C}^{(n)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*}\sum_{\substack{\omega_1,\ldots,\omega'_n\\ \varepsilon_1,\ldots,\varepsilon'_n}}\sum_{\substack{\alpha_1,\ldots,\alpha_n:\\ \sum_i\alpha_i>0}}\times \\ & \times\mathcal{E}^T((\psi^{\varepsilon_1}_{\omega_1}\partial^{\alpha_1}\psi^{\varepsilon'_1}_{\omega'_1})(\mathbf{x}_1);\cdots;(\psi^{\varepsilon_n}_{\omega_n}\partial^{\alpha_n}\psi^{\varepsilon'_n}_{\omega'_n})(\mathbf{x}_n))\ , \end{aligned} \tag{4.33}$$

where in the last sum $\alpha_i\in\{0,1\}$. Once again, $\mathbf{x}_i$ is one of the sites of bond b_i. Using the definition of truncated expectation and Wick's rule, the second

line can be evaluated as

$$- \sum_{\pi \text{ on } \{2,\dots,n\}} \delta_{\omega'_1,\omega_{\pi(2)}} \cdots \delta_{\omega'_{\pi(n)},\omega_1} \delta_{\varepsilon'_1,-\varepsilon_{\pi(2)}} \cdots \delta_{\varepsilon'_{\pi(n)},-\varepsilon_1} \times \tag{4.34}$$
$$\times\, \partial^{\alpha_1} G_{\omega'_1\omega'_1}(\mathbf{x}_1 - \mathbf{x}_{\pi(2)}) \cdots \partial^{\alpha_2} G_{\omega_1\omega_1}(\mathbf{x}_{\pi(n)} - \mathbf{x}_1) \ ,$$

where the sum runs over the permutations of $\{2,\dots,n\}$. Finally, using Proposition 1 and substituting the result back into (4.33), we find:

$$|D'_n(\boldsymbol{\xi},\boldsymbol{\eta})| \le C_n \sum_{b_1 \in \mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}} \cdots \sum_{b_n \in \mathcal{C}^{(n)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}} \sum_{\pi \text{ on } \{2,\dots,n\}} \times \tag{4.35}$$
$$\times \sum_{j=1}^{n} \frac{1}{|\mathbf{x}_1 - \mathbf{x}_{\pi(2)}|} \cdots \frac{1}{|\mathbf{x}_{\pi(j)} - \mathbf{x}_{\pi(j+1)}|^2} \cdots \frac{1}{|\mathbf{x}_{\pi(n)} - \mathbf{x}_1|}$$

where the square in one of the denominators is due to the fact that at least one propagator comes with a derivative.

As for $\mathcal{E}^T(\sigma_{b_1}\mathcal{A}_{b_1};\dots;\sigma_{b_n}\mathcal{A}_{b_n})$, going back to (4.16)-(4.17) we see that we can distinguish two contributions: one that collects all terms without oscillating pre-factors $(-1)^{x_i}, i = 1,2$ and one that contains at least one term with oscillating factor.

Let us look at the latter first. When we sum over $b_1,\dots,b_n$, we remarked in Section 4.2.1 that the effect of an oscillating factor $(-1)^{x_j}$ is the same as a discrete derivative ∂_j acting on a propagator. Therefore, the contribution to the n-th cumulant can be bounded exactly like $D'_n(\boldsymbol{\xi},\boldsymbol{\eta})$ in (4.35), with a different constant prefactor C'_n.

Next, we look at the term without oscillating factors. In analogy with the derivation of the first term in the r.h.s. of (4.18), one can check that we get

$$2^n \cdot 2 \cdot \mathrm{Re}\Big[(-i)^n \, \Delta z_{b_1} \dots \Delta z_{b_n} \mathcal{E}^T(\psi^+_{\mathbf{x}_1,1}\psi^-_{\mathbf{x}_1,1};\cdots;\psi^+_{\mathbf{x}_n,1}\psi^-_{\mathbf{x}_n,1})\Big] \ . \tag{4.36}$$

The truncated expectation in (4.36) can be evaluated via Wick's rule as:

$$\mathcal{E}^T(\psi^+_{\mathbf{x}_1,1}\psi^-_{\mathbf{x}_1,1};\cdots;\psi^+_{\mathbf{x}_n,1}\psi^-_{\mathbf{x}_n,1}) = \tag{4.37}$$
$$= - \sum_{\pi \text{ on } \{2,\dots,n\}} G_{++}(\mathbf{x}_1 - \mathbf{x}_{\pi(2)}) \cdots G_{++}(\mathbf{x}_{\pi(n)} - \mathbf{x}_1) \ .$$

Plugging the decomposition (3.9) into (4.37) (and recalling that $m = 0$) gives

$$- \sum_{\pi \text{ on } \{2,\dots,n\}} \mathfrak{g}^{(0)}_{++}(\mathbf{x}_1 - \mathbf{x}_{\pi(2)}) \cdots \mathfrak{g}^{(0)}_{++}(\mathbf{x}_{\pi(n)} - \mathbf{x}_1) + R'(\mathbf{x}_1,\dots,\mathbf{x}_n) \ ,$$

where R' collects all the terms involving at least one factor $R(\mathbf{x}-\mathbf{x}')$ from (3.9). Now, a well known combinatorial identity (see e.g. [28, Eq. (D.29)]) states that, if $n \ge 3$ and $\mathbf{x}_1,\dots,\mathbf{x}_n$ are all distinct, then

$$\sum_{\pi \text{ on } \{2,\dots,n\}} \mathfrak{g}^{(0)}_{++}(\mathbf{x}_1 - \mathbf{x}_{\pi(2)}) \cdots \mathfrak{g}^{(0)}_{++}(\mathbf{x}_{\pi(n)} - \mathbf{x}_1) = 0 \ . \tag{4.38}$$

Therefore, the only non vanishing contributions to the expression in (4.36) come from the terms involving at least one factor $R(\mathbf{x}-\mathbf{x}')$. If we use (3.12), we see that the expression in braces in (4.36) is bounded qualitatively in the same way as (4.35), possibly with C_n replaced by a new constant C''_n.

In order to evaluate the sum (4.35), we resort to a multiscale decomposition and a tree expansion that are typical of constructive quantum field theory. While in the non-interacting case $\lambda = 0$ this could be avoided, this is the right approach that can be generalized to the interacting case.

First we bound the r.h.s. by using the following "multiscale decomposition" (valid for $|\mathbf{x}| \geq 1$ and $s > 0$):

$$\frac{1}{|\mathbf{x}|^s} \leq \kappa_s \sum_{h \leq 0} 2^{hs} e^{-2^h |\mathbf{x}|} \,, \tag{4.39}$$

which implies

$$|D'_n(\boldsymbol{\xi}, \boldsymbol{\eta})| \leq C_n \sum_{b_1 \in \mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}} \cdots \sum_{b_n \in \mathcal{C}^{(n)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}} \sum_{\pi \text{ on } \{2,\dots,n\}} \times \tag{4.40}$$
$$\times \sum_{j=1}^{n} \sum_{h_1,\dots,h_n \leq 0} 2^{h_j} \Big[\prod_{k=1}^{n} 2^{h_k} e^{-2^{h_k} |\mathbf{x}_{\pi(k)} - \mathbf{x}_{\pi(k+1)}|} \Big] \,,$$

where $\pi(1)$ and $\pi(n+1)$ should be interpreted as being equal to 1. The label h_k has the interpretation of "scale label" of the propagator from $\mathbf{x}_{\pi(k)}$ to $\mathbf{x}_{\pi(k+1)}$, and the quantity $2^{h_k} e^{-2^{h_k} |\mathbf{x}_{\pi(k)} - \mathbf{x}_{\pi(k+1)}|}$ is called its "dimensional contribution". Note that the r.h.s. of (4.40) has a natural graphical interpretation in terms of the "sun" diagram in Fig. 8, representing n vertices of type $-J_b E_b$ (see Fig. 4), to be called $v_1, \dots, v_n$, connected by propagators (the solid lines) in a loop.

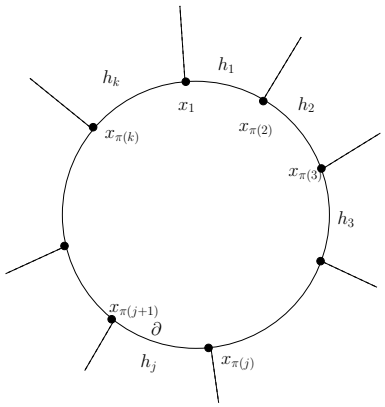

FIGURE 8. The graphical interpretation of the r.h.s. of (4.40): the figure represents n vertices of type $-J_b E_b$ connected by propagators in a loop. Each propagator is associated with a scale label h_k. The symbol ∂ on the propagator connecting $\mathbf{x}_{\pi(j)}$ with $\mathbf{x}_{\pi(j+1)}$ recalls that such propagator, from a dimensional point of view, behaves like (the single-scale contribution to) $\partial G_{++}(\mathbf{x}_{\pi(j)} - \mathbf{x}_{\pi(j+1)})$.

Now we can sum over $b_1, \dots, b_n$ (which is the same as summing over $\mathbf{x}_1, \dots, \mathbf{x}_n$), observing that each sum is one-dimensional (b_i and, therefore, $\mathbf{x}_i$ run along the path $\mathcal{C}^{(i)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$) and that, thanks to the way the paths were

chosen, $|\mathbf{x}_i - \mathbf{x}_j| \geq c_n(d_i + d_j)$, with $d_i = \min\{d(\mathbf{x}_i, \boldsymbol{\xi}), d(\mathbf{x}_i, \boldsymbol{\eta})\}$ and e.g. $d(\mathbf{x}_i, \boldsymbol{\xi})$ the distance between $\boldsymbol{\xi}$ and $\mathbf{x}_i$ along $\mathcal{C}^{(i)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$. Then,

$$\prod_{k=1}^{n} e^{-2^{h_k}|\mathbf{x}_{\pi(k)}-\mathbf{x}_{\pi(k+1)}|} \leq \prod_{k=1}^{n} e^{-c_n d_{\pi(k)}(2^{h_k}+2^{h_{k-1}})} , \tag{4.41}$$

where h_0 should be interpreted as being equal to h_n (i.e., the ordered sequence $(h_1, \dots, h_n)$ should be thought of as having "periodic boundary conditions", as suggested by Fig. 8). The k-th factor can now be easily summed over $b_{\pi(k)}$ and gives:

$$\sum_{b_{\pi(k)} \in \mathcal{C}^{(\pi(k))}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}} e^{-c_n d_{\pi(k)}(2^{h_k}+2^{h_{k-1}})} \leq 2\sum_{d=0}^{\infty} e^{-c_n d\cdot(2^{h_k}+2^{h_{k-1}})} \leq C'_n 2^{-\max\{h_k,h_{k-1}\}} . \tag{4.42}$$

Plugging these bounds into (4.40) gives

$$|D'_n(\boldsymbol{\xi}, \boldsymbol{\eta})| \leq C''_n \sum_{h_1,\dots,h_n \leq 0} 2^{h^*} \Big[\prod_{k=1}^{n} 2^{h_k} 2^{-\max\{h_k,h_{k-1}\}} \Big] , \tag{4.43}$$

where $h^* := \max_{j=1,\dots,n} h_j$. The sum over the h_i's in the r.h.s. of (4.43) can be performed in various ways. We follow a specific strategy (possibly not the most straightforward), which admits a natural generalization to the interacting case.

We say that a group of vertices forms a cluster on scale h if:

- the vertices are connected in the sub-graph where only lines on scale $h' \geq h$ are drawn;
- the group of vertices is maximal (i.e. no other vertex can be added while keeping the first property).

With this definition, every cluster contains at least 2 vertices. Note that the same group of vertices can be a cluster on various different scales. Every choice of $(h_1, \dots, h_n)$ defines a set of clusters, which are partially ordered in the natural sense induced by the subset relation: if a cluster v on scale h *strictly* contains a cluster v' on scale h', then $h' > h$, and we shall say that v' follows v. More in general, if v on scale h contains a cluster v' on scale $h' > h$, we say that v' follows v. In this sense, every choice of $(h_1, \dots, h_n)$ defines a cluster structure. An example is shown in Fig. 9.

The partial ordering introduced above allows to represent a cluster structure as a tree, see Fig. 10. The tree can be drawn on a grid of vertical lines, each associated with its scale label, and ordered from left to right, from the scale of the root (which is by convention one unit smaller than $\min_j h_j$) to 1. Vertices v_i correspond to endpoints (leaves) of the tree, which are all drawn by convention on the vertical line of scale 1. The intersections between the vertical lines and the tree are called nodes. All the nodes followed by at least two endpoints correspond to clusters: the cluster of scale h_v associated with such a node v is the set of endpoints following v on τ; in terms of this definition, it is natural to think the endpoints, as well as the nodes followed by just one endpoint, as (trivial) clusters. Given the tree, the cluster structure can be reconstructed unambiguously. If we identify trees obtained from

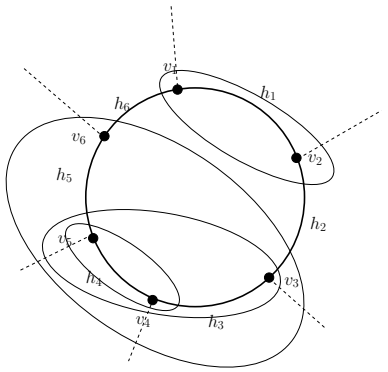

FIGURE 9. An example of a graph with (part of) its cluster structure. In the picture it is assumed that $h_6 < h_2 < h_1$ and $h_2 < h_5 < h_3 < h_4$. The cluster on scale h_2 is not indicated explicitly.

each other by pivoting the branches on the branching points, then the trees are in one-to-one correspondence with the cluster structures.

In Section 5, when analyzing the interacting model, we will need a more general class of trees.

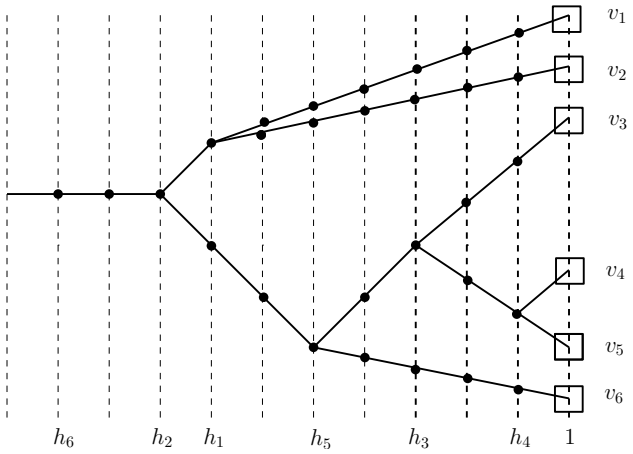

FIGURE 10. The tree representing the hierarchical cluster structure of the labelled graph in Fig. 9. Dots are nodes, squares are endpoints.

Every labelled tree can be naturally thought of as a "topological" (i.e., unlabeled) tree, together with its scale labels. The idea is to first sum over the scale labels at fixed topological tree, and then sum over the topological trees. This can be done very easily: calling $\mathcal{T}^0_{h;n}$ the family of labelled trees with n endpoints and root on scale h that we just introduced, (4.43) implies

$$|D'_n(\boldsymbol{\xi},\boldsymbol{\eta})| \le C'''_n \sum_{h<0} \sum_{\tau\in\mathcal{T}^0_{h;n}} 2^{h^*_\tau} \prod_{v\in V(\tau)} 2^{h_v \tilde{n}_v} \prod_{v\in V_{nt}(\tau)} 2^{-h_v \bar{m}^J_v} , \qquad (4.44)$$

where: (i) $V(\tau)$ is the set of nodes of τ that are neither endpoints nor the root; (ii) $V_{nt}(\tau)$ is the set of branching points of τ; (iii) $h^*_\tau = \max_{v\in V_{nt}(\tau)} h_v$;

(iv) $\tilde{n}_v$ is the number of propagators contained in the cluster v but not in any other cluster $v' > v$ [we say that a propagator is contained in a cluster v of scale h_v if it connects two endpoints in v, and if its scale is $\geq h_v$]; (v) if $v \in V_{nt}(\tau)$, then $\bar{m}_v^J$ is the number of endpoints contained in the cluster v but not in any other cluster $v' \in V_{nt}(\tau)$ such that $v' > v$. The exponent in the last product can be rewritten as follows. First note that, given a function f_v on $V(\tau)$ one has

$$\sum_{v\in V(\tau)} h_v f_v = h \sum_{v\in V(\tau)} f_v + \sum_{v\in V(\tau)} \sum_{\substack{w\in V(\tau):\\ w\geq v}} f_w \tag{4.45}$$

with h as usual the scale of the root. Similarly,

$$\sum_{v\in V_{nt}(\tau)} h_v f_v = h \sum_{v\in V_{nt}(\tau)} f_v + \sum_{v\in V_{nt}(\tau)} (h_v - h_{v'}) \sum_{\substack{w\in V_{nt}(\tau):\\ w\geq v}} f_w, \tag{4.46}$$

where, given $v \in V_{nt}(\tau)$, we denoted by v' the rightmost node in $V_{nt}(\tau)$ preceding v on τ (if v is the leftmost node in $V_{nt}(\tau)$, then we let $h_{v'} = h$). On the other hand, if n_v^e is the number of solid lines exiting from the cluster v in the Feynman diagram, see Fig. 8, and m_v^J is the number of endpoints following v, one has

$$\sum_{\substack{v\in V(\tau):\\ v\geq w}} \tilde{n}_v = m_w^J - \frac{n_w^e}{2} \ , \tag{4.47}$$

which can be easily proved by induction. Similarly, if $w \in V_{nt}(\tau)$, then $\sum_{\substack{v\in V_{nt}(\tau):\\ v\geq w}} \bar{m}_v^J = m_w^J$. Then, one deduces

$$\sum_{v\in V(\tau)} h_v \tilde{n}_v - \sum_{v\in V_{nt}(\tau)} h_v \bar{m}_v^J = - \sum_{w\in V^*(\tau)} n_w^e/2 \ , \tag{4.48}$$

where $V^*(\tau) = \{v \in V(\tau) : m_v^J > 1\}$ and we used the fact that $\sum_{v\in V(\tau)} \tilde{n}_v = \sum_{v\in V_{nt}(\tau)} \bar{m}_v^J$. Moreover, $n_v^e = 2$ for every cluster except the one at scale $h+1$ (just look at Fig. 8). Therefore, plugging (4.48) back into (4.44) gives

$$|D'_n(\boldsymbol{\xi}, \boldsymbol{\eta})| \leq 2C'''_n \sum_{h<0} \sum_{\tau\in\mathcal{T}^0_{h;n}} 2^{h^*_\tau} \prod_{v\in V^*(\tau)} 2^{-1} \ , \tag{4.49}$$

which readily shows that the sum over the scale labels is convergent (first sum over the scale labels h_v at fixed h^*_τ, and then over $h^*_\tau \leq 0$): finally, we multiply by the number of topological trees with n endpoints, which is a constant depending only on n, so that

$$|D'_n(\boldsymbol{\xi}, \boldsymbol{\eta})| \leq C''''_n \ , \tag{4.50}$$

as desired.

We are left with the rest $R_n(\boldsymbol{\xi}, \boldsymbol{\eta})$ in (4.31), which is easier to analyze. In order to estimate it, we do not even need to use the cancellation (4.38).

Proceeding as above[3], we find the analogue of (4.35):

$$|R_n(\boldsymbol{\xi},\boldsymbol{\eta})| \le C_n \overset{\circ}{\sum_{\substack{b_i\in\mathcal{C}^{(i)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}\\ i=1,\dots,n}}} \sum_{\pi \text{ on } \{2,\dots,n\}} \frac{1}{1+|\mathbf{x}_1-\mathbf{x}_{\pi(2)}|}\cdots\frac{1}{1+|\mathbf{x}_{\pi(n)}-\mathbf{x}_1|}\,, \tag{4.51}$$

where the $\circ$ on the sum indicates the constraint that at least one coordinate belongs to $B_{r_n}(\boldsymbol{\xi})\cup B_{r_n}(\boldsymbol{\eta})$. Note that, contrary to (4.35), here all the decay powers in the factors in the r.h.s. are equal to 1: in (4.35) one of the propagators decayed faster (as $(dist)^{-2}$), as a consequence of the cancellation (4.38) that we are not using here. However, the constraint that at least one coordinate belongs to an $O(1)$ region (with respect to $|\boldsymbol{\xi}-\boldsymbol{\eta}|$) is enough for inducing a bound analogous to (4.50). Indeed, as in the derivation of (4.40),

$$|R_n(\boldsymbol{\xi},\boldsymbol{\eta})| \le C'_n \overset{\circ}{\sum_{\substack{b_i\in\mathcal{C}^{(i)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}\\ i=1,\dots,n}}} \sum_{\pi \text{ on } \{2,\dots,n\}} \sum_{h_1,\dots,h_n\le 0} \Big[\prod_{k=1}^{n} 2^{h_k} e^{-2^{h_k}|\mathbf{x}_{\pi(k)}-\mathbf{x}_{\pi(k+1)}|}\Big]\,. \tag{4.52}$$

After summing over $b_1,\dots,b_n$, we get

$$|R_n(\boldsymbol{\xi},\boldsymbol{\eta})| \le C''_n \sum_{h_1,\dots,h_n\le 0} 2^{\max_j h_j}\Big[\prod_{k=1}^{n} 2^{h_k} 2^{-\max\{h_k,h_{k-1}\}}\Big]\,, \tag{4.53}$$

where the gain factor $2^{\max_j h_j}$ arises from the fact that at least one of the coordinates $\mathbf{x}_i$ is not summed over (or, more precisely, is summed over a region of size r_n) and, therefore, at least one of the factors $2^{-\max\{h_k,h_{k-1}\}}$ in the right side of (4.53) in reality should not be there (in fact, recall that these factors come from (4.42); if the sum over d from 0 to ∞ in (4.42) is replaced by a sum over a finite set of nonnegative integers, then the right side of (4.42) can be replaced by a constant C'_n). The right side of (4.53) is the same as (4.43) and, therefore, leads to the analogue of (4.50): $|R_n(\boldsymbol{\xi},\boldsymbol{\eta})|\le C_n$. This concludes the proof of the main theorem in the case $\lambda = 0$.

5. The interacting case: constructive RG

5.1. **Failure of Feynman graph expansion.** We have now to compute the n-point dimer correlation given by (4.3) in the $\lambda\neq 0$ case, which requires computing the kernels $S_{\Lambda,n}(b_1,\dots,b_n)$ in (4.6). As explained in Section 4, the dimer correlations can be written as power series in $\alpha = e^{\lambda}-1$ by using (4.6), and each order of the expansion can be represented diagrammatically as a finite sum over Feynman diagrams. Such diagrams are more complicated but qualitatively similar to the ones appearing in §4.2.2, and one can bound them by using a multiscale decomposition of the propagator similar in spirit to (4.39) (except that it will be an identity instead of a bound) together with an organization in trees analogous to those in Fig. 10; in the interacting case, these trees are known as *Gallavotti-Nicolò* trees, first

[3]To be precise, when applying (4.8) one should take into account the multiplicity of the coinciding bonds. Since these multiplicities are bounded by n, this only changes the constants C_n below.

introduced in [33] for studying the renormalization theory of the φ^4_4 Quantum Field Theory (QFT), and later applied to several other problems in statistical mechanics and field theory (for a detailed derivation of the tree expansion, see e.g. [31] and the more recent reviews [34, 36, 49]; a description of its main features is summarized below, for completeness). Natural as it appears, such a strategy *does not work*, and actually perturbation expansion in Feynman diagrams does not provide any information on the interacting dimer correlations. As this is a key point in order to understand the motivations of the more elaborate analysis in the following sections, it is convenient to explain why the power series expansion in Feynman diagrams does not work, i.e., it cannot be proved directly to be convergent.

As discussed after (4.6), the n-point dimer correlations $S_{\Lambda,n}(b_1,\dots,b_n)$ in the interacting case can be written as sums of Feynman graphs like the ones in Fig. 5. Among the allowed Feynman graphs there are both the simple "sun" graphs appearing in the non interacting case, such as that in Fig. 8, and the much more complicated diagrams (e.g. Fig. 6) obtained by contracting in all possible connected ways the vertices in Fig. 3-4. In order to estimate the value of a generic Feynman graph, we use a multiscale decomposition of the propagator analogous to (4.39). More precisely, assume that $L^{-1} \ll m \ll 1$ and let $h^* = \lfloor \log_2 m \rfloor$. We rewrite the propagator $g(\mathbf{x},\mathbf{y})$ as in (3.2) and then, via (3.5) and (3.6), we re-express it in terms of the propagator $G(\mathbf{x})$ defined in (3.7) (or, better, defined as the finite volume counterpart of (3.7) with boundary conditions $(\theta,\tau)=(1,1)$, in which case the integrals over $\mathbf{k}$ are sums[4] in $\mathcal{D}^{(11)}_\Lambda$). Next we decompose $G(\mathbf{x})$ as

$$G(\mathbf{x}) = \sum_{h=h^*+1}^{0} G^{(h)}(\mathbf{x}) + G^{(\le h^*)}(\mathbf{x}) \,, \tag{5.1}$$

where $G^{(h)}$ (resp. $G^{(\le h^*)}$) is as in (3.7), except that $\chi(\mathbf{k})$ is replaced by

$$f_h(\mathbf{k}) := \chi_h(\mathbf{k}) - \chi_{h-1}(\mathbf{k}) \tag{5.2}$$

(resp. by $\chi_{h^*}(\mathbf{k})$); here $\chi_0(\mathbf{k}) := \bar\chi(\mathbf{k})$, while $\chi_h(\mathbf{k}) := \chi(2^{-h}\mathbf{k})$, $\forall h<0$ (functions $\bar\chi(\cdot)$ and $\chi(\cdot)$ were defined in Section 3). Observe that f_h (resp. χ_{h^*}) is in the Gevrey class of order 2 with compact support contained in

$$S_h := \{\mathbf{k}\in\mathbb{T}^2 :\ c2^h \le \sin^2 k_1 + \sin^2 k_2 \le C2^h\} \,, \tag{5.3}$$

(resp. in $\cup_{h\le h^*} S_h$) for suitable constants $c, C>0$, and that $\sum_{h=h^*+1}^0 f_h(\mathbf{k}) + \chi_{h^*}(\mathbf{k}) = \bar\chi(\mathbf{k})$. Moreover, $\bar\chi(\mathbf{k})$ and $\chi(\mathbf{k})$ can be chosen in such a way that

$$f_{h_1}(\mathbf{k}) f_{h_2}(\mathbf{k}) = 0 \,, \quad \text{if} \quad |h_1-h_2|>1 \tag{5.4}$$

(it is enough to use the explicit functions $\bar\chi(\mathbf{k})$ and $\chi(\mathbf{k})$ constructed after (3.4) and choose the parameter ε small enough); in order to simplify certain technical issues, we shall assume the validity of (5.4) from now on.

Observe also that for $h^* < h \le 0$ the matrix $G^{(h)}(\mathbf{x})$ satisfies

$$\|G^{(h)}(\mathbf{x})\| \le C2^h e^{-c\sqrt{2^h|\mathbf{x}|}} \,. \tag{5.5}$$

[4]From now on, unless explicitly stated, we shall write integrals over $\mathbf{k}$ just as shorthands for the corresponding finite volume sums. All the equations and estimates written formally in the thermodynamic limit are valid at finite volume as well, uniformly in Λ.

The propagator $G^{(\le h^*)}$ satisfies the same estimate, with h replaced by h^* (thanks to the presence of the infrared cutoff induced by the mass $m \sim 2^{h^*}$). The off-diagonal elements of $G^{(h)}$ satisfy a similar estimate, times an extra dimensional gain proportional to $m2^{-h} \sim 2^{h^*-h}$.

As we mentioned, the "pressure" $E_\Lambda(\lambda, m)$ and the kernels $S_{\Lambda,k}(b_1, \dots, b_k)$ can be expressed, via (4.6) and the rules explained after (4.7), as a sum over Feynman diagrams, whose values involve suitable products of propagators G. Each propagator can then be decomposed as in (5.1) and in this way we obtain "labelled Feynman graphs" with solid lines (propagators) each carrying a scale label $h^* \le h \le 0$, the label $h = h^*$ corresponding to $G^{(\le h^*)}$. Any labelled graph has a corresponding cluster structure, which can be conveniently represented by a Gallavotti-Nicolò tree, similar to the one in Fig. 10, except that the endpoints can now be of different type, depending on whether they are associated with a contribution of type $V_\Lambda(\psi)$ (i.e., of type $\xi(\gamma)$, see (2.20)), in which case the endpoints will be called "normal", or of type $\mathcal{B}_\Lambda(\psi, \mathbf{J})$ (i.e., of type $-J_b E_b^{(m)}$ or of type $\tilde\xi(\gamma; R)$ with $|\gamma| \ge 2$, see (2.29)), in which case they will be called "special". Note that in the non-interacting case $\alpha = 0$ we had $V_\Lambda(\psi) = 0$ and $\mathcal{B}_\Lambda(\psi, \mathbf{J}) = -\sum_{b\subset\Lambda} J_b E_b^{(m)}$, so that all the endpoints were special.

Proceeding as in §4.2.2, we can first sum over the labelled diagrams associated with a given tree, and then over the trees. It is important to realize that, given a labeled tree (including possibly the labels that specify the order in α of the endpoints), there may be many Feynman diagrams compatible with it, see e.g. Figure 11.

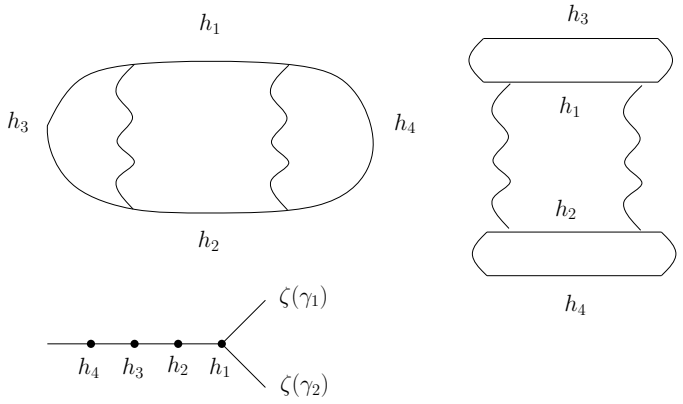

FIGURE 11. Two different labelled Feynman graphs, coming from the contractions of two vertices of type $\zeta(\gamma_i), i = 1, 2$ with $|\gamma_i| = 2$, giving the same tree. Here, $h_4 < h_3 < h_2 < h_1$.

5.1.1. *The tree and the labelled Feynman diagram expansions.* To explain precisely how to express $E_\Lambda(\lambda, m)$ and $S_{\Lambda,k}(b_1, \dots, b_k)$ as a sum over trees and over Feynman diagrams compatible with the trees, we need to make a small detour about the main features and definitions of the Gallavotti-Nicolò (GN) trees. The trees introduced in this section are called "non-renormalized trees", as opposed to the "renormalized trees" that will be introduced in the following sections. Let us also remark that some of the conventions

introduced here are slightly different from those used in Section 4, such as the rule for identifying trees, and the meaning of the word "vertex".

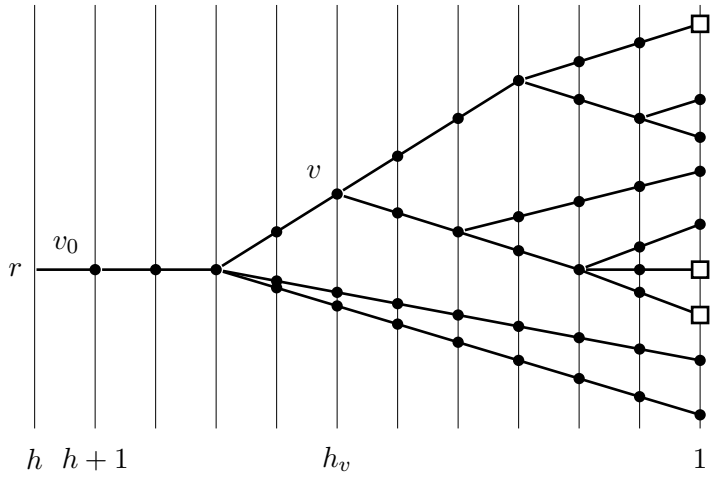

FIGURE 12. A tree $\tau \in \tilde{\mathcal{T}}^{(h)}_{N,n}$ with $N = 6$ and $n = 3$: the root is on scale h and the endpoints are all on scale 1.

(1) Consider the family of all trees which can be constructed by joining a point r, the *root*, with an ordered set of $N + n \geq 1$ points, the *endpoints* of the *unlabeled tree*, so that r is not a branching point. The endpoints can be of two types, either normal or special, the former drawn as dots, the latter as squares, see Fig.12; N and n are the number of normal and special endpoints, respectively. The branching points will be called the *non trivial vertices*. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order. Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. We shall also consider the *labelled trees* (to be called simply trees in the following); they are defined by associating some labels with the unlabeled trees, as explained in the following items.

(2) We associate a label $h^* - 1 \leq h < 0$ with the root and we denote by $\tilde{\mathcal{T}}^{(h)}_{N,n}$ the corresponding set of labelled trees with N normal and n special endpoints (the tilde in $\tilde{\mathcal{T}}^{(h)}_{N,n}$ reminds that the trees are non-renormalized). Moreover, we introduce a family of vertical lines, labeled by an integer taking values in $[h, 1]$, and we represent any tree $\tau \in \tilde{\mathcal{T}}^{(h)}_{N,n}$ so that, if v is an endpoint, it is contained in the vertical line with index $h_v = 1$, while if it is a non trivial vertex, it is contained in a vertical line with index $h < h_v \leq 0$, to be called the *scale* of v; the root r is on the line with index h. In general, the tree will intersect the vertical lines in set of points different from the root, the endpoints and the branching points; these points will be called *trivial vertices*. The set of the *vertices* will be the union of

the endpoints, of the trivial vertices and of the non trivial vertices; note that the root is not a vertex. Every vertex v of a tree will be associated to its scale label h_v, defined, as above, as the label of the vertical line whom v belongs to. Note that, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$.

(3) There is only one vertex immediately following the root, called v_0 and with scale label equal to $h + 1$.

(4) Given a vertex v of $\tau \in \tilde{\mathcal{T}}^{(h)}_{N,n}$ that is not an endpoint, we can consider the subtrees of τ with root v, which correspond to the connected components of the restriction of τ to the vertices $w \geq v$. If a subtree with root v contains only v and one endpoint on scale $h_v + 1$, it will be called a *trivial subtree* (and in this case $h_v = 0$).

(5) If v is not and endpoint, the cluster associated with it is the set of endpoints following v on τ; if v is an endpoint, it is itself a (trivial) cluster. The tree provides an organization of endpoints into a labelled hierarchy of clusters (the cluster structure).

(6) Normal endpoints are associated with (one of the monomials contributing to) $V_\Lambda(\psi)$, while special endpoints are associated with (one of the monomials contributing to) $\mathcal{B}_\Lambda(\psi, \mathbf{J})$, both thought of as functions of the Majorana fields $\psi_{\mathbf{x},\gamma}$, with $\mathbf{x} \in \Lambda$ and $\gamma = 1, \dots, 4$.

In order to distinguish the various contributions arising from the choices of the monomials in the factors $V_\Lambda(\psi)$ and $\mathcal{B}_\Lambda(\psi, \mathbf{J})$ associated with the endpoints, as well as the scale at which each field in these monomials is contracted, we need a few more definitions. We introduce a *field label* f to distinguish the field variables appearing in the monomials associated with the endpoints; the set of field labels associated with the endpoint v will be called I_v; if v is not an endpoint, we shall call I_v the set of field labels associated with the endpoints following the vertex v. Note that every field can be either of type J or ψ: correspondingly, we denote by I^J_v and I^ψ_v the set of field labels of type J and ψ, respectively, associated with v. Furthermore, we denote by $\mathbf{x}(f)$ the point of the field variable with label f; if $f \in I^J_v$, we denote by $b(f)$ the bond label of the corresponding J field, and we let $\mathbf{x}(f)$ and $j(f)$ be such that $b(f) = (\mathbf{x}(f), \mathbf{x}(f) + \hat{e}_{j(f)})$; if $f \in I^\psi_v$, we denote by $\gamma(f)$ the Majorana "quasi-particle" label of the corresponding Grassmann field. Similarly, we let $\mathbf{x}_v := \cup_{f \in I_v} \mathbf{x}(f)$, etc. Note that, given an endpoint v and the labels $P_v = I_v$, $\mathbf{x}_v$, etc., the value of the endpoint is uniquely specified, and we denote it by $K_v(\mathbf{x}_v, P_v) J(P^J_v) \psi(P^\psi_v)$, where K_v is the *kernel* of v, while $J(P) = \prod_{f \in P} J_{b(f)}$ and $\psi(P) = \prod_{f \in P} \psi_{\mathbf{x}(f),\gamma(f)}$.

We associate with any vertex v of the tree a subset P_v of I_v, the *external fields* of v; their cardinality is the analogue of the quantity n^e_v introduced right after (4.45). We further denote by P^J_v and P^ψ_v the subsets of P_v of fields of type J and ψ, respectively (of course, $P^J_v \cap P^\psi_v = \emptyset$ and $P^J_v \cup P^\psi_v = P_v$). These subsets must satisfy various constraints. First of all, if v is not an endpoint and $v_1, \dots, v_{s_v}$ are the $s_v \geq 1$ vertices immediately following it on τ, then $P_v \subseteq \cup_i P_{v_i}$; if v is an endpoint, $P_v = I_v$. If v is not an endpoint, we shall denote by Q_{v_i} the intersection of P_v and P_{v_i}; this definition implies that $P_v = \cup_i Q_{v_i}$. The union of the subsets $P_{v_i} \setminus Q_{v_i}$ is, by definition, the

set of the *internal fields* of v, and is non empty if $s_v > 1$. Similar definitions are valid for Q^ψ_v, Q^J_v, etc. Note that $P^J_v = Q^J_v$ for all v, simply because J is, by definition, an external field. Given $\tau \in \tilde{\mathcal{T}}^{(h)}_{N,n}$, there are many possible choices of the subsets P_v, $v \in \tau$, compatible with all the constraints. We shall denote by $\mathcal{P}_\tau$ the family of all these choices and by $\mathbf{P}$ the elements of $\mathcal{P}_\tau$. For every τ and $\mathbf{P} \in \mathcal{P}_\tau$, we let $\Gamma(\mathbf{P}, \tau)$ be the set of labelled Feynman diagrams compatible with the tree and the choice of the field labels.

In terms of these trees and labels, the generating function for correlations in (4.6) can be written as (see e.g. [34, Section 6])

$$\log \mathcal{Z}^{(11)}_\Lambda(\lambda, m, \mathbf{A}) = \sum_{\substack{N,n\geq 0:\\ N+n\geq 1}} \sum_{h=h^*-1}^{-1} \sum_{\tau\in\tilde{\mathcal{T}}^{(h)}_{N,n}} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau:\\ P^\psi_{v_0}=\emptyset}}^{*} \sum_{\mathbf{x}_{v_0}} J(P^J_{v_0}) \times \tag{5.6}$$

$$\times \Big[\prod_{v\in E(\tau)} K_v(\mathbf{x}_v, P_v) \Big] \Big[\prod_{v\in V(\tau)} \frac{1}{s_v!}\, \mathcal{E}^T_{h_v} \left(\psi(P_{v_1}\backslash Q_{v_1}), \dots, \psi(P_{v_{s_v}}\backslash Q_{v_{s_v}}) \right) \Big]$$

where the $*$ on the sum over $\mathbf{P}$ indicates the constraint that $P_{v_0} = \emptyset$ and the set of internal fields of v_0 is non empty. Moreover, $\mathcal{E}^T_h$ indicates truncated expectation with respect to the propagator $G^{(h)}$, if $h > h^*$, or $G^{(\leq h^*)}$, if $h = h^*$. Finally, $E(\tau)$ is the set of endpoints of τ, $V(\tau)$ is the set of vertices of τ that are not in $E(\tau)$; for each $v \in V(\tau)$, we indicated by $v_1, \dots, v_{s_v}$ the vertices immediately following v on τ. After having re-expressed the truncated expectations in the right side as a sum over Feynman diagrams, we obtain the desired representation of the generating function in terms of a double sum over trees and labelled Feynman diagrams:

$$\log \mathcal{Z}^{(11)}_\Lambda(\lambda, m, \mathbf{A}) = \sum_{\substack{N,n\geq 0:\\ N+n\geq 1}} \sum_{h=h^*-1}^{-1} \sum_{\tau\in\tilde{\mathcal{T}}^{(h)}_{N,n}} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau:\\ P^\psi_{v_0}=\emptyset}}^{*} \sum_{\mathbf{x}_{v_0}} \sum_{\mathcal{G}\in\Gamma(\tau,\mathbf{P})} J(P^J_{v_0}) \mathrm{Val}(\mathcal{G}) \,. \tag{5.7}$$

To obtain the multiscale expansion for $E_\Lambda(\lambda, m)$ it is enough to compute this expression for $\mathbf{J} = \mathbf{0}$. Similarly, to obtain $S_{\Lambda,k}(b_1, \dots, b_k)$ it is enough to derive with respect to $J_{b_1}, \dots, J_{b_k}$ and then take $\mathbf{J} = \mathbf{0}$.

5.1.2. *Dimensional estimates.* At this point we can discuss how to obtain estimates on the generic term of the non-renormalized expansion just introduced, and see whether the resulting upper bound is summable or not over all the labels and the trees.

Let us consider for simplicity a contribution to $E_\Lambda(\lambda, m)$. That is, consider $\mathcal{G} \in \Gamma(\tau, \mathbf{P})$, where $\tau \in \tilde{\mathcal{T}}^{(h)}_{N,0}$ and $P_{v_0} = \emptyset$; note that $P_v = P^\psi_v$, because $n = 0$. In order to estimate $\mathrm{Val}(\mathcal{G})$ we use that, from (5.5),

$$\|G^{(h)}(\cdot)\|_1 := \sum_{\mathbf{x}\in\Lambda} \|G^{(h)}(\mathbf{x})\| \leq C 2^{-h} \,, \qquad \|G^{(h)}(\cdot)\|_\infty \leq C 2^h \,. \tag{5.8}$$

Moreover, given $v \in E(\tau)$ and an arbitrary field label $f^* \in P_v$,

$$\sum_{\mathbf{x}_v\backslash\mathbf{x}(f^*)} |K_v(\mathbf{x}_v, P_v)| \leq C^{|P_v|} \alpha^{|P_v|/2-1} \,, \tag{5.9}$$

as it follows from the very definition of the kernel v. Therefore,

$$\sum_{\mathbf{x}_{v_0}} |\mathrm{Val}(\mathcal{G})| \le |\Lambda| \Big[\prod_{v\in E(\tau)} (C')^{|P_v|}\, \alpha^{|P_v|/2-1} \Big] \Big[\prod_{v\in V(\tau)} \frac{1}{s_v!} 2^{h_v \tilde n_v - 2h_v(s_v-1)} \Big] , \tag{5.10}$$

where $\tilde n_v = (\sum_{i=1}^{s_v} |P_{v_i}| - |P_v|)/2$ was already introduced after (4.44), i.e., it is the number of propagators contained in v but not in any $v' > v$ or, equivalently, the number of propagators obtained by contracting the internal fields of v. We also recall that s_v is the number of vertices immediately following v on τ (i.e., the number of clusters contained in v but not in any other cluster $w > v$). To understand (5.10) note that the factor $|\Lambda|$ in (5.10) comes from translation invariance (i.e. from the sum over the location of the cluster at scale $h+1$) and that the factor associated with the product over the endpoints comes from (5.9). Moreover, the factor associated with the product over $V(\tau)$ comes from the following argument: for any vertex $v \in V(\tau)$ with s_v descendants $v_1, \dots, v_{s_v}$, we select a minimal number, $s_v - 1$, of propagators at scale h_v connecting them; all the non-selected lines are estimated in the ℓ_∞ norm and give $C2^{h_v}$ each, by the second of (5.8); when the relative positions of $v_1, \dots, v_{s_v}$ are summed over, each selected line gives instead $C2^{-h_v}$ by the first of (5.8).

Then we proceed as in (4.48), and in particular we use (4.45) for $f_v = \tilde n_v - 2(s_v - 1)$ and the analogues of (4.47), namely

$$\begin{aligned} &\sum_{\substack{v\in V(\tau):\\ v\ge w}} \tilde n_v = \frac12 \sum_{\substack{v\in V(\tau):\\ v\ge w}} \Big(\sum_{i=1}^{s_v} |P_{v_i}| - |P_v| \Big) = \frac12 \big(|I_w| - |P_w| \big) , \\ &\sum_{\substack{v\in V(\tau):\\ v\ge w}} (s_v - 1) = m_w - 1 , \end{aligned} \tag{5.11}$$

where m_w is the number of normal endpoints following w on τ, and we get

$$\sum_{\mathbf{x}_{v_0}} |\mathrm{Val}(\mathcal{G})| \le |\Lambda| (C')^{|I_{v_0}|}\, \alpha^{\frac{|I_{v_0}|}{2} - N}\, 2^{h(2 + \frac{|I_{v_0}|}{2} - 2m_{v_0})} \prod_{v\in V(\tau)} \frac{1}{s_v!} 2^{2 - \frac{|P_v|}{2} + \frac{|I_v|}{2} - 2m_v},$$

where we used the fact that P_{v_0} is empty. Next we note that

$$h m_{v_0} + \sum_{v\in V(\tau)} m_v = h|I_{v_0}| + \sum_{v\in V(\tau)} |I_v| = 0 , \tag{5.12}$$

thanks to the fact that the vertices immediately preceding the endpoints on τ are all on scale 0 (otherwise, we would have e.g. $h m_{v_0} + \sum_{v\in V(\tau)} m_v = \sum_{v\in E(\tau)} h_{v'}$ with v' the vertex immediately preceding v on τ). Therefore,

$$\sum_{\mathbf{x}_{v_0}} |\mathrm{Val}(\mathcal{G})| \le |\Lambda| (C')^{|I^\psi_{v_0}|}\, \alpha^{\frac{|I^\psi_{v_0}|}{2} - N}\, 2^{2h} \Big[\prod_{v\in V(\tau)} \frac{1}{s_v!} 2^{2 - \frac{|P^\psi_v|}{2}} \Big], \tag{5.13}$$

where the apex ψ on I_v and P_v is inserted to recall that in the case considered so far $P_v = P^\psi_v$. A similar estimate is valid for the contributions to $S_{\Lambda,k}(b_1, \dots, b_k)$ from the graphs $\Gamma \in \mathcal{G}(\tau, \mathbf{P})$ with $\tau \in \tilde{\mathcal{T}}^{(h)}_{N,n}$, with the only

important difference that the *scaling dimension* $2 - |P_v^\psi|/2$ is replaced by $2 - |P_v^\psi|/2 - |P_v^J|$, to be denoted by d_v.

If we could assume that the scaling dimensions $d_v = 2 - |P_v^\psi|/2 - |P_v^J|$ are ≤ 0 for all $v \in V(\tau)$ and strictly negative for $v \in V^*(\tau)$ (here $V^*(\tau)$ is the subset of vertices of $V(\tau)$ that are followed by at least two endpoints), then (5.13) would be summable over the scale labels, and after summation we would get a bound proportional to $\alpha^{\frac{|I_{v_0}^\psi|}{2} - N}$, as desired. However, there are trees τ and graphs $\Gamma \in \mathcal{G}(\tau, \mathbf{P})$ with vertices $v \in V(\tau)$ such that d_v is either 0 or 1: this happens for $(|P_v^\psi|, |P_v^J|) = (2,0), (4,0), (2,1)$, in which case (5.13) is not summable, uniformly in L, over the trees in $\tilde{\mathcal{T}}_{N,0}^{(h)}$ and on the scale label $h < 0$. In the Renormalization Group language, clusters with scaling dimension 0 are called *marginal*, and those with scaling dimension 1 are called (linearly) *relevant*. Note that in the non-interacting case there were neither marginal, nor relevant clusters in $V^*(\tau)$, simply because $|P_v^J| \ge 2$ for such vertices; moreover, all vertices followed by exactly one endpoint had $(|P_v^\psi|, |P_v^J|) = (2,1)$, so that $d_v = 0$ for such vertices; as a consequence we could safely sum over the scale labels. On the other hand, in the interacting case the presence of trees and graphs containing marginal or relevant clusters is inevitable, and this makes the Feynman diagram expansion useless, because it leads to the bounds on e.g. $|E_\Lambda(\lambda, m)|$ that diverge as $m \to 0$ and $L \to \infty$ (recall that, in the sums on scales, $|h_v|$ ranges from 0 to a constant times $-\log m \gg -\log L$). In other words, the Feynman graph expansion is *not* sufficient for gaining control on the perturbative expansion at $\alpha \neq 0$, not even order by order in α.

On top of the problem of divergence of Feynman diagrams outlined above, there is also a combinatorial issue to be faced: even if we could sum every single Feynman diagram over the scale labels, we should still sum over the Feynman diagrams. However, assuming for definiteness that $n = 0$, the number of Feynman diagrams is at least $(\text{const.})^N (N!)^2$, where N is the total number of normal endpoints and we used the fact that every endpoint is associated with a vertex with 4 or more fermionic fields (i.e. half-lines), as well as the fact that the number of Feynman diagrams is equal to the number of possible Wick contractions of such fields (it is easy to see that the number of possible contractions of the half-lines exiting from N vertices, each with 4 external half-lines, scales like $(\text{const.})^N (N!)^2$, and even faster if we allow vertices with more than 4 external half-lines). On the other hand, the factor $\prod_{v \in V(\tau)} 1/s_v!$ in (5.13) behaves like $1/N!$ at large N, which means that the bound on the total contribution of order N grows like (assuming for simplicity that all endpoints have 4 external lines) $\alpha^N N!$, which is not summable in N, even for α small.

These two problems are the counterparts of analogous difficulties emerging in Quantum Field Theory. The divergence of Feynman diagrams with m as $m \to 0$ is called the *infrared problem*, and it signals that an expansion in α is not suitable for treating the interacting system at hand. Rather, we need to introduce scale-dependent parameters λ_h, Z_h which measure the effective strength of the interaction and of the propagator at scale h (in the language of field theory, Z_h is called "wave function renormalization"). The

theory depends analytically on λ_h, Z_h, so that all the potential divergences of the theory are "absorbed" into the definition of the *running coupling constants* λ_h, Z_h, whose behavior can be studied in terms of a *finite-dimensional* discrete flow equation. For example, the iterative equation for λ_h leads a priori (i.e., on the basis of dimensional estimates of the contributions to $\beta_h^\lambda := \lambda_h - \lambda_{h-1}$, which are also expressed as a perturbation series in λ_h, Z_h) to a divergence of λ_h as $h \to -\infty$ (dimensionally, the divergence is linear in $|h|$); however, remarkable cancellations in the *beta function* β_h^λ allow one to show that λ_h reaches a fixed point close to α as $h \to -\infty$. The same cancellations are of course (a posteriori) present also in the original naive power series expansion, but are much less visible there. The phenomenon is similar (even though more complicated) to what happens in KAM theory: even though apparently divergent, the Lindstedt series for the conjugation function is analytic in the perturbation parameter; however, in order to prove analyticity, one must first isolate the potentially divergent contributions, then re-group (or re-sum) them, and finally show that all the divergent contributions within each family of re-grouped terms cancel among each other; see e.g. [32, Ch.8-9] for a review of KAM theory in this perspective.

Finally, let us comment about the combinatorial divergence due to the large number of diagrams: this divergence indicates that we should not simply expand in a sum over Feynman graphs, but rather over (resummed) families of such diagrams; in the fermionic context, the regrouping of Feynman diagrams into families leads to a *determinant* or Pfaffian expansion, which is better behaved combinatorially than the original expansion (see for instance [34, Section 4]). Roughly speaking, using the signs from the fermionic Wick rule we can regroup families of Feynman diagrams into determinants; the sum over Feynman diagrams is obtained by expanding the determinant along a row or column; however, it is better to estimate the determinant of an $n \times n$ matrix in terms of the maximal eigenvalue, rather than in terms of the sum over the $n!$ terms in the definition of the determinant.

Using well known methods coming from constructive QFT one can solve the above difficulties, as explained in the following sections.

5.2. **The multiscale renormalized expansion.** As discussed in the previous subsection, the perturbation theory in Feynman diagrams for the pressure and correlation functions of the model does not appear to be convergent in α, uniformly in L and m. In this section we show that at finite L and m we can reorganize the expansion, thus obtaining a new series, the *renormalized expansion*, which is not a power series in α anymore and has better convergence properties. In particular, it will allow us to show that the observables of interest are well-defined and analytic in α, uniformly as $L \to \infty$ and $m \to 0$. The renormalized expansion is obtained by performing the integration of the infrared degrees of freedom in a multiscale fashion, as described below. Roughly speaking we proceed as follows: we first decompose the propagator in a way similar to (5.1) and we integrate step by step the propagator on scale $h = 0, -1, -2, \dots$ At each step, before integrating the next scale, we properly resum the expansion at hand, by isolating the divergent parts of the relevant and marginal contributions from the rest (the irrelevant terms); the relevant and marginal divergent parts are proportional to the

running coupling constants, already mentioned at the end of the previous subsection. Moreover, at each step we express the effect of the integration on scale h in a way similar to (5.6), with the important difference that the truncated expectation in the right side of (5.6) is not written as a sum over Feynman diagrams, but rather as a sum over Pfaffians, each of which collects several contributions arising from different pairings. The resulting expansion takes the form of a multiscale Pfaffian expansion, expressed in terms of the running coupling constants, rather than in terms of α.

We start from

$$\mathcal{Z}_\Lambda^{(11)}(\lambda, m, \mathbf{A}) = \int P_\Lambda(d\psi) e^{V_\Lambda^{(0)}(\psi) + \mathcal{B}_\Lambda^{(0)}(\psi, \mathbf{J})}, \tag{5.14}$$

where $V_\Lambda^{(0)} + \mathcal{B}_\Lambda^{(0)}$ is obtained from $V_\Lambda + \mathcal{B}_\Lambda$ by re-expressing the original Grassmann fields in terms of Majorana fields, via (3.5). For notational simplicity, in the right side we dropped the label (11), and we shall do the same in the following, implicitly assuming anti-periodic boundary conditions on the Grassmann variables in both coordinate directions, unless otherwise specified. After the integration of the fields on scales $0, -1, \dots, h+1$, we recast (5.14) into the form:

$$\mathcal{Z}_\Lambda = e^{E_\Lambda^{(h)} + S_\Lambda^{(h)}(\mathbf{J})} \int P_{Z_h, m_h, \chi_h}(d\psi^{(\le h)}) e^{V_\Lambda^{(h)}(\sqrt{Z_h}\psi^{(\le h)}) + \mathcal{B}_\Lambda^{(h)}(\sqrt{Z_h}\psi^{(\le h)}, \mathbf{J})}, \tag{5.15}$$

which we now prove by induction, by properly defining $E_\Lambda^{(h)}$, $S_\Lambda^{(h)}(\mathbf{J})$, etc. At the initial step, $E_\Lambda^{(0)} = S_\Lambda^{(0)}(\mathbf{J}) = 0$, $Z_0 = 1$, $m_0(\mathbf{k}) = m \cos k_1$, and $P_{Z_0, m_0, \chi_0}(d\psi)$ is the same as $P_\Lambda(d\psi)$, once written in the basis of the Majorana fields ψ_γ. At scale h, the gaussian integration $P_{Z_h, m_h, \chi_h}(d\psi^{(\le h)})$ has propagator

$$\frac{g^{(\le h)}(\mathbf{x}-\mathbf{y})}{Z_h} := \int_{\mathbb{T}^2} \frac{d\mathbf{k}}{(2\pi)^2} \frac{\chi_h(\mathbf{k})}{Z_h} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} \begin{pmatrix} \hat{G}_{m_h}(\mathbf{k}) & 0 \\ 0 & \hat{G}_{m_h}(\mathbf{k}) \end{pmatrix} \tag{5.16}$$

where

$$\hat{G}_{m_h}(\mathbf{k}) = \frac{1}{2} \begin{pmatrix} -i \sin k_1 + \sin k_2 & i m_h(\mathbf{k}) \\ -i m_h(\mathbf{k}) & -i \sin k_1 - \sin k_2 \end{pmatrix}^{-1}. \tag{5.17}$$

The functions $V_\Lambda^{(h)}(\psi)$ and $\mathcal{B}_\Lambda^{(h)}(\psi, \mathbf{J})$ are called the effective potential and effective source term, respectively. They admit the following representation (that will follow from the induction, too):

$$V_\Lambda^{(h)}(\psi) = \sum_{\substack{n \ge 1: \\ n \text{ even}}} \sum_{\gamma_1, \dots, \gamma_n} \sum_{\mathbf{x}_1, \dots, \mathbf{x}_n} W_{n, \boldsymbol{\gamma}}^{(h)}(\mathbf{x}_1, \dots, \mathbf{x}_n) \Big[\prod_{i=1}^n e^{i \mathbf{p}_{\gamma_i} \mathbf{x}_i} \psi_{\mathbf{x}_i, \gamma_i} \Big] \tag{5.18}$$

where $\boldsymbol{\gamma}$ is a shorthand for $(\gamma_1, \dots, \gamma_n)$. Similarly,

$$\begin{aligned} \mathcal{B}_\Lambda^{(h)}(\psi, \mathbf{J}) \;=\; & \sum_{\substack{n \ge 1: \\ n \text{ even}}} \sum_{q \ge 1} \sum_{\substack{\gamma_1, \dots, \gamma_n, \\ j_1, \dots, j_q}} \sum_{\substack{\mathbf{x}_1, \dots, \mathbf{x}_n \\ \mathbf{y}_1, \dots, \mathbf{y}_q}} W_{n, q, \boldsymbol{\gamma}, \mathbf{j}}^{(h)}(\mathbf{x}_1, \dots, \mathbf{x}_n; \mathbf{y}_1, \dots, \mathbf{y}_q) \times \\ & \times \Big[\prod_{i=1}^n e^{i \mathbf{p}_{\gamma_i} \mathbf{x}_i} \psi_{\mathbf{x}_i, \gamma_i} \Big] \Big[\prod_{i=1}^q J_{\mathbf{y}_i, j_i} \Big], \end{aligned} \tag{5.19}$$

where $J_{\mathbf{x},j}$ is an alternative symbol for J_b, with $b = (\mathbf{x}, \mathbf{x}+\hat{e}_j)$. At the initial step, $h = 0$, the kernels $W^{(0)}_{n,q,\boldsymbol{\gamma},\mathbf{j}}$ and $W^{(0)}_{n,\boldsymbol{\gamma}}$ are obtained from (2.20) and (2.29) after re-expressing the field ψ in the Majorana basis, via (3.5). Because of the factors $t_b^{(m)}$ entering the definition of $E_b^{(m)}$, these kernels are not translation invariant. However, the non-translation invariant terms vanish at $m = 0$ (see e.g. the quartic terms in the right side of (2.21) as an illustration): therefore, $\mathcal{P}_0 W^{(0)}_{n,q,\boldsymbol{\gamma},\mathbf{j}} := W^{(0)}_{n,q,\boldsymbol{\gamma},\mathbf{j}}\Big|_{m=0}$ *is* translation invariant (similarly for $\mathcal{P}_0 W^{(0)}_{n,\boldsymbol{\gamma}}$), a fact that will be useful in the following. Similarly, for later convenience, we introduce the operator $\mathcal{P}_1$, which extracts the linear part in m from the kernel it acts on: $\mathcal{P}_1 W^{(0)}_{n,q,\boldsymbol{\gamma},\mathbf{j}} := m\partial_m W^{(0)}_{n,q,\boldsymbol{\gamma},\mathbf{j}}\Big|_{m=0}$. It is easy to see that the kernels $\mathcal{P}_1 W^{(0)}_{n,q,\boldsymbol{\gamma},\mathbf{j}}(\mathbf{x}_1,\dots,\mathbf{x}_n;\mathbf{y}_1,\dots,\mathbf{y}_q)$ are translation invariant, *up to an overall oscillatory factor* $(-1)^{(\mathbf{x}_1)_1}$ (see again (2.21)). The same properties are valid for the kernels at lower scales, as it will follow from the induction below. It will also follow that the kernels $W^{(h)}_{n,q,\boldsymbol{\gamma},\mathbf{j}}(\mathbf{x}_1,\dots,\mathbf{x}_n;\mathbf{y}_1,\dots,\mathbf{y}_q)$ satisfy natural dimensional estimates, which are the counterparts of the multiscale dimensional estimates (5.13) and will be discussed in Section 5.4.

5.3. **Localization and renormalization.** Let us now turn to the inductive proof if (5.15). We already discussed the validity of (5.15) at the first step, $h = 0$. We now need to show how to go from scale h to $h-1$. The first key step that we have to perform at each iteration is the localization procedure, which consists in isolating the potentially divergent contributions in $V^{(h)}_\Lambda$ and $\mathcal{B}^{(h)}_\Lambda$ from the rest; next we will rescale the Grassmann fields and finally we will integrate out the (rescaled) fields on scale h. Let us describe the first step, i.e., the localization. We write:

$$V^{(h)}_\Lambda = \mathcal{L}V^{(h)}_\Lambda + \mathcal{R}V^{(h)}_\Lambda\,, \qquad \mathcal{B}^{(h)}_\Lambda = \mathcal{L}\mathcal{B}^{(h)}_\Lambda + \mathcal{R}\mathcal{B}^{(h)}_\Lambda \tag{5.20}$$

where $\mathcal{L}$, the localization operator, is a projection operator that acts linearly on the effective potential as described in the following. The operator $\mathcal{R}$ is called the renormalization operator: it extracts from $V^{(h)}_\Lambda + \mathcal{B}^{(h)}_\Lambda$ the well-behaved ("irrelevant") part. For simplicity, in the following we spell out the definitions of $\mathcal{L}$ and $\mathcal{R}$ in the $L \to \infty$ case only, the finite volume case being treatable in a similar, even though notationally more cumbersome, way, see e.g. [12, Eqs.(2.74)-(2.75)]. Recall that the only potentially divergent diagrams in the multiscale expansion are those with $(|P^\psi_v|, |P^J_v|) = (2,0), (4,0), (2,1)$ (see the discussion after (5.13)), the $(2,0)$ terms being relevant, and $(4,0), (2,1)$ being marginal: therefore, $\mathcal{L}$ acts non trivially only on these terms. More precisely, denoting by $V^{(h)}_{n,\Lambda}$ the n-legged contribution to the effective potential, i.e.,

$$V^{(h)}_{n,\Lambda}(\psi) = \sum_{\substack{\mathbf{x}_1,\dots,\mathbf{x}_n \\ \boldsymbol{\gamma}}} W^{(h)}_{n,\boldsymbol{\gamma}}(\mathbf{x}_1,\dots,\mathbf{x}_n)\Big[\prod_{i=1}^n e^{i\mathbf{p}_{\gamma_i}\mathbf{x}_i}\psi_{\mathbf{x}_i,\gamma_i}\Big]\,. \tag{5.21}$$

we let (dropping the Λ label to indicate that we are formally giving the definition in the $L \to \infty$ case only)

$$\begin{aligned} \mathcal{L}V_2^{(h)}(\psi) &= \sum_{\substack{\mathbf{x},\mathbf{y}\\ \gamma,\gamma'}} e^{i\mathbf{p}_\gamma \mathbf{x}} \psi_{\mathbf{x},\gamma} \mathcal{P}_0 W^{(h)}_{2,(\gamma,\gamma')}(\mathbf{x},\mathbf{y}) e^{i\mathbf{p}_{\gamma'}\mathbf{y}} \big[1 + (\mathbf{y}-\mathbf{x})\cdot \hat{\boldsymbol{\partial}}\big] \psi_{\mathbf{x},\gamma'} \\ &+ \sum_{\substack{\mathbf{x},\mathbf{y}\\ \gamma,\gamma'}} e^{i\mathbf{p}_\gamma \mathbf{x}} \psi_{\mathbf{x},\gamma} \mathcal{P}_1 W^{(h)}_{2,(\gamma,\gamma')}(\mathbf{x},\mathbf{y}) e^{i\mathbf{p}_{\gamma'}\mathbf{y}} \psi_{\mathbf{x},\gamma'} \end{aligned} \tag{5.22}$$

and

$$\mathcal{L}V_4^{(h)}(\psi) = \sum_{\substack{\mathbf{x}_1,\dots,\mathbf{x}_4\\ \gamma_1,\dots,\gamma_4}} \mathcal{P}_0 K^{(h)}_{4,\boldsymbol{\gamma}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) \Big[\prod_{i=1}^4 e^{i\mathbf{p}_{\gamma_i}\mathbf{x}_i} \psi_{\mathbf{x}_1,\gamma_i}\Big] , \tag{5.23}$$

while $\mathcal{L}V_n^{(h)}(\psi) = 0$, $\forall n > 4$. In the first line of (5.22), $\hat{\boldsymbol{\partial}}$ indicates the symmetric discrete gradient, whose i-th component acts on lattice functions as $\hat\partial_i f(\mathbf{x}) = \frac{1}{2}(f(\mathbf{x}+\hat e_i) - f(\mathbf{x}-\hat e_i))$. Note that all the fields appearing in these formulas are *localized* at the same point, or at two points at a distance 1, which justifies the name of *localization operator* for $\mathcal{L}$. The action of $\mathcal{L}$ on the source term is defined similarly (and it acts non-trivially only on the term with two ψ fields):

$$\mathcal{L}\mathcal{B}^{(h)}(\psi) = \sum_{\substack{\mathbf{x},\mathbf{y},\mathbf{z}\\ \gamma,\gamma',j}} e^{i\mathbf{p}_\gamma \mathbf{x}} \psi_{\mathbf{z},\gamma} \mathcal{P}_0 W^{(h)}_{2,1,(\gamma,\gamma'),j}(\mathbf{x},\mathbf{y};\mathbf{z}) e^{i\mathbf{p}_{\gamma'}\mathbf{y}} \psi_{\mathbf{z},\gamma'} J_{\mathbf{z},j} \, . \tag{5.24}$$

5.3.1. *The meaning of the localization procedure.* Let us explain the rationale behind the definition of $\mathcal{L}$ given above. The key fact to be noticed is that it guarantees that: (1) the action of $\mathcal{R} = 1 - \mathcal{L}$ on the kernels produces a dimensional gain, which is enough to make the analogue of the dimensional estimate (5.13) for renormalized graphs (i.e. graphs such that each non-trivial subgraph is renormalized by the action of $\mathcal{R}$) convergent; (2) the algebraic structure of $\mathcal{L}$ is sufficiently simple that the linear space spanned by $\mathcal{L}(V + \mathcal{B})$ is finite dimensional, i.e., it can be parametrized by a finite number of constants.

Let us first discuss item (1), by focusing on the quartic kernels. Think of a quartic kernel together with its "external fields" $\psi_{\mathbf{x}_1,\gamma_1}, \dots, \psi_{\mathbf{x}_4,\gamma_4}$, i.e., consider the combination

$$V_4^{(h)}(\psi) = \sum_{\substack{\mathbf{x}_1,\dots,\mathbf{x}_4\\ \gamma_1,\dots,\gamma_4}} W^{(h)}_{4,\boldsymbol{\gamma}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) \Big[\prod_{i=1}^4 e^{i\mathbf{p}_{\gamma_i}\mathbf{x}_i} \psi_{\mathbf{x}_i,\gamma_i}\Big] . \tag{5.25}$$

When such a term appears in the multiscale procedure, the external fields $\psi_{\mathbf{x}_i,\gamma_i}$ should be thought of as fields on scales $\le h$, see (5.15); in other words, in the multiscale integration procedure, such fields will be contracted on scales $h_1, \dots, h_4$ smaller or equal than h. We let h' be the maximum among $h_1, \dots, h_4$. Moreover, by proceeding in a way similar to the one described in Section 5.1, $W^{(h)}_{4,\boldsymbol{\gamma}}$ can be written as a sum over τ and $\mathbf{P} \in \mathcal{P}_\tau$ of terms $W^{(h)}_{4,\boldsymbol{\gamma}}(\tau, \mathbf{P}; \mathbf{x}_1, \dots, \mathbf{x}_4)$, where τ specifies the cluster structure of the labelled diagrams contributing to it, while $\mathbf{P} = \{P_v\}_{v \in V(\tau) \cup E(\tau)}$ specifies

the field labels associated with the vertices v of τ (recall that $|P_v|$ represents the number of fields external to the subdiagram associated with v). We denote by $V_4^{(h)}(\tau, \mathbf{P})$ the analogue of (5.25) at fixed τ and $\mathbf{P}$. Given τ and $\mathbf{P}$, we let h'' be the smallest scale of a propagator entering the computation of $W_{4,\boldsymbol{\gamma}}^{(h)}(\tau, \mathbf{P}; \mathbf{x}_1, \dots, \mathbf{x}_4)$; note that $h'' > h$. Recall that the quartic kernels have scaling dimension zero; therefore, in order to turn them into convergent subdiagrams, it is enough to "gain one dimension", i.e., to gain a dimensional factor proportional to $2^{h'-h''}$. To see that the dimensional estimate of $\mathcal{R}V_4^{(h)}(\tau, \mathbf{P})$ is better than the one of $V_4^{(h)}(\tau, \mathbf{P})$ precisely by such a factor, we rewrite (denoting $\underline{\mathbf{x}} = (\mathbf{x}_1, \dots, \mathbf{x}_4)$)

$$\begin{aligned}
&\mathcal{R}V_4^{(h)}(\tau, \mathbf{P}; \psi) = \qquad (5.26)\\
&= \sum_{\underline{\mathbf{x}}, \boldsymbol{\gamma}} \Big[\prod_{i=1}^4 e^{i\mathbf{p}_{\gamma_i}\mathbf{x}_i} \Big] \Big\{ (1-\mathcal{P}_0) W_{4,\boldsymbol{\gamma}}^{(h)}(\tau, \mathbf{P}; \underline{\mathbf{x}}) \Big[\prod_{i=1}^4 \psi_{\mathbf{x}_i,\gamma_i} \Big] + \\
&+ \mathcal{P}_0 W_{4,\boldsymbol{\gamma}}^{(h)}(\tau, \mathbf{P}; \underline{\mathbf{x}}) \Big[\psi_{\mathbf{x}_1,\gamma_1} \psi_{\mathbf{x}_2,\gamma_2} \psi_{\mathbf{x}_3,\gamma_3} (\psi_{\mathbf{x}_4,\gamma_4} - \psi_{\mathbf{x}_1,\gamma_4}) + \cdots \\
&\qquad \cdots + \psi_{\mathbf{x}_1,\gamma_1} (\psi_{\mathbf{x}_2,\gamma_2} - \psi_{\mathbf{x}_1,\gamma_2}) \psi_{\mathbf{x}_1,\gamma_3} \psi_{\mathbf{x}_1,\gamma_4} \Big] \Big\} .
\end{aligned}$$

The kernel in the second line has an operator $1 - \mathcal{P}_0$ acting on it, which extracts its m-dependent part, i.e., it extracts the m-dependent part from at least one of the propagators contributing to its value; recalling that every m extracted from a propagator $G^{(h_v)}$ comes with a dimensional gain of the order $2^{h^*-h_v}$ (see the discussion after (5.5); note that the terms linear in m originate necessarily from the non-diagonal part of some propagator), we see that this term has the desired dimensional gain, simply because $2^{h^*-h_v} \le 2^{h'-h''}$. The other terms in the right side involve a difference between two fields at different locations, of the form $\psi_{\mathbf{x}_i,\gamma_i} - \psi_{\mathbf{x}_1,\gamma_i}$, which is formally (i.e., forgetting lattice effects) the same as

$$\psi_{\mathbf{x}_i,\gamma_i} - \psi_{\mathbf{x}_1,\gamma_i} \simeq (\mathbf{x}_i - \mathbf{x}_1) \cdot \int_0^1 ds \; \boldsymbol{\partial} \psi_{\mathbf{x}_1 + s(\mathbf{x}_i - \mathbf{x}_1),\gamma_i} \; . \qquad (5.27)$$

Now note that the factor $\mathbf{x}_i - \mathbf{x}_1$ goes together with $\mathcal{P}_0 W_{4,\boldsymbol{\gamma}}^{(h)}(\tau, \mathbf{P}; \underline{\mathbf{x}})$, which decays over a typical length scale 2^{-h} or, more precisely, $2^{-h''}$ (in fact $\mathcal{P}_0 W_{4,\boldsymbol{\gamma}}^{(h)}(\tau, \mathbf{P}; \underline{\mathbf{x}})$ is a combination of propagators, each having a scale $h_{prop} \ge h''$ and, therefore, decaying over a length scale $2^{-h_{prop}}$). Therefore, $\mathbf{x}_i - \mathbf{x}_1$ corresponds to a dimensional contribution proportional to $2^{-h''}$, just because all propagators involved in the computation of $W_{4,\boldsymbol{\gamma}}^{(h)}(\tau, \mathbf{P}; \mathbf{x}_1, \dots, \mathbf{x}_4)$ are on scale $k \ge h''$ and therefore are essentially zero beyond distances of order $2^{-k} \le 2^{-h''}$. Similarly, the derivative $\boldsymbol{\partial}$ acting on $\psi_{\cdot,\gamma_i}$ corresponds to a dimensional contribution proportional to $2^{h_i} \le 2^{h'}$, simply because the derivative of $G^{(h_i)}$ satisfies the same qualitative estimates as $G^{(h_i)}$ (see (5.5)) times an extra 2^{h_i}. In conclusion, all the terms appearing in $\mathcal{R}V_4^{(h)}(\tau, \mathbf{P})$ are associated with a gain factor $2^{h'-h''}$, which is enough to renormalize the (marginal) quartic terms. Note that the action of $\mathcal{L}$ essentially corresponds to extracting the zero order term in a Taylor expansion of the kernel with respect to m, and of the fields with respect to $\mathbf{x}_1$; conversely, the action of

$\mathcal{R}$ corresponds to taking the rest of first order of the same Taylor expansion. The rest of order 1 has an improved estimate by a factor $2^{h'-h''}$ as compared to the original kernel. By proceeding similarly, one can show that the rest of order 2 has a dimensional gain $\propto 2^{2(h'-h'')}$, etc. The rationale behind the definition of $\mathcal{L}$ should now be clear: if it acts on a marginal term, it extracts the zero-th order term in the aforementioned Taylor expansion, so that the renormalized part has a gain $2^{h'-h''}$, which is enough to make it convergent; if it acts on a linearly relevant term (i.e., a term with scaling dimension 1), it extracts the zero-th plus first order terms in the Taylor expansion (this is precisely the choice done in (5.22)), so that the renormalized part (which is a Taylor rest of order 2) has a gain $2^{2(h'-h'')}$.

Let us now discuss item (2) at the beginning of this subsection; i.e., let us discuss the explicit structure of the local terms, and let us show that they are parametrized by a finite number of constants. We define

$$K^{(h)}_{2,\boldsymbol{\gamma}}(\mathbf{x}-\mathbf{y}) := \mathcal{P}_0 W^{(h)}_{2,\boldsymbol{\gamma}}(\mathbf{x},\mathbf{y}) \,, \tag{5.28}$$

$$K^{(h)}_{4,\boldsymbol{\gamma}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) := \mathcal{P}_0 W^{(h)}_{4,\boldsymbol{\gamma}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) \,, \tag{5.29}$$

$$M^{(h)}_{2,\boldsymbol{\gamma}}(\mathbf{x}-\mathbf{y}) := (-1)^{y_1}\mathcal{P}_1 W^{(h)}_{2,\boldsymbol{\gamma}}(\mathbf{x},\mathbf{y}) \,, \tag{5.30}$$

$$B^{(h)}_{2,1,\boldsymbol{\gamma},j}(\mathbf{x}-\mathbf{y},\mathbf{x}-\mathbf{z}) := \mathcal{P}_0 W^{(h)}_{2,1,\boldsymbol{\gamma},j}(\mathbf{x},\mathbf{y};\mathbf{z}) \,, \tag{5.31}$$

so that $K^{(h)}_{2,\boldsymbol{\gamma}}$, $K^{(h)}_{4,\boldsymbol{\gamma}}$ and $B^{(h)}_{2,1,\boldsymbol{\gamma},j}$ are independent of m, while $M^{(h)}_{2,\boldsymbol{\gamma}}$ is linear in m. They are all translation invariant. Let us separately rewrite in a more compact way the contributions to the local part of the effective potential associated with these kernels. As an illustration, let us consider the contribution to the local part of the effective potential associated with $K^{(h)}_{2,\boldsymbol{\gamma}}$, which can be rewritten as (again, we provide formulas only in the $L\to\infty$ limit; we also add the apex ($\le h$) to the fields to recall that they are on scale $\le h$)

$$\begin{aligned}
&\sum_{\mathbf{x},\mathbf{y}}\sum_{\gamma,\gamma'} e^{i\mathbf{p}_\gamma\mathbf{x}}\psi^{(\le h)}_{\mathbf{x},\gamma} K^{(h)}_{2,(\gamma,\gamma')}(\mathbf{x}-\mathbf{y}) e^{i\mathbf{p}_{\gamma'}\mathbf{y}}\big[1+(\mathbf{y}-\mathbf{x})\cdot\hat{\boldsymbol{\partial}}\big]\psi^{(\le h)}_{\mathbf{x},\gamma'} = \\
&= \sum_{\gamma,\gamma'}\int \frac{d\mathbf{k}\,d\mathbf{k}'}{(2\pi)^2}\Big(\hat K^{(h)}_{2,(\gamma,\gamma')}(\mathbf{p}_{\gamma'}) + \sum_{j=1}^{2}\sin k'_j\partial_{k_j}\hat K^{(h)}_{2,(\gamma,\gamma')}(\mathbf{p}_{\gamma'})\Big)\times \\
&\qquad \times\hat\psi^{(\le h)}_{-\mathbf{k},\gamma}\hat\psi^{(\le h)}_{\mathbf{k}',\gamma'}\delta(\mathbf{k}+\mathbf{p}_\gamma-\mathbf{k}'-\mathbf{p}_{\gamma'}) \,,
\end{aligned} \tag{5.32}$$

where the integrals over $\mathbf{k},\mathbf{k}'$ run as usual over the torus $\mathbb{T}^2$, δ is a periodic Dirac delta over the torus,

$$\hat\psi^{(\le h)}_{\mathbf{k},\gamma} := \sum_{\mathbf{x}} e^{i\mathbf{k}\mathbf{x}}\psi^{(\le h)}_{\mathbf{x},\gamma} \tag{5.33}$$

and $\hat K^{(h)}_{2,(\gamma,\gamma')}(\mathbf{k}) := \sum_{\mathbf{x}} e^{i\mathbf{k}\mathbf{x}} K^{(h)}_{2,(\gamma,\gamma')}(\mathbf{x})$. Now, recalling that the field $\hat\psi^{(\le h)}_{\mathbf{k},\gamma}$ has the same support as $\chi_h(\mathbf{k}) = \chi(2^{-h}\mathbf{k})$ for all $h\le -1$ (in the sense that its propagator has this support; remember from the discussion after (3.3) that the support of $\chi(\cdot)$ is essentially $\{\mathbf{k}\in\mathbb{T}^2 : ||\mathbf{k}||\le \pi/2\}$ where $||\cdot||$ is the Euclidean distance on $\mathbb{T}^2$), then the only non-vanishing terms in (5.32) are the diagonal ones, i.e., those with $\gamma=\gamma'$. In a similar way we find that

the contribution to the local part of the effective potential associated with $M^{(h)}_{2,\boldsymbol{\gamma}}$ can be rewritten as

$$\sum_{\mathbf{x},\mathbf{y}}\sum_{\gamma,\gamma'} e^{i\mathbf{p}_\gamma \mathbf{x}}\psi^{(\leq h)}_{\mathbf{x},\gamma} M^{(h)}_{2,(\gamma,\gamma')}(\mathbf{x}-\mathbf{y}) e^{i(\mathbf{p}_{\gamma'}+(\pi,0))\mathbf{y}}\psi^{(\leq h)}_{\mathbf{x},\gamma'} = \tag{5.34}$$

$$= \sum_{\gamma,\gamma'}\int \frac{d\mathbf{k}\, d\mathbf{k}'}{(2\pi)^2}\hat\psi^{(\leq h)}_{-\mathbf{k},\gamma}\hat M^{(h)}_{2,(\gamma,\gamma')}(\mathbf{p}_{\gamma'}+(\pi,0))\hat\psi^{(\leq h)}_{\mathbf{k}',\gamma'}\delta(\mathbf{k}+\mathbf{p}_\gamma-\mathbf{k}'-\mathbf{p}_{\gamma'}-(\pi,0)) \ .$$

Thanks to the above mentioned properties of the support of the field $\hat\psi^{(\leq h)}_{\mathbf{k},\gamma}$, the only non-vanishing terms in (5.34) are those with $(\gamma,\gamma') = (1,2),(2,1),(3,4),(4,3)$. In terms of these definitions and properties we can rewrite

$$\mathcal{L}V^{(h)}_2(\psi) = \int \frac{d\mathbf{k}}{(2\pi)^2}\hat\psi^T_{-\mathbf{k}} C_h(\mathbf{k})\hat\psi_{\mathbf{k}} \tag{5.35}$$

where $\hat\psi_{\mathbf{k}}$ is a column vector with components $\hat\psi_{\mathbf{k},\gamma}$, $\gamma = 1,2,3,4$, and $C_h(\mathbf{k})$ is a block-diagonal matrix of the form:

$$C_h(\mathbf{k}) = \begin{pmatrix} c_h(\mathbf{k}) & 0 \\ 0 & d_h(\mathbf{k}) \end{pmatrix} \tag{5.36}$$

with

$$c_h(\mathbf{k}) = \begin{pmatrix} a^{(h)}_1 \sin k_1 + b^{(h)}_1 \sin k_2 & \sigma^{(h)}_{1,2} \\ \sigma^{(h)}_{2,1} & a^{(h)}_2 \sin k_1 + b^{(h)}_2 \sin k_2 \end{pmatrix} \tag{5.37}$$

$$d_h(\mathbf{k}) = \begin{pmatrix} a^{(h)}_3 \sin k_1 + b^{(h)}_3 \sin k_2 & \sigma^{(h)}_{3,4} \\ \sigma^{(h)}_{4,3} & a^{(h)}_4 \sin k_1 + b^{(h)}_4 \sin k_2 \end{pmatrix} \tag{5.38}$$

and: $a^{(h)}_\gamma = \partial_{k_1}\hat K^{(h)}_{2,(\gamma,\gamma)}(\mathbf{p}_\gamma)$, $b^{(h)}_\gamma = \partial_{k_2}\hat K^{(h)}_{2,(\gamma,\gamma)}(\mathbf{p}_\gamma)$, $\sigma^{(h)}_{\gamma,\gamma'} = \frac{1}{2}(\hat M^{(h)}_{2,(\gamma,\gamma')}(\mathbf{p}_\gamma) - \hat M^{(h)}_{2,(\gamma',\gamma)}(\mathbf{p}_{\gamma'}))$. Even more: by using the symmetries of the Grassmann action and of the propagator, one can check (see Appendix B for details) that:

- $a^{(h)}_\gamma$ is independent of γ and purely imaginary: i.e., $a^{(h)}_\gamma = iz_h$ for some real constant z_h;
- $b^{(h)}_\gamma = (-1)^\gamma i a^{(h)}_\gamma$, so that $b^{(h)}_\gamma = (-1)^{\gamma-1} z_h$, for the same constant z_h;
- $\sigma^{(h)}_{1,2} = -\sigma^{(h)}_{2,1} = \sigma^{(h)}_{3,4} = -\sigma^{(h)}_{4,3} = i\sigma_h$, for some real constant σ_h.

Therefore, in (5.36), $c_h(\mathbf{k}) = d_h(\mathbf{k})$, with

$$c_h(\mathbf{k}) = -\begin{pmatrix} z_h(-i\sin k_1 + \sin k_2) & i\sigma_h \\ -i\sigma_h & z_h(-i\sin k_1 - \sin k_2) \end{pmatrix} \ , \tag{5.39}$$

that is, $\mathcal{L}V^{(h)}_2$ has the same structure as the inverse of the propagator in (3.6)-(3.7), and it is parametrized just by two real constants z_h and σ_h.

As far as the quartic and source local terms are concerned, we find:

$$\mathcal{L}V^{(h)}_4(\psi) = l_h \sum_{\mathbf{x}} \psi_{\mathbf{x},1}\psi_{\mathbf{x},2}\psi_{\mathbf{x},3}\psi_{\mathbf{x},4} \ , \tag{5.40}$$

$$\mathcal{L}\mathcal{B}^{(h)}(\psi) = \sum_{\gamma<\gamma'}\sum_j Z_{h;(\gamma,\gamma'),j}\sum_{\mathbf{z}} e^{i(\mathbf{p}_\gamma+\mathbf{p}_{\gamma'})\mathbf{z}}\psi_{\mathbf{z},\gamma}\psi_{\mathbf{z},\gamma'}J_{\mathbf{z},j} \ , \tag{5.41}$$

where:

- In (5.40), the constant l_h is

$$l_h = \sum_{\substack{\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4 \\ \pi\in S_4}} (-1)^\pi K^{(h)}_{4,(\pi(1),\pi(2),\pi(3),\pi(4))}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) \prod_{i=1}^4 e^{i\mathbf{p}_{\pi(i)}\mathbf{x}_i} \ , \quad (5.42)$$

  where S_4 is the set of permutations of $(1,2,3,4)$.
- In (5.41), the constants $Z_{h;(\gamma,\gamma'),j}$ are

$$Z_{h;(\gamma,\gamma'),j} = \hat B^{(h)}_{2,1,(\gamma,\gamma'),j}(\mathbf{p}_{\gamma'},\mathbf{p}_\gamma-\mathbf{p}_{\gamma'}) - \hat B^{(h)}_{2,1,(\gamma',\gamma),j}(\mathbf{p}_\gamma,\mathbf{p}_{\gamma'}-\mathbf{p}_\gamma) \ . \quad (5.43)$$

Using again the symmetries of the Grassmann action given in Appendix B we find that the constant l_h is real, while the constants $Z_{h;(\gamma,\gamma'),j}$ are such that the source term takes the following form:

$$\mathcal{LB}^{(h)}(\psi) = \frac{Z^{(1)}_h}{Z_h} F_1(\psi,\mathbf{J}) + \frac{Z^{(2)}_h}{Z_h} F_2(\psi,\mathbf{J}) \ , \quad (5.44)$$

where $Z^{(1)}_h$, $Z^{(2)}_h$, Z_h are real constants (Z_h is the same as in (5.15)-(5.16) and is inserted here for later convenience), and

$$\begin{aligned} F_1(\psi,\mathbf{J}) = 2i\sum_{\mathbf{z}} (-1)^{z_1+z_2}\Big[& J_{\mathbf{z},1}(\psi_{\mathbf{z},1}\psi_{\mathbf{z},3}+\psi_{\mathbf{z},2}\psi_{\mathbf{z},4}) + \\ & + iJ_{\mathbf{z},2}(\psi_{\mathbf{z},1}\psi_{\mathbf{z},3}-\psi_{\mathbf{z},2}\psi_{\mathbf{z},4})\Big] \ , \\ F_2(\psi,\mathbf{J}) = 2i\sum_{\mathbf{z}} \Big[& J_{\mathbf{z},1}(-1)^{z_1+1}(\psi_{\mathbf{z},1}\psi_{\mathbf{z},2}+\psi_{\mathbf{z},3}\psi_{\mathbf{z},4}) + \\ & + J_{\mathbf{z},2}(-1)^{z_2}(\psi_{\mathbf{z},1}\psi_{\mathbf{z},4}+\psi_{\mathbf{z},2}\psi_{\mathbf{z},3})\Big] \ . \end{aligned}$$

In conclusion, thanks to the way $\mathcal{L}$ is defined and to the symmetry of the theory, the local part of the effective potential is parametrized by 5 real constants, namely $z_h, \sigma_h, l_h, Z^{(1)}_h, Z^{(2)}_h$. These constants are all independent of m, except σ_h, which is exactly linear in m. As we shall see in the following, the terms proportional to z_h and σ_h are inserted step by step into the gaussian integration, thus "dressing" iteratively the propagator at scale h.

Remark 4. *The only constant contributing to the effective mass at scale h is σ_h, see* (5.48) *below. Since it is linear in m, we find that the propagator at $m=0$ is massless: this is an instance of the fact that our theory remains critical at $m=0$, irrespective of the value of the interaction λ.*

5.3.2. *The integration of the fields on scale h.* Now that the definitions of local and renormalized parts are given, we can proceed with the inductive proof of (5.15) (we refer the reader for instance to [12, 34] for more details). We assume the representation to be valid at scale h, and we rewrite (5.15) as

$$\mathcal{Z}_\Lambda = e^{E^{(h)}_\Lambda + t^{(h)}_\Lambda + S^{(h)}_\Lambda(\mathbf{J})} \int P_{\tilde Z_{h-1},m_{h-1},\chi_h}(d\psi^{(\le h)}) \times \quad (5.45)$$
$$\times e^{\mathcal{L}V^{(h)}_{4,\Lambda}(\sqrt{Z_h}\psi^{(\le h)}) + \mathcal{LB}^{(h)}_\Lambda(\sqrt{Z_h}\psi^{(\le h)},\mathbf{J}) + \mathcal{R}V^{(h)}_\Lambda(\sqrt{Z_h}\psi^{(\le h)}) + \mathcal{RB}^{(h)}_\Lambda(\sqrt{Z_h}\psi^{(\le h)},\mathbf{J})} \ ,$$

where

$$e^{t_\Lambda^{(h)}} P_{\tilde{Z}_{h-1},m_{h-1},\chi_h}(d\psi^{(\le h)}) = P_{Z_h,m_h,\chi_h}(d\psi^{(\le h)}) e^{\mathcal{L}V^{(h)}_{2,\Lambda}(\sqrt{Z_h}\psi^{(\le h)})} \tag{5.46}$$

and $t_\Lambda^{(h)}$ accounts for the change in the normalization of the two gaussian Grassmann integrations. The "dressed" measure $P_{\tilde{Z}_{h-1},m_{h-1},\chi_h}(d\psi^{(\le h)})$ has a propagator similar to (5.16), namely

$$\frac{\tilde{g}^{(\le h)}(\mathbf{x})}{Z_{h-1}} = \int_{\mathbb{T}^2} \frac{d\mathbf{k}}{(2\pi)^2} \frac{\chi_h(\mathbf{k})}{\tilde{Z}_{h-1}(\mathbf{k})} e^{-i\mathbf{k}\mathbf{x}} \begin{pmatrix} \hat{G}_{m_{h-1}}(\mathbf{k}) & 0 \\ 0 & \hat{G}_{m_{h-1}}(\mathbf{k}) \end{pmatrix} , \tag{5.47}$$

where

$$\tilde{Z}_{h-1}(\mathbf{k}) = Z_h(1 + z_h\chi_h(\mathbf{k})) , \qquad m_{h-1}(\mathbf{k}) = \frac{Z_h}{\tilde{Z}_{h-1}(\mathbf{k})}(m_h(\mathbf{k}) + \sigma_h\chi_h(\mathbf{k})) \tag{5.48}$$

and Z_{h-1} denotes the value of $\tilde{Z}_{h-1}(\mathbf{k})$ at $\mathbf{k} = \mathbf{0}$. We now split $P_{\tilde{Z}_{h-1},m_{h-1},\chi_h}$ as:

$$P_{\tilde{Z}_{h-1},m_{h-1},\chi_h}(d\psi^{(\le h)}) = P_{Z_{h-1},m_{h-1},\chi_{h-1}}(d\psi^{(\le h-1)}) P_{Z_{h-1},m_{h-1},\tilde{f}_h}(d\psi^{(h)}) \tag{5.49}$$

where

$$\tilde{f}_h(\mathbf{k}) = Z_{h-1}\Big[\frac{\chi_h(\mathbf{k})}{\tilde{Z}_{h-1}(\mathbf{k})} - \frac{\chi_{h-1}(\mathbf{k})}{Z_{h-1}}\Big] . \tag{5.50}$$

Note that $\tilde{f}_h(\mathbf{k})$ has the same support of $f_h(\mathbf{k})$, in fact (using the fact that $\chi_{h-1}(\mathbf{k})\chi_h(\mathbf{k}) = \chi_{h-1}(\mathbf{k})$)

$$\tilde{f}_h(\mathbf{k}) = f_h(\mathbf{k}) \frac{1 + z_h}{1 + z_h\chi_h(\mathbf{k})} . \tag{5.51}$$

Coherently with the above notations, the propagator of $P_{Z_{h-1},m_{h-1},\tilde{f}_h}(d\psi^{(h)})$ is

$$\frac{g^{(h)}(\mathbf{x})}{Z_{h-1}} = \int_{\mathbb{T}^2} \frac{d\mathbf{k}}{(2\pi)^2} \frac{\tilde{f}_h(\mathbf{k})}{Z_{h-1}} e^{-i\mathbf{k}\mathbf{x}} \begin{pmatrix} \hat{G}_{m_{h-1}}(\mathbf{k}) & 0 \\ 0 & \hat{G}_{m_{h-1}}(\mathbf{k}) \end{pmatrix} \tag{5.52}$$

Note that $g^{(h)}$ satisfies the same estimate as (5.5), possibly with different constants C, c, provided that z_h in (5.51) stays uniformly small for all scales $h \le 0$. We now rescale the fields so that

$$\begin{aligned} \widehat{V}^{(h)}_\Lambda(\sqrt{Z_{h-1}}\psi^{(\le h)}) &:= \mathcal{L}V^{(h)}_{4,\Lambda}(\sqrt{Z_h}\psi^{(\le h)}) + \mathcal{R}V^{(h)}_\Lambda(\sqrt{Z_h}\psi^{(\le h)}) , \\ \widehat{\mathcal{B}}^{(h)}_\Lambda(\sqrt{Z_{h-1}}\psi^{(\le h)}) &:= \mathcal{B}^{(h)}_\Lambda(\sqrt{Z_h}\psi^{(\le h)}) . \end{aligned} \tag{5.53}$$

It follows that

$$\mathcal{L}\widehat{V}^{(h)}(\psi) = \lambda_h \sum_{\mathbf{x}} \psi_{\mathbf{x},1}\psi_{\mathbf{x},2}\psi_{\mathbf{x},3}\psi_{\mathbf{x},4} =: \lambda_h F_\lambda(\psi) , \tag{5.54}$$

$$\mathcal{L}\widehat{\mathcal{B}}^{(h)}(\psi) = \frac{Z_h^{(1)}}{Z_{h-1}} F_1(\psi, \mathbf{J}) + \frac{Z_h^{(2)}}{Z_{h-1}} F_2(\psi, \mathbf{J}) , \tag{5.55}$$

with

$$\lambda_h = \Big(\frac{Z_h}{Z_{h-1}}\Big)^2 l_h . \tag{5.56}$$

We now define

$$e^{V_\Lambda^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)})+\mathcal{B}_\Lambda^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)},\mathbf{J})+\tilde E_\Lambda^{(h-1)}+\tilde S_\Lambda^{(h-1)}(\mathbf{J})} = \tag{5.57}$$
$$= \int P_{Z_{h-1},m_{h-1},\tilde f_h}(d\psi^{(h)})\; e^{\lambda_h F_\lambda(\sqrt{Z_{h-1}}\psi^{(\le h)})+\sum_{j=1}^2 Z_h^{(j)}F_j(\psi^{(\le h)},\mathbf{J})}\times$$
$$\times\, e^{\mathcal{R}\widehat V_\Lambda^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})+\mathcal{R}\widehat{\mathcal{B}}_\Lambda^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)},\mathbf{J})}$$

with the constants fixed by the convention that $\mathcal{B}_\Lambda^{(h-1)}(\psi,\mathbf{0}) = V_\Lambda^{(h-1)}(0) = \tilde S_\Lambda^{(h-1)}(\mathbf{0}) = 0$, which proves (5.15) with h replaced by $h-1$, if one sets $E_\Lambda^{(h-1)} = E_\Lambda^{(h)} + t_\Lambda^{(h)} + \tilde E_\Lambda^{(h-1)}$ and $S_\Lambda^{(h-1)}(\mathbf{J}) = S_\Lambda^{(h)}(\mathbf{J}) + \tilde S_\Lambda^{(h-1)}(\mathbf{J})$.

Using (5.57) and the definition of truncated expectation, we can rewrite

$$V_\Lambda^{(h-1)}(\sqrt{Z_{h-1}}\psi) + \mathcal{B}_\Lambda^{(h-1)}(\sqrt{Z_{h-1}}\psi,\mathbf{J}) + \tilde E_\Lambda^{(h-1)} + \tilde S_\Lambda^{(h-1)}(\mathbf{J}) = \tag{5.58}$$
$$= \sum_{s\ge 1}\frac{1}{s!}\mathcal{E}_h^T\Big(\underbrace{\widehat V_\Lambda^{(h)}(\sqrt{Z_{h-1}}(\psi+\psi^{(h)}),\mathbf{J});\cdots;\widehat V_\Lambda^{(h)}(\sqrt{Z_{h-1}}(\psi+\psi^{(h)}),\mathbf{J})}_{s\text{ times}}\Big),$$

where $\mathcal{E}_h^T$ is the truncated expectation with respect to the propagator $Z_{h-1}^{-1}g^{(h)}$ of the field $\psi^{(h)}$ (cf. (5.52)), $\forall h > h^*$, and h^* is the first scale (with respect to the ordering $h = 0,-1,-2,\ldots$) such that[5] $m_h(\mathbf{0}) > 2^h$. Moreover, $\widehat V_\Lambda^{(h)}(\psi,\mathbf{J})$ is a shorthand for $\widehat V_\Lambda^{(h)}(\psi) + \widehat{\mathcal{B}}_\Lambda^{(h)}(\psi,\mathbf{J})$. When we reach scale h^* we perform the last integration with respect to the propagator $Z_{h^*-1}^{-1}\tilde g^{(\le h^*)}$ (see (5.47)) and we compute the result at $\psi = 0$, thus obtaining the contribution to the pressure and to the generating function from the last scale, $\tilde E_\Lambda^{(h^*)} + \tilde S_\Lambda^{(h^*)}(\mathbf{J})$.

5.3.3. *The Beta function.* Note that the above procedure allows us to write the *effective constants* $(\lambda_h, Z_h, m_h(\mathbf{0}), Z_h^{(1)}, Z_h^{(2)})$ with $h \le 0$, in terms of (λ_k, Z_k) with $h < k \le 0$:

$$\lambda_{h-1} = \lambda_h + \beta_h^\lambda\,,\qquad \frac{Z_{h-1}}{Z_h} = 1+\beta_h^Z\,,\qquad \frac{m_{h-1}(\mathbf{0})}{m_h(\mathbf{0})} = 1+\beta_h^m\,,$$
$$\frac{Z_{h-1}^{(1)}}{Z_h^{(1)}} = 1+\beta_h^{Z,1}\,,\qquad \frac{Z_{h-1}^{(2)}}{Z_h^{(2)}} = 1+\beta_h^{Z,2}\,, \tag{5.59}$$

where $\beta_h^\# = \beta_h^\#\big((\lambda_h,Z_h),\ldots,(\lambda_0,Z_0)\big)$ is the so–called *Beta function*. Of course, applying iteratively (5.59), one can also see $\beta_h^\#$ as a function of λ.

As we discussed in Section 5.4, the effective potential and therefore the beta function at any scale h are expressed via a tree expansion, induced by the iterative procedure above. If the expansion is well defined and absolutely summable (uniformly in m,Λ), then it immediately follows that it is *analytic* in λ. The crucial point, that will follow from formulas like (5.70), is that absolute summability follows if we have that λ_h and $(Z_h/Z_{h-1})-1$ remain small uniformly in h. It is then essential to study the dynamical system defined by $\beta_h^\#$. The flow induced by (5.59) is non trivial and it has been

[5]Note that this definition is slightly different from the one given at the beginning of Section 5.1, which referred to the non-renormalized expansion, where the mass was not dressed iteratively under the RG flow. The correct one, used from now on and keeping into account the mass renormalization, is the current one. With some abuse of notation we indicate it by the same symbol.

investigated in a series of works from the mid 1990s to the mid 2000s for very similar models (cf. [11, 13, 14, 48, 49] among others), by combining the use of the Schwinger-Dyson equation with local Ward Identites. The conceptual scheme used for studying the flow is reviewed in Section 5.5 below. The key results are the following.

(1) The beta function for λ_h is asymptotically vanishing as $h \to -\infty$, i.e.,
$$|\beta_h^\lambda| \le C|\lambda|^2 2^{\theta h}, \tag{5.60}$$
for a suitable $0 < \theta < 1$. As a consequence, $\lim_{h\to-\infty} \lambda_h = \lambda_{-\infty}(\lambda)$, which is analytically close to λ_0. Moreover, the limit is reached exponentially fast, $|\lambda_h(\lambda) - \lambda_{-\infty}(\lambda)| \le C|\lambda^2|2^{\theta h}$. The reason why this is true is discussed briefly in Section 5.5.

(2) the ratio Z_h/Z_{h-1} stays analytically close to 1 uniformly in h, so that
$$\big|\frac{Z_h}{Z_{h-1}} - 1\big| \le C|\lambda|\,, \tag{5.61}$$

(3) The "wave function renormalization" Z_h, the two renormalization constants $Z_h^{(i)}$, $i = 1, 2$, and the effective mass $m_h(\mathbf{0})$ develop critical exponents in the infrared, namely
$$Z_h \sim 2^{\eta(\lambda)h}\,, \qquad Z_h^{(i)} \sim 2^{\eta_i(\lambda)h}\,, \qquad m_h(\mathbf{0}) \sim m\, 2^{\eta_m(\lambda)h}\,, \tag{5.62}$$
where $\sim$ means that the ratio of the two sides is bounded from above and below by two universal positive constants, uniformly in h. Moreover, $\eta(\lambda)$, $\eta_1(\lambda)$, $\eta_2(\lambda)$ and $\eta_m(\lambda)$ are analytic functions of λ, vanishing at $\lambda = 0$.

Next we need to explain how to prove the analyticity of the effective potentials and of the beta function. Of course the point is to properly compute and estimate the series of truncated expectations in the right side of (5.58), as well as the one appearing in the definition of beta function. In principle, in order to compute it, we could use the representation in terms of Feynman diagrams; however, this choice would lead us to a renormalized expansion affected by the same combinatorial divergences discussed at the end of Section 5.1. As already mentioned there, in order to solve the combinatorial problem associated with the proliferation of Feynman diagrams, we have to devise an alternative expansion scheme, known as the determinant or Pfaffian expansion, which is discussed in the next subsection.

5.3.4. *The Pfaffian expansion.* In order to express scale by scale the series in the right side of (5.58) in the form of a convergent expansion, we use the following representation for the truncated expectation, originally due to Battle, Brydges and Federbush [5, 16, 17], later improved and simplified [18, 1] and re-derived in several review papers, see e.g. [34, 36]: using a notation similar to (4.4) and (4.7) we get
$$\mathcal{E}_h^T\big(X_1(\psi);\dots;X_s(\psi)\big) = c_1\cdots c_s\, Z_{h-1}^{-p} \sum_{T\in\mathbf{T}} \alpha_T \prod_{\ell\in T} g_\ell^{(h)} \int dP_T(\mathbf{t})\, \mathrm{Pf}(M^{h,T}(\mathbf{t}))\,. \tag{5.63}$$

Here:

- the constants c_i are those appearing in the definition of X_i and $2p = \sum_{i=1}^s n_i$ (recall that n_i is the order of the monomial X_i);
- the first sum runs over set of lines forming a *spanning tree* between the s vertices corresponding to the monomials $X_1, \dots, X_s$, i.e., T is a set of lines that becomes a tree if one identifies all the points in the same clusters;
- α_T is a sign (irrelevant for the subsequent bounds);
- $g_\ell^{(h)}$ is a shorthand for $g^{(h)}_{\gamma(\ell),\gamma'(\ell)}(\mathbf{x}(\ell) - \mathbf{x}'(\ell))$, where $\gamma(\ell), \gamma'(\ell)$ and $\mathbf{x}(\ell), \mathbf{x}'(\ell)$ are the γ and $\mathbf{x}$ indices associated with the two ends of the line ℓ, which should be thought of as being obtained from the pairing (contraction) of two fields $\psi_{\mathbf{x}(\ell),\gamma(\ell)}$ and $\psi_{\mathbf{x}'(\ell),\gamma'(\ell)}$;
- if $\mathbf{t} = \{t_{i,i'} \in [0,1], 1 \le i, i' \le s\}$, then $dP_T(\mathbf{t})$ is a probability measure with support on a set of $\mathbf{t}$ such that $t_{i,i'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i = \mathbf{u}_i(\mathbf{t}) \in \mathbb{R}^s$ of unit norm;
- $M^{h,T}(\mathbf{t})$ is an antisymmetric $(2p - 2s + 2) \times (2p - 2s + 2)$ matrix, whose elements are given by $M^{h,T}_{f,f'} = t_{i(f),i(f')} g^{(h)}_{\ell(f,f')}$, where: $f, f' \notin \cup_{\ell \in T}\{f^1_\ell, f^2_\ell\}$ and f^1_ℓ, f^2_ℓ are two field labels associated with the two (entering and exiting) half-lines contracted into ℓ; $i(f) \in \{1, \dots, s\}$ is s.t. $f \in P_{v_{i(f)}}$; $g^{(h)}_{\ell(f,f')}$ is the propagator associated with the line obtained by contracting the two half-lines with indices f and f'.

If $s = 1$ the sum over T is empty, but we can still use Eq.(5.63) by interpreting the r.h.s. as equal to 0 if P_1 is empty and equal to $\mathrm{Pf} M^{h,T}(\mathbf{1})$ otherwise. Note that if the Pfaffian is expanded by using its definition (2.4), then (5.63) reduces to the usual representation of the truncated expectation in terms of connected Feynman diagrams. The spanning trees in (5.63) guarantee the minimal connection among the vertices $X_1, \dots, X_s$ and the Pfaffian can be thought of as a resummation of all the Feynman diagrams obtained by pairing (contracting) in all possible ways the fields outside the spanning tree, with the rule that each contracted pair $(\psi_{\mathbf{x},\gamma}, \psi_{\mathbf{y},\gamma'})$ is replaced by $Z_{h-1}^{-1} g^{(h)}_{\gamma,\gamma'}(\mathbf{x} - \mathbf{y})$; the interpolation in $\mathbf{t}$ is necessary in order to avoid an over-counting of the diagrams.

Eq.(5.63) has the advantage that the Pfaffian $\mathrm{Pf}(M^{h,T}(\mathbf{t}))$ can be bounded by using the *Gram-Hadamard inequality* [34], which leads to an estimate scaling like $(C2^h)^{p-s+1}$; here $2(p - s + 1)$ is the size of the antisymmetric matrix $M^{h,T}$ and C is the constant appearing in the estimate of $g^{(h)}$, see the lines following (5.52). This is in contrast with the estimate scaling like $(p - s + 1)!(C2^h)^{p-s+1}$ that we would get via the Feynman expansion. Morally speaking, recalling that $\mathrm{Pf}^2 M = \det M$, the Gram-Hadamard inequality is similar in spirit to bounding the determinant of a $k \times k$ matrix by the largest eigenvalue to the power k (which is combinatorially optimal), rather than by the number of terms in the determinant times the maximum of the matrix elements. Finally, the number of spanning trees is bounded by $s! C^p$, with p the total number of fields appearing in $X_1, \dots, X_s$ [34, Appendix A3.3]. Note that, if formula (5.63) is applied to the right side of (5.58), then the number of spanning trees $\propto s!$ is compensated by the factor $1/s!$ appearing there. This is the key combinatorial advantage provided by

the Pfaffian expansion, allowing us to prove that the kernels of the effective potential, as obtained from the iterative procedure above, are expressed by absolutely convergent series in the sequence $\{\lambda_h, Z_h/Z_{h-1}\}_{h\le 0}$, or equivalently in λ. A convenient representation, which plays an important role in the proof of convergence, is the renormalized GN tree expansion.

5.4. **The renormalized tree expansion.** The iterative construction described above naturally leads (cf. [12, 34]) to a representation of $V_\Lambda^{(h)}$, $\mathcal{B}_\Lambda^{(h)}$, etc., in terms of renormalized GN trees, which are defined in a way very similar to the one described in Section 5.1.1, with the following important differences.

(1) A renormalized tree τ contributing to $V_\Lambda^{(h)}$, $\mathcal{B}_\Lambda^{(h)}$, $\tilde{E}_\Lambda^{(h+1)}$, or $\tilde{S}^{(h+1)}(\mathbf{J})$ has root on scale h and can have endpoints on all possible scales between $h+2$ and $+1$. The endpoints v on scales $h_v \le 0$ are preceded by a node of τ (on scale h_v-1) that is necessarily a branching point.
(2) Normal endpoints on scale h_v are associated with $\lambda_{h_v-1}F_\lambda$, if $h_v \le 0$; they are associated with (one of the monomials contributing to) $\widehat{V}^{(0)}$, if $h_v = 1$. Similarly, special endpoints on scale h_v are associated with $(Z^{(j)}_{h_v-1}/Z_{h_v-2})F_j$ (cf. (5.44)), where either $j=1$ or $j=2$, if $h_v \le 0$; they are associated with (one of the monomials contributing to) $\widehat{\mathcal{B}}^{(0)}$, if $h_v = 1$.
(3) Each vertex of the tree that is not an endpoint and that is not the special vertex v_0 (the leftmost vertex of the tree, immediately following the root on τ) is associated with the action of an $\mathcal{R}$ operator.

The family of renormalized trees with root on scale h, N normal endpoints and n special endpoints will be denoted by $\mathcal{T}^{(h)}_{N,n}$. In terms of renormalized trees, the left side of (5.58) can be written as (replacing $h-1$ by h)

$$V_\Lambda^{(h)}(\sqrt{Z_h}\psi^{(\le h)}) + \mathcal{B}_\Lambda^{(h)}(\sqrt{Z_h}\psi^{(\le h)}, \mathbf{J}) + \tilde{E}_\Lambda^{(h)} + \tilde{S}_\Lambda^{(h)}(\mathbf{J}) = \\ = \sum_{\substack{N,n\ge 0\\ N+n\ge 1}} \sum_{\tau\in\mathcal{T}^{(h)}_{N,n}} V^{(h)}(\tau, \sqrt{Z_h}\psi^{(\le h)}, \mathbf{J}) \,, \tag{5.64}$$

where $V^{(h)}(\tau, \sqrt{Z_h}\psi^{(\le h)}, \mathbf{J})$ is defined iteratively: if v_0 is the first vertex of τ, if $\tau_1, \dots, \tau_s$ ($s = s_{v_0}$) are the subtrees of τ with root v_0, and if $\mathcal{E}^T_{h+1}$ is the truncated expectation associated with the propagator $Z_h^{-1}g^{(h+1)}$,

$$V^{(h)}(\tau, \sqrt{Z_h}\psi^{(\le h)}, \mathbf{J}) = \tag{5.65}$$
$$= \frac{1}{s!}\mathcal{E}^T_{h+1}\big(\overline{V}^{(h+1)}(\tau_1, \sqrt{Z_h}\psi^{(\le h+1)}, \mathbf{J}); \dots; \overline{V}^{(h+1)}(\tau_s, \sqrt{Z_h}\psi^{(\le h+1)}, \mathbf{J})\big) \,,$$

where $\psi^{(\le h+1)} = \psi^{(\le h)} + \psi^{(h+1)}$ and $\overline{V}^{(h+1)}(\tau_i, \sqrt{Z_h}\psi^{(\le h+1)}, \mathbf{J})$:

- is equal to $\mathcal{R}\widehat{V}^{(h+1)}(\tau_i, \sqrt{Z_h}\psi^{(\le h+1)}, \mathbf{J}))$ if τ_i is non trivial. Here $\mathcal{R}$ is the linear operator induced by the definitions (5.20)–(5.24), and $\widehat{V}^{(h+1)}(\tau_i, \sqrt{Z_h}\psi^{(\le h+1)}, \mathbf{J})$ is defined in analogy with (5.53), that is
$$\widehat{V}^{(h+1)}(\tau_i, \sqrt{Z_h}\psi^{(\le h+1)}, \mathbf{J}) := V^{(h+1)}(\tau_i, \sqrt{Z_{h+1}}\psi^{(\le h+1)}, \mathbf{J}) \,;$$
- is equal to $\lambda_{h+1}F_\lambda(\sqrt{Z_h}\psi^{(\le h+1)})$ if τ_i is trivial, $h < -1$ and the endpoint of τ_i is normal;

- is equal to $\sum_{j=1}^{2}(Z_{h+1}^{(j)}/Z_h)F_j(\sqrt{Z_h}\psi^{(\le h+1)},\mathbf{J})$ if τ_i is trivial, $h<-1$ and the endpoint of τ_i is special;
- is equal to $\widehat{V}^{(0)}(\sqrt{Z_{-1}}\psi^{(\le 0)})$ (resp. $\widehat{\mathcal{B}}^{(0)}(\sqrt{Z_{-1}}\psi^{(\le 0)},\mathbf{J})$) if τ_i is trivial, $h=-1$ and the endpoint of τ_i is normal (resp. special).

In order to compute as explicitly as possible the tree values $V^{(h)}(\tau,\psi,\mathbf{J})$, we can inductively apply (5.65) and use the Pfaffian representation for the truncated expectation in its right side. In the process of doing that, we can naturally distinguish the various contributions arising from the choices of the monomials in the factors $\widehat{V}^{(0)}$ and $\widehat{\mathcal{B}}^{(0)}$ associated with the endpoints on scale 1, as well as the scale at which each field in these monomials is contracted (we can keep track of these informations via the labels $\mathbf{P}$ attached to the trees, as explained in Section 5.1.1).

The resulting formula has a natural structure, slightly complicated by the presence of the $\mathcal{R}$ operators acting at all vertices of the tree that are not endpoints. Therefore, in order to make it as transparent as possible, let us *temporarily neglect the action of the renormalization operator*, i.e., let us temporarily pretend that the action of $\mathcal{R}$ on the nodes of τ that are not endpoints is replaced by the identity. Then the result of the iteration would lead to the following relation (the reader can easily convince himself of the formula by induction, or consult the aforementioned reviews for more details, see in particular [12, 34]):

$$V_*^{(h)}(\tau,\sqrt{Z_h}\psi,\mathbf{J})=\sum_{\mathbf{P}\in\mathcal{P}_\tau}\sqrt{Z_h}^{|P_{v_0}^\psi|}\sum_{T\in\mathbf{T}}\int d\mathbf{x}_{v_0}W^*_{\tau,\mathbf{P},\mathbf{T}}(\mathbf{x}_{v_0})\psi(P_{v_0}^\psi)J(P_{v_0}^J)\ , \tag{5.66}$$

where $T=\bigcup_{v\ \text{not e.p.}}T_v$ is the union of the spanning trees T_v associated with all the nodes that are not endpoints in τ, which arise from the inductive application of the Pfaffian formula (5.63). Moreover, $W^*_{\tau,\mathbf{P},\mathbf{T}}$ is given by

$$\begin{aligned}W^*_{\tau,\mathbf{P},\mathbf{T}}(\mathbf{x}_{v_0})=&\Big[\prod_{v\ \text{not e.p.}}(Z_{h_v}/Z_{h_v-1})^{|P_v^\psi|/2}\Big]\Big[\prod_{v\ \text{e.p.}}K_v^{(h_v)}(\mathbf{x}_v)\Big]\times\\ &\times\Big\{\prod_{v\ \text{not e.p.}}\frac{1}{s_v!}\int dP_{T_v}(\mathbf{t}_v)\mathrm{Pf}(M^{h_v,T_v}(\mathbf{t}_v))\Big[\prod_{\ell\in T_v}g_\ell^{(h_v)}\Big]\Big\}\ ,\end{aligned} \tag{5.67}$$

where $K_v^{(h_v)}(\mathbf{x}_v)$ is equal to: λ_{h_v-1}, if v is a normal endpoint on scale $h_v<1$; $(Z_{h_v-1}^{(j_v)}/Z_{h_v-2})$, if v is a special endpoint on scale $h_v<1$; the kernel of the monomial of $\widehat{V}^{(0)}$ (resp. $\widehat{\mathcal{B}}^{(0)}$) compatible with the assignment of external fields P_v, if v is a normal (resp. special) endpoint on scale $h_v=1$.

The analogous formula for $V^{(h)}(\tau,\sqrt{Z_h}\psi,\mathbf{J})$, in which we do not neglect the action of $\mathcal{R}$ on all the vertices of τ that are not endpoints, can be written in the form

$$\begin{aligned}&V^{(h)}(\tau,\sqrt{Z_h}\psi,\mathbf{J})=\\ &\quad=\sum_{\mathbf{P}\in\mathcal{P}_\tau}\sqrt{Z_h}^{|P_{v_0}^\psi|}\sum_{T\in\mathbf{T}}\sum_{\beta\in B_T}\int d\mathbf{x}_{v_0}W_{\tau,\mathbf{P},\mathbf{T},\beta}(\mathbf{x}_{v_0})\big[\psi(P_{v_0}^\psi)\big]_\beta\,J(P_{v_0}^J)\ ,\end{aligned} \tag{5.68}$$

where β is a multi-index that keeps track of the various terms arising from the action of $\mathcal{R}$: see e.g. (5.26), which shows that the action of $\mathcal{R}$ on a

four-legged kernel produces 4 different terms. Moreover,

$$\big[\psi(P^{\psi}_{v_0})\big]_{\beta} = \prod_{f\in P^{\psi}_{v_0}} \partial^{q_\beta(f)}_{j_\beta(f)} \psi^{(\le h)}_{\mathbf{x}(f),\omega(f)} \ , \tag{5.69}$$

where $j_\beta(f) \in \{1,2\}$ and q_β is a nonnegative integer ≤ 2; the action of a derivative on the fields arises from the interpolation formula (5.27), see [12] for details. The expressions (5.66)–(5.68) can be bounded dimensionally by using inductively the bound on the Pfaffians mentioned in Section 5.3.4. In the absence of the action of the $\mathcal{R}$ operators the resulting bound has the same structure as the final bound of Section 5.1, see (5.13) and following discussion, modulo an improved combinatorial factor due to the use of the Pfaffians rather than of the Feynman diagrams. If, on the contrary, we take the action of $\mathcal{R}$ into account, the dimensional factors are improved by the gain factors discussed after (5.27) and the result is

$$\frac{1}{|\Lambda|}\int d\mathbf{x}_{v_0} |W_{\tau,\mathbf{P},T,\beta}(\mathbf{x}_{v_0})| \le C^{N+n}\Big[\prod_{v \text{ e.p.}} \|K_v\|_\infty\Big] 2^{h(2-\frac{1}{2}|P^{\psi}_{v_0}|-|P^{J}_{v_0}|-Q_\beta)} \times$$

$$\times\Big[\prod_{\substack{v \text{ not}\\ \text{e.p.}}} \frac{C^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|}}{s_v!}\Big(\frac{Z_{h_v}}{Z_{h_v-1}}\Big)^{\frac{|P^{\psi}_v|}{2}} 2^{2-\frac{1}{2}|P^{\psi}_v|-|P^{J}_v|-z(P_v)}\Big] \ , \tag{5.70}$$

where $\|K_v\|_\infty$ is the ℓ_∞ norm of the function $K^{(h_v)}_v(\mathbf{x}_v)$ in (5.67): it is equal to $\big|Z^{(j_v)}_{h_v-1}/Z_{h_v-2}\big|$, if v is a special endpoint on scale ≤ 0; it is equal to $|\lambda_{h_v-1}|$ if it is a normal endpoint on scale ≤ 0; it is equal to $(C|\lambda|)^{|P^{\psi}_v|/2-1}$ if it is on scale $+1$. Moreover, $Q_\beta = \sum_{f\in P^{\Psi}_{v_0}} q_\beta(f)$, and (recall (5.62) for the definition of η_m)

$$z(P_v) = \begin{cases} 1-|\eta_m| & \text{if} \quad (P^{\psi}_v, P^{J}_v) = (4,0),(2,1) \\ 2(1-|\eta_m|) & \text{if} \quad (P^{\psi}_v, P^{J}_v) = (2,0) \\ 0 & \text{otherwise,} \end{cases} \tag{5.71}$$

where the dependence upon η_m arises from the non-trivial renormalization of the effective mass[6]. This is the analogue of [12, Eq.(3.110)] and the details of its proof can be found there. Note that the renormalized scaling dimension $2-\frac{1}{2}|P^{\psi}_v|-|P^{J}_v|-z(P_v)$ appearing at exponent in the last factor of (5.70) satisfies

$$2-\frac{1}{2}|P^{\psi}_v|-|P^{J}_v|-z(P_v) \le -1+2|\eta_m| \tag{5.72}$$

for any vertex v, which is smaller than $-1+\epsilon$ for any fixed $\epsilon>0$ if λ is small enough (as we always assume; recall that η_m is analytically close to 0 for λ close to 0).

From Eqs. (5.60) and (5.61) it follows that

$$|\lambda_h| + \Big|\frac{Z_h}{Z_{h-1}} - 1\Big| \le C|\lambda| \ , \tag{5.73}$$

[6]In the lines following (5.26) we explained that every factor m extracted from $G^{(h_v)}$ gives a gain $\propto 2^{h^*-h}$. This is slightly incorrect, in that it neglects the critical exponent in the third of (5.62). In fact, every factor m extracted from $G^{(h_v)}$ corresponds dimensionally to a gain $\propto m_h(\mathbf{0})/2^h$. Using the fact that $m_h(\mathbf{0}) \sim m 2^{\eta_m h}$ and $m_{h^*}(\mathbf{0}) \sim 2^{h^*}$, we find that $m_h(\mathbf{0})/2^h \sim 2^{(h^*-h)(1-\eta_m)}$, which is certainly smaller than $2^{(h^*-h)(1-|\eta_m|)}$.

uniformly in h, which (together with (5.72)) implies that the right side of (5.70) is summable over $\beta, T, \mathbf{P}$ and τ (with the constraint that h, $|P_{v_0}^{\psi}|$, $|P_{v_0}^{J}|$ and Q_β are fixed): that is, the tree expansion for the effective potential at scale h is absolutely summable, uniformly in m, Λ. The same holds for the beta function controlling the flow of the running constants $\lambda_h, Z_h/Z_{h-1}$: therefore, these constants are analytic functions of λ, uniformly in h. Finally, since all the terms in the expansion are analytic in λ, one concludes that the kernels of the effective potential on scale h are analytic in λ.

Remark 5 (The short memory property)**.** *Given that*

$$2 - \frac{1}{2}|P_v^{\psi}| - |P_v^{J}| - z(P_v) + \frac{|P_v^{\psi}|}{2} \times \left| \log \frac{Z_h}{Z_{h-1}} \right| \tag{5.74}$$

is smaller than, say, $-1+\epsilon$ for every vertex and for every h, (if λ is small enough), we have not only that the tree expansion converges, but also that the sum restricted to the trees τ with root on scale h and at least one vertex on scale $k > h$ is improved by a factor $C_\theta 2^{\theta(h-k)}$, for any fixed $0 < \theta < 1$ and λ small enough, with respect to the sum over all *trees with root on scale h. This improved bound is usually referred to as the* short memory property *(i.e., trees with long branches are exponentially suppressed).*

5.5. **Emerging Dirac description.** A few comments are in order. The multiscale construction and the control on the flow of the effective renormalization constants are robust under small changes in the original energy function of the model; e.g., they remain valid in the presence of a finite range rather than purely nearest neighbor interaction. A small analytical change in the weights entering the definition of the model induces a small analytical change in the values of $\lambda_{-\infty}(\lambda)$, $\eta(\lambda)$ and $\eta_i(\lambda)$. In this sense, these functions are *non-universal*, i.e., they are model-dependent. However, the critical exponents η, η_1 and η_2 are *universal* (i.e., model-independent) functions of $\lambda_{-\infty}(\lambda)$. While we will not need the detailed proof of this fact in the following, it is important to give at least an idea of the reason why it is true, since the computation of the multi-point dimer correlation functions is crucially based on this. Moreover, the conceptual scheme used in the proof of universality of the critical exponents, as functions of $\lambda_{-\infty}$, is the same used in the very proof of items (1)–(3) after (5.59), i.e., in the control of the flows of λ_h, Z_h, etc., driven by the beta function. Roughly speaking, the idea is the following: we show that the dimer model is associated with a reference continuum (as opposed to lattice) model with the same long-distance (infrared) behavior; this is proved by observing that the tree values of the two models are asymptotically the same as $h \to -\infty$, with h the scale of the root of the tree. The reference model is the same for a large class of statistical mechanics or field theory models, including the Luttinger model [14], the XXZ spin chain [12], the repulsive 1D Hubbard model [8], the 8 vertex model [15], the Ashkin-Teller model at criticality [47, 37], etc.; in this sense, these models are all in the same *universality class*, as far as the long-distance behavior of the correlation functions and their critical exponents are concerned. The flow of the running coupling constants is asymptotically the same for the dimer and the reference model (and, a posteriori, also for

the Luttinger model, for the XXZ chain, etc). The key fact is that the reference model displays more symmetries (as mentioned below) than the dimer model: these can be used to show that the beta function for the analogue of λ_h in the reference model is zero, asymptotically as $h \to -\infty$; as a consequence, the same property remains true for the dimer model, as well as for the other models mentioned above. Let us now describe more technically how this idea is implemented.

At each step of the multiscale integration procedure, we can decompose the single scale propagator (5.52) as the sum of a massless relativistic propagator plus a rest:

$$\frac{g^{(h)}(\mathbf{x})}{Z_{h-1}} = \frac{1}{Z_{h-1}} \Big(g_R^{(h)}(\mathbf{x}) + r^{(h)}(\mathbf{x}) \Big) \,, \tag{5.75}$$

where

$$g_R^{(h)}(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^2} e^{-i\mathbf{k}\mathbf{x}} \tilde{f}_h(\mathbf{k}) (-ik_1 + Jk_2)^{-1} \tag{5.76}$$

and J is the diagonal matrix with diagonal elements $(1, -1, 1, -1)$. Note that the rest satisfies improved dimensional estimates as compared to $g^{(h)}$ and $g_R^{(h)}$: i.e., $||r^{(h)}(\mathbf{x})||$ satisfies an estimate similar to (5.5) with an extra factor $2^h + 2^{(h^*-h)(1-|\eta_m|)}$ in the right side.

Remark 6. *Any observable on scale h can be naturally decomposed in the sum of a dominant part plus a rest: the dominant part is expressed in terms of Gallavotti-Nicolò trees with all the endpoints on scale ≤ 0 and their values computed by replacing all the single-scale propagators by their massless relativistic approximation; the rest can be written as a sum of trees, each of which either has at least one endpoint on scale 1, or it has at least one single scale propagator of type $r^{(k)}$ for some $k \geq h$. It is easy to see that the rest satisfies a better dimensional estimate than the dominant part (better by an exponential factor $2^{\theta h} + 2^{\theta(h^*-h)}$, with $0 < \theta < 1$, in the infrared limit). To see this, use the estimate above for $||r^{(k)}(\mathbf{x})||$ and the short memory property (Remark 5): just note that $2^k 2^{\theta(h-k)} \leq 2^{\theta h}$.*

The dominant parts of the effective potentials and of the effective sources of the dimer model are exponentially close to the corresponding contributions computed in a suitable *reference model* of massless Dirac fermions in the continuum with an ultraviolet (i.e. large momenta) cut-off. This reference model can be thought of as a regularization of the Thirring model and it is defined in terms of the following functional integral, related to the formal scaling limit of the (fermionic representation of the) dimer model. The generating function $\mathcal{W}(\mathbf{J})$ of correlations of the reference model is defined as:

$$e^{\mathcal{W}(\mathbf{J})} = \int P_{Z_M}(d\psi^{(\leq M)}) e^{\mathcal{V}(\sqrt{Z_M}\psi^{(\leq N)}) + \mathcal{B}(\sqrt{Z_M}\psi^{(\leq M)}, \mathbf{J})} \tag{5.77}$$

where the Grassmann field is $\{\psi^{(\le M)\pm}_{\mathbf{x},\omega}\}^{\omega=\pm}_{\mathbf{x}\in\mathbb{R}^2}$, $P_{Z_M}(d\psi^{(\le M)})$ is the Grassmann gaussian integration with propagator

$$\begin{aligned}\frac{g^{(\le M)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y})}{Z_M} &= \int P_{Z_M}(d\psi^{(\le M)})\psi^{(\le M)-}_{\mathbf{x},\omega}\psi^{(\le M)+}_{\mathbf{y},\omega'} \\ &= \frac{\delta_{\omega,\omega'}}{Z_M}\int \frac{d\mathbf{k}}{(2\pi)^2}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{\chi_M(\mathbf{k})}{-ik_1+\omega k_2}\,, \qquad (5.78)\end{aligned}$$

and the label M indicates an ultraviolet cutoff (coherently with our previous notations, it is a function that vanishes say for $\|\mathbf{k}\|\ge 2^M$), to be eventually removed, $M\to+\infty$. Moreover,

$$\mathcal{V}(\psi)=\lambda_M\int d\mathbf{x}d\mathbf{y}v(\mathbf{x}-\mathbf{y})\psi^+_{\mathbf{x},1}\psi^-_{\mathbf{x},1}\psi^+_{\mathbf{y},-1}\psi^-_{\mathbf{y},-1} \qquad (5.79)$$

with $v(\mathbf{x}-\mathbf{y})$ a smooth short-range potential (decaying on scale of order 1), and

$$\begin{aligned}\mathcal{B}(\psi,\mathbf{J}) &= Z^{(1)}_M\sum_\omega\int d\mathbf{x}\,J^{(1)}_\omega(\mathbf{x})\psi^{(\le M)+}_{\mathbf{x},\omega}\psi^{(\le M)-}_{\mathbf{x},\omega} \qquad (5.80)\\ &+ Z^{(2)}_M\sum_\omega\int d\mathbf{x}\,J^{(2)}_\omega(\mathbf{x})\psi^{(\le M)+}_{\mathbf{x},\omega}\psi^{(\le M)-}_{\mathbf{x},-\omega}. \qquad (5.81)\end{aligned}$$

The correspondence between the observables of the dimer and of the ($M\to\infty$ limit of the) reference model is established via the identification between the fields $(\psi^\pm_{\mathbf{x},\omega})_{\mathbf{x}\in\mathbb{R}^2}$ of the relativistic continuum model and the Dirac fields $(\psi^\pm_{\mathbf{x},\omega})_{\mathbf{x}\in\Lambda}$ (see (3.16)) of the dimer model. The identification is approximate because the former are defined on the continuum and the latter on the lattice; however, asymptotically at large distances, the two fields have the same correlation functions (just compare the small-momentum behavior of (5.78) and of (3.18), cf. also (3.7)).

The generating function $\mathcal{W}(\mathbf{J})$ can be expressed in terms of trees whose values are essentially the same as those of the dominant trees (i.e. those where the propagators $g^{(h)}$ are replaced by the relativistic propagators $g^{(h)}_R$, as discussed above) contributing to the generating function of the dimer model. In particular both types of trees are associated only with endpoints of type λ_h, $Z^{(1)}_h$, or $Z^{(2)}_h$, and the single-scale propagators have exactly the same form, once the identification between Dirac fields of continuum and discrete models is used.

A minor difference between the contributions associated with the tree expansion for the reference model and the dominant contributions of the dimer model lies in the fact that the trees contributing to $\mathcal{W}(\mathbf{J})$ have endpoints on all scales $\le M$, rather than ≤ 0; moreover, the sequence of running coupling constants $\lambda_{h,R}, Z_{h,R}, Z^{(1)}_{h,R}, Z^{(2)}_{h,R}$ of the reference model, corresponding to the initial data $\lambda_M, Z_M, Z^{(1)}_M, Z^{(2)}_M$ is different in general from the corresponding sequence of the dimer model. However, the key observation is that the initial data $\lambda_M, Z_M, Z^{(1)}_M, Z^{(2)}_M$ of the reference model can be properly adjusted, so that the $\lambda_{h,R}, Z_{h,R}, Z^{(1)}_{h,R}, Z^{(2)}_{h,R}$ are asymptotically the same as the constants

of the dimer model, as $h \to -\infty$, namely

$$|\lambda_h - \lambda_{h,R}| + \Big|\frac{Z_{h-1}}{Z_h} - \frac{Z_{R,h-1}}{Z_{R,h}}\Big| \le C_\theta |\lambda|^2 2^{\theta h} \ , \qquad \Big|\frac{Z^{(i)}_{h-1}}{Z^{(i)}_h} - \frac{Z^{(i)}_{R,h-1}}{Z^{(i)}_{R,h}}\Big| \le C_\theta |\lambda| 2^{\theta h} \ , \tag{5.82}$$

for some $0 < \theta < 1$ and a suitable $C_\theta > 0$. In particular, the infrared fixed point of $\lambda_{h,R}$ is the same as the one of λ_h: $\lambda_{-\infty,R} = \lambda_{-\infty}$. If this wise choice of the initial data on scale M is made, then the values of the trees of the reference model are asymptotically the same as the dominant trees of the dimer model, up to exponentially small relative errors. In particular, the critical exponents of the dimer model are exactly the same as those of the reference model, if expressed in terms of $\lambda_{-\infty}$.

As anticipated above, the reason why it is useful to introduce the reference model at all is that it has more symmetries than the dimer model. In particular, its "action" $\mathcal{V} + \mathcal{B}$ is formally covariant under a "local chiral gauge transformation" $\psi^\pm_{\mathbf{x},\omega} \mapsto e^{\pm i\alpha_\omega(\mathbf{x})}\psi^\pm_{\mathbf{x},\omega}$ (here, "local" refers to the fact that the phase transformation depends on the point, "chiral" to the fact that it depends on ω, while "formally" means "up to corrections due to the ultraviolet regularization $\chi_M(\mathbf{k})$"). The latter induces exact identities (known as Ward Identities) between the correlation functions of the reference model, which in turn induce asymptotic identities between the correlations of the dimer model. By playing with these identities one can prove [14], among other things, the already mentioned asymptotically vanishing of the beta function for $\lambda_{h,R}$ as $h \to -\infty$ (which implies a similar property for the beta function of λ_h in the dimer model), and that the critical exponent η_1 of $Z^{(1)}_h$ is *equal* to the one of Z_h, that we called η in (5.62), see [12]. More implications of the correspondence between the dimer and the reference model are exploited in the next section.

The tree expansion for the reference model has been extensively analyzed by Renormalization Group methods; it has been proven in [7, Theorem 1.1] (see in particular [7, Eqs.(1.16) and (4.46)]) that, in the limit $M \to \infty$,

$$\langle \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega}; \psi^+_{\mathbf{y},\omega'}\psi^-_{\mathbf{y},\omega'}\rangle_{R,\lambda_{-\infty}} = \frac{\delta_{\omega,\omega'}}{(4\pi)^2} \frac{1 + A_1(\lambda_{-\infty})}{\big((x_1-y_1) + i\omega(x_2-y_2)\big)^2} \ , \tag{5.83}$$

$$\langle \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},-\omega}; \psi^+_{\mathbf{y},-\omega'}\psi^-_{\mathbf{y},\omega'}\rangle_{R,\lambda_{-\infty}} = \frac{\delta_{\omega,\omega'}}{(4\pi)^2} \frac{1 + A_2(\lambda_{-\infty})}{|\mathbf{x}-\mathbf{y}|^{2+2\eta'_2(\lambda_{-\infty})}} \ , \tag{5.84}$$

where $\langle\cdot\rangle_{R,\lambda_{-\infty}}$ denotes the average with respect to the ($M \to \infty$ limit of the) reference model, with bare parameters λ_M, Z_M etc fixed in such a way that the infrared limit of the effective coupling constant is equal to the same $\lambda_{-\infty}$ as the one of the dimer model. Moreover, A_1, A_2 and η'_2 are analytic functions of their argument, vanishing at $\lambda_{-\infty} = 0$, and $\eta'_2 = \eta_2 - \eta$, where η_2 and η are the critical exponents of $Z^{(2)}_h$ and Z_h, respectively; see (5.62).

If $q > 2$, there is a remarkable cancellation which can be inferred from [7, Theorem 1.1] and is the analogue of the cancellation (4.38) that we already used in the analysis of the non-interacting dimer model. In fact, one has

$$\langle \psi^+_{\mathbf{x}_1,\omega_1}\psi^-_{\mathbf{x}_1,\omega_1}; \cdots ; \psi^+_{\mathbf{x}_q,\omega_q}\psi^-_{\mathbf{x}_q,\omega_q}\rangle_{R,\lambda_{-\infty}} = 0 \ . \tag{5.85}$$

6. Height fluctuations in the interacting case

6.1. **Tree expansion for the correlation functions.** The multiscale construction described in the previous section induces a representation of the multipoint dimer correlation functions in terms of a renormalized tree expansion. We limit ourselves to the discussion of the correlations at distinct bonds, the general case being treatable in a similar manner. Using (2.25) and the discussion in Section 2.5, we find:

$$\langle \mathbb{1}_{b_1}; \cdots ; \mathbb{1}_{b_q} \rangle_\lambda := \lim_{m\to 0} \lim_{\Lambda \nearrow \mathbb{Z}^2} \langle \mathbb{1}_{b_1}; \cdots ; \mathbb{1}_{b_q} \rangle_{\Lambda;\lambda,m} = \tag{6.1}$$

$$= \lim_{m\to 0} \lim_{\Lambda \nearrow \mathbb{Z}^2} \frac{\partial^k}{\partial A_{b_1} \cdots \partial A_{b_q}} \log \mathcal{Z}^{(11)}_\Lambda(\lambda, m, \mathbf{A})\Big|_{\mathbf{A}=\mathbf{0}} =$$

$$= \frac{\partial^q}{\partial J_{b_1} \cdots \partial J_{b_q}} S(\mathbf{J})\Big|_{\mathbf{J}=\mathbf{0}} , \tag{6.2}$$

where $S(\mathbf{J}) = \sum_{h\le 0} \tilde S^{(h)}(\mathbf{J})$ is the ($m \to \infty$ limit of the $\Lambda \nearrow \mathbb{Z}^2$ limit of the) generating function computed via the renormalized expansion described above. Note that in the last line of (6.2) we exchanged a derivative with the limits $\Lambda \nearrow \mathbb{Z}^2$, $m \to 0$. This is justified by the fact that the derivatives of $S(\mathbf{J})$ can be expressed via an absolutely convergent expansion, uniformly in Λ and m, as already anticipated above (see below for more details about the bounds on the tree values contributing to the correlation functions). For what follows, note that the limit $m \to 0$ corresponds to $h^* \to -\infty$; therefore, in the following formulas, we shall always replace h^* by $-\infty$ (to do things more pedantically, one should first perform all the estimates at h^* finite, check that the resulting bounds are uniform in h^* and then send $h^* \to -\infty$, but we spare these details to the reader).

The single-scale contribution to $S(\mathbf{J})$ can be written in a way similar to (5.19):

$$\tilde S^{(h)}(\mathbf{J}) = \sum_{q\ge 1} \sum_{j_1,\dots,j_q} \sum_{\mathbf{y}_1,\dots,\mathbf{y}_q} S^{(h)}_{q,\mathbf{j}}(\mathbf{y}_1, \dots, \mathbf{y}_q) \prod_{i=1}^q J_{\mathbf{y}_i, j_i} , \tag{6.3}$$

where $S^{(h)}_{q;\mathbf{j}}(\mathbf{y}_1, \dots, \mathbf{y}_q)$ collects the contributions to $W^{(h)}_{q;\mathbf{j}}(\mathbf{y}_1, \dots, \mathbf{y}_q)$ involving at least one propagator on scale $h+1$. Therefore,

$$\langle \mathbb{1}_{(\mathbf{y}_1, \mathbf{y}_1+\hat e_{j_1})}; \cdots ; \mathbb{1}_{(\mathbf{y}_q, \mathbf{y}_q+\hat e_{j_q})} \rangle_\lambda = q! \sum_{h\le 0} S^{(h)}_{q,\mathbf{j}}(\mathbf{y}_1, \dots, \mathbf{y}_q) \tag{6.4}$$

and, as explained in the previous sections, $S^{(h)}_{q,\mathbf{j}}(\mathbf{y}_1, \dots, \mathbf{y}_q)$ can be expressed by a sum over trees $\tau \in \mathcal{T}^{(h)}_{N,n}$ with $n \le q$ *special endpoints*[7] and $N \ge 0$

[7] The reason why $n \le q$ rather than $n = q$ is that some special endpoints - those on scale 1 - could be associated with contributions of order two or more in the $\mathbf{J}$ fields, such as the one depicted graphically in Fig.4.

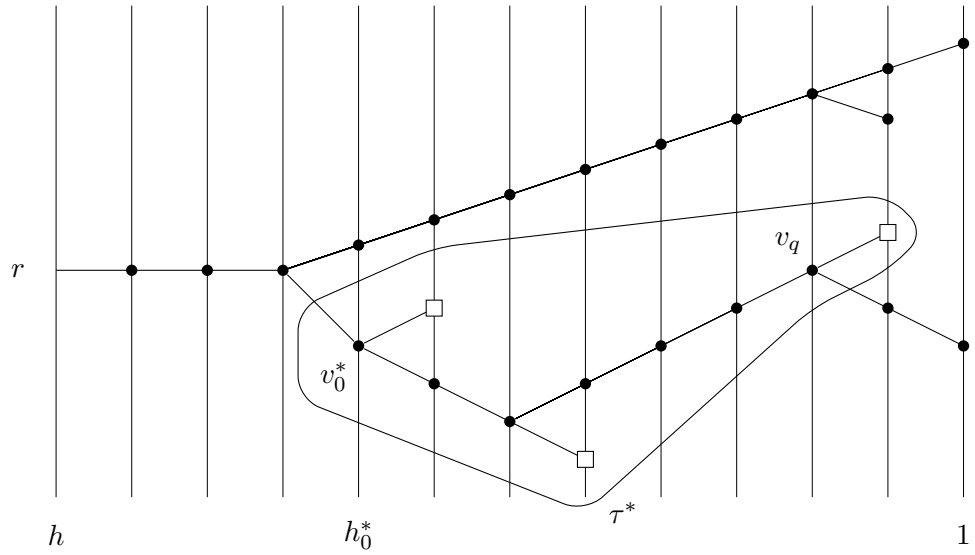

FIGURE 13. Example of a tree $\tau \in \mathcal{T}^{(h)}_{N,n}$ appearing in the expansion for the m points correlation function, with $N = 3$ and $n = q = 3$. The subtree τ^* associated with τ and with a choice of $\mathbf{P}$ such that $|P_{\bar{v}_0}| = |P^J_{\bar{v}_0}| = 3$ is highlighted.

normal end-points:

$$\langle \mathbb{1}_{(\mathbf{y}_1,\mathbf{y}_1+\hat{e}_{j_1})}; \cdots ; \mathbb{1}_{(\mathbf{y}_q,\mathbf{y}_q+\hat{e}_{j_q})} \rangle_\lambda = \tag{6.5}$$
$$= \sum_{h\le 0} \sum_{N\ge 0} \sum_{n=1}^{q} \sum_{\tau\in\mathcal{T}^{(h)}_{N,n}} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau:\\ |P_{v_0}|=|P^J_{v_0}|=q}} S_{\tau,\mathbf{P}}(\mathbf{y}_1, j_1; \cdots ; \mathbf{y}_q, j_q) \, .$$

Here $S_{\tau,\mathbf{P}}(\mathbf{y}_1, j_1; \cdots ; \mathbf{y}_m, j_m)$ is the tree value, which can be bounded in a way similar to Eq.(5.70). Given $\tau \in \mathcal{T}^{(h)}_{N,n}$, let us denote by τ^* the minimal subtree of τ connecting all its special endpoints. For each $v \in \tau$, let s^*_v be the number of vertices immediately following v on τ such that $|P^J_v| \ge 1$ (i.e., the number of descendants of v in τ^*). Moreover, let $V_{nt}(\tau^*)$ be the set of vertices in τ^* with $s^*_v > 1$, which are the branching points of τ^*. For future reference, we also define v^*_0 to be the leftmost vertex on τ^* and h^*_0 its scale. See Fig.13.

Given these definitions, we can write the bound for $S_{\tau,\mathbf{P}}(\mathbf{y}_1, j_1; \cdots ; \mathbf{y}_q, j_q)$ as

$$|S_{\tau,\mathbf{P}}(\mathbf{y}_1, j_1; \cdots ; \mathbf{y}_q, j_q)| \le \tag{6.6}$$
$$\le q! C^{N+n} \Big[\prod_{v \text{ e.p.}} \|K_v\|_\infty \Big] 2^{h(2-|P^J_{v_0}|)} \Big[\prod_{v\in V_{nt}(\tau^*)} 2^{2(s^*_v-1)h_v} e^{-c\sqrt{2^{h_v}\delta_v}} \Big] \times$$
$$\times \Big[\prod_{\substack{v \text{ not}\\ \text{e.p.}}} \frac{C^{\sum_{i=1}^{s_v} |P_{v_i}|-|P_v|}}{s_v!} \Big(\frac{Z_{h_v}}{Z_{h_v-1}} \Big)^{\frac{|P^\psi_v|}{2}} 2^{2-\frac{1}{2}|P^\psi_v|-|P^J_v|-z(P_v)} \Big] \, ,$$

which is very similar to the bound Eq.(5.70) for the renormalized kernels of the effective potential, modulo the presence of the product over the vertices $v \in V_{nt}(\tau^*)$ of $2^{2(s_v^*-1)h_v} e^{-c\sqrt{2^{h_v}\delta_v}}$, where $\delta_v = \delta_v(\mathbf{x}_1, \dots, \mathbf{x}_m)$ is the tree distance of the set $\mathbf{x}_v^* := \cup_{f \in P_v^J} \{\mathbf{x}(f)\}$, i.e. the length of the shortest tree graph on $\mathbb{Z}^2$ connecting the points of $\mathbf{x}_v^*$. In this product, the factors $2^{2(s_v^*-1)h_v}$ take into account the dimensional gain coming from the fact that we are *not* integrating over the space labels $\mathbf{y}_i$ of the external fields (the gain is meant in comparison with Eq.(5.70) where, on the contrary, we integrated over all the field variables); moreover, the factors $e^{-c\sqrt{2^{h_v}\delta_v}}$ come from the decaying factors $e^{-c\sqrt{2^{h_w}|\mathbf{x}(\ell)-\mathbf{x}'(\ell)|}}$ (similar to those in (5.5)) associated with the propagators $g_\ell^{(h_w)}$, $w \geq v$, see [6, Section 2.3] for a few more details about how to extract these factors starting from the bounds on the kernels of the effective potential.

As in Section 5.5, the tree expansion (6.5) can be refined by decomposing the single scale propagators as in (5.75). The "refined tree expansion" brings along an extra set of labels, which allows one to distinguish between relativistic propagators $g_R^{(h)}$ (the dominant terms, exponentially close to those of the relativistic continuum model) and non-relativistic propagators $r^{(h)}$. In the perspective of computing the height fluctuations, it will be also convenient to distinguish two classes of terms among the dominant ones: those with all the special endpoints of type $Z_{h_v}^{(1)}$, and the rest. The final decomposition we shall use takes the following form:

$$\begin{aligned} &\langle \mathbb{1}_{(\mathbf{y}_1,\mathbf{y}_1+\hat{e}_{j_1})}; \cdots ; \mathbb{1}_{(\mathbf{y}_q,\mathbf{y}_q+\hat{e}_{j_q})} \rangle_\lambda = \\ &= \mathcal{S}^{(1)}_{q,\mathbf{j}}(\mathbf{y}_1, \dots, \mathbf{y}_q) + \mathcal{S}^{(2)}_{q,\mathbf{j}}(\mathbf{y}_1, \dots, \mathbf{y}_q) + \mathcal{S}^{(3)}_{q,\mathbf{j}}(\mathbf{y}_1, \dots, \mathbf{y}_q) \ , \end{aligned} \tag{6.7}$$

where: $\mathcal{S}^{(1)}$ collects all the dominant contributions (i.e. all propagators are relativistic ones, $g_R^{(h)}$, and all the endpoints are on scales ≤ 0) from trees whose special endpoints are all of type $Z_h^{(1)}$; $\mathcal{S}^{(2)}$ collects all the dominant contributions from trees with at least one special endpoint of type $Z_h^{(2)}$; $\mathcal{S}^{(3)}$ collects all the subdominant contributions, i.e., the contributions from trees with at least one endpoint on scale 1, or at least one propagator of type $r^{(h)}$.

6.2. **The two-point function and the height variance.** If $q = 2$, (6.7) reads:

$$\langle \mathbb{1}_{(\mathbf{x},\mathbf{x}+\hat{e}_j)}; \mathbb{1}_{(\mathbf{y},\mathbf{y}+\hat{e}_{j'})} \rangle_\lambda = \mathcal{S}^{(1)}_{2,(j,j')}(\mathbf{x},\mathbf{y}) + \mathcal{S}^{(2)}_{2,(j,j')}(\mathbf{x},\mathbf{y}) + \mathcal{S}^{(3)}_{2,(j,j')}(\mathbf{x},\mathbf{y}) \ . \tag{6.8}$$

We calculate the analog of $\mathcal{S}^{(a)}_{2,(j,j')}(\mathbf{x},\mathbf{y}), a = 1, 2$ for the ($M \to \infty$ limit of the) continuum reference model, to be called $\mathcal{S}^{(a)}_{R;2,(j,j')}(\mathbf{x},\mathbf{y})$. For this purpose, it is convenient to note that the local contributions (5.44) associated with the endpoints of type $Z_h^{(1)}$ or $Z_h^{(2)}$ appearing in the tree expansion of the dimer model can be rewritten in terms of the Dirac fields via (3.16) in

the following way:

$$F_1 = 2\sum_{\mathbf{x}}(-1)^{\mathbf{x}}\Big[J_{\mathbf{x},1}\sum_{\omega}\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} + iJ_{\mathbf{x},2}\sum_{\omega}\omega\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega}\Big]\ , \tag{6.9}$$

$$F_2 = 2\sum_{\mathbf{x}}\Big[J_{\mathbf{x},1}(-1)^{x_1}\sum_{\omega}\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},-\omega} + iJ_{\mathbf{x},2}(-1)^{x_2}\sum_{\omega}\omega\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},-\omega}\Big] \tag{6.10}$$

Then, we have for instance from (6.9)

$$\mathcal{S}^{(1)}_{R;2,(1,2)}(\mathbf{x},\mathbf{y}) = 4i(-1)^{\mathbf{x}-\mathbf{y}}\langle\sum_{\omega}\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega};\sum_{\omega'}\omega'\psi^+_{\mathbf{y},\omega'}\psi^-_{\mathbf{y},\omega'}\rangle_{R,\lambda_{-\infty}} \tag{6.11}$$

and similar expressions for the others. After some algebra, we find from (6.9), (6.10), (5.83) and (5.84):

$$\mathcal{S}^{(1)}_{R;2,(j,j')}(\mathbf{x},\mathbf{0}) = -\frac{1+A_1(\lambda_{-\infty})}{2\pi^2}(-1)^{\mathbf{x}}\,\mathrm{Re}\frac{(i)^{j+j'}}{(x_1+ix_2)^2}\ , \tag{6.12}$$

$$\mathcal{S}^{(2)}_{R;2,(j,j')}(\mathbf{x},\mathbf{0}) = \delta_{j,j'}\frac{1+A_2(\lambda_{-\infty})}{2\pi^2}(-1)^{x_j}\frac{1}{|\mathbf{x}|^{2+2\eta_2'}}\ . \tag{6.13}$$

On the other hand, it is an immediate byproduct of the Renormalization Group analysis (see e.g. [15, Lemma 2.1]) that, once the bare parameters of the relativistic model are chosen in such a way that the infrared limits of λ_h and $\lambda_{h,R}$ are the same, the difference between the dimer-dimer correlation and its relativistic counterpart is bounded as:

$$\sum_{i=1,2}\big|\mathcal{S}^{(i)}_{R;2,(j,j')}(\mathbf{x},\mathbf{y}) - \mathcal{S}^{(i)}_{2,(j,j')}(\mathbf{x},\mathbf{y})\big| + \big|\mathcal{S}^{(3)}_{2,(j,j')}(\mathbf{x},\mathbf{y})\big| \le \frac{C_\theta}{|\mathbf{x}-\mathbf{y}|^{2+\theta}}\ , \tag{6.14}$$

for a suitable $0<\theta<1$ and $C_\theta>0$. Plugging (6.12)–(6.14) into (6.8) we obtain an asymptotic formula for the two-point dimer correlation, that we can use to compute the height variance.

Using (4.2) and (6.12)–(6.14), we see that after taking the limit $\Lambda \nearrow \mathbb{Z}^2$ first, and then $m\to 0$, we get (assuming that the bonds b_1, b_2 appearing in the sum are all mutually disjoint):

$$\langle h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}; h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}\rangle = \sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}\sum_{b_2\in\mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}\sigma_{b_1}\sigma_{b_2}\Big[-\frac{K_1}{2\pi^2}\,\mathrm{Re}\frac{(-1)^{\mathbf{x}_1-\mathbf{x}_2}(i)^{j_1+j_2}}{(z_{\mathbf{x}_1}-z_{\mathbf{x}_2})^2}$$

$$+\delta_{j_1,j_2}\frac{K_2}{2\pi^2}\frac{(-1)^{(\mathbf{x}_1-\mathbf{x}_2)_{j_1}}}{|\mathbf{x}_1-\mathbf{x}_2|^{2+2\eta_2'}} + R(\mathbf{x}_1,\mathbf{x}_2)\Big] \tag{6.15}$$

with

$$|R(\mathbf{x}_1,\mathbf{x}_2)| \le \frac{C_\theta}{|\mathbf{x}_1-\mathbf{x}_2|^{2+\theta}}$$

where $\mathbf{x}_1, \mathbf{x}_2, j_1, j_2$ are such that $b_1 = (\mathbf{x}_1, \mathbf{x}_1+\hat{e}_{j_1})$ and $b_2 = (\mathbf{x}_2, \mathbf{x}_2+\hat{e}_{j_2})$, and K_1, K_2 are shorthands for $1+A_1(\lambda_{-\infty}), 1+A_2(\lambda_{-\infty})$. Recall now that $\sigma_{b_i} = \alpha_{b_i}(-1)^{\mathbf{x}_i}(-1)^{j_i}$, with α_b defined just after (4.11). Using this explicit

expression for σ_b into (6.15), we can rewrite the term in the first line as:

$$-\frac{K_1}{2\pi^2}\sum_{\substack{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}\\ b_2\in\mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}}\sigma_{b_1}\sigma_{b_2}\,\mathrm{Re}\frac{(-1)^{\mathbf{x}_1-\mathbf{x}_2}(i)^{j_1+j_2}}{(z_{\mathbf{x}_1}-z_{\mathbf{x}_2})^2}=-\frac{K_1}{2\pi^2}\sum_{\substack{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}\\ b_2\in\mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}}\mathrm{Re}\frac{\Delta z_{b_1}\Delta z_{b_2}}{(z_{\mathbf{x}_1}-z_{\mathbf{x}_2})^2}\ , \tag{6.16}$$

where Δz_{b_i} is the displacement associated with the elementary portion of the path $\mathcal{C}^{(i)}$ crossing b_i, thought of as a complex vector of modulus 1. Just compare with the first term in the r.h.s. of (4.24) in the non-interacting case. Exactly like in the $\lambda=0$ situation, the r.h.s. of (6.16) is K_1 times the integral in (4.26) (which is the desired dominant contribution to the variance of the height), plus a rest that is uniformly bounded in $|\boldsymbol{\xi}-\boldsymbol{\eta}|$.

Let us now estimate the contributions to the variance coming from the two terms in the second line of (6.15). The last term immediately leads to a contribution that is uniformly bounded in $|\boldsymbol{\xi}-\boldsymbol{\eta}|$. Regarding the second term, note that $\sigma_{b_1}\sigma_{b_2}(-1)^{(\mathbf{x}_1-\mathbf{x}_2)_{j_1}}=\alpha_{b_1}\alpha_{b_2}(-1)^{(\mathbf{x}_1-\mathbf{x}_2)_{3-j_1}}$: namely, the oscillatory $\sigma_{b_1}\sigma_{b_2}$ do not compensate the oscillatory factor $(-1)^{(\mathbf{x}_1-\mathbf{x}_2)_{j_1}}$. Once summed over the path, and using the fact that the paths $\mathcal{C}^{(i)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ consist of union of straight portions, each of which is formed by an even number of bonds, we see that the oscillatory factor $(-1)^{(\mathbf{x}_1-\mathbf{x}_2)_{3-j_1}}$ has the same effect as a discrete derivative (we are sketchy here, but the very same argument was used in the non-interacting model just after (4.24)):

$$\Big|\sum_{\substack{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}\\ b_2\in\mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}}\alpha_{b_1}\alpha_{b_2}\delta_{j_1,j_2}\frac{(-1)^{(\mathbf{x}_1-\mathbf{x}_2)_{3-j_1}}}{|\mathbf{x}_1-\mathbf{x}_2|^{2+2\eta_2'}}\Big|\le c\sum_{\substack{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}\\ b_2\in\mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}}\Big|\partial_{3-j_1}\frac{1}{|\mathbf{x}_1-\mathbf{x}_2|^{2+2\eta_2'}}\Big|\ , \tag{6.17}$$

which shows that also this term is bounded uniformly in $|\boldsymbol{\xi}-\boldsymbol{\eta}|$. This concludes the proof of the main theorem for $q=2$ (recall again that notationally here q plays the same role as the integer n in the main theorem and in formula (4.2)).

6.3. **The multipoint dimer correlation and higher cumulants of the height.** To compute the cumulant of order $q>2$ of the height difference we use (4.2) (with n replaced by q) and sharp bounds on $\langle\mathbb{1}_{b_1};\cdots;\mathbb{1}_{b_q}\rangle$. Proceeding as in (4.31) in Section 4.2.2, in the q-fold sum over the bonds $b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$, …, $b_q\in\mathcal{C}^{(q)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$, we distinguish a contribution that includes the terms where all the bonds are outside the two balls $B_{r_q}(\boldsymbol{\xi})$, $B_{r_q}(\boldsymbol{\eta})$, from the rest. By construction, the first contribution involves bond configurations such that the bonds are all mutually disjoint, and is the most difficult to bound. After explaining how to deal with it, it will be clear that the rest gives a finite contribution to the height fluctuation, bounded uniformly in $|\boldsymbol{\xi}-\boldsymbol{\eta}|$. Let us then focus on the contribution where all the bonds are outside the two aforementioned balls (this constraint will be indicated by a $*$ on the sum over the bonds $b_i\in\mathcal{C}^{(i)}$, as in (4.31)). Within these contributions, after decomposing the q-point dimer correlation as in (6.7), we can distinguish three classes of

terms, those coming from $\mathcal{S}^{(1)}_{q,\mathbf{j}}$, those from $\mathcal{S}^{(2)}_{q,\mathbf{j}}$, and those from $\mathcal{S}^{(3)}_{q,\mathbf{j}}$. We consider $\mathcal{S}^{(1)}_{q,\mathbf{j}}$ first, and then we discuss the other two contributions.

The contributions of type $\mathcal{S}^{(1)}_{q,\mathbf{j}}$. Notice first of all that, if $q > 2$, for the relativistic model the cancellation (5.85) implies that

$$\mathcal{S}^{(1)}_{R;q,\mathbf{j}}(\mathbf{x}_1, \dots, \mathbf{x}_q) = 0 \ . \tag{6.18}$$

Therefore, writing

$$\mathcal{S}^{(1)}_{q,\mathbf{j}}(\mathbf{x}_1, \dots, \mathbf{x}_q) = \mathcal{S}^{(1)}_{q,\mathbf{j}}(\mathbf{x}_1, \dots, \mathbf{x}_q) - \mathcal{S}^{(1)}_{R;q,\mathbf{j}}(\mathbf{x}_1, \dots, \mathbf{x}_q) \ , \tag{6.19}$$

we can bound this difference as

$$\begin{aligned} &\sum_h \sum_{N\geq 0} \sum_{\tau\in\mathcal{T}^{(h)}_{N,q}}^{(1)} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau: \\ |P_{v_0}|=|P^J_{v_0}|=q}} \sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*} \cdots \sum_{b_q\in\mathcal{C}^{(q)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*} \times \\ &\times |S^{dom}_{\tau,\mathbf{P}}(\mathbf{x}_1, j_1; \cdots ; \mathbf{x}_q, j_q) - S^{R}_{\tau,\mathbf{P}}(\mathbf{x}_1, j_1; \cdots ; \mathbf{x}_q, j_q)| \ , \end{aligned} \tag{6.20}$$

where, once again $\mathbf{x}_i, j_i$ are such that $b_i = (\mathbf{x}_i, \mathbf{x}_i + \hat{e}_{j_i})$. Moreover, the apex (1) on the sum over the trees recalls that we are summing over the contributions associated with $\mathcal{S}^{(1)}_{q,\mathbf{j}}$. We denoted by $S^{dom}_{\tau,\mathbf{P}}$ the dominant contribution to the value of the tree τ (i.e. the contribution coming from replacing each propagator $g^{(h)}$ with $g^{(h)}_R$, see (5.75)), and by $S^{R}_{\tau,\mathbf{P}}$ the tree value computed in the relativistic reference model. The sum over h ranges between $-\infty$ and M, where M is the ultraviolet cutoff of the reference model, to be eventually sent to infinity.

We distinguish three types of contributions, that we treat separately:

(a) those associated with the trees with endpoints all on scales ≤ 0, each of which comes in the form of a difference between the dominant contribution of the tree value in the dimer model, and the corresponding tree value in the reference model; these contributions are essentially the same, modulo the fact that the effective constants associated with the endpoints of the tree for the dimer model are $\lambda_h, Z_h, Z^{(1)}_h$, while those in the tree for the reference model are $\lambda_{R,h}, Z_{R,h}, Z^{(1)}_{R,h}$. Recalling (5.82), we find that each tree contribution to (6.19) is bounded in a way similar to (6.6), times an extra $2^{\theta h_w}$ factor, with w the right-most endpoint of the tree.

(b) those associated with the trees that have root at scale $h < 0$ but have at least one endpoint on scale $h_v \geq 1$. Since these terms do not appear (by definition) in $\mathcal{S}^{(1)}_{q,\mathbf{j}}$, we have $|S^{dom}_{\tau,\mathbf{P}} - S^{R}_{\tau,\mathbf{P}}| = |S^{R}_{\tau,\mathbf{P}}|$. These terms will turn out to be negligible due to the short memory property.

(c) those associated with trees with root at scale $h \geq 0$. Also in this case, $S^{dom}_{\tau,\mathbf{P}} = 0$.

We claim that the sum (6.20) can be bounded by

$$C_q \sum_{h=-\infty}^{+\infty} 2^{h(2-q)} \min\{2^{\theta' h}, e^{-c'\,2^{h/2}}\} \sum_{N\geq 0} C^N |\lambda|^N \sum_{\tau\in\mathcal{T}^{(h)}_{N,q}}^{(1)} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau:\\ |P_{v_0}|=|P^J_{v_0}|=q}} \times$$
$$\times \sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*} \cdots \sum_{b_q\in\mathcal{C}^{(q)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*} \Big[\prod_{v \text{ special e.p.}} \frac{Z^{(1)}_{h_v-1}}{Z_{h_v-1}} \Big] \Big[\prod_{v\in V_{nt}(\tau^*)} 2^{2(s^*_v-1)h_v} e^{-c\sqrt{2^{h_v}\delta_v}} \Big] \times$$
$$\times \Big[\prod_{v \text{ not e.p.}} \Big(\frac{Z_{h_v}}{Z_{h_v-1}} \Big)^{|P^\psi_v|/2} 2^{2-\frac{1}{2}|P^\psi_v|-|P^J_v|-z(P_v)+\theta'} \Big] \tag{6.21}$$

for some positive small $\theta', c' > 0$, where we recall that the "pruned tree" τ^* was defined after (6.5), see Figure 13. Let us see why. First consider the terms of type (a), for which $h < 0$: as we explained, each of these satisfies the estimate (6.6), times an extra $2^{\theta h_w}$ factor, with w the right-most endpoint of the tree. We can replace $2^{\theta h_w}$ by $2^{\theta' h}$ with some $0 < \theta' < \theta$, provided we add θ' to the exponent $2 - \frac{1}{2}|P^\psi_v| - |P^J_v| - z(P_v)$ at each vertex that is not an endpoint. Of course, we will choose θ' sufficiently small so that the exponents remain strictly negative at each vertex (recall (5.72)). Next consider terms of type (c), for which $h \geq 0$. In this case, the dimensional gain arises from the factors $e^{-c\sqrt{2^{h_v}\delta_v}}$ in the second line of (6.6), that is smaller than $e^{-(c/2)\sqrt{2^{h_v}\delta_v}} e^{-(c/2)2^{h/2}}$, simply because $\delta_v \geq 1$ and $h_v \geq h$. As for the terms of type (b), the corresponding trees either have root on scale close to zero, or they are very long and in that case the dimensional gain comes from the short memory property. More precisely, given that there is at least a vertex on scale 0 we can extract from the bound (6.6) a factor $2^{\theta' h}$ provided we add θ' to every exponent $2 - \frac{1}{2}|P^\psi_v| - |P^J_v| - z(P_v)$.

To prove that the $q > 2$ cumulants of the height differences do not diverge with distance, it remains to show that (6.21) is bounded by some constant depending only on q. Using the fact that, as mentioned above, the critical exponent of $Z^{(1)}_h$ is *equal* to the one of Z_h we can bound the ratios $Z^{(1)}_{h_v-1}/Z_{h_v-1}$ by a positive constant, independent of h_v. Moreover, from (5.61) we have $Z_{h_v}/Z_{h_v-1} \leq 1 + C|\lambda|$. Also, by proceeding as in the proof of (4.43), we find that, for a suitable $C'_q > 0$,

$$\sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*} \cdots \sum_{b_q\in\mathcal{C}^{(q)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*} \prod_{v\in V_{nt}(\tau^*)} e^{-c'\sqrt{2^{h_v}\delta_v}} \leq C'_q \prod_{v\in V_{nt}(\tau^*)} 2^{-h_v \bar{m}^J_v}\,, \tag{6.22}$$

where $\bar m_v^J$ is the number of special endpoints contained in the cluster v but not in any other cluster $v' > v$. In conclusion, (6.20) is bounded by

$$C_q'' \sum_{h=-\infty}^{+\infty} 2^{h(2-q)} \min\{2^{\theta' h}, e^{-c'\, 2^{h/2}}\} \sum_{N\geq 0} C^N |\lambda|^N \times \tag{6.23}$$
$$\times \sum_{\tau\in\mathcal{T}_{N,q}^{(h)}}^{(1)} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau:\\ |P_{v_0}|=|P_{v_0}^J|=q}} \Big[\prod_{v\in V_{nt}(\tau^*)} 2^{h_v(2s_v^*-2-\bar m_v^J)}\Big]\Big[\prod_{v \text{ not e.p.}} 2^{\bar d_v(P_v)}\Big]$$

where $\bar d_v(P_v) = 2 - |P_v^\psi|/2(1-C|\lambda|) - |P_v^J| - z_v + \theta'$ which, from (5.72), is negative and actually smaller than $-1+\epsilon$ if λ and θ' are small enough.

By proceeding as in the proof of (4.47), we find

$$\prod_{v\in V_{nt}(\tau^*)} 2^{h_v(2s_v^*-2-\bar m_v^J)} = 2^{h_0^*(q-2)} \prod_{v\in V(\tau^*)} 2^{|P_v^J|-2} \tag{6.24}$$

where $V(\tau^*)$ is the set of vertices of τ^* that are not endpoints and h_0^*-1 is the scale of the root of τ^*. Note that the factor $2^{h_0^*(q-2)}$, multiplied by the factor $2^{h(2-q)}$ that is in (6.23), equals a product of $2^{q-2} = 2^{|P_v^J|-2}$ over all vertices on the branch joining the root of τ with the root of τ^*. Therefore, (6.23) is bounded by

$$C_q'' \sum_{h=-\infty}^{+\infty} \min\{2^{\theta' h}, e^{-c'\, 2^{h/2}}\} \sum_{N\geq 0} C^N |\lambda|^N \sum_{\tau\in\mathcal{T}_{N,q}^{(h)}}^{(1)} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau:\\ |P_{v_0}|=|P_{v_0}^J|=q}} \prod_{v \text{ not e.p.}} 2^{\hat d_v(P_v)}\,, \tag{6.25}$$

where

$$\hat d_v(P_v) = \begin{cases} -|P_v^\psi|/2(1-C|\lambda|) - z_v + \theta' & \text{if} \quad |P_v^J| > 0 \\ \bar d_v(P_v) & \text{otherwise.} \end{cases} \tag{6.26}$$

In fact, note that when $P_v^J = 0$, the vertex v is neither in τ^* nor in the path between v_0 and v_0^* and, therefore, and therefore it does not appear in the product in the r.h.s. of (6.24). Note that $\hat d_v \leq -a < 0$ for every v. From this, it follows that (6.25) is summable over $\mathbf{P}, \tau$ and h, the result being a finite, q-dependent, constant, as desired.

The contributions of type $\mathcal{S}_{q,\mathbf{j}}^{(2)}$. Let us now consider $\mathcal{S}_{q,\mathbf{j}}^{(2)}$, which is apriori very dangerous, in that each of the trees contributing to it is bounded as in (6.6), without any extra obvious gain. Nevertheless, as was the case also for $q=2$ in Section 6.3, the dimensional gain arises from oscillating factors, when summing over the bonds b_j in the paths $\mathcal{C}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}^{(j)}$, $j \leq q$.

The contribution to the q-th cumulant of the height difference from terms of type $\mathcal{S}_{q,\mathbf{j}}^{(2)}$ is of the form (with notations similar to (6.20))

$$\sum_{\substack{h\leq 0\\ N\geq 0}} \sum_{\tau\in\mathcal{T}_{N,q}^{(h)}}^{(2)} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau:\\ |P_{v_0}|=|P_{v_0}^J|=q}} \sum_{b_1\in\mathcal{C}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}^{(1)}}^{*} \cdots \sum_{b_q\in\mathcal{C}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}^{(q)}}^{*} \sigma_{b_1}\cdots\sigma_{b_q} S_{\tau,\mathbf{P}}^{dom}(\mathbf{x}_1, j_1; \cdots; \mathbf{x}_q, j_q)\,, \tag{6.27}$$

where the notation is analogous to the one used above for the contributions of type $\mathcal{S}^{(1)}_{q,\mathbf{j}}$. An important difference is that here we do not take absolute values, since we want to take advantage of the signs σ_b.

Note in fact that the oscillatory factor (arising from (6.9) and (6.10)) of a dominant tree (i.e., of one of the trees contributing to $\mathcal{S}^{(1)}_{q,\mathbf{j}}$ or to $\mathcal{S}^{(2)}_{q,\mathbf{j}}$) is equal to the product of the oscillatory factors of the type $(-1)^{\mathbf{x}}$ or $(-1)^{x_i}$ associated with the special endpoints of these trees. The value of a dominant tree equals this oscillatory factor times a "non-oscillatory" value (see below for more details), obtained by contracting via relativistic propagators (which by definition have no oscillatory factors attached) the contributions that are left attached to all the endpoints. Now, it is apparent from (6.9) that all the trees contributing to $\mathcal{S}^{(1)}_{q,\mathbf{j}}(\mathbf{x}_1,\dots,\mathbf{x}_q)$ have the same oscillatory factor, equal to $(-1)^{\mathbf{x}_1+\dots+\mathbf{x}_q}$. This compensates exactly with the factor $(-1)^{\mathbf{x}_1+\dots+\mathbf{x}_q}$ from the product of σ_b, see (4.11).

The situation is different for $\mathcal{S}^{(2)}_{q,\mathbf{j}}(\mathbf{x}_1,\dots,\mathbf{x}_q)$: we recall that the trees involved in this expression have at least one special endpoint of type $Z^{(2)}_h$. If we denote by $\{(\mathbf{x}_i,j_i)\}_{i\in I_2}$ the set of points and directions associated with the endpoints of type $Z^{(2)}_h$ (here $I_2\subseteq\{1,\dots,q\}$ is a suitable *nonempty* index set), then $S^{dom}_{\tau,\mathbf{P}}(\mathbf{x}_1,j_1;\cdots;\mathbf{x}_q,j_q)$ comes with the oscillatory factor $\Big[\prod_{i\in I_2}(-1)^{(\mathbf{x}_i)_{j_i}}\Big]\Big[\prod_{i\in I_2^c}(-1)^{\mathbf{x}_i}\Big]$, where $I_2^c=\{1,\dots,q\}\setminus I_2$. This means that

$$\begin{aligned} &S^{dom}_{\tau,\mathbf{P}}(\mathbf{x}_1,j_1;\cdots;\mathbf{x}_q,j_q)= \\ &\quad=\Big[\prod_{i\in I_2}(-1)^{(\mathbf{x}_i)_{j_i}}\Big]\Big[\prod_{i\in I_2^c}(-1)^{\mathbf{x}_i}\Big]\tilde S^{dom}_{\tau,\mathbf{P}}(\mathbf{x}_1,j_1;\cdots;\mathbf{x}_q,j_q)\ , \end{aligned} \tag{6.28}$$

where $\tilde S^{dom}_{\tau,\mathbf{P}}$ is a "non-oscillatory" function, in the sense that it satisfies the following natural scaling properties: if $\mathbf{n}=(n_1,n_2)$ and $\partial^{\mathbf{n}}_{\mathbf{x}}=\partial^{n_1}_{x_1}\partial^{n_2}_{x_2}$ with ∂_{x_i} the discrete derivative in the i-th coordinate direction,

$$\begin{aligned} &\Big|\Big[\prod_{i=1}^q\partial^{\mathbf{n}_i}_{\mathbf{x}_i}\Big]\tilde S^{dom}_{\tau,\mathbf{P}}(\mathbf{x}_1,j_1;\cdots;\mathbf{x}_q,j_q)\Big|\le q!C^{N+q}|\lambda|^N2^{h(2-q)}\times \\ &\quad\times\Big[\prod_{v\ \text{special e.p.}}C_{n_v}2^{h_vn_v}\frac{Z^{(i_v)}_{h_v-1}}{Z_{h_v-1}}\Big]\Big[\prod_{v\in V_{nt}(\tau^*)}2^{2(s^*_v-1)h_v}e^{-c\sqrt{2^{h_v}\delta_v}}\Big]\times \\ &\quad\times\Big[\prod_{v\ \text{not e.p.}}\Big(\frac{Z_{h_v}}{Z_{h_v-1}}\Big)^{|P^\psi_v|/2}2^{2-\frac12|P^\psi_v|-|P^J_v|-z(P_v)}\Big] \end{aligned} \tag{6.29}$$

where, if v is a special endpoint, then $n_v=(n_v)_1+(n_v)_2$. This bound differs from (6.6) just by the dimensional factors $2^{h_vn_v}$, which arise from the action of the derivatives $\partial^{\mathbf{n}_i}_{\mathbf{x}_i}$ on a relativistic propagator $g^{(h_v)}_R$, cf. (5.76).

Now, using (4.11) and (6.28), we rewrite (6.27) as

$$\sum_{\substack{h\le 0\\ N\ge 0}} \sum_{\tau\in\mathcal{T}^{(h)}_{N,q}}^{(2)} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau:\\ |P_{v_0}|=|P^J_{v_0}|=q}} \sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*} \cdots \sum_{b_q\in\mathcal{C}^{(q)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*} \alpha_{b_1}(-1)^{j_1}\cdots\alpha_{b_q}(-1)^{j_q} \times$$
$$\times\Big[\prod_{i\in I_2}(-1)^{(\mathbf{x}_i)_{3-j_i}}\Big]\tilde S^{dom}_{\tau,\mathbf{P}}(\mathbf{x}_1,j_1;\cdots;\mathbf{x}_q,j_q)\ . \tag{6.30}$$

Using the fact that the paths $\mathcal{C}^{(i)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ consist of straight portions, each of which is formed by an even number of bonds, we find that

$$\Big|\sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*}\cdots\sum_{b_q\in\mathcal{C}^{(q)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*}\Big[\prod_{i=1}^{q}\alpha_{b_i}(-1)^{j_i}\Big]\Big[\prod_{i\in I_2}(-1)^{(\mathbf{x}_i)_{3-j_i}}\Big]\tilde S^{dom}_{\tau,\mathbf{P}}(\mathbf{x}_1,j_1;\cdots;\mathbf{x}_q,j_q)\Big|\le$$
$$\le\sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*}\cdots\sum_{b_q\in\mathcal{C}^{(q)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*}\Big|\Big[\prod_{i\in I_2}\partial_{(\mathbf{x}_i)_{3-j_i}}\Big]\tilde S^{dom}_{\tau,\mathbf{P}}(\mathbf{x}_1,j_1;\cdots;\mathbf{x}_q,j_q)\Big|\ . \tag{6.31}$$

Finally, we recognize that the summand in the right side of this equation can be bounded by the right side of (6.29), with the factor

$$\Big[\prod_{v\ \text{special e.p.}} C_{n_v}2^{h_v n_v}\frac{Z^{(i_v)}_{h_v}}{Z_{h_v}}\Big]$$

replaced in this specific case by

$$\Big[\prod_{\substack{v\ \text{special e.p.:}\\ i_v=1}}\frac{Z^{(1)}_{h_v}}{Z_{h_v}}\Big]\Big[\prod_{\substack{v\ \text{special e.p.:}\\ i_v=2}}C_1 2^{h_v}\frac{Z^{(2)}_{h_v}}{Z_{h_v}}\Big]\le C^q\Big[\prod_{\substack{v\ \text{special e.p.:}\\ i_v=2}}2^{h_v(1-|\eta'|)}\Big]\ , \tag{6.32}$$

where we used the fact that $Z^{(1)}_h/Z_h\le C$ and $Z^{(2)}_h/Z_h\le C2^{-|\eta'|h}$. Since the number of special endpoints of type $Z^{(2)}_h$ is at least 1, the product in the right side of this equation is smaller than $2^{\theta\bar h}$, where θ is a suitable constant between zero and one, and $\bar h$ is the largest among the scales of the special endpoints of type $Z^{(2)}_h$. As we did for the terms of type (a) of $\mathcal{S}^{(1)}_{q,\mathbf{j}}$, the dimensional gain $2^{\theta\bar h}$ can be "transferred to the root", i.e. transformed into $2^{\theta' h}$ provided θ' is added to theexponent $2-|P^\psi_v|/2-|P^J_v|-z(P_v)$ of each node. At that point, one proceeds like after (6.21).

The contributions of type $\mathcal{S}^{(3)}_{q,\mathbf{j}}$. We are finally left with $\mathcal{S}^{(3)}_{q,\mathbf{j}}$, which can be treated in a way similar to (and actually simpler than) the previous cases: the trees contributing to it either contain a non-relativistic propagator $r^{(h_w)}$, which produce an extra factor $2^{\theta h_w}$ (which can be "transferred to the root" by using the short memory property, as for the terms of type (a) of $\mathcal{S}^{(1)}_{q,\mathbf{j}}$); or contain endpoints on scale 1, in which case the short memory property produces an extra factor $2^{\theta h}$. In addition to these gains, one should take into account that all the special endpoint of type 2, possibly appearing in a tree contributing to $\mathcal{S}^{(3)}_{q,\mathbf{j}}$, whose presence could a priori produce a bad dimensional factor $2^{-h|\eta'|}$, are associated with an oscillatory factor

that effectively acts as a derivative operator, thus improving the bad factor $2^{-h|\eta'|}$ into $2^{h(1-|\eta'|)}$, precisely as discussed for $\mathcal{S}^{(2)}_{q,\mathbf{j}}$. Details are left to the reader. Summarizing, also the contributions of type $\mathcal{S}^{(3)}_{q,\mathbf{j}}$ give rise to a finite (q-dependent) constant, which concludes the proof of the main theorem.

Appendix A. The free propagator

In this section we compute the free propagator with (θ,τ) boundary conditions, as in (2.22). To this purpose, we need to diagonalize the Grassmann quadratic form $S = S(\psi) = -\frac{1}{2}\sum_{\mathbf{x},\mathbf{y}\in\Lambda}\psi_{\mathbf{x}}(K^{(\theta\tau)}_{\mathbf{t}^{(\mathbf{m})}})_{\mathbf{x},\mathbf{y}}\psi_{\mathbf{y}}$.

For $\mathbf{k}\in\mathcal{D}^{(\theta\tau)}_\Lambda$, we let

$$\hat\psi_{\mathbf{k}} = \sum_{\mathbf{x}\in\Lambda}\psi_{\mathbf{x}}e^{i\mathbf{k}\mathbf{x}} \tag{A.1}$$

so that

$$\psi_{\mathbf{x}} = \frac{1}{L^2}\sum_{\mathbf{k}\in\mathcal{D}^{(\theta,\tau)}_\Lambda}\hat\psi_{\mathbf{k}}e^{-i\mathbf{k}\mathbf{x}}. \tag{A.2}$$

Note that (A.2) holds also when one of the two coordinates of $\mathbf{x}$ equals $L/2+1$, in which case it gives the correct boundary condition $\psi_{(L/2+1,y)} = (-1)^\theta\psi_{(-L/2+1,y)}$ and $\psi_{(x,L/2+1)} = (-1)^\tau\psi_{(x,-L/2+1)}$.

Plugging (A.2) into the definition of S and using the anticommutation relation $\hat\psi_{\mathbf{k}}\hat\psi_{\mathbf{k}'} = -\hat\psi_{\mathbf{k}'}\hat\psi_{\mathbf{k}}$ one finds with standard computations

$$S(\psi) = \frac{1}{L^2}\sum_{\mathbf{k}\in\mathcal{D}^{(\theta\tau)}_\Lambda}\Big\{\hat\psi_{\mathbf{k}}\hat\psi_{-\mathbf{k}}(-i\sin k_1+\sin k_2) + m\hat\psi_{\mathbf{k}}\hat\psi_{-\mathbf{k}+(\pi,0)}\cos k_1\Big\} \tag{A.3}$$

where it is understood that, if $-\mathbf{k}$ does not belong to $\mathcal{D}^{(\theta\tau)}_\Lambda$, one should interpret $-\mathbf{k}$ as $-\mathbf{k}+2\pi(n_1,n_2)\in\mathcal{D}^{(\theta\tau)}_\Lambda$ for the suitable choice of $n_i\in\mathbb{Z}$ (similarly for $-\mathbf{k}+(\pi,0)$).

We have rewritten S as $-(1/2)\sum_{\mathbf{k},\mathbf{k}'}\hat\psi_{\mathbf{k}}A_{\mathbf{k},\mathbf{k}'}\hat\psi_{\mathbf{k}'}$ where the matrix A connects $\mathbf{k}$ only with $-\mathbf{k}$ and with $-\mathbf{k}+(\pi,0)$. To apply (2.8) it remains only to invert A. It is easy to check that the only non-zero elements of A^{-1} are

$$A^{-1}_{\mathbf{k},-\mathbf{k}} = L^2\,\frac{i\sin k_1+\sin k_2}{2D(\mathbf{k},m)} \tag{A.4}$$

$$A^{-1}_{\mathbf{k},-\mathbf{k}+(\pi,0)} = L^2\,\frac{m\cos k_1}{2D(\mathbf{k},m)}. \tag{A.5}$$

Then, formula (2.22) is obtained simply from (A.2), (2.8) and

$$\int_{(\theta\tau)}P^{(\theta\tau)}_\Lambda(d\psi)\hat\psi_{\mathbf{k}}\hat\psi_{\mathbf{k}'} = A^{-1}_{\mathbf{k},\mathbf{k}'}.$$

A.1. **Large distance behavior of $G(\mathbf{x})$.** Here we prove Proposition 1. Take $\mathbf{x}\neq\mathbf{0}$ and let $\hat G(\mathbf{k})$ be $1/(2D(\mathbf{k},m))$ times the 2×2 matrix in the

integral (3.7) and $\hat{\mathfrak{g}}(\mathbf{k})$ be $1/(2\bar{D}(\mathbf{k},m))$ times the matrix in (3.10). We have

$$G(\mathbf{x}) = \int\limits_{[-\pi,\pi]^2} \frac{d\mathbf{k}}{(2\pi)^2}\chi(\mathbf{k})e^{-i\mathbf{k}\mathbf{x}}\hat{\mathfrak{g}}(\mathbf{k}) + R_1(\mathbf{x}). \tag{A.6}$$

The diagonal matrix elements of $\hat{G}(\mathbf{k}) - \hat{\mathfrak{g}}(\mathbf{k})$ are C^∞ functions on the support of $\chi(\cdot)$, uniformly in m small, except at the origin. At $\mathbf{k} = \mathbf{0}$, one can easily check that they have bounded first derivatives and that the second derivatives are bounded by $O(1/|\mathbf{k}|)$. As a consequence (given that $\chi(\mathbf{k})$ is C^∞), an integration by parts argument shows that the remainder $R_1(\mathbf{x})$ decay as fast as $|\mathbf{x}|^{-2}$, uniformly in m small. Next, we rewrite the first term in the r.h.s. of (A.6) as

$$\int\limits_{\mathbb{R}^2} \frac{d\mathbf{k}}{(2\pi)^2}e^{-i\mathbf{k}\mathbf{x}}\hat{\mathfrak{g}}(\mathbf{k}) + R_2(\mathbf{x}) \tag{A.7}$$

where in R_2 the integral also spans on $\mathbb{R}^2$ but the integrand is multiplied by $(1-\chi(\mathbf{k}))$. Since $(1-\chi(\mathbf{k}))$ is C^∞ and vanishes in a neighborhood of 0, where $\hat{\mathfrak{g}}$ is singular, again (via integrations by parts) it is easy to see that $R_2(\mathbf{x})$ decays faster than any inverse power of $|\mathbf{x}|$, uniformly in m small. The sum $R_1 + R_2$ produces the remainder R in (3.9).

It remains to evaluate the integral in (A.7). We first integrate over k_2 using the residue theorem and we get (assume to fix ideas that $x_2 > 0$; recall that $\mathbf{x} \neq \mathbf{0}$)

$$\frac{i}{8\pi}\int_{\mathbb{R}} dk_1 \frac{e^{-ik_1x_1 - x_2a(k_1,m)}}{a(k_1,m)} \begin{pmatrix} k_1 - a(k_1,m) & m \\ -m & k_1 + a(k_1,m) \end{pmatrix} \tag{A.8}$$

with $a(k_1,m) = \sqrt{m^2 + (1-m^2)k_1^2}$. When $m = 0$ the integral w.r.t. k_1 is immediately computed and (A.8) gives the matrix $\mathfrak{g}^{(0)}(\mathbf{x})$ defined in (3.13).

A.2. **Finite-size corrections for the non-interacting system.** Here we prove that, as long as $m > 0$, the finite-L corrections to the free propagator $g_\Lambda^{(\theta,\tau)}$ are exponentially small, and that the ratio of Pfaffians (2.32) tends to 1 exponentially fast.

Let us start with the Poisson summation formula, that in our notations we can write as follows: if $\hat{F}$ is a smooth function on the torus $\mathbb{T}^2$ and L is an even integer, then

$$\frac{1}{L^2}\sum_{\mathbf{k}\in\mathcal{D}_\Lambda^{(\theta,\tau)}} \hat{F}(\mathbf{k}) = \sum_{\ell_1,\ell_2\in\mathbb{Z}} F(\ell_1 L, \ell_2 L)(-1)^{\theta\ell_1+\tau\ell_2} \tag{A.9}$$

where $\theta, \tau \in \{0,1\}$ and

$$F(\mathbf{x}) = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} \hat{F}(\mathbf{k})e^{-i\mathbf{k}\mathbf{x}}d\mathbf{k}. \tag{A.10}$$

If $\hat{F}(\mathbf{k}) = e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}N(\mathbf{k},m,y_1)/(2D(\mathbf{k},m))$ then the l.h.s. is exactly $g_\Lambda^{(\theta,\tau)}$, cf. (2.23). The term $(\ell_1,\ell_2) = (0,0)$ in the r.h.s. is $g(\mathbf{x},\mathbf{y})$ (cf. (2.24)), while the terms $(\ell_1,\ell_2) \neq (0,0)$ give a contribution exponentially small in L, since the Fourier transform of the analytic function $\hat{F}(\mathbf{k})$ decays exponentially fast.

As for (2.32), from the definition of Pfaffian and the explicit form of $K_\Lambda^{(\theta\tau)}$ (see (A.3)) we have

$$\frac{1}{L^2}\log \mathrm{Pf} K_\Lambda^{(\theta\tau)} = \frac{1}{4L^2}\sum_{\mathbf{k}\in\mathcal{D}_\Lambda^{(\theta\tau)}} \log[4D(\mathbf{k},m)]. \tag{A.11}$$

Using again the Poisson summation formula and the smoothness of $D(\mathbf{k},m)$ on the torus, the right-hand side gives

$$\frac{1}{4(2\pi)^2}\int_{\mathbb{T}^2} d\mathbf{k}\log\Big[4D(\mathbf{k},m)\Big] + O(\exp(-c(m)L)) \tag{A.12}$$

for some $c(m)>0$ and the claim on the ratios of Pfaffians follows.

Appendix B. Symmetry properties

In this Appendix we list the symmetry properties of the Grassmann action required for proving the properties of the coefficients $a_\gamma^{(h)}$, $b_\gamma^{(h)}$, $\sigma_{\gamma,\gamma'}^{(h)}$, l_h and $Z_{h;(\gamma,\gamma'),j}$ listed after (5.36), after (5.42) and after (5.43). It is straightforward to check that the gaussian integration $P_\Lambda(d\psi)$, the interaction $V_\Lambda^{(0)}$ and the source term $\mathcal{B}_\Lambda^{(0)}$ are separately invariant under the following symmetry transformations, irrespective of the Grassmann boundary conditions.

(1) *Parity*: $\psi_{\mathbf{x},\gamma}\to i\psi_{-\mathbf{x},\gamma}$ and $J_{\mathbf{x},j}\to J_{-\mathbf{x}-\hat e_j,j}$.

(2) *Reflections around the horizontal axis*: First change $\kappa\to\kappa^*$, where κ is a generic coefficient in the polynomials $V_\Lambda^{(0)}$, $\mathcal{B}_\Lambda^{(0)}$, and in the quadratic action entering the definition of $P_\Lambda(d\psi)$; then

$$(\psi_{\mathbf{x},1},\psi_{\mathbf{x},2},\psi_{\mathbf{x},3},\psi_{\mathbf{x},4})\mapsto(\psi_{\tilde{\mathbf{x}},1},-\psi_{\tilde{\mathbf{x}},2},-\psi_{\tilde{\mathbf{x}},3},\psi_{\tilde{\mathbf{x}},4})\ ,\qquad \text{with } \tilde{\mathbf{x}}=(x_1,-x_2)\ , \tag{B.1}$$

and $J_{\mathbf{x},1}\to J_{\tilde{\mathbf{x}},1}$, $J_{\mathbf{x},2}\to J_{\tilde{\mathbf{x}}-\hat e_2,2}$.

(3) *Quasi-particle interchange* #1:

$$(\psi_{\mathbf{x},1},\psi_{\mathbf{x},2},\psi_{\mathbf{x},3},\psi_{\mathbf{x},4})\mapsto(-\psi_{\mathbf{x},3},-\psi_{\mathbf{x},4},\psi_{\mathbf{x},1},\psi_{\mathbf{x},2}) \tag{B.2}$$

while $J_{\mathbf{x},j}$ is left unchanged.

(4) *Quasi-particle interchange* #2:

$$(\psi_{\mathbf{x},1},\psi_{\mathbf{x},2},\psi_{\mathbf{x},3},\psi_{\mathbf{x},4})\mapsto(-\psi_{\tilde{\mathbf{x}},2},\psi_{\tilde{\mathbf{x}},1},-\psi_{\tilde{\mathbf{x}},4},\psi_{\tilde{\mathbf{x}},3})\ ,\qquad \text{with } \tilde{\mathbf{x}}=(x_1,-x_2)\ , \tag{B.3}$$

and $J_{\mathbf{x},1}\to J_{\tilde{\mathbf{x}},1}$, $J_{\mathbf{x},2}\to J_{\tilde{\mathbf{x}}-\hat e_2,2}$.

(5) If in addition $m=0$, invariance under *reflections in a diagonal line*: First change $\kappa\to\kappa^*$, then transform the Grassmann fields as:

$$(\psi_{\mathbf{x},1},\psi_{\mathbf{x},2},\psi_{\mathbf{x},3},\psi_{\mathbf{x},4})\mapsto\sqrt{i}(\psi_{\tilde{\mathbf{x}},1},-i\psi_{\tilde{\mathbf{x}},4},-\psi_{\tilde{\mathbf{x}},3},-i\psi_{\tilde{\mathbf{x}},2})\ ,\text{with } \tilde{\mathbf{x}}=(x_2,x_1)\ , \tag{B.4}$$

and the external sources as $J_{\mathbf{x},1}\longleftrightarrow J_{\tilde{\mathbf{x}},2}$.

It is easy to check that the symmetries above are preserved by the multiscale integration. Therefore, the effective potentials and effective source terms on scales $h=-1,-2$, etc, are also invariant under the same symmetries. We can then use these symmetries in order to infer suitable symmetry properties of the kernels of $V^{(h)}$ and $\mathcal{B}^{(h)}$, which in particular imply the desired properties listed after (5.36), (5.42) and (5.43).

As an illustration of the general method used to infer properties on the renormalization constants from the symmetries above, let us discuss the consequences of symmetry (5) on the structure of the diagonal terms in $\mathcal{L}V_2^{(h)}$. The diagonal terms in $\mathcal{P}_0 V_2^{(h)}(\psi)$ have the form

$$\sum_{\gamma} \int \frac{d\mathbf{k}}{(2\pi)^2} \hat{\psi}_{-\mathbf{k},\gamma} \hat{K}^{(h)}_{2,(\gamma,\gamma)}(\mathbf{k}+\mathbf{p}_\gamma) \hat{\psi}_{\mathbf{k},\gamma} \tag{B.5}$$

and they are left invariant by symmetry (5), which is applicable since in $\mathcal{P}_0 V_2^{(h)}(\psi)$ the mass m is set to zero. That is, (B.5) is equal to

$$\sum_{\gamma} i(-1)^{\gamma+1} \int \frac{d\mathbf{k}}{(2\pi)^2} \hat{\psi}_{\tilde{\mathbf{k}},\tilde{\gamma}} \big[\hat{K}^{(h)}_{2,(\gamma,\gamma)}(\mathbf{k}+\mathbf{p}_\gamma)\big]^* \hat{\psi}_{-\tilde{\mathbf{k}},\tilde{\gamma}} \tag{B.6}$$

where $\tilde{1}=1$, $\tilde{2}=4$, $\tilde{3}=3$ and $\tilde{4}=2$. Therefore,

$$\hat{K}^{(h)}_{2,(\gamma,\gamma)}(\mathbf{k}+\mathbf{p}_\gamma) = i(-1)^{\gamma} \big[\hat{K}^{(h)}_{2,(\gamma,\gamma)}(\tilde{\mathbf{k}}+\mathbf{p}_{\tilde{\gamma}})\big]^* , \tag{B.7}$$

which implies that $a^{(h)}_\gamma = i(-1)^\gamma \big[b^{(h)}_{\tilde{\gamma}}\big]^*$. To give another example, symmetry (2) implies $a^{(h)}_\gamma = -\big[a^{(h)}_\gamma\big]^*$ and $b^{(h)}_\gamma = \big[b^{(h)}_\gamma\big]^*$. The remaining reality and symmetry properties of the renormalization constants can be derived similarly and their proof is left to the reader. It is important to observe that, to prove the desired properties of $\sigma^{(h)}_{\gamma,\gamma'}$, that depend on the mass m, one does not need to use symmetry (5) that holds only for $m=0$.

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References

[1] A. Abdesselam, V. Rivasseau: *Explicit Fermionic Tree Expansions*, Lett. Math. Phys. **44**, 77-88 (1998).

[2] F. Alet, J. Jacobsen, G. Misguich, V. Pasquier, F. Mila, M. Troyer: *Interacting Classical Dimers on the Square Lattice*, Phys. Rev. Lett. **94**, 235702 (2005).

[3] P. W. Anderson: *The Resonating Valence Bond State in La_2CuO_4 and Superconductivity* , Science **235**, 1196-1198 (1987).

[4] M. Bander, C. Itzykson: *Quantum-field-theory calculation of the two-dimensional Ising model correlation function*, Phys. Rev. D **15**, 463-469 (1977).

[5] G. Battle and P. Federbush: *A note on cluster expansions, tree graph identities, extra 1/n! factors!!!*, Letters in Mathematical Physics **8**, 55-57 (1984).

[6] G. Benfatto, P. Falco, and V. Mastropietro: *Functional integral construction of the massive Thirring model: verification of axioms and massless limit*, Comm. Math. Phys. **273**, 67–118 (2007).

[7] G. Benfatto, P. Falco, V. Mastropietro: *Massless Sine-Gordon and Massive Thirring Models: Proof of Colemans Equivalence*, Comm. Math. Phys. **285**, 713-762 (2009).

[8] G. Benfatto, P. Falco, V. Mastropietro: *Universality of one-dimensional Fermi systems, I and II*, Comm. Math. Phys. **330**, 153-215 and 217-282 (2014).

[9] G. Benfatto, G. Gallavotti: *Perturbation theory of the Fermi surface in a quantum liquid. A general quasiparticle formalism and one-dimensional systems*, J. Stat. Phys. **59**, 541-664 (1990).

[10] G. Benfatto, G. Gallavotti: *Renormalization Group*, Princeton University Press, 1995.

[11] G. Benfatto, G. Gallavotti, A. Procacci, B. Scoppola: *Beta function and Schwinger functions for a many fermions system in one dimension. Anomaly of the Fermi surface*, Comm. Math. Phys. **160**, 93-171 (1994).
[12] G. Benfatto, V. Mastropietro: *Renormalization group, hidden symmetries and approximate Ward identities in the XYZ model*, Rev. Math. Phys. **13**, 1323-1435 (2001).
[13] G. Benfatto, V. Mastropietro: *On the density-density critical indices in interacting Fermi systems*, Comm. Math. Phys. **231**, 97-134 (2002).
[14] G. Benfatto, V. Mastropietro: *Ward identities and chiral anomaly in the Luttinger liquid*, Comm. Math. Phys. **258**, 609-655 (2005).
[15] G. Benfatto, V. Mastropietro: *Universality relations in non-solvable quantum spin chains*, J. Stat. Phys. **138**, 1084-1108 (2010).
[16] D. Brydges: *A short course on cluster expansions*, in *Phénomènes critiques, systèmes aléatoires, théories de jauge*, Les Houches summer school session 43, pages 129-183, North-Holland (1986).
[17] D. Brydges, P. Federbush: *A new form of the Mayer expansion in classical statistical mechanics*, J. Math. Phys. **19**, 2064-2067 (1978).
[18] D. Brydges, T. Kennedy: *Mayer Expansions and the Hamilton-Jacobi Equation*, J. Stat. Phys. **48**, 19-49 (1987).
[19] D. Charrier, F. Alet, P. Pujol: *Gauge Theory Picture of an Ordering Transition in a Dimer Model*, Phys. Rev. Lett. **101**, 167205 (2008).
[20] V. S. Dotsenko, V. S. Dotsenko: *Critical behaviour of the phase transition in the 2D Ising Model with impurities*, Adv. Phys. **32**, 129-172 (1983).
[21] J. Dubedat: *Dimers and analytic torsion I*, preprint (arXiv:1110.2808).
[22] J. Dubedat, *Exact bosonization of the Ising model*, arXiv:1112.4399
[23] P. Falco: *Interacting fermions picture for dimer models*, Phys. Rev. E **87**, 060101(R) (2013).
[24] P. Falco: *Arrow-arrow correlations for the six-vertex model*, Phys. Rev. E **88**, 030103(R) (2013).
[25] P. Fendley, R. Moessner, S. L. Sondhi: *Classical dimers on the triangular lattice*, Phys. Rev. B **66**, 214513 (2002).
[26] M. E. Fisher: *Statistical mechanics of dimers on a plane lattice*, Phys. Rev. **124**, 1664-1672 (1961).
[27] M. E. Fisher, J. Stephenson: *Statistical Mechanics of Dimers on a Plane Lattice. II. Dimer Correlations and Monomers*, Phys. Rev. **132**, 1411-1431 (1963).
[28] J. Fröhlich, R. Götschmann, P. A. Marchetti: *The effective gauge field action of a system of non-relativistic electrons*, Comm. Math. Phys. **173**, 417-452 (1995).
[29] J. Fröhlich, T. Spencer, *The Kosterlitz-Thouless transition in two-dimensional Abelian spin systems and the Coulomb gas*, Comm. Math. Phys. **81** (1981), 527–602.
[30] C. Itzykson, J. Drouffe: *Statistical Field Theory: From Brownian Motion to Renormalization and Lattice Gauge Theory*, Cambridge University Press (1991).
[31] G. Gallavotti: *Renormalization theory and ultraviolet stability for scalar fields via renormalization group methods*, Rev. Mod. Phys. **57**, 471-562 (1985).
[32] G. Gallavotti, F. Bonetto, G. Gentile. *Aspects of the ergodic, qualitative and statistical theory of motion*, Springer Verlag, Berlin (2004).
[33] G. Gallavotti, F. Nicolò: *Renormalization theory in four-dimensional scalar fields. Part I*, Comm. Math. Phys. **100**, 545-590 (1985).
[34] G. Gentile, V. Mastropietro: *Renormalization group for one-dimensional fermions. A review on mathematical results*, Phys. Rep. **352**, 273-438 (2001).
[35] G. Giacomin, S. Olla, H. Spohn: *Equilibrium uctuations for the $\nabla\phi$ interface model*, Ann. Probab. **29**, 1138-1172 (2001).
[36] A. Giuliani: *The ground state construction of the two-dimensional hubbard model on the honeycomb lattice*, in *Quantum Theory from Small to Large Scales*, Lecture Notes of the Les Houches Summer School, Vol. 95, Oxford Univ. Press (2012).

[37] A. Giuliani, V. Mastropietro: *Anomalous universality in the anisotropic Ashkin-Teller model*, Comm. Math. Phys. **256**, 681-735 (2005); *Anomalous critical exponents in the anisotropic Ashkin-Teller model*, Phys. Rev. Lett. **93**, 190603 (2004).

[38] A. Giuliani, R. L. Greenblatt, V. Mastropietro, *The scaling limit of the energy correlations in non integrable Ising models* , Jour. Math. Phys. **53**, 095214 (2012).

[39] L. P. Kadanoff: *Connections between the Critical Behavior of the Planar Model and That of the Eight-Vertex Model*, Phys. Rev. Lett. **39**, 903-905 (1977).

[40] P. W. Kasteleyn: *The statistics of dimers on a lattice: I. The number of dimer arrangements on a quadratic lattice*, Physica **27**, 1209-1225 (1961); and J. Math. Phys. **4**, 287-293 (1963).

[41] R. Kenyon: *Local statistics of lattice dimers*, Ann. Inst. H. Poincaré, Prob Stat. **33**, 591-618 (1997).

[42] R. Kenyon: *Conformal Invariance of Domino Tiling*, Ann. Probab. **28**, 759-795 (2000).

[43] R. Kenyon: *Dominos and the Gaussian Free Field*, Ann. Probab. **29**, 1128-1137 (2001).

[44] R. Kenyon: *Lectures on dimers*, Park City Math Institute Lectures, available at arXiv:0910.3129.

[45] R. Kenyon, A. Okounkov, S. Sheffield: *Dimers and amoebae*, Ann. Math. **163**, 1019-1056 (2006).

[46] B. Laslier, F. L. Toninelli: *How quickly can we sample a uniform domino tiling of the* $2L \times 2L$ *square via Glauber dynamics?*, Prob. Theory Rel. Fields, in press (available at arXiv:1210.5456).

[47] V. Mastropietro: *Ising models with four spin interaction at criticality*, Comm.Math. Phys. **244**, 595-642 (2004).

[48] V.Mastropietro: *Non perturbative Adler-Bardeen Theorem*, J. Math. Phys. **48** 22302-22334 (2007).

[49] V. Mastropietro: *Non-perturbative renormalization*, World Scientific (2008).

[50] J. Miller, *Fluctuations for the Ginzburg-Landau* $\nabla\phi$ *Interface Model on a Bounded Domain*, Comm. Math. Phys. **308**, 591-639 (2011).

[51] R. Moessner, K. S. Raman: *Quantum dimer models*, Lecture notes, Trieste 2007, available at arXiv:0809.3051.

[52] R. Moessner, S. L. Sondhi: *Resonating Valence Bond Phase in the Triangular Lattice Quantum Dimer Model*, Phys. Rev. Lett. **86**, 1881-1884 (2001).

[53] A. Naddaf, T. Spencer: *On homogenization and scaling limit of some gradient perturbations of a massless free field*, Comm. Math. Phys. **183**, 55-84 (1997).

[54] J. W. Negele, H. Orland: *Quantum Many-particle Systems*, Westview Press (2008).

[55] S. Papanikolaou, E. Luijten, E. Fradkin: *Quantum criticality, lines of fixed points, and phase separation in doped two-dimensional quantum dimer models*, Phys. Rev. B **76**, 134514 (2007).

[56] D.S. Rokhsar, S.A. Kivelson: *Superconductivity and the Quantum Hard-Core Dimer Gas*, Phys. Rev. Lett. **61**, 2376-2379 (1988).

[57] S. Samuel: *The use of anticommuting variable integrals in statistical mechanics. I. The computation of partition functions*, J. Math. Phys. **21**, 2806-2814 (1980); and II. The computation of correlation functions, J. Math. Phys. **21**, 2815-2819 (1980).

[58] S. Sheffield: *Gaussian free fields for mathematicians*, Prob. Theory Rel. Fields **139**, 521-541 (2007).

[59] S. Smirnov: *Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits percolation*, C. R. Acad. Sci. I - Math. **333**, 239-244 (2001).

[60] S. Smirnov: *Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model* Ann. Math. **172**, 1435-1467 (2010).

[61] N.V. Temperley, M. E. Fisher: *Dimer problem in statistical mechanics-an exact result*, Philos. Mag. **6**, 1061-1063 (1961).

[62] J. B. Zuber, C. Itzykson: *Quantum field theory and the two-dimensional Ising model*, Phys. Rev. D **15**, 2875-2884 (1977).

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