The Boltzmann–Grad Limit of a Hard Sphere System: Analysis of the Correlation Error

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In memory of Oscar Erasmus Lanford III

Abstract. We present a quantitative analysis of the Boltzmann–Grad (low–density) limit of a hard sphere system. We introduce and study a set of functions (correlation errors) measuring the deviations in time from the statistical independence of particles (propagation of chaos). In the context of the BBGKY hierarchy, a correlation error of order k measures the event where k tagged particles are connected by a chain of interactions preventing the factorization. We prove that, provided k is not too large, such an error flows to zero with the hard spheres diameter ε, for short times, as $\varepsilon^{\gamma k}$, for some $\gamma > 0$. This requires a new analysis of many–recollision events, and improves previous estimates of high order correlation functions.

Keywords. Kinetic theory, low–density limit, BBGKY hierarchy, Boltzmann equation, propagation of chaos

Contents

1 Introduction

In 1975 O. E. Lanford III presented his celebrated proof of the mathematical validity of the Boltzmann equation for hard spheres, in a time interval small enough [17]. To remind his result, let us consider a system of identical hard spheres of diameter ε moving in the whole space $\mathbb{R}^3$, with collisions governed by the usual laws of elastic reflection.

A state of the system over the grand canonical phase space $\Gamma = \cup_{n\geq 0}(\mathbb{R}^3\times\mathbb{R}^3)^n$ is characterized via the symmetric (in the exchange of the particles) probability densities $(1/n!)W^{\varepsilon}_{0,n}$ of finding exactly n spheres with given positions and velocities. If $p_n = (1/n!)\int W^{\varepsilon}_{0,n}$, then $\sum_n p_n = 1$ and the average number of particles is $\langle N\rangle = \sum_n n\, p_n$.

We are interested in analyzing a low–density limit, namely the *Boltzmann–Grad limit* [11, 12], defined by

$$\langle N\rangle \to \infty, \quad \varepsilon \to 0 \quad \text{and} \quad \langle N\rangle\varepsilon^2 \to \lambda^{-1} > 0\ , \tag{1.1}$$

where λ is a fixed constant proportional to the mean free path. Since $\langle N\rangle$ and ε are related in the Boltzmann–Grad limit, let us use a single parameter, say ε, and rescale properly $\langle N\rangle = \varepsilon^{-2}\lambda^{-1}$.

From the densities $W^{\varepsilon}_{0,n}$ we construct the *rescaled correlation functions* (r.c.f. in the sequel) $f^{\varepsilon}_{0,j} = \varepsilon^{2j}\rho^{\varepsilon}_{0,j}$ for $j = 1, 2, \cdots$, where $\rho^{\varepsilon}_{0,j}$ are the usual correlation functions defined in the following way. For any subset Δ of $\mathbb{R}^3\times\mathbb{R}^3$, the integral $\int_\Delta \rho^{\varepsilon}_{0,j}$ expresses the mean value of the product $N_\Delta(N_\Delta - 1)\cdots(N_\Delta - j + 1)$, where N_Δ is the number of particles in Δ. Note that the rescaling ε^{2j} allows to see finite quantities as $\varepsilon \to 0$. The r.c.f. differ from the marginals only for proper normalization factors.

We focus on the quantities $f^{\varepsilon}_j(t)$, namely the corresponding r.c.f. evolved at time t in accord to the hard sphere dynamics.

Lanford proved that, if the initial state factorizes in this limit, meaning that

$$f^{\varepsilon}_{0,j} \to f_0^{\otimes j} \tag{1.2}$$

as $\varepsilon \to 0$, where f_0 is a given one–particle probability density, then there exists $\bar{t} > 0$ such that, in the same limit,

$$f^{\varepsilon}_j(t) \to f(t)^{\otimes j} \qquad \text{for} \quad t < \bar{t} \tag{1.3}$$

almost everywhere in $(\mathbb{R}^3\times\mathbb{R}^3)^j$, where $f(t)$ is a solution of the Boltzmann equation with initial datum f_0.

Note that we found convenient to recall the theorem as stated in [16] (or also in [2, 25]), namely without fixing the total number of particles. The advantage of this formulation in our context will be discussed later on.

It may be worth to mention that the only validity result holding globally in time refers to the special situation of a rare cloud of gas expanding in the vacuum, [13, 14].

We remind here the *Boltzmann equation* for the unknown $f = f(x, v, t)$, with hard sphere kernel and mean free path λ [4],

$$(\partial_t + v\cdot\nabla_x) f(x,v,t) = \lambda^{-1} \int_{\mathbb{R}^3\times S^2_+} dv_1 d\omega \, (v - v_1)\cdot\omega \Big\{ f(x, v_1', t) f(x, v', t) - f(x, v_1, t) f(x, v, t) \Big\} \tag{1.4}$$

where $S^2_+ = \{\omega \in S^2 | \, (v - v_1)\cdot\omega \geq 0\}$, S^2 is the unit sphere in $\mathbb{R}^3$ (with surface measure $d\omega$), (v, v_1) is a pair of velocities in incoming collision configuration and (v', v_1') is the corresponding pair of outgoing velocities defined by the elastic reflection rules

$$\begin{cases} v' = v - \omega[\omega\cdot(v - v_1)] \\ v_1' = v_1 + \omega[\omega\cdot(v - v_1)] \end{cases} . \tag{1.5}$$

The proof of [17] was carried out by assuming suitable uniform estimates on the family of r.c.f. at time zero. The available estimates deteriorate in time in such a way that, at time $\bar{t}$, any possibility of a uniform control is lost. Indeed the strategy of Lanford was based on an expansion of $f^\varepsilon_j(t)$, for given $j > 0$, in terms of a series (involving only the initial data $f^\varepsilon_{0,j}$) which is absolutely convergent, uniformly in ε, only for a short time interval. To complete the proof it was enough to exploit the term by term convergence holding by virtue of geometric and measure–zero arguments (see also [23, 28, 27, 9, 29]).

The original argument of Lanford was qualitative, in the sense that (1.3) was shown without an explicit rate of convergence. Recently, explicit estimates on the rate of convergence have been obtained in [10] (see also [19] for a different class of potentials).

Furthermore the explicit control of the error has been used to reach hydrodynamic regimes: see [3], where the heat equation is derived from the hard sphere dynamics in a low–density regime, by studying one tagged particle in a gas close to equilibrium.

In the present paper we give a different and quantitative proof of Lanford's result. We build up a method for a systematic analysis of the error term and for a uniform control of the state extended up to high order r.c.f., i.e. marginals with large values of j.

Let us explain more precisely what we mean by the previous statement.

In the preceding literature [10, 19] the proof of Lanford has been worked out with explicit estimates showing a variety of contributions, and a quite elaborate final expression of the error. On the other hand, it is often stated that *propagation of chaos* is the core of convergence towards the Boltzmann equation, i.e. the conservation in time of the factorization of the state of the system, provided that it holds at time zero. In this work, we aim first of all to clarify and make quantitative this statement. We shall focus our approach to the Boltzmann–Grad limit on a careful control of the delicate mechanism of propagation of chaos.

The breakdown of the statistical independence is mostly due to mechanical effects. First, one should keep in mind that any given state of the real system (in particular, whatever choice of the time–zero state) cannot be exactly factorized, because of the simple hard core exclusion. This is a static, essentially trivial, feature. Secondly, and most importantly, correlations between particles are generated by the dynamics itself. In the context of the above references, the events responsible for these dynamical correlations are called *recollisions*. Their effective control is quite complicated since they depend on the full particle dynamics.

In our strategy, the first goal is to identify and strictly isolate the dynamical events which destroy propagation of chaos. In doing so, we end up quite naturally with a representation of the j–particle r.c.f. in terms of a sum of contributions of factorized states of different sizes.

More precisely, let a configuration of j particles be $\mathbf{z}_j = (z_1, \cdots, z_j)$, where $z_i = (x_i, v_i)$ are the position and the velocity of particle i respectively. Then we introduce a new sequence of ($\varepsilon-$dependent) functions $E_j = E_j(\mathbf{z}_j, t)$, $j = 1, 2, \cdots$, defined implicitly by

$$f_J^\varepsilon(t) = \sum_{K \subset J} (f_1^\varepsilon(t))^{\otimes K} E_{J \setminus K}(t) \ , \tag{1.6}$$

where $J = \{1, 2, \cdots, j\}$, K is a subset of indices in J ($\emptyset$ and J are included in the sum, with the convention $E_\emptyset = 1$) and, if $Q = \{i_1, \cdots, i_q\}$, one denotes

$$\begin{aligned} &f_Q^\varepsilon(\mathbf{z}_Q, t) = f_q^\varepsilon(z_{i_1}, \cdots, z_{i_q}, t) \ , \\ &E_Q(\mathbf{z}_Q, t) = E_q(z_{i_1}, \cdots, z_{i_q}, t) \ , \\ &(f_1^\varepsilon(t))^{\otimes Q} = f_1^\varepsilon(z_{i_1}, t) f_1^\varepsilon(z_{i_2}, t) \cdots f_1^\varepsilon(z_{i_q}, t) \ . \end{aligned} \tag{1.7}$$

Eq. (1.6) can be inverted to give

$$E_J(t) = \sum_{K \subset J} (-1)^k (f_1^\varepsilon(t))^{\otimes K} f_{J \setminus K}^\varepsilon(t) \ , \tag{1.8}$$

having put $k = |K|$, where $|K|$ is the cardinality of the set K. Here we shall call $E_J(t)$ the *correlation error* of order j, since its size is a measure of how far are the r.c.f. from a product state.

Similar formulas are quite familiar in Equilibrium Statistical Mechanics, when one deals with cluster properties of a state. These are typical in gas phases, e.g. the classical analysis of equilibrium of a dilute system in chapter 4.4 of [21]. Notice, however, that (1.6) is not the usual decomposition of the classical cluster expansion, since it does not take into account possible uncorrelations among subclusters of $J \setminus K$. The cluster property which is the object of our study differs from 4.4.3 of [21] because, in our case, the correlation of *each* pair of particles is vanishing in the limit.

The propagation of chaos amounts to say that, for any given j, as $\varepsilon \to 0$

$$E_j(0) \to 0 \quad \Longrightarrow \quad E_j(t) \to 0$$

for $t > 0$.

Actually we are going to prove that, in the Boltzmann–Grad limit, there exists a constant $\gamma > 0$ such that, if t is small enough, then

$$\int dv_1 \cdots dv_j |E_j(z_1, \cdots, z_j, t)| \leq \varepsilon^{\gamma j} , \tag{1.9}$$

for a given configuration $x_1, \cdots, x_j$ of distinct points and ε sufficiently small. Of course an estimate similar to (1.9) has to be assumed at time zero, together with uniform estimates on the family of r.c.f. (as in Lanford's theorem). We shall construct explicit examples of physically relevant initial states satisfying such hypotheses.

Beyond the important time limitation in (1.9), there is another intrinsic restriction. Clearly, estimate (1.9) is not expected to hold for very large j since, for instance, if $j \simeq \varepsilon^{-2}$, $f^\varepsilon_J(t)$ will be very different from a product state. As a matter of fact, we shall prove that (1.9) holds for $j \leq \varepsilon^{-\alpha}$, for some $\alpha \in (0, 1)$. That is, j may diverge, but gently.

Let us comment on the physical idea underlying formula (1.6). Given j particles, say particles $1, \cdots, j$, their joint distribution does not factorize because the clusters of particles influencing the dynamics of each of the j tagged particles are, generally speaking, not disjoint. Nevertheless we may expect that, in a rarefied gas, this event is unlikely, at least for a given j, small ε, and t small enough.

To give a mathematically precise statement of this fact, we look at the BBGKY hierarchy. The hierarchy gives an expression of $f^\varepsilon_j(t)$ in terms of a series expansion depending only on the initial data. Each term of this expansion is in one–to–one correspondence with a special (unphysical, virtual) trajectory of clusters of particles flowing backwards in time. Looking at such flows one may single out precisely the (re-)collisions that generate correlations. Roughly speaking, formulas (1.6), (1.9) are then constructed by systematically replacing such collision–events with "free overlap–events" where the two considered particles ignore and cross each other freely, and estimating the consequent errors. Technically, this will be done by means of (i) a suitable cluster expansion, and (ii) geometric estimates for many–recollision events.

The final result expresses $f_j^\varepsilon(t)$ as a sum of contributions. The first, $O(1)$, is just the product state. Then, we sum over all possible ways of choosing two correlated particles, the remaining $j-2$ particles being uncorrelated. This events are $O(\varepsilon^{2\gamma})$. (Clearly, a single particle cannot be correlated: note that $E_1 \equiv 0$.) Then we pass to the events in which three particles are correlated, which give a contribution $O(\varepsilon^{3\gamma})$, and so on.

In this paper we will derive the bound on E_j, as roughly explained above, exploiting the series expansion for f_j^ε. Another possibility is to use (1.8) and the evolution equations for f_j^ε and f_1^ε, but a closed evolution equation for E_j seems to be difficult to write and to handle with.

We finally remark that a control on the correlation error $E_j(t)$ allows a control of interesting physical quantities, such as the fluctuations of extensive observables from their mean value. For instance it is easy to show that, denoting by n_Δ the fraction of particles falling in the measurable set $\Delta \subset \mathbb{R}^3 \times \mathbb{R}^3$ then, for a finite sequence $\Delta_1, \cdots, \Delta_j$ of disjoint sets, we have

$$\Big\langle \prod_{i=1}^{j} (n_{\Delta_i} - \langle n_{\Delta_i} \rangle_t) \Big\rangle_t = \int_{\Delta_1 \times \cdots \times \Delta_j} dz_1 \cdots dz_j E_j(z_1, \cdots, z_j, t) \ , \tag{1.10}$$

where $\langle \cdot \rangle_t$ is the average with respect to the time evolved state at time t. This equation specifies the quantitative information given by the estimate of $E_j(t)$, on how the statistical independence is achieved. For previous results on the fluctuation field in the Boltzmann–Grad limit, see [2, 25, 26].

Note that no quantity associated to the Boltzmann equation appears yet in the previous formulas. Indeed the functions E_j describe a part, but not all, of the total dynamical correlation between particles, the remainder being encoded in the one–point marginal f_1^ε. Working again in terms of virtual backwards flows, one may extract from the definition of f_1^ε a second (and last) class of recollision–events. This operation leads to define another interesting sequence of quantities, that is $E_j^{\mathscr{E}}(\mathbf{z}_j, t)$, $j = 1, 2, \cdots$, given by

$$f_J^\varepsilon(t) = \sum_{K \subset J} (g^\varepsilon(t))^{\otimes K} E_{J \setminus K}^{\mathscr{E}}(t) \tag{1.11}$$

(where we extend in an obvious way the notations of (1.7)) or by

$$E_J^{\mathscr{E}}(t) = \sum_{K \subset J} (-1)^k (g^\varepsilon(t))^{\otimes K} f_{J \setminus K}^\varepsilon(t) \ , \tag{1.12}$$

where $g^\varepsilon(t)$ is defined by an explicit expression that does *not* involve any more correlations among particles. Namely, $g^\varepsilon(t)$ is the series solution to the *Enskog equation*, which we recall:

$$(\partial_t + v\cdot\nabla_x)g^\varepsilon(x,v,t) = \lambda^{-1}\int_{\mathbb{R}^3\times S^2_+} dv_1 d\omega\ (v-v_1)\cdot\omega \\ \times\Big\{g^\varepsilon(x-\omega\varepsilon,v_1',t)g^\varepsilon(x,v',t) - g^\varepsilon(x+\omega\varepsilon,v_1,t)g^\varepsilon(x,v,t)\Big\}\ , \tag{1.13}$$

where we used the notations introduced next to (1.4).

Note that if $f_j^\varepsilon(t)$ factorizes strictly, i.e. $f_J^\varepsilon(t) = (f_1^\varepsilon(t))^{\otimes J}$, then

$$E_J^{\mathscr{E}}(t) = ((f_1^\varepsilon - g^\varepsilon)(t))^{\otimes J}\ .$$

More generally, the size of $E_J^{\mathscr{E}}(t)$ is a measure of both the breakdown of propagation of chaos and the error in the convergence of f_1^ε to g^ε. We will show that $E_j^{\mathscr{E}}(t)$ can be bounded as $E_j(t)$, i.e.

$$\int dv_1\cdots dv_j |E_j^{\mathscr{E}}(t)| \le \varepsilon^{\gamma j} \tag{1.14}$$

for t small enough and $j\le\varepsilon^{-\alpha}$, as soon as $f_1^\varepsilon(0)$ is assumed to converge uniformly as a power of ε to the initial datum for the Enskog equation.

In our framework, the Enskog equation appears as a *bridge* between the hard sphere dynamics and the Boltzmann equation. In particular, to obtain the representation (1.11)–(1.14), no regularity property has to be assumed for the state of the system. The Enskog picture is what emerges from the mechanical system once we eliminate all the sources of correlation, including both the dynamical correlation and the static correlation of the time–zero state.

The difference between the Enskog system described by $g^\varepsilon(t)$ and the Boltzmann system described by $f(t)$ and (1.4) (with same initial data), is that interactions between particles are described as occurring at distance ε instead that at distance zero. A simple continuity property (assumed for the initial data) implies then the following final formulas:

$$f_J^\varepsilon(t) = \sum_{K\subset J}(f(t))^{\otimes K}E^{\mathcal{B}}_{J\setminus K}(t)\ ,$$

$$\int dv_1\cdots dv_j|E_J^{\mathcal{B}}(t)| \le \varepsilon^{\gamma j} \tag{1.15}$$

for t small enough and $j\le\varepsilon^{-\alpha}$. Note that Equation (1.15) is a reformulation of Lanford's result together with an explicit representation of the error.

The quantities $E_J^{\mathcal{B}}(t)$, under the name "v–functions", were previously introduced in [5, 6, 7] in dealing with kinetic limits of stochastic particle systems.

We note finally that the result (1.15) provides a further information. Suppose to have a nice solution to the Boltzmann equation satisfying the uniform estimate $|f(t)| \leq G$. Then (1.15) implies that, if $\gamma' < \gamma$,

$$|f_j^\varepsilon(t)| \leq \Big(|f(t)| + \varepsilon^{\gamma'}\Big)^j \leq (2G)^j \tag{1.16}$$

outside a set of arbitrarily small measure, for t small enough, $j \leq \varepsilon^{-\alpha}$ and ε sufficiently small. Using (1.9), the same result may be deduced starting from an estimate $|f_1^\varepsilon(t)| \leq G$. Note that, from the bounds of the previous literature [10, 19], an analogous control can be obtained only up to values of j diverging logarithmically with ε (since one has $|f_j^\varepsilon - f^{\otimes j}| \leq C^j \varepsilon^{\gamma'}$, where C depends on the initial data).

As pointed out already by Lanford himself [17], the restriction to short times of the convergence result (1.3) can be removed if one has the *a priori* estimate on the f_j^ε showing that singularities are not developed in time (which of course, even if true, seems to be hard to control in a general context). Also our correlation error estimates could well be true for $t > \bar{t}$ if, in addition to (1.16), one disposes of a uniform estimate on the f_j^ε for $j > \varepsilon^{-\alpha}$. A proof of such a statement is however missing (see also Remark 2 on page 33).

The plan of the paper is the following. In Section 2 we introduce the preliminaries of our analysis, i.e. the model, the hierarchies, the "tree expansion"–representations which are our basic tool, and the family of initial states we consider. Section 3 is devoted to the precise statement of our results and the proofs.

2 Hierarchies

In this section we describe precisely our setting. First, we introduce the hard sphere model and recall some basic results on the dynamics. Then, we explain how to describe the evolution of a statistical state of the system (Sec. 2.1). An explicit representation can be given in terms of a tree expansion and of a class of special flows of particles evolving backwards in time. An analogous description is also possible for the Boltzmann evolution equation. These well known expressions, which will be our starting point, are introduced in Sec. 2.2 together with some new expansions that will have the role of an intermediate object in the transition towards the kinetic limit. Finally, in Sec. 2.3 we discuss our assumptions on the state at time zero.

2.1 Basic definitions

2.1.1 The hard sphere system

We consider a *system of hard spheres* of unit mass and of diameter $\varepsilon > 0$ moving, for simplicity, in the whole space $\mathbb{R}^3$. We will denote

$$z_i = (x_i, v_i) \in \mathbb{R}^6$$

the state of the i–th particle, $i = 1, 2, \cdots$. For groups of particles we shall use the notation

$$\mathbf{z}_j = z_1, \cdots, z_j, \qquad \mathbf{z}_{j,n} = z_{j+1}, \cdots, z_{j+n},$$

and call "particle i" a particle whose configuration is labelled by the index i.

We will work over the grand canonical *phase space*

$$\mathcal{M}(\varepsilon) = \cup_{n \geq 0} \mathcal{M}_n(\varepsilon) \;, \tag{2.1.1}$$

where

$$\mathcal{M}_n(\varepsilon) = \left\{ \mathbf{z}_n \in \mathbb{R}^{6n}, \;\; |x_i - x_j| > \varepsilon, \; i \neq j \right\} , \qquad \mathcal{M}_0(\varepsilon) = \emptyset. \tag{2.1.2}$$

Unless necessary we drop, for simplicity, the dependence on ε of the spaces defined above.

Notice that $\mathcal{M}_N$, with $N \sim \varepsilon^{-2}$, is the canonical N–particle phase space used in [17] and in most of the subsequent literature on the Boltzmann–Grad limit. In this paper we find convenient to consider a more general class of measures where the exact number of particles is not necessarily fixed. The advantage of this picture will be clarified in the remark after Eq. (2.2.18).

The equations of motion for the n–particle system are defined as follows. Between collisions each particle moves on a straight line with constant velocity. When two hard spheres collide with positions x_i, x_j (at distance ε), normalized relative distance

$$\omega = (x_i - x_j)/|x_i - x_j| = (x_i - x_j)/\varepsilon \in S^2$$

and incoming velocities v_i, v_j (that means $(v_i - v_j) \cdot \omega < 0$), these are instantaneously transformed to outgoing velocities v_i', v_j' (with $(v_i' - v_j') \cdot \omega > 0$) through the relations

$$\begin{aligned} v_i' &= v_i - \omega[\omega \cdot (v_i - v_j)] \;, \\ v_j' &= v_j + \omega[\omega \cdot (v_i - v_j)] \;. \end{aligned} \tag{2.1.3}$$

The collision transformation is invertible and preserves the Lebesgue measure on $\mathbb{R}^6$.

The above prescription defines the *flow of the n–particle dynamics*, $t \mapsto \mathtt{T}_n^\varepsilon(t)\mathbf{z}_n$. Observe that these rules do not cover all possible situations, e.g. triple collisions are excluded. Nevertheless, as proved by Alexander in [1], there exists a full–measure subset of $\mathcal{M}_n$, over which $\mathtt{T}_n^\varepsilon(t)$ is uniquely defined for all t (see also [18, 9]). Thus $\mathtt{T}_n^\varepsilon(t)$ can be defined as a one–parameter group of Borel maps on $\mathcal{M}_n$, leaving invariant the Lebesgue measure.

Notice that the flow $\mathtt{T}_n^\varepsilon(t)$ is piecewise continuous in t (we do not identify outgoing and incoming configurations). If necessary, we may distinguish the limit from the future (+) and the limit from the past (−) by writing $\mathtt{T}_n^\varepsilon(t^\pm)\mathbf{z}_n = \lim_{\varepsilon\to 0^+}\mathtt{T}_n^\varepsilon(t\pm\varepsilon)\mathbf{z}_n$. Moreover, we shall fix the convention of right–continuity of the flow, $\mathtt{T}_n^\varepsilon(t)\mathbf{z}_n = \mathtt{T}_n^\varepsilon(t^+)\mathbf{z}_n$.

2.1.2 Statistical states

Let us turn now to the precise statistical description of our system. We shall adopt a general formulation, in the spirit of classical Statistical Mechanics [20].

We introduce the set of density functions over $\mathcal{M}$, denoted $\mathbf{W}_0^\varepsilon = \{W_{0,n}^\varepsilon\}_{n\geq 0}$, where $W_{0,n}^\varepsilon : \mathcal{M}_n \to \mathbb{R}^+$ is a positive Borel function symmetric in the particle labels. The quantity $(1/n!)W_{0,n}^\varepsilon(\mathbf{z}_n)$ gives the probability density of finding exactly n particles, in a state $z_1, \cdots, z_n$. We refer to $\mathbf{W}_0^\varepsilon$ as the *state of the particle system.*

Note that n, the total number of particles, is a random variable, and $(1/n!)\int W_{0,n}^\varepsilon$ is its distribution. The normalization condition reads

$$\sum_{n=0}^{\infty}\frac{1}{n!}\int W_{0,n}^\varepsilon = 1\ . \tag{2.1.4}$$

Given an initial measure over $\mathcal{M}$ with density specified by $\mathbf{W}_0^\varepsilon$, its evolution at time t is given by the *Liouville equation*

$$W_n^\varepsilon(\mathbf{z}_n,t) = W_{0,n}^\varepsilon(\mathtt{T}_n^\varepsilon(-t)\mathbf{z}_n), \tag{2.1.5}$$

to be valid almost everywhere in $\mathcal{M}_n$. This defines $\mathbf{W}^\varepsilon(t)$, the state at time t.

Next we define the vector of *correlation functions* over $\mathcal{M}$ as $\boldsymbol{\rho}^\varepsilon(t) = \{\rho_j^\varepsilon(t)\}_{j\geq 0}$, $t \geq 0$, by

$$\rho_j^\varepsilon(\mathbf{z}_j,t) = \sum_{k=0}^{\infty}\frac{1}{k!}\int_{\mathcal{M}_k\cap S(\mathbf{x}_j)^k} dz_{j+1}\cdots dz_{j+k}W_{j+k}^\varepsilon(\mathbf{z}_{j+k},t)\ , \tag{2.1.6}$$

where

$$S(\mathbf{x}_j) = \left\{z=(x,v)\in\mathbb{R}^6 \;\middle|\; |x-x_k|>\varepsilon \ \text{ for all } k=1,\cdots,j\right\}. \tag{2.1.7}$$

We say that *a state admits correlation functions* when the series in the right hand side of (2.1.6) is convergent, together with the series in the inverse formula

$$W_j^\varepsilon(\mathbf{z}_j, t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{\mathcal{M}_k \cap S(\mathbf{x}_j)^k} dz_{j+1} \cdots dz_{j+k} \rho_{j+k}^\varepsilon(\mathbf{z}_{j+k}, t) \ . \tag{2.1.8}$$

In this case, for any finite ε, the set of functions $\boldsymbol{\rho}^\varepsilon(t)$ describes all the properties of the system. Later on, we will assume explicit estimates ensuring convergence of the series.

The normalization condition for the correlation functions is

$$\int_{\mathcal{M}_j} \rho_j^\varepsilon(\mathbf{z}_j, t) d\mathbf{z}_j = \mathbb{E}_t(n(n-1)\dots(n-j+1)) = \mathbb{E}_0(n(n-1)\dots(n-j+1)) \ , \tag{2.1.9}$$

where n is the total number of particles, and the expectation $\mathbb{E}_t$ is done with respect to the state $\mathbf{W}^\varepsilon(t)$.

In this setting, the *Boltzmann–Grad scaling* is given by the following condition: the *average* number of particles has to diverge as ε^{-2}, that is

$$\lim_{\varepsilon \to 0} \varepsilon^2 \int_{\mathbb{R}^6} \rho_1^\varepsilon(z_1, t) = \lambda^{-1} \ , \tag{2.1.10}$$

where $\lambda > 0$ is proportional to the mean free path. From now on, we shall fix

$$\lambda = 1$$

for notational simplicity.

The central object of our study becomes the collection of *rescaled correlation functions* (r.c.f.) defined by

$$f_j^\varepsilon(\mathbf{z}_j, t) = \varepsilon^{2j} \rho_j^\varepsilon(\mathbf{z}_j, t) \ . \tag{2.1.11}$$

The above framework includes:

– the *canonical* states, $W_{0,n}^\varepsilon = 0$ for $n \neq N$, in which case the normalization condition is

$$\int \rho_j^\varepsilon = N(N-1)\cdots(N-j+1) \ , \tag{2.1.12}$$

and the scaling (2.1.10) reduces to $\varepsilon^2 N \to \lambda^{-1} = 1$;

– a variety of *grand canonical* states (non fixed number of particles), as presented in the introduction. In this case the scaling is $\varepsilon^2 \sum n p_n \to \lambda^{-1} = 1$, where $p_n = (1/n!) \int W_{0,n}^\varepsilon$.

2.1.3 BBGKY hierarchy

Before starting the analysis of the low–density regime, let us describe the time evolution for any fixed $\varepsilon > 0$. The evolution equations for the considered quantities were first derived formally by Cercignani in [8].

Assuming some explicit bound and sufficient smoothness, he deduced the hard sphere version of the well known *BBGKY hierarchy* of equations, which for the rescaled correlation functions takes the form

$$\left(\partial_t + \sum_{i=1}^{j} v_i \cdot \nabla_{x_i}\right) f_j^\varepsilon(\mathbf{z}_j, t) = \sum_{i=1}^{j} \int_{S^2\times\mathbb{R}^3} d\omega\, dv_{j+1}\, B^\varepsilon(\omega; v_{j+i}-v_i)\, f_{j+1}^\varepsilon(\mathbf{z}_j, x_i+\varepsilon\omega, v_{j+1}, t)\ , \tag{2.1.13}$$

where

$$B^\varepsilon(\omega; v_{j+i} - v_i) = \omega \cdot (v_{j+i} - v_i)\mathbb{1}_{\{\min_{\ell=1,\cdots,j;\ell\neq i} |x_i+\omega\varepsilon-x_\ell|>\varepsilon\}}(\omega)\ . \tag{2.1.14}$$

Here $\mathbb{1}_A$ denotes the indicator function of the set A.

Notice that the difference of this formula with respect to the hierarchy written for marginals in the canonical setting (as for instance in [8]), is that a factor $\varepsilon^2(N-j)$ is absent in the right hand side (N = fixed total number of particles). This is a small notational advantage in using correlation functions.

The *series solution* of the hierarchy (obtained from integration and repeated iteration of the above formula) is

$$\begin{aligned} f_j^\varepsilon(t) = \sum_{n\geq 0} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \\ \mathcal{S}_j^\varepsilon(t-t_1)\mathcal{C}_{j+1}^\varepsilon \mathcal{S}_{j+1}^\varepsilon(t_1-t_2)\cdots \mathcal{C}_{j+n}^\varepsilon \mathcal{S}_{j+n}^\varepsilon(t_n) f_{j+n}^\varepsilon(0)\ , \end{aligned} \tag{2.1.15}$$

where we used the definitions of *interacting flow operator* $\mathcal{S}_j^\varepsilon(t)$ and *BBGKY collision operator* $\mathcal{C}_{j+1}^\varepsilon$, i.e. respectively

$$\mathcal{S}_j^\varepsilon(t) f_j^\varepsilon(\mathbf{z}_j, \cdot) = f_j^\varepsilon(\mathtt{T}_j^\varepsilon(-t)\mathbf{z}_j, \cdot) \tag{2.1.16}$$

and

$$\mathcal{C}_{j+1}^\varepsilon = \sum_{k=1}^{j} \mathcal{C}_{k,j+1}^\varepsilon \tag{2.1.17}$$

$$\mathcal{C}_{k,j+1}^\varepsilon f_{j+1}^\varepsilon(\mathbf{z}_j, \cdot) = \int_{S^2\times\mathbb{R}^3} d\omega dv_{j+1} B^\varepsilon(\omega; v_{j+1} - v_k) f_{j+1}^\varepsilon(\mathbf{z}_j, x_k + \omega\varepsilon, v_{j+1}, \cdot)\ .$$

Rigorous derivations of the hard sphere hierarchy, under rather weak assumptions on the initial measure, have been discussed later on, e.g. [24, 15, 22]. The latter references focus mainly on the validity of the series expansion (2.1.15).

Let us formulate the result in a form useful for our analysis. We shall assume that there exist constants $z, \beta > 0$ and a function $h \in L^1(\mathbb{R}^3; \mathbb{R}^+)$ with $\sup_x h(x) = z$, such that the rescaled functions at time zero, $f_j^\varepsilon(\cdot, 0) \equiv f_{0,j}^\varepsilon$, satisfy the bound

$$f_{0,j}^\varepsilon(\mathbf{z}_j) \leq h(x_1) \cdots h(x_j)\ e^{-(\beta/2)\sum_{i=1}^j v_i^2} \leq z^j\ e^{-(\beta/2)\sum_{i=1}^j v_i^2}\ . \tag{2.1.18}$$

Then we have the following

Proposition 1 (BBGKY series expansion) *Let $\mathbf{W}_0^\varepsilon$ be a state of the hard sphere system, with rescaled correlation functions $f_{0,j}^\varepsilon$ satisfying* (2.1.18)*. Then the measure at any time $t > 0$ has rescaled correlation functions $f_j^\varepsilon(t)$ given by Eq.* (2.1.15)*, for almost all points in $\mathcal{M}_j$.*

For a complete proof of the validity result as formulated above, we refer to [22][1].

Proposition 1 is the starting point of our analysis. All the formulas involving the r.c.f. at positive times will be valid only almost everywhere.

2.1.4 Boltzmann hierarchy

Now we want to give a picture of the Boltzmann equation which can be conveniently compared to (2.1.15).

Suppose that f is a solution to the Boltzmann equation (1.4) (with $\lambda = 1$). Consider the products

$$f_j(\mathbf{z}_j, t) = f(t)^{\otimes j}(\mathbf{z}_j) = f(z_1, t) f(z_2, t) \cdots f(z_j, t)\ . \tag{2.1.19}$$

The family of f_j solves then the hierarchy of equations

$$\left(\partial_t + \sum_{i=1}^j v_i \cdot \nabla_{x_i}\right) f_j = \mathcal{C}_{j+1} f_{j+1}\ ,$$

[1]Note that the quoted result of [22] (Corollary 2) is stated for a system of particles in a finite box. Once assumed the explicit assumption on the spatial decay, Eq. (2.1.18), the result can be easily established on the full space along the same lines.

where

$$\mathcal{C}_{j+1} = \sum_{k=1}^{j} \mathcal{C}_{k,j+1} \tag{2.1.20}$$

$$\mathcal{C}_{k,j+1} = \mathcal{C}^+_{k,j+1} - \mathcal{C}^-_{k,j+1}$$

$$\mathcal{C}^+_{k,j+1} f_{j+1}(\mathbf{z}_j,\cdot) = \int_{S^2_+ \times \mathbb{R}^3} d\omega dv_{j+1} (v_k - v_{j+1}) \cdot \omega f_{j+1}(z_1, \cdots, x_k, v'_k, \cdots, z_j, x_k, v'_{j+1}, \cdot)$$

$$\mathcal{C}^-_{k,j+1} f_{j+1}(\mathbf{z}_j,\cdot) = \int_{S^2_+ \times \mathbb{R}^3} d\omega dv_{j+1} (v_k - v_{j+1}) \cdot \omega f_{j+1}(z_1, \cdots, x_k, v_k, \cdots, z_j, x_k, v_{j+1}, \cdot) \ ,$$

with

$$\begin{cases} v'_k = v_k - \omega[\omega \cdot (v_k - v_{j+1})] \\ v'_{j+1} = v_{j+1} + \omega[\omega \cdot (v_k - v_{j+1})] \end{cases} \tag{2.1.21}$$

and

$$S^2_+ = \{\omega \mid (v_k - v_{j+1}) \cdot \omega \geq 0\} \ . \tag{2.1.22}$$

The corresponding series solution reads

$$f_j(t) = \sum_{n \geq 0} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$
$$\cdot \mathcal{S}_j(t - t_1) \mathcal{C}_{j+1} \mathcal{S}_{j+1}(t_1 - t_2) \cdots \mathcal{C}_{j+n} \mathcal{S}_{j+n}(t_n) f_{0,j+n} \ , \tag{2.1.23}$$

where now $\mathcal{S}_j(t)$ is the *free flow operator*, defined as

$$\mathcal{S}_j(t) f_j(\mathbf{z}_j,\cdot) = f_j(x_1 - v_1 t, v_1, \cdots, x_j - v_j t, v_j, \cdot) \ , \tag{2.1.24}$$

and

$$f_{0,j} = f_0^{\otimes j} \tag{2.1.25}$$

are the initial data.

The absolute convergence of this formula has been discussed in [17] and holds (over all $\mathbb{R}^6$) only for a sufficiently small time. We shall give a proof, for completeness, in Section 3.1.2 (Proposition 3). This implies, in particular, local existence and uniqueness of the solution to the time–integrated version of the Boltzmann hierarchy in the class of continuous functions such that $|f_j(t)| \leq c^j e^{-c' \sum_{i=1}^j v_i^2}$ for some $c, c' > 0$. Moreover, in the case of initial product states, factorization is propagated in time, each factor being the local solution to the time–integrated Boltzmann equation (see formula (2.2.34) in the following section).

Note that the similarity of (2.1.23) and (2.1.15) is still not so evident at this stage. In particular, in (2.1.20) the Boltzmann operator is decomposed explicitly into its positive and negative part, while the BBGKY collision operator in (2.1.17) has been written in its original compact form. The representation of the two series expansions given in section 2.2 will make more apparent their relationship.

2.1.5 Enskog hierarchy

We conclude this section by providing an intermediate item between the BBGKY hierarchy and the Boltzmann hierarchy, that is the so called *Enskog hierarchy.* As mentioned in the introduction, this will turn to be useful in the sequel.

Let g^ε be a solution to the *Enskog Equation* (1.13) (with $\lambda = 1$). Proceeding as above, the products

$$g_j^\varepsilon(\mathbf{z}_j, t) = g^\varepsilon(t)^{\otimes j}(\mathbf{z}_j) \tag{2.1.26}$$

satisfy

$$\left(\partial_t + \sum_{i=1}^{j} v_i \cdot \nabla_{x_i}\right) g_j^\varepsilon = \mathcal{C}_{j+1}^{\mathscr{E}} g_{j+1}^\varepsilon \ , \tag{2.1.27}$$

where the definition of $\mathcal{C}_{j+1}^{\mathscr{E}}$ is induced by that of the collision operator on the right hand side of (1.13) (the symbol $\mathscr{E}$ stands for "Enskog", while we drop the dependence on ε).

Deriving the corresponding series solution, and performing a change of variables $\omega \to -\omega$ inside the positive part of $\mathcal{C}_{j+1}^{\mathscr{E}}$ (see the next section for details), one obtains easily

$$\begin{aligned} g_j^\varepsilon(t) = \sum_{n\geq 0} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \\ \cdot \mathcal{S}_j(t-t_1)\mathcal{C}_{j+1}^{\mathscr{E}}\mathcal{S}_{j+1}(t_1-t_2)\cdots\mathcal{C}_{j+n}^{\mathscr{E}}\mathcal{S}_{j+n}(t_n) g_{0,j+n}^\varepsilon \ , \end{aligned} \tag{2.1.28}$$

where the collision operator can be written as

$$\mathcal{C}_{j+1}^{\mathscr{E}} g_{j+1}^\varepsilon(\mathbf{z}_j, \cdot) = \sum_{k=1}^{j} \int_{S^2\times\mathbb{R}^3} d\omega dv_{j+1} (v_{j+1} - v_k)\cdot\omega \, g_{j+1}^\varepsilon(\mathbf{z}_j, x_k + \omega\varepsilon, v_{j+1}, \cdot) \ , \tag{2.1.29}$$

and

$$g_{0,j}^\varepsilon = f_0^{\otimes j} \tag{2.1.30}$$

are the initial data (which in this paper will be assumed, for simplicity, equal to the initial data of the Boltzmann hierarchy).

Notice that the operator $\mathcal{C}_{j+1}^{\mathscr{E}}$ is identical to $\mathcal{C}_{j+1}^{\varepsilon}$ introduced in (2.1.17), except for the fact that the former allows particles to overlap (i.e. particles at distance smaller than ε).

Local existence, uniqueness and propagation of chaos are discussed exactly as for the Boltzmann hierarchy (see the comment after (2.1.25), and formula (2.2.28) in the next section).

2.2 The tree expansion

In the first part of this section we shall follow mainly [19] Sec. 6, adapting discussions and notation therein to the simpler case of hard spheres. Our purpose is to rewrite formulas (2.1.15) and (2.1.23) in a convenient and more explicit way.

We start from (2.1.15), which we write as

$$f_j^\varepsilon(t) = \sum_{n\geq 0} \sum_{\mathbf{k}_n}^{*} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \\ \cdot \mathcal{S}_j^\varepsilon(t-t_1)\mathcal{C}^\varepsilon_{k_1,j+1}\mathcal{S}^\varepsilon_{j+1}(t_1-t_2)\cdots\mathcal{C}^\varepsilon_{k_n,j+n}\mathcal{S}^\varepsilon_{j+n}(t_n)f^\varepsilon_{0,j+n}\ , \tag{2.2.1}$$

where

$$\sum_{\mathbf{k}_n}^{*} = \sum_{k_1=1}^{j}\sum_{k_2=1}^{j+1}\cdots\sum_{k_n=1}^{j+n-1}\ . \tag{2.2.2}$$

We introduce the $n-$***collision,*** $j-$***particle tree***, denoted $\Gamma(j,n)$, as the collection of integers $k_1,\cdots,k_n$ that are present in the sum (2.2.2), i.e.

$$k_1\in I_j, k_2\in I_{j+1},\cdots,k_n\in I_{j+n-1}\ , \qquad \text{with} \qquad I_s=\{1,2,\cdots,s\}, \tag{2.2.3}$$

so that

$$\sum_{\mathbf{k}_n}^{*} = \sum_{\Gamma(j,n)}\ . \tag{2.2.4}$$

The name "tree" is justified by its natural graphical representation, which we explain by means of an example: see Figure 1 corresponding to $\Gamma(2,5)$ given by $1,2,1,3,2$. In the figure, we have also drawn a time arrow in order to associate times to the nodes of the trees: at time t_i the line $j+i$ is "created". Lines 1 and 2 of the example, existing for all times, are called "root lines".

2.2.1 The interacting backwards flow (IBF)

Given a j–particle tree $\Gamma(j,n)$ and fixed a value of all the integration variables in the expansion (2.2.1) (times, unit vectors, velocities), we associate to them a special ($\varepsilon-$dependent) trajectory of particles, which we call *interacting backwards flow* (IBF in the following), since it will be naturally defined by going backwards in time. The rules for the construction of this evolution are explained in what follows.

First, we introduce a notation for the configuration of particles in the IBF, by making use of Greek alphabet, i.e. $\boldsymbol{\zeta}^\varepsilon(s)$, where $s\in[0,t]$ is the time. Note that there is no

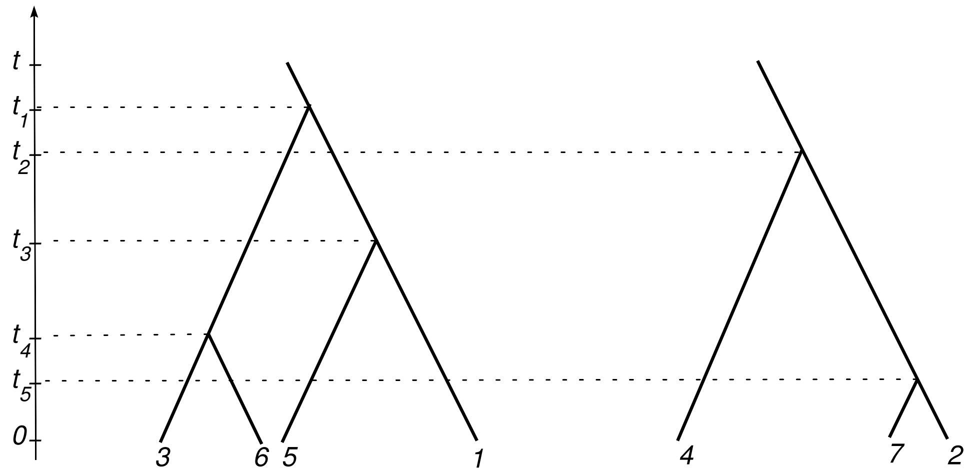

Figure 1: The two–particle tree $\Gamma(2,5) = (1,2,1,3,2)$. The tree associated to 1 is $\Gamma_1 = (1,1,3)$, while $\Gamma_2 = (2,2)$.

label specifying the number of particles. This number depends indeed on the time. If $s \in (t_{r+1}, t_r)$ (with the convention $t_0 = t, t_{n+1} = 0$), there are exactly $j + r$ particles:

$$\boldsymbol{\zeta}^\varepsilon(s) = (\zeta_1^\varepsilon(s), \cdots, \zeta_{j+r}^\varepsilon(s)) \in \mathcal{M}_{j+r} \qquad \text{for } s \in (t_{r+1}, t_r) \;, \tag{2.2.5}$$

with

$$\zeta_i^\varepsilon(s) = (\xi_i^\varepsilon(s), \eta_i^\varepsilon(s)) \;, \tag{2.2.6}$$

the positions and velocities of the particles being respectively

$$\begin{aligned} \boldsymbol{\xi}^\varepsilon(s) &= (\xi_1^\varepsilon(s), \cdots, \xi_{j+r}^\varepsilon(s)) \;, \\ \boldsymbol{\eta}^\varepsilon(s) &= (\eta_1^\varepsilon(s), \ldots, \eta_{j+r}^\varepsilon(s)) \;. \end{aligned} \tag{2.2.7}$$

Our final goal is to write Eq. (2.2.1) in terms of the IBF (to be defined below), i.e.:

$$f_j^\varepsilon(\mathbf{z}_j, t) = \sum_{n\geq 0} \sum_{\Gamma(j,n)} \mathcal{T}^\varepsilon(\mathbf{z}_j, t) \tag{2.2.8}$$

where $\mathcal{T}^\varepsilon(\mathbf{z}_j, t)$ is the *value of the tree* $\Gamma(j,n)$ *with configuration* $\mathbf{z}_j$ *at time* t, *for the interacting flow*,

$$\mathcal{T}^\varepsilon(\mathbf{z}_j, t) = \int d\Lambda(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n}) \prod_{i=1}^n B^\varepsilon(\omega_i; v_{j+i} - \eta_{k_i}^\varepsilon(t_i)) f_{0,j+n}^\varepsilon(\boldsymbol{\zeta}^\varepsilon(0)) \;, \tag{2.2.9}$$

$$\begin{aligned} \mathbf{t}_n &= t_1, \cdots, t_n \;, \\ \boldsymbol{\omega}_n &= \omega_1, \cdots, \omega_n \;, \\ \mathbf{v}_{j,n} &= v_{j+1}, \cdots, v_{j+n} \;, \end{aligned} \tag{2.2.10}$$

$d\Lambda$ is the measure on $\mathbb{R}^n \times S^{2n} \times \mathbb{R}^{3n}$

$$d\Lambda(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n}) = \mathbb{1}_{\{t>t_1>t_2\cdots>t_n>0\}} dt_1 \dots dt_n d\omega_1 \dots d\omega_n dv_{j+1} \dots dv_{j+n} \, , \tag{2.2.11}$$

and we use the shorthand notation

$$B^\varepsilon(\omega_i; v_{j+i} - \eta^\varepsilon_{k_i}(t_i)) = \omega_i \cdot (v_{j+i} - \eta^\varepsilon_{k_i}(t_i)) \mathbb{1}_{\{|\xi^\varepsilon_{j+i}(t_i) - \xi^\varepsilon_k(t_i)| > \varepsilon \ \forall k \neq k_i\}} \, . \tag{2.2.12}$$

In other words, in the generic term $\mathcal{T}^\varepsilon(\mathbf{z}_j, t)$, the initial datum $f^\varepsilon_{0,j+n}$ is integrated, with the suitable weight, over all the possible time–zero states of the IBF associated to $\Gamma(j,n)$.

In formula (2.2.9), the triple (t_i, ω_i, v_{j+i}) may be thought as associated to the node of $\Gamma(j,n)$ where line $j+i$ is created (see Figure 1). In the rest of the paper, we shall abbreviate further

$$\int d\Lambda(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n}) \prod B^\varepsilon = \int d\Lambda(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n}) \prod_{i=1}^n B^\varepsilon(\omega_i; v_{j+i} - \eta^\varepsilon_{k_i}(t_i)) \, , \tag{2.2.13}$$

where the $\eta^\varepsilon_{k_i}(t_i)$ in the factors B^ε have to be computed through the rules specified below, starting from the set of variables $(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n})$, the corresponding j–particle tree (whose nodes are labeled by $(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n})$), together with the associated value of $\mathbf{z}_j, t$.

Let us construct $\boldsymbol{\zeta}^\varepsilon(s)$ for a fixed collection of variables $\Gamma(j,n), \mathbf{z}_j, \mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n}$, with

$$t \equiv t_0 > t_1 > t_2 > \cdots > t_n > t_{n+1} \equiv 0 \, , \tag{2.2.14}$$

and $\boldsymbol{\omega}_n$ satisfying a further constraint that will be specified soon. The root lines of the j–particle tree are associated to the first j particles, with configuration $\zeta^\varepsilon_1, \cdots, \zeta^\varepsilon_j$. Each branch $j+\ell$ $(\ell = 1, \cdots, n)$ represents a new particle with the same label, and state $\zeta^\varepsilon_{j+\ell}$. This new particle appears, going backwards in time, at time t_ℓ in a collision state with a previous particle (branch) $k_\ell \in \{1, \cdots, j+\ell-1\}$, with either incoming or outgoing velocity.

More precisely, in the time interval (t_r, t_{r-1}) particles $1, \cdots, j+r-1$ flow according to the usual dynamics $\mathtt{T}^\varepsilon_{j+r-1}$. This defines $\boldsymbol{\zeta}^\varepsilon_{j+r-1}(s)$ starting from $\boldsymbol{\zeta}^\varepsilon_{j+r-1}(t_{r-1})$. At time t_r the particle $j+r$ is "*created*" by particle k_r in the position

$$\xi^\varepsilon_{j+r}(t_r) = \xi^\varepsilon_{k_r}(t_r) + \omega_r \varepsilon \tag{2.2.15}$$

and with velocity v_{j+r}. This defines $\boldsymbol{\zeta}^\varepsilon(t_r) = (\zeta^\varepsilon_1(t_r), \cdots, \zeta^\varepsilon_{j+r}(t_r))$.

The characteristic function in the collision operator (2.1.17)–(2.1.14) (or the characteristic function in (2.2.12)), is a constraint on ω_r ensuring that two hard spheres cannot be at distance smaller than ε.

Next, the evolution in (t_{r+1}, t_r) is contructed applying to this configuration the dynamics $\mathtt{T}^{\varepsilon}_{j+r}$ (with negative times). We have two cases. If $\omega_r \cdot (v_{j+r} - \eta^{\varepsilon}_{k_r}(t_r)) \leq 0$, then the velocities are incoming and no scattering occurs, namely for times $s < t_r$ the pair of particles moves backwards freely with velocities $\eta^{\varepsilon}_{k_r}(t_r)$ and v_{j+r}. If instead $\omega_r \cdot (v_{j+r} - \eta^{\varepsilon}_{k_r}(t_r)) \geq 0$, the pair is post–collisional. Then the presence of the interaction in the flow $\mathtt{T}^{\varepsilon}_{j+r}$ forces the pair to perform a (backwards) instantaneous collision. The two situations are depicted in Fig. 2.

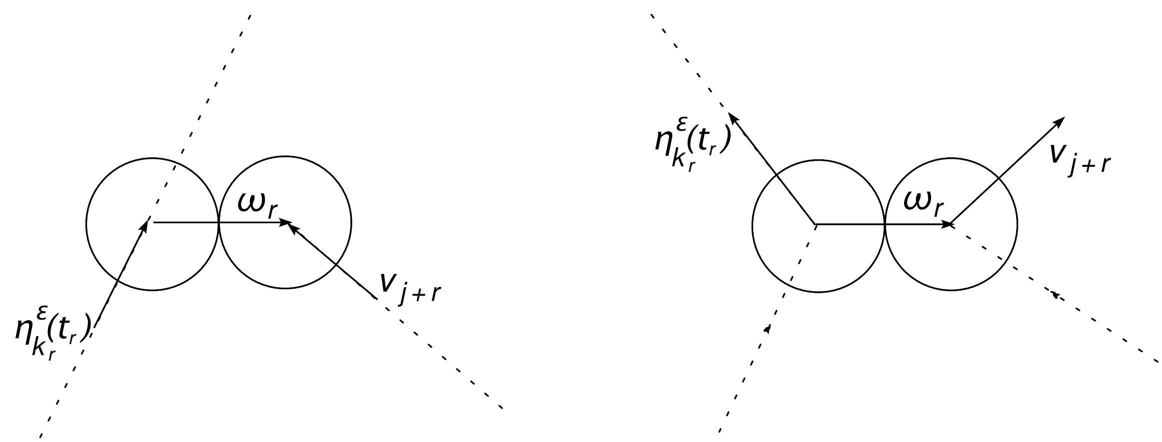

Figure 2: At time t_r, particle $j+r$ is *created* by particle k_r, either in incoming $(\omega_r \cdot (v_{j+r} - \eta^{\varepsilon}_{k_r}(t_r)) \leq 0)$ or in outgoing $(\omega_r \cdot (v_{j+r} - \eta^{\varepsilon}_{k_r}(t_r)) \geq 0)$ collision configuration. Particle k_r is called the *progenitor* of particle $j+r$.

Proceeding inductively, the IBF is constructed for all times $s \in [0, t]$.

2.2.2 Recollisions and factorization

Observe that between two creation times t_r, t_{r+1} any pair of particles among the existing $j+r$ can possibly interact. These interactions are called *recollisions*, because they may involve particles that have already interacted at some creation time (in the future) with another particle of the IBF. In our language, recollisions are the "interactions different from creations".

Let us focus now in more detail on the structure of the backwards flow and on the mechanisms of correlation.

We observe preliminarily that the graphical representation of a $n-$collision, $j-$particle tree $\Gamma(j,n) = (k_1, \cdots, k_n)$ consists of j connected components. Each of these components is associated to a root line $i \in \{1, 2, \cdots, j\}$ and collects n_i nodes $i_1, i_2, \cdots, i_{n_i}$. In particular, we have the following map:

$$\begin{aligned} &\Gamma(j,n) \longrightarrow \mathbf{\Gamma}_j = \Gamma_1, \cdots, \Gamma_j\ , \\ &\Gamma_i = (k^i_1, \cdots, k^i_{n_i})\ , \qquad k^i_h = k_{i_h}\ . \end{aligned} \tag{2.2.16}$$

In the sequel we will call simply **tree** *(generated by i) the collection of integers* Γ_i*.* In the example of Figure 1 one has $\Gamma_1 = (1,1,3), \Gamma_2 = (2,2)$.

Note that the map (2.2.16) is not invertible, since the collection $\mathbf{\Gamma}_j$ does *not* specify the ordering of particles belonging to different trees. A one–to–one correspondence is instead the following:

$$n,\ \Gamma(j,n),\ \mathbf{t}_n \longleftrightarrow n_1, \Gamma_1, \mathbf{t}^1_{n_1}, \cdots, n_j, \Gamma_j, \mathbf{t}^j_{n_j}\ , \tag{2.2.17}$$

where

$$\mathbf{t}^i_{n_i} = t^i_1, \cdots, t^i_{n_i}\ , \qquad t^i_h = t_{i_h}\ .$$

Clearly $n = \sum_i n_i$.

For a given sequence of trees $\mathbf{\Gamma}_j$, there are several $j-$particle trees $\Gamma(j,n)$ having $\mathbf{\Gamma}_j$ as image of the map (2.2.16). However summing the time–ordered product over such trees $\Gamma(j,n)$ is equivalent to a free time integration leaving only the partial ordering dictated by the sequence $\mathbf{\Gamma}_j$. Namely there holds:

$$\sum_{\Gamma(j,n)} \int \mathbb{1}_{\{t>t_1>t_2\cdots>t_n>0\}} dt_1 \dots dt_n\, F = \prod_{i=1}^{j} \sum_{\Gamma_i} \int \mathbb{1}_{\{t>t^i_1>t^i_2\cdots>t^i_{n_i}>0\}} dt^i_1 \dots dt^i_{n_i}\, F$$

where $F = F(\mathbf{\Gamma}_j, \mathbf{t}_n)$.

Applying this property to the expansion (2.2.8), we obtain the following factorization result:

$$\begin{aligned} f^\varepsilon_j(\mathbf{z}_j,t) &= \sum_{n\geq 0} \sum_{\Gamma(j,n)} \int d\Lambda(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n}) \prod B^\varepsilon f^\varepsilon_{0,j+n}(\boldsymbol{\zeta}^\varepsilon(0)) \\ &= \prod_{i=1}^{j} \left(\sum_{n_i,\Gamma_i} \int d\Lambda(\mathbf{t}^i_{n_i}, \boldsymbol{\omega}^i_{n_i}, \mathbf{v}^i_{1,n_i}) \right) \prod B^\varepsilon f^\varepsilon_{0,j+n}(\boldsymbol{\zeta}^\varepsilon(0))\ . \end{aligned} \tag{2.2.18}$$

In (2.2.18), the triples in $(\mathbf{t}^i_{n_i}, \boldsymbol{\omega}^i_{n_i}, \mathbf{v}^i_{1,n_i})$ are associated to the nodes of the tree Γ_i, while the IBF (hence the integrand $\prod B^\varepsilon f^\varepsilon_{0,j+n}$) has to be computed with the rules specified in the previous section.

With the notations introduced above (see in particular Figure 1), it should be clear that each particle of the IBF "belongs" to exactly one tree Γ_i. Therefore we may distinguish two types of recollisions. The *internal recollisions*, occurring among particles of the same tree and the *external recollisions*, occurring between particles belonging to different trees. Because of the external recollisions, we say that different trees are *correlated*, in the sense that their interacting backwards flows are not pairwise independent.

Remark. Formula (2.2.18) shows a *partial factorization*: a full factorization is prevented by the correlations of the initial datum $f^{\varepsilon}_{0,j+n}$ and, more importantly, of the external recollisions in the IBF. If we simply ignore external recollisions and replace $f^{\varepsilon}_{0,j+n}$ with a tensor product, then (2.2.18) becomes a completely factorized expression. Note, however, that this is consequence of our choice of having the total number of particles non fixed. In the canonical setting, where the total number of particles N is fixed, one would have the additional correlation given by the constraint $\sum n_i \leq N - j$. This is the main advantage of using a grand canonical formalism.

From now on, in handling formula (2.2.18) and similar ones established in the sequel, we will use intensively the notations

$$\Gamma_i = \text{tree generated by particle } i \in \{1, \cdots, j\} \;, \tag{2.2.19}$$

which is a (n_i–collision) tree with associated configuration z_i at time t,

$$(\mathbf{t}^i_{n_i}, \boldsymbol{\omega}^i_{n_i}, \mathbf{v}^i_{1,n_i}) = \text{collection of triples associated to the nodes of } \Gamma_i \;, \tag{2.2.20}$$

and

$$S(i) = \text{set of particles associated to } \Gamma_i \;. \tag{2.2.21}$$

Moreover,

$$S(K) = \cup_i S(i) \;, \tag{2.2.22}$$

where K is any subset of $\{1, \cdots, j\}$.

Using the symmetry of the state, we could change notation in the integrals (2.2.18), by substituting $\boldsymbol{\zeta}^{\varepsilon}(0)$ with $\Big(\boldsymbol{\zeta}^{\varepsilon}_{S(1)}(0), \cdots, \boldsymbol{\zeta}^{\varepsilon}_{S(j)}(0)\Big)$, where $\boldsymbol{\zeta}^{\varepsilon}_{S(i)} = \{\zeta^{\varepsilon}_k \;;\; k \in S(i)\}$. As already pointed out, configurations $\boldsymbol{\zeta}^{\varepsilon}_{S(i)}$ with different values of i are correlated through the external recollisions.

2.2.3 The *uncorrelated* IBF

Let us introduce a different notion of backwards flow, in which such correlations are ignored. Suppose that we want the tree Γ_i to be "uncorrelated". Then, for all $k \in S(i)$, we substitute the IBF $\zeta^{\varepsilon}_k(s)$ with the evolution

$$\tilde{\zeta}^{\varepsilon}_k(s) \;, \tag{2.2.23}$$

to be constructed as $\zeta_k^\varepsilon(s)$ with the additional prescription that *its external recollisions are ignored*. The constraint excluding overlaps of created particles in Γ_i with particles of different trees at the moment of creation, has to be also ignored. Notice that (2.2.23) is a function of the only $z_i, \Gamma_i, \mathbf{t}^i_{n_i}, \boldsymbol{\omega}^i_{n_i}, \mathbf{v}^i_{1,n_i}$.

If we require that *all* the trees in the expansions (2.2.18) are uncorrelated, i.e. we replace $\boldsymbol{\zeta}^\varepsilon(s) \to \tilde{\boldsymbol{\zeta}}^\varepsilon(s)$ inside the formula, we have that:
– factors B^ε associated to different trees become completely independent;
– the initial data are evaluated in the time–zero state

$$\tilde{\boldsymbol{\zeta}}^\varepsilon(0) = \left(\tilde{\boldsymbol{\zeta}}^\varepsilon_{S(1)}(0), \cdots, \tilde{\boldsymbol{\zeta}}^\varepsilon_{S(j)}(0)\right) \in \mathcal{M}_{1+n_1} \times \cdots \times \mathcal{M}_{1+n_j}$$

(with $\tilde{\boldsymbol{\zeta}}^\varepsilon_{S(i)} = \{\tilde{\zeta}^\varepsilon_k \ ; \ k \in S(i)\}$), that is a collection of j independent objects. Therefore, we end up with the following formula

$$\prod_{i=1}^{j} \left(\sum_{n_i, \Gamma_i} \int d\Lambda(\mathbf{t}^i_{n_i}, \boldsymbol{\omega}^i_{n_i}, \mathbf{v}^i_{1,n_i}) \prod B^\varepsilon \right) f^\varepsilon_{0,j+n}(\tilde{\boldsymbol{\zeta}}^\varepsilon(0)) \tag{2.2.24}$$

(where $\prod B^\varepsilon$ is now inside the big brackets), which differs from the tensorized product $f_1^\varepsilon(t)^{\otimes j}(\mathbf{z}_j)$ *only* because of the correlations assumed for the initial r.c.f. $f^\varepsilon_{0,j+n}$.

2.2.4 The Enskog backwards flow (EBF)

In the last formula there is still a nontrivial correlation among particles of the *same* tree. This is due to the internal recollisions in $\tilde{\boldsymbol{\zeta}}^\varepsilon$, among particles of each set $S(i)$. To get rid of them, one has to introduce the completely uncorrelated backwards flow

$$\zeta_k^{\mathscr{E}}(s) \tag{2.2.25}$$

(where $\mathscr{E}$ stands for "Enskog") for all $k \in S(i)$, to be constructed as $\tilde{\zeta}^\varepsilon_k(s)$ with the additional prescription that *its internal recollisions are ignored*, together with the constraint excluding overlaps of created particles at the moment of creation.

The evolution $\boldsymbol{\zeta}^{\mathscr{E}}$ will be called *Enskog backwards flow* (EBF). In this flow particles are created at distance ε (from their progenitor), but they may reach a distance smaller than ε during the evolution (in particular, its time–zero state $\boldsymbol{\zeta}^{\mathscr{E}}(0)$ varies in $\mathbb{R}^{6(j+\sum_i n_i)}$).

Alternatively, we may say that the EBF is constructed exactly as the IBF, except for the following differences:
- the interacting dynamics $\mathtt{T}^\varepsilon$ is replaced by the simple free dynamics;
- there is no constraint on ω_r.

The name "Enskog" is due to the obvious connection with the Enskog equation. Indeed, Eq. (2.1.28)–(2.1.29) can be written explicitly

$$g_j^\varepsilon(\mathbf{z}_j,t)=\sum_{n=0}^{\infty}\sum_{\Gamma(j,n)}\int d\Lambda(\mathbf{t}_n,\boldsymbol{\omega}_n,\mathbf{v}_{j,n})\prod B^{\mathscr{E}}g^\varepsilon_{0,j+n}(\boldsymbol{\zeta}^{\mathscr{E}}(0))$$
$$=\prod_{i=1}^{j}\left(\sum_{n_i,\Gamma_i}\int d\Lambda(\mathbf{t}^i_{n_i},\boldsymbol{\omega}^i_{n_i},\mathbf{v}^i_{1,n_i})\prod B^{\mathscr{E}}\right)g^\varepsilon_{0,j+n}(\boldsymbol{\zeta}^{\mathscr{E}}(0))\ , \qquad (2.2.26)$$

where $\prod B^{\mathscr{E}}=\prod_{i=1}^n B(\omega_i;v_{j+i}-\eta^{\mathscr{E}}_{k_i}(t_i))$,

$$B(\omega_i;v_{j+i}-\eta^{\mathscr{E}}_{k_i}(t_i))=\omega_i\cdot(v_{j+i}-\eta^{\mathscr{E}}_{k_i}(t_i))\ . \qquad (2.2.27)$$

Note that the EBF allows a complete factorization, whenever the initial datum does. Namely if $g^\varepsilon_{0,j}=(f_0)^{\otimes j}$ for all j, the expansion above gives immediately

$$g_j^\varepsilon(\mathbf{z}_j,t)=\prod_{i=1}^{j}\left(\sum_{n_i,\Gamma_i}\int d\Lambda(\mathbf{t}^i_{n_i},\boldsymbol{\omega}^i_{n_i},\mathbf{v}^i_{1,n_i})\prod B^{\mathscr{E}}g^\varepsilon_{0,1+n_i}(\boldsymbol{\zeta}^{\mathscr{E}}_{S(i)}(0))\right)\ , \qquad (2.2.28)$$

where $\boldsymbol{\zeta}^{\mathscr{E}}_{S(i)}=\{\zeta^{\mathscr{E}}_k\ ;\ k\in S(i)\}$.

2.2.5 The Boltzmann backwards flow (BBF)

Finally, the above discussions can be repeated, with minor changes, for the case of the Boltzmann series (2.1.23). The interacting backwards flow is now substituted by the *Boltzmann backwards flow* (BBF) $\boldsymbol{\zeta}(s)$. For it, we use the same notations of (2.2.5)–(2.2.7) with the superscript ε omitted.

Since the collision operator (2.1.20) is splitted into a gain and a loss term, together with the sum over $\Gamma(j,n)$, we have an additional $\sum_{\boldsymbol{\sigma}_n}$ with $\boldsymbol{\sigma}_n=(\sigma_1,\cdots,\sigma_n)$, $\sigma_i=\pm$. To have a compact expression, we change variables $\omega\to-\omega$ inside the positive part of the collision operators. As a result, in each term of the expansion, σ_i fixes the sign of the product $\omega_i\cdot(v_{j+i}-\eta_{k_i}(t_i))$ (where the relative velocity at the moment of creation appears). Note that the same procedure has to be followed when deriving (2.1.28)–(2.1.29) from (2.1.26).

The BBF turns out to be defined exactly as the IBF, except for the following differences:
- the interacting dynamics $\mathtt{T}^\varepsilon$ is replaced by the simple free dynamics;
- in the right hand side of (2.2.15) the second term is missing, i.e. the created particle appears at the same position of its progenitor: $\xi_{j+r}(t_r)=\xi_{k_r}(t_r)$;
- there is no constraint on ω_r other than the one implied by the value of σ_r;

- if $\sigma_r = +$, to determine the state of particles in (t_{r+1}, t_r), *before* applying free evolution we have to change velocities according to $(\eta_{k_r}(t_r^+), v_{j+r}) \to (\eta_{k_r}(t_r^-), \eta_{j+r}(t_r^-))$, where $\to$ denotes the elastic scattering rule. We recall that, in our conventions, $\eta_{k_r}(t_r) \equiv \eta_{k_r}(t_r^+)$ (which indicates the limit from the future, while $\eta_{k_r}(t_r^-)$ indicates the limit from the past).

Eq. (2.1.23) can then be rewritten:

$$f_j(\mathbf{z}_j, t) = \sum_{n=0}^{\infty} \sum_{\Gamma(j,n)} \mathcal{T}(\mathbf{z}_j, t) \ , \tag{2.2.29}$$

where $\mathcal{T}(\mathbf{z}_j, t)$ is the *value of the tree* $\Gamma(j,n)$ *with configuration* $\mathbf{z}_j$ *at time* t, *for the Boltzmann flow*,

$$\mathcal{T}(\mathbf{z}_j, t) = \int d\Lambda(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n}) \prod B \, f_{0,j+n}(\boldsymbol{\zeta}(0)) \ , \tag{2.2.30}$$

with $\prod B = \prod_{i=1}^n B(\omega_i; v_{j+i} - \eta_{k_i}(t_i))$ and

$$\begin{aligned} B(\omega_i; v_{j+i} - \eta_{k_i}(t_i)) &= \sum_{\sigma_i} \sigma_i |\omega_i \cdot (v_{j+i} - \eta_{k_i}(t_i))| \mathbb{1}_{\{\sigma_i \omega_i \cdot (v_{j+i} - \eta_{k_i}(t_i)) \geq 0\}} \\ &= \omega_i \cdot (v_{j+i} - \eta_{k_i}(t_i)) \ . \end{aligned} \tag{2.2.31}$$

Note that, in the final formula, the difference between gain and loss collision operators is hidden inside the *rule* for the construction of the BBF, which depends, as explained above, on the sign of each product $\omega_i \cdot (v_{j+i} - \eta_{k_i}(t_i))$. (A similar consideration can be made for the case of Enskog.)

Note also that

$$\boldsymbol{\eta} = \boldsymbol{\eta}^{\mathscr{E}} \tag{2.2.32}$$

and

$$B = B^{\mathscr{E}} \ , \tag{2.2.33}$$

the only difference in the BBF and the EBF being due to the position in space of created particles.

As before, (2.2.29) can be immediately written in the form

$$f_j(\mathbf{z}_j, t) = \prod_{i=1}^{j} \left(\sum_{n_i, \Gamma_i} \int d\Lambda(\mathbf{t}_{n_i}^i, \boldsymbol{\omega}_{n_i}^i, \mathbf{v}_{1,n_i}^i) \prod B \right) f_{0,j+n}(\boldsymbol{\zeta}(0)) \ , \tag{2.2.34}$$

which shows a complete factorization in the case of factorized initial data.

2.2.6 Summary

We conclude the section with a summary. We have introduced:

(i) the interacting backwards flow, $\boldsymbol{\zeta}^{\varepsilon}(s)$, expressing the evolution of the rescaled correlation functions of the hard sphere system;

(ii) the partially uncorrelated flow $\tilde{\boldsymbol{\zeta}}^{\varepsilon}(s)$, obtained from the IBF by ignoring the external recollisions;

(iii) the Enskog backwards flow $\boldsymbol{\zeta}^{\mathscr{E}}(s)$, obtained from the IBF by ignoring all the recollisions;

(iv) the Boltzmann backwards flow, $\boldsymbol{\zeta}(s)$, describing the evolution of functions obeying the Boltzmann hierarchy.

The flows in (iii) and (iv) will be used to prove convergence of the hard sphere system to the Enskog and the Boltzmann equation, while (ii) will be enough for the proof of propagation of chaos.

2.3 Initial data

In this section we describe the class of initial states considered in the present paper.

2.3.1 Additional notations

To begin with, we introduce some notations that will be repeatedly used in the rest of the paper.

We agree to use capital latin letters

$$J, K, L, \cdots$$

for subsets of indices of $\{1, 2, 3, \cdots\}$, and corresponding small letters for the cardinality of the same sets

$$j = |J|,\ k = |K|,\ l = |L|, \cdots .$$

Unless specified, the letter J will indicate the set of the first j positive integers:

$$J = (1, 2, \cdots, j).$$

For a generic vector of elements with indices given by $A = (a_1, a_2, \cdots, a_{|A|})$ we will use the notation

$$\mathbf{V}_A = (V_{a_1}, V_{a_2}, \cdots, V_{a_{|A|}}).$$

If $g_{|A|}$ is a symmetric function over $\mathbb{R}^{6|A|}$, then we set

$$g_A(\mathbf{z}_A) = g_{|A|}(z_{a_1}, \cdots, z_{a_{|A|}}).$$

Note that the symbol g_A (with the argument dropped) specifies the indices of the variables in which the function is evaluated.

If g is a function over $\mathbb{R}^6$, we abbreviate

$$g^{\otimes A}(\mathbf{z}_A) = g(z_{a_1})g(z_{a_2}) \cdots g(z_{a_{|A|}}).$$

Finally, we use the conventions

$$g_\emptyset = 1, \qquad g^{\otimes\emptyset} = 1 \ . \tag{2.3.1}$$

2.3.2 Assumptions

The state of the particle system at time zero, $\mathbf{W}_0^\varepsilon$, admits, as r.c.f., the collection $f_{0,j}^\varepsilon : \mathcal{M}_j \to \mathbb{R}^+$, $j \geq 0$, satisfying the bound (2.1.18). By definition, the r.c.f. are Borel functions symmetric in the permutation of particles.

In addition, we assume:

Hypothesis 1 *There exist two positive constants $\alpha_0 < 1, \gamma_0$ such that the initial r.c.f. admit the following representation:*

$$f_{0,J}^\varepsilon = \sum_{H \subset J} (f_{0,1}^\varepsilon)^{\otimes H} E_{J \setminus H}^0 \tag{2.3.2}$$

with $E_\emptyset^0 = 1$, $E_k^0 : \mathcal{M}_k \to \mathbb{R}$ and, for ε small enough,

$$|E_K^0| \leq \varepsilon^{\gamma_0 k} z^k \ e^{-(\beta/2)\sum_{i\in K} v_i^2} \qquad \forall \ k < \varepsilon^{-\alpha_0} \ . \tag{2.3.3}$$

The bound (2.3.3) holds uniformly in $\mathcal{M}_k(\varepsilon)$. Observe that (2.3.2) can be inverted and

$$E_K^0 = \sum_{H \subset K} (-f_{0,1}^\varepsilon)^{\otimes H} f_{0,K\setminus H}^\varepsilon \ . \tag{2.3.4}$$

Observe also that $E_1^0 = 0$.

Let us turn to the initial data for the Boltzmann and Enskog hierarchies. This is given by Eq.s (2.1.25) and (2.1.30), where $f_0 = f_0(x, v)$ is a probability density over $\mathbb{R}^6$ ($\int_{\mathbb{R}^6} f_0 = 1$).

Hypothesis 2 *There exists a positive constant γ_0 such that, for ε small enough,*

$$| \left(f_{0,1}^{\varepsilon} - f_0\right)(x,v)| \leq \varepsilon^{\gamma_0} z e^{-(\beta/2)v^2} \ . \tag{2.3.5}$$

The constant γ_0 has been chosen equal to the one in Hypothesis 1 for notational simplicity.

Putting together the hypotheses 1 and 2, it follows that the r.c.f. admit as well the following representation:

$$f_{0,J}^{\varepsilon} = \sum_{H\subset J} f_0^{\otimes H} E_{J\setminus H}^{\mathcal{B},0} \ , \tag{2.3.6}$$

with $E_k^{\mathcal{B},0} : \mathcal{M}_k \to \mathbb{R}$ satisfying

$$|E_K^{\mathcal{B},0}| \leq \varepsilon^{\gamma_0' k} z^k \ e^{-(\beta/2)\sum_{i\in K} v_i^2} \qquad \forall \ k < \varepsilon^{-\alpha_0} \ , \tag{2.3.7}$$

for some $\gamma_0' > 0$ and ε small enough.

Actually the two representations given, Hypotheses 1 and (2.3.6)–(2.3.7), are equivalent as soon as we assume Hypothesis 2. Indeed, starting from (2.3.6), setting $f_0^{\otimes H} = (f_{0,1}^{\varepsilon} - E_1^{\mathcal{B},0})^{\otimes H}$ and expanding, one finds formula (2.3.2) with

$$E_K^0 = \sum_{Q\subset K} (-1)^q (E_1^{\mathcal{B},0})^{\otimes Q} E_{K\setminus Q}^{\mathcal{B},0} \ , \tag{2.3.8}$$

hence (2.3.7) implies $|E_K^0| \leq 2^k \varepsilon^{\gamma_0' k} z^k \ e^{-(\beta/2)\sum_{i\in K} v_i^2} < \varepsilon^{\gamma_0 k} z^k \ e^{-(\beta/2)\sum_{i\in K} v_i^2}$ for any $\gamma_0 < \gamma_0'$ (and ε small). The proof of the inverse statement is similar (one finds $\gamma_0' < \gamma_0$).

Remark. With respect to the usual hypotheses of the Lanford's theorem, we are asking the additional explicit information (2.3.2)–(2.3.3). In section 2.3.4 below we will show some natural concrete example of state admitting this representation.

2.3.3 An equivalent representation

In the proof of our main result (see Theorem 1 below), we apply the tree expansion technique described in Section 2.2, to the initial system of r.c.f. satisfying the representation (2.3.2)–(2.3.3). To combine these two structures we shall use a different, more convenient, "tree–dependent" representation of the initial data, given in terms of clusters of particles $\{S_1, S_2, \cdots\}$, where S_k is the group of particles associated to the one–particle tree generated by k (i.e. $S_k = S(k)$ in the notation (2.2.21)). Namely we introduce an expansion on products of higher order, not only 1–point, rescaled correlation functions.

Consider a partition of the set J into nontrivial clusters,

$$\mathcal{J} = (S_1, \cdots, S_{|\mathcal{J}|}) \ . \tag{2.3.9}$$

That is, $J = \cup_{i=1}^{|\mathcal{J}|} S_i$, $S_i \cap S_{i'} = \emptyset$, $|S_i| > 0$. We will use a calligraphic capital letter, e.g. $\mathcal{A}$, for a partition in *clusters* of the set of indices A:

$$A = \bigcup_{S \in \mathcal{A}} S \, .$$

Lemma 1 *The following assertion is equivalent to Hypothesis 1. There exist two positive constants $\alpha_0 < 1, \gamma_0''$ such that the initial r.c.f. admit the following collection of representations:*

$$f_{0,J}^{\varepsilon} = \sum_{\mathcal{H} \subset \mathcal{J}} \left(\prod_{S \in \mathcal{H}} f_{0,S}^{\varepsilon} \right) \bar{E}_{\mathcal{J} \setminus \mathcal{H}}^{0} \, , \tag{2.3.10}$$

where $\mathcal{J}$ is any given partition of the set J, $\bar{E}_{\emptyset}^{0} = 1$, $\bar{E}_{\mathcal{K}}^{0} : \mathcal{M}_{|K|} \to \mathbb{R}$ and, for ε small enough,

$$|\bar{E}_{\mathcal{K}}^{0}| \leq \varepsilon^{\gamma_0''|\mathcal{K}|} z^k \; e^{-(\beta/2)\sum_{i \in K} v_i^2} \qquad \forall \; k < \varepsilon^{-\alpha_0} \, , \tag{2.3.11}$$

with $|\mathcal{K}|$ = total number of clusters in $\mathcal{K}$, and $k = |K|$ = total number of indices in $\mathcal{K}$.

Proof. Equation (2.3.10) is a generalization of (2.3.2)–(2.3.3), which is obtained by taking $\mathcal{J} = J$.

Let us show that (2.3.2)–(2.3.3) implies (2.3.11). Inverting (2.3.10) we find

$$\bar{E}_{\mathcal{K}}^{0} = \sum_{\mathcal{Q} \subset \mathcal{K}} (-1)^{|\mathcal{Q}|} \left(\prod_{S \in \mathcal{Q}} f_{0,S}^{\varepsilon} \right) f_{0,K \setminus Q}^{\varepsilon} \, . \tag{2.3.12}$$

By using (2.3.2), it follows that

$$\bar{E}_{\mathcal{K}}^{0} = \sum_{\mathcal{Q} \subset \mathcal{K}} (-1)^{|\mathcal{Q}|} \sum_{\substack{L_1, \cdots, L_{|\mathcal{Q}|} \\ L_r \subset S_{i_r}}} \prod_{r=1}^{|\mathcal{Q}|} E_{L_r}^{0} \sum_{L_0 \subset K \setminus Q} E_{L_0}^{0} \left(f_{0,1}^{\varepsilon} \right)^{\otimes L^c} \, , \tag{2.3.13}$$

where $i_1, \cdots, i_{|\mathcal{Q}|}$ are the indices of the clusters in $\mathcal{Q}$, and $L^c = K \setminus \cup_{r=0}^{|\mathcal{Q}|} L_r$. Note that the first sum is over subsets of clusters while the other sums run over subsets of indices. Setting $L = \cup_{r=0}^{|\mathcal{Q}|} L_r$ we obtain

$$\bar{E}_{\mathcal{K}}^{0} = \sum_{L \subset K} \left(f_{0,1}^{\varepsilon} \right)^{\otimes K \setminus L} \sum_{\mathcal{Q} \subset \mathcal{K}} (-1)^{|\mathcal{Q}|} E_{L \cap (K \setminus Q)}^{0} \prod_{r=1}^{|\mathcal{Q}|} E_{L \cap S_{i_r}}^{0} \, . \tag{2.3.14}$$

Now observe that in the above sum it must be $|L \cap S_i| > 0$ for all $S_i \in \mathcal{K}$. Otherwise if $L \cap S_i = \emptyset$ for some i, setting $S^* = S_i$, since $E^0_{L\cap S_i} = 1$

$$\begin{aligned}
&\sum_{\substack{\mathcal{Q}\subset\mathcal{K}\\ S^*\in\mathcal{Q}}} (-1)^{|\mathcal{Q}|} E^0_{L\cap(K\setminus Q)} \prod_{r=1}^{|\mathcal{Q}|} E^0_{L\cap S_{i_r}} + \sum_{\substack{\mathcal{Q}\subset\mathcal{K}\\ S^*\notin\mathcal{Q}}} (-1)^{|\mathcal{Q}|} E^0_{L\cap(K\setminus Q)} \prod_{r=1}^{|\mathcal{Q}|} E^0_{L\cap S_{i_r}} \\
&= -\sum_{\mathcal{Q}\subset\mathcal{K}\setminus\{S^*\}} (-1)^{|\mathcal{Q}|} E^0_{L\cap(K\setminus Q)} \prod_{r=1}^{|\mathcal{Q}|} E^0_{L\cap S_{i_r}} + \sum_{\mathcal{Q}\subset\mathcal{K}\setminus\{S^*\}} (-1)^{|\mathcal{Q}|} E^0_{L\cap(K\setminus Q)} \prod_{r=1}^{|\mathcal{Q}|} E^0_{L\cap S_{i_r}} \\
&= 0\ . \qquad (2.3.15)
\end{aligned}$$

As a consequence, (2.3.11) follows easily from (2.1.18) and (2.3.3), by choosing $\gamma_0'' < \gamma_0$. ■

2.3.4 Explicit examples

Let us verify the assumptions introduced in the previous section, for a grand canonical state $\mathbf{W}_0^\varepsilon$ with a system of densities

$$\frac{1}{n!} W^\varepsilon_{0,n}(\mathbf{z}_n) = \frac{1}{\mathcal{Z}_\varepsilon} \frac{e^{-\mu_\varepsilon}\mu_\varepsilon^n}{n!} f_0^{\otimes n}(\mathbf{z}_n)\ , \qquad (2.3.16)$$

where $\varepsilon^2\mu_\varepsilon = 1$,

$$\mathcal{Z}_\varepsilon = \sum_{n\geq 0} \frac{e^{-\mu_\varepsilon}\mu_\varepsilon^n}{n!} \mathcal{Z}_n^{can}\ , \qquad (2.3.17)$$

and where the "canonical" normalization constant is

$$\mathcal{Z}_n^{can} = \int_{\mathcal{M}_n} d\mathbf{z}_n f_0^{\otimes n}(\mathbf{z}_n) = \int_{\mathbb{R}^{6n}} d\mathbf{z}_n f_0^{\otimes n}(\mathbf{z}_n) \prod_{1\leq i<k\leq n} \bar{\chi}^0_{i,k}\ , \qquad (2.3.18)$$

($\mathcal{Z}_0^{can} = 1$) with

$$\begin{aligned}
&\bar{\chi}^0_{i,k} = 1 - \chi^0_{i,k} \\
&\chi^0_{i,k} = \chi(x_i - x_k) \\
&\chi(x) = \begin{cases} 1 & |x| \leq \varepsilon \\ 0 & |x| > \varepsilon \end{cases}\ .
\end{aligned} \qquad (2.3.19)$$

Note that we drop the dependence on ε of characteristic functions for notational simplicity. The function f_0 can be any probability density over $\mathbb{R}^6$ satisfying $f_0(x,v) \leq (h(x)/2)e^{-(\beta/2)v^2}$, for some $h \in L^1(\mathbb{R}^3;\mathbb{R}^+)$ with $\sup_x h(x) = z$, and $z, \beta > 0$.

The asymptotic behaviour of the normalization constants can be easily proved to be $\mathcal{Z}_n^{can} \sim e^{-Cn^2\varepsilon^3}$ ($n >> \varepsilon^{-2}$, $C > 0$) and $\mathcal{Z}_\varepsilon \sim e^{-C\varepsilon^{-1}}$ (see e.g. [19]) .

Proposition 2 *The state of the system defined by* (2.3.16) *admits r.c.f. satisfying Hypotheses 1 and 2, and the estimate* (2.1.18)*.*

Proof. It suffices to check (2.3.6)–(2.3.7).

By (2.1.6) and (2.1.11), the rescaled correlation functions are

$$f^{\varepsilon}_{0,j}(\mathbf{z}_j) = \frac{F^{\varepsilon}(\mathbf{z}_j)}{\mathcal{Z}_{\varepsilon}} f_0^{\otimes j}(\mathbf{z}_j) \;, \tag{2.3.20}$$

where

$$F^{\varepsilon}(\mathbf{z}_j) = \sum_{n \geq 0} \frac{e^{-\mu_\varepsilon}\mu_\varepsilon^n}{n!} F^{j+n}_{can}(\mathbf{z}_j) \tag{2.3.21}$$

and

$$\begin{aligned} F^{j+n}_{can}(\mathbf{z}_j) &= \int_{\mathcal{M}_n \cap S(\mathbf{x}_j)^n} d\mathbf{z}_{j,n} f_0^{\otimes n}(\mathbf{z}_{j,n}) \\ &= \int_{\mathbb{R}^{6n}} d\mathbf{z}_{j,n} f_0^{\otimes n}(\mathbf{z}_{j,n}) \left(\prod_{i=1}^{j} \prod_{k=j+1}^{j+n} \bar{\chi}^0_{i,k} \right) \left(\prod_{j+1 \leq i < k \leq j+n} \bar{\chi}^0_{i,k} \right) \end{aligned} \tag{2.3.22}$$

($F^j_{can}(\mathbf{z}_j) = \mathcal{Z}^{can}_n$).

For any $j, n \geq 1$, we rewrite $F^{j+n}_{can}(\mathbf{z}_j)$ by using

$$\prod_{i=1}^{j} \prod_{k=j+1}^{j+n} \bar{\chi}^0_{i,k} = \prod_{i=1}^{j} (1 - \chi^0_{i,J^c}) \tag{2.3.23}$$

where $J^c = \{j+1, \cdots, j+n\}$ and

$$\begin{aligned} \chi^0_{i,J^c} &= \left(1 - \prod_{k=j+1}^{j+n} \bar{\chi}^0_{i,k} \right) \\ &= \mathbb{1}_{\{\mathbf{z}_{j,n} \,\mid\, \exists k \in J^c \;\text{ such that }\; |x_i - x_k| \leq \varepsilon\}} \;. \end{aligned}$$

We remind that $\mathbb{1}_A$ denotes the indicator function of the event A.

Expanding the product in (2.3.23), we find

$$\prod_{i=1}^{j} \prod_{k=j+1}^{j+n} \bar{\chi}^0_{i,k} = \sum_{K \subset J} (-1)^k \chi^0_{K,J^c} \;, \tag{2.3.24}$$

with

$$\chi^0_{K,J^c} = \prod_{i \in K} \chi^0_{i,J^c} \;.$$

This, substituted into (2.3.22), leads to

$$f^{\varepsilon}_{0,j}(\mathbf{z}_j) = \sum_{L\subset J} f_0^{\otimes L}(\mathbf{z}_L) E^{\mathcal{B},0}_{J\setminus L}(\mathbf{z}_{J\setminus L}) \;, \tag{2.3.25}$$

where $E^{\mathcal{B},0}_{\emptyset} = 1$ and, for $k \geq 1$,

$$E^{\mathcal{B},0}_{K}(\mathbf{z}_k) = (-f_0)^{\otimes k}(\mathbf{z}_k)\frac{1}{\mathcal{Z}_\varepsilon}\sum_{n\geq 1}\frac{e^{-\mu_\varepsilon}\mu_\varepsilon^n}{n!}\int d\mathbf{z}_{k,n} f_0^{\otimes n}(\mathbf{z}_{k,n})\;\chi^0_{K,K^c}\prod_{k+1\leq i<h\leq k+n}\bar{\chi}^0_{i,h}\;. \tag{2.3.26}$$

To bound the error, we observe that

$$\chi^0_{K,K^c} \leq \prod_{i\in K}\sum_{i'\in K^c}\chi^0_{i,i'} = \sum_{j_1,\cdots,j_k\in K^c}\prod_{i=1}^k \chi^0_{i,j_i}\;. \tag{2.3.27}$$

The last product of characteristic functions means the following condition: k spheres, with configuration $(z_1,\cdots,z_k)\in\mathcal{M}_k$, overlap with the spheres centered in $x_{j_1},\cdots,x_{j_k}$ respectively .

Let $\overline{\kappa}$ be the maximum number of three–dimensional hard spheres that can be simultaneously overlapped by a single one. Then, we need at least $k/\overline{\kappa}$ *different* spheres in K^c in order to satisfy the condition. Namely, it must be $n\geq k/\overline{\kappa}$ in (2.3.26), so that (2.3.27) becomes

$$\chi^0_{K,K^c} \leq \sum_{q\geq k/\overline{\kappa}}^{n}\;\sum_{\mathbf{j}_k}^{(q)}\prod_{i=1}^k\chi^0_{i,j_i}\;, \tag{2.3.28}$$

where $\sum^{(q)}$ is the sum restricted to $\mathbf{j}_k\in K^c$ that include exactly q different values, say $j^{(1)},\cdots,j^{(q)}$.

Each one of the integrals in $z_{j^{(1)}},\cdots,z_{j^{(q)}}$, is bounded by $z(2\pi/\beta)^{3/2}kB\varepsilon^3$, where B is the volume of the unit ball and we used the uniform bound for f_0. The number of terms in $\sum^{(q)}$ is bounded by $(Cn)^q$, $C>0$. Therefore, we obtain

$$|E^{\mathcal{B},0}_K(\mathbf{z}_k)| \leq\; (h/2)^{\otimes K}\, e^{-(\beta/2)\sum_{i\in K}v_i^2}\frac{1}{\mathcal{Z}_\varepsilon}\sum_{q\geq k/\overline{\kappa}}(C'k\varepsilon^3)^q\sum_{n\geq q}\frac{e^{-\mu_\varepsilon}\mu_\varepsilon^n}{n!}n^q\mathcal{Z}^{can}_{n-q}\;, \tag{2.3.29}$$

for some suitable constant $C'>0$.

The last sum is equal to

$$\begin{aligned}&\sum_{n\geq 0}\frac{e^{-\mu_\varepsilon}\mu_\varepsilon^{n+q}}{(n+q)!}(n+q)^q\mathcal{Z}^{can}_n\\ &= \mu_\varepsilon^q\sum_{n\geq 0}\frac{e^{-\mu_\varepsilon}\mu_\varepsilon^n}{n!}\frac{(n+q)^q}{(n+q)(n+q-1)\cdots(n+1)}\mathcal{Z}^{can}_n\\ &\leq \mu_\varepsilon^q e^q\mathcal{Z}_\varepsilon\;.\end{aligned} \tag{2.3.30}$$

Hence, for $k < \varepsilon^{-\alpha_0}$,

$$
\begin{aligned}
|E_K^{\mathcal{B},0}(\mathbf{z}_k)| &\le (h/2)^{\otimes K}\, e^{-(\beta/2)\sum_{i\in K} v_i^2} \sum_{q\ge k/\overline{\kappa}} (C' e\, \varepsilon^{1-\alpha_0})^q \\
&\le (C''\varepsilon^{1-\alpha_0})^{k/\overline{\kappa}}\ (h/2)^{\otimes K}\, e^{-(\beta/2)\sum_{i\in K} v_i^2}\,, \qquad (2.3.31)
\end{aligned}
$$

that implies (2.3.7) by choosing $\gamma_0' < 1/\overline{\kappa}(1-\alpha_0)$ and ε small enough.

The estimate (2.1.18) follows immediately. ■

To conclude we observe that the state (2.3.16) fulfills our hypotheses, because it satisfies the Boltzmann–Grad scaling (2.1.10). This follows also immediately from (2.3.6) and (2.3.31).

Remark. We have deliberately chosen to discuss above the simplest possible example. This is inspired from Gibbsian equilibrium, and corresponds to a state where the probability of finding n particles is $p_n = \mathcal{Z}_n^{can}\mathcal{Z}_\varepsilon^{-1}(1/n!)e^{-\mu_\varepsilon}\mu_\varepsilon^n$ and the distribution of the n particles $(\mathcal{Z}_n^{can})^{-1} f_0^{\otimes n}$. The factor multiplying the Poissonian distribution measures the probability of having n non–overlapping spheres.

We have not verified the representation (2.3.6)–(2.3.7) for the corresponding canonical state (see (2.1.12)), for which a more elaborate expansion than (2.3.23) seems to be necessary.

3 Main results

Our main results are collected in the following theorem. In the statement, we use the definitions given in Section 2.1, notation and assumptions given in Section 2.3 and the additional definition:

$$
\mathcal{M}_n^x(\delta) = \left\{ \mathbf{x}_n \in \mathbb{R}^{3n},\ \ |x_i - x_j| > \delta,\ i \ne j \right\}. \qquad (3.1)
$$

Here δ is an $\varepsilon-$dependent, positive quantity. With our estimates, it can be chosen as

$$
\delta = \varepsilon^\theta \qquad (3.2)
$$

for some $\theta < 1$.

Theorem 1 *Let $\mathbf{W}_0^\varepsilon$ be a state of the hard sphere system with rescaled correlation functions $f_{0,j}^\varepsilon$ satisfying the bound* (2.1.18) *and Hypothesis 1. Let $\mathbf{W}^\varepsilon(t)$ be the state evolved at time $t > 0$, with r.c.f. $f_j^\varepsilon(t)$. Then there exist two positive constants α, γ, a time $t^* > 0$ and*

$\varepsilon_0 > 0$ *such that, for any* $t < t^*$ *and* $\varepsilon < \varepsilon_0$,

$$f_J^\varepsilon(t) = \sum_{H \subset J} (f_1^\varepsilon(t))^{\otimes H} E_{J \setminus H}(t) \ , \tag{3.3}$$

$$\sup_{\mathcal{M}_k^x(\delta)} \int_{\mathbb{R}^{3k}} d\mathbf{v}_k |E_K(t)| \leq \varepsilon^{\gamma k} \qquad \forall \ k < \varepsilon^{-\alpha} \ . \tag{3.4}$$

Moreover, let f_0 *be the initial datum for the Enskog and the Boltzmann equation, and* $g^\varepsilon(t)$, $f(t)$ *respectively their (series) solutions (given by* (2.1.28) *and* (2.1.23) *with* $j = 1$*).*

If f_0 *satisfies Hypothesis 2, then there exists a positive constant* γ' *such that, for any* $t < t^*$ *and* $\varepsilon < \varepsilon_0$,

$$f_J^\varepsilon(t) = \sum_{H \subset J} (g^\varepsilon(t))^{\otimes H} E^{\mathscr{E}}_{J \setminus H}(t) \ , \tag{3.5}$$

$$\sup_{\mathcal{M}_k^x(\delta)} \int_{\mathbb{R}^{3k}} d\mathbf{v}_k |E^{\mathscr{E}}_K(t)| \leq \varepsilon^{\gamma' k} \qquad \forall \ k < \varepsilon^{-\alpha} \ . \tag{3.6}$$

If, additionally, f_0 *is Lipschitz continuous with respect to the space variables, with Lipschitz constant* $Le^{-(\beta/2)v^2}$, $L > 0$, *then there exists a positive constant* γ'' *such that, for any* $t < t^*$ *and* $\varepsilon < \varepsilon_0$,

$$f_J^\varepsilon(t) = \sum_{H \subset J} (f(t))^{\otimes H} E^{\mathcal{B}}_{J \setminus H}(t) \ , \tag{3.7}$$

$$\sup_{\mathcal{M}_k^x(\delta)} \int_{\mathbb{R}^{3k}} d\mathbf{v}_k |E^{\mathcal{B}}_K(t)| \leq \varepsilon^{\gamma'' k} \qquad \forall \ k < \varepsilon^{-\alpha} \ . \tag{3.8}$$

We remind that, in the above expansions, we use the convention (2.3.1).

Equation (3.3)–(3.4) is an expression for the propagation of chaos, with an explicit representation of the error, while Equations (3.5)–(3.6) and (3.7)–(3.8) express in addition the asymptotic equivalence of the r.c.f. with the solution of the Enskog and the Boltzmann hierarchies.

In this paper we are not concerned with optimal bounds on rates of convergence nor with the optimality of the coefficient α. Improvements in this direction would complicate considerably the proof presented in the following sections. The values of the coefficients are, of course, also limited by the corresponding γ_0, α_0 appearing in the estimates of the initial data (see Section 2.3).

The limiting time t^* is obtained by imposing the absolute (uniform in ε) convergence of the series expansions appearing in the proof, and is determined only by z, β (see estimate (2.1.18)).

Remarks.
(1) The use of the $\left(\sup_{\mathcal{M}_k^x(\delta)} \int_{\mathbb{R}^{3k}} d\mathbf{v}_k\right)$ −norm simplifies considerably the proof of the estimate of many–recollision events (Lemmas 5 and 6 in Section 3.2), which is essential to obtain a $k-$dependent rate of convergence in (3.4). A Chebyshev's inequality implies also that $|E_K(t)| \leq \varepsilon^{\bar{\gamma}k}$ for some $\bar{\gamma} > 0$ (and similar estimates for $E^{\mathscr{E}}$ and $E^{\mathcal{B}}$), outside some subset of $\mathcal{M}_k$ of measure smaller than $\varepsilon^{(\gamma-\bar{\gamma})k}$.
(2) In particular, the comparison with the uniform estimate of Hypothesis 1 shows that the set of convergence deteriorates in time. This is a feature of the Boltzmann–Grad limit. In fact, since the external recollisions prevent factorization in formula (2.2.18), the propagation of chaos $f_J^\varepsilon(t) \to (f_1^\varepsilon(t))^{\otimes J}$ necessarily fails over the time–dependent set

$$\left\{\mathbf{z}_J \;\middle|\; \min_{i,k\in J} \min_{s\in(0,t)} [(x_i - x_k) - (v_i - v_k)\, s] = 0\right\}.$$

Actually it can be proved that, pointwise outside this null–measure set, $E_K(t) = O(\varepsilon^\eta)$ for some $\eta > 0$ (see [19], where this is done for a system of smoothly interacting particles). In Theorem 1, our main point is the dependence on k of the estimates, rather than the precise set of convergence. Observe that this loss of information prevents us to iterate the result up to times larger than t^*, even in the assumption of nice a priori estimates on the family of r.c.f..
(3) It is worth to note also that, even though all the correlation errors considered in our theorem are implicitly defined over the sets $\mathcal{M}_k(\varepsilon)$, the definitions could be well extended over $\mathbb{R}^{6k}$ by simply putting $f_J^\varepsilon(\mathbf{z}_J, t) = 0$ when two particles in $\mathbf{z}_J$ are at distance smaller than ε, see (1.8). Inside the "corridors", it is possible to have only estimates of type $|E_K(t)| \leq (const.)^k$. In this paper, we will never use informations on the correlation errors outside $\mathcal{M}_k$.

3.1 Proof: preliminaries

3.1.1 Plan of the proof

We start by the tree expansion (2.2.18) on which we shall select the recollision events preventing the factorization. Roughly speaking, if k trees are recolliding, where $k \leq j$ and j is the initial number of particles appearing in the r.c.f., one gains a factor $O(\varepsilon^\gamma)$ for each recolliding couple: see the geometric estimate of Lemma 5 (where a single external recollision is analysed) and the ordering argument in Section 3.2.7 below, needed to control the many–recollision event.

Unfortunately a rough estimate on the number of pairs of (potentially) recolliding trees, yields a factor of order 2^{k^2} for large k. The resulting $\varepsilon^{\gamma k/2} 2^{k^2}$ is small only for

$k \leq O(-\log \varepsilon)$. To extend the validity range of our correlation error estimate to larger $k \sim O(\varepsilon^{-\alpha})$, we use a graph expansion technique: see Section 3.1.3 below. This turns out to be useful also to control the propagation of the initial correlations.

More precisely, the proof of Theorem 1 is organized as follows.

In Section 3.1.2 we state some basic inequality that will be used to simplify the functional series introduced in Section 2.2 and to ensure their uniform, absolute convergence. In Section 3.1.3 we prove the algebraic lemma which is the core of our method of expansion.

Section 3.2 is devoted to the proof of estimate (3.4). First we define, in terms of backwards flows, the class of events (recollisions/overlaps) that will appear in the explicit expression of the correlation error E_K (Sections 3.2.1 and 3.2.2). Secondly we derive the expression of E_K from the series (2.2.18), by applying the graph expansion technique (Section 3.2.3). Then we turn to the estimate of E_K. In Sections 3.2.4–3.2.6 we deal with the combinatorics produced by the expansions and reduce the problem to the estimate of an integral over events with a fixed number of (recollision– or overlap–) constraints. The control of these many–recollision integrals is worked out in Sections 3.2.7 and 3.2.8.

Finally, in Sections 3.3 and 3.4 we derive the estimates (3.6) and (3.8) respectively. This is a simple corollary of the previous analysis.

3.1.2 Basic estimates

We begin by asserting a well known property of the BBGKY, Enskog and Boltzmann series expansions introduced in Section 2.2.

Proposition 3 (short time estimates) *If the initial data $f^{\varepsilon}_{0,j}$ and $f_{0,j}$ are bounded as in* (2.1.18)*, then there holds absolute convergence of the expansions* (2.2.8)*,* (2.2.26) *(uniformly in ε) and* (2.2.29)*, for any $t < \bar{t} = \bar{t}(z,\beta)$.*

This result was already discussed in the original paper of Lanford [17]. The proof, reported here for completeness, is reduced to the bound given by the following lemma (case $a = 1$). For future use, this is stated in a somewhat general form.

Lemma 2 *Let $a = 1, 2$. There exist constants $\bar{t}, \bar{C} > 0$ such that, for any $t < \bar{t}$, the following estimate holds:*

$$\sum_{n\geq 0} z^{j+n} \sum_{\Gamma(j,n)} \int d\Lambda(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n}) \left(\prod |B^{\varepsilon}|\right)^a \; e^{-(\beta/2)\sum_{i\in S(J)}(\eta^{\varepsilon}_i(0))^2} \leq \bar{C}^j e^{-(\beta/4)\sum_{i\in J} v_i^2} \, . \tag{3.2.1}$$

The same result holds when $B^\varepsilon, \boldsymbol{\zeta}^\varepsilon$ are replaced by $B^{\mathscr{E}}, \boldsymbol{\zeta}^{\mathscr{E}}$ (Enskog flow) or $B, \boldsymbol{\zeta}$ (Boltzmann flow).

We remind that, by the notation (2.2.22), $S(J) = \{1, 2, \cdots, j+n\}$.

In the case $a = 1$, this shows that the expansions of Proposition 3 are also absolutely convergent in the norm $\sup_{\mathcal{M}_j^x(\delta)} \int_{\mathbb{R}^{3j}} d\mathbf{v}_j$.

The case $a = 2$ in the above lemma implies the following result, which will be used (precisely, in Section 3.2.5) to simplify the expression of formulas in the recollision estimates. This procedure was already applied in [19].

Corollary 1 *Let $F \leq 1$ be any positive measurable function of the variables $\mathbf{z}_j$, $\mathbf{t}_n$, $\boldsymbol{\omega}_n$, $\mathbf{v}_{j,n}$. Let $N > 0$ and $\theta_1 > 0$. There exists $\bar{C}' > 0$ such that, for any $t < \bar{t}$,*

$$\int d\mathbf{v}_j \sum_{n=0}^{N} z^{j+n} \sum_{\Gamma(j,n)} \int d\Lambda(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n}) \left(\prod |B^\varepsilon|\right) \, e^{-(\beta/2)\sum_{i\in S(J)}(\eta_i^\varepsilon(0))^2} \, F$$
$$\leq (\bar{C}')^j \varepsilon^{\theta_1 j} + \varepsilon^{-\theta_1 j} \sum_{n=0}^{N} z^{j+n} \sum_{\Gamma(j,n)} \int d\mathbf{v}_j d\Lambda(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n})$$
$$\cdot e^{-(\beta/2)\sum_{i\in S(J)}(\eta_i^\varepsilon(0))^2} \, \mathbb{1}_{\{|\xi^\varepsilon_{j+i}(t_i) - \xi^\varepsilon_k(t_i)| > \varepsilon \ \forall k \neq k_i\}} \, F \, . \tag{3.2.2}$$

The result holds also when $B^\varepsilon, \boldsymbol{\zeta}^\varepsilon$ are replaced by $B^{\mathscr{E}}, \boldsymbol{\zeta}^{\mathscr{E}}$ (Enskog flow) or $B, \boldsymbol{\zeta}$ (Boltzmann flow).

To deduce the corollary, it is enough to observe that the integral on the l.h.s., when restricted to the set such that $\prod |B^\varepsilon| > \varepsilon^{-\theta_1 j}$, is bounded by $\varepsilon^{\theta_1 j}$ times the $\int d\mathbf{v}_j$ of the left hand side in (3.2.1) with $a = 2$. Applying the lemma, we obtain the result by taking $\bar{C}' = \bar{C}(4\pi/\beta)^{3/2}$.

Proof of Lemma 2. The conservation of energy at collisions implies

$$\sum_{i\in S(J)} (\eta_i^\varepsilon(0))^2 = \sum_{i=1}^{j+n} v_i^2 \, . \tag{3.2.3}$$

In particular $\sum_{k_i=1}^{j+i-1} \left(\eta^\varepsilon_{k_i}(t_i)\right)^2 \leq \sum_{i=1}^{j+n} v_i^2$. Then, using the expression of B^ε (see (2.2.12)) and that of the sum over trees (see (2.2.2)), we find

$$\sum_{\Gamma(j,n)} \left(\prod |B^\varepsilon|\right)^a \leq a^n \prod_{i=1}^{n} \left[(j+n)|v_{j+i}|^a + \left(\sum_{l=1}^{j+n} v_l^2\right)^{\frac{a}{2}}\right] . \tag{3.2.4}$$

Moreover,

$$\Big(\sum_{l=1}^{j+n} v_l^2\Big)^{\frac{a}{2}} e^{-\frac{\beta}{4n}\sum_{i=1}^{j+n} v_i^2} \leq \frac{4n}{e\beta}\,. \tag{3.2.5}$$

Replacing these estimates in the l.h.s. of (3.2.1), it follows that we can bound it by

$$e^{-(\beta/4)\sum_{i\in J} v_i^2} \sum_{n\geq 0} 2^n z^{n+j} \int d\Lambda \prod_{i=1}^{n} \Big((j+n) v_{j+i}^a e^{-\frac{\beta}{4} v_{j+i}^2} + \frac{4n}{e\beta} e^{-\frac{\beta}{4} v_{j+i}^2} \Big)\,. \tag{3.2.6}$$

The integral on the velocities factorizes so that

$$(3.2.6) \leq e^{-(\beta/4)\sum_{i\in J} v_i^2} \sum_{n\geq 0} C(z,\beta)^{j+n} \frac{t^n}{n!} (j+n)^n \tag{3.2.7}$$

for a suitable constant $C(z,\beta) > 0$ (which can be explicitly written in terms of gaussian integrals). Since

$$\frac{(j+n)^n}{n!} \leq \frac{(j+n)^{j+n}}{(j+n)!} \leq e^{j+n}\,, \tag{3.2.8}$$

we have that (3.2.7) is bounded by a geometric series. Hence choosing

$$\bar{t} < \frac{1}{C(z,\beta)e}\,, \tag{3.2.9}$$

we obtain (3.2.1).

The cases of the Enskog and of the Boltzmann flow are treated in the same way. ■

Remark. Lemma 2 and Proposition 3 imply immediately that the correlation errors $E_K, E_K^{\mathscr{E}}, E_K^{\mathcal{B}}$ introduced in Theorem 1 are bounded as a $(const.)^k$, uniformly in ε, for all $t < \bar{t}$. Note that this is also true in the regions $\mathbb{R}^{6k} \setminus \mathcal{M}_k$, as soon as the definition of the r.c.f. is extended there as $f_J^\varepsilon(\mathbf{z}_J, t) = 0$.

3.1.3 A graph expansion

We shall introduce here a method of expansion that will be applied to the BBGKY series (2.2.18) in order to reconstruct the representation (3.3) and obtain an explicit expression for the $E_K(t)$. We find convenient to discuss first this result in the following abstract formulation, since it will be used twice in Section 3.2.3 below, i.e. in dealing both with the initial data and with the control of the external recollisions.

Before proceeding, let us comment again on the meaning of the representation (3.3). Its purpose is to give a detailed expression of the r.c.f. in terms of uncorrelated quantities. As already discussed, correlations are essentially due to some constraints expressing "forbidden" events. For instance, any admissible state does not allow hard spheres to overlap

$(|x_i - x_k| < \varepsilon)$. A similar situation occurs when looking at the external recollisions between particles of different trees, namely the most important source of dynamical correlation. We want to remove these constraints in the effort of making independent the events of interest, and of showing that they produce a (hopefully small) perturbation of the identity. This can be done by means of the expansion introduced below.

Let us start with some classical definitions.

Definition 1

(i) A graph over a set $\mathcal{I} = \{a_1 \dots a_n\}$ *of vertices, is a collection of edges (links)* $\{\ell_{i,j}\}_{i \neq j}$*, where* $\ell_{i,j}$ *takes the values* $1, 0$ *if the vertices* a_i *and* a_j *are connected or not respectively.*

(ii) $\mathcal{G}$ *is the family of all graphs over* $\mathcal{I}$*.*

(iii) We introduce the following characteristic functions on $\mathcal{G}$*:*

$$\chi_{i,K} = 1$$

if and only if the vertex a_i *is connected with some vertex in* $K \subset \mathcal{I}$*;*

$$\bar{\chi}_{i,K} = 1 - \chi_{i,K} \ ;$$

and, for $H \subset \mathcal{I}$*,*

$$\chi_{H,K} = \prod_{i \in H} \chi_{i,K} \ ,$$

$$\bar{\chi}_{H,K} = \prod_{i \in H} \bar{\chi}_{i,K} \ .$$

Observe that $\chi_{H,K} = 1$ if and only if any vertex of H is connected with some vertex in K, and $\bar{\chi}_{H,K} = 1$ if and only if any vertex in H is not connected with any vertex in K. Note also that a vertex can not be self connected, i.e. $\chi_{i,i} = 0$ and $\bar{\chi}_{i,i} = 1$.

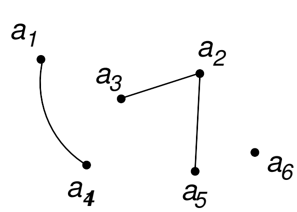

Figure 3: Graph $\ell_{1,4} = \ell_{2,3} = \ell_{2,5} = 1$, and different $\ell_{i,j} = 0$.

With these definitions, we have the following

Lemma 3 *Let $L \subset \mathcal{I}$ and $L_0 = \mathcal{I} \setminus L$. Then,*

$$\bar{\chi}_{L,L\cup L_0} = \sum_{Q\subset L} R(Q, L_0) \tag{3.2.10}$$

where, for some pure constant $C > 0$,

$$|R(Q, L_0)| \le C^q\, q!\, \chi_{Q,Q\cup L_0} \ . \tag{3.2.11}$$

We remind that we are using the notation $q = |Q|$. Moreover, we use the convention $\bar{\chi}_{\emptyset,\cdot} = \chi_{\emptyset,\cdot} = 1$.

Note that each term of the expansion (3.2.10) does not depend on $L \setminus Q$, i.e. there is no condition on the vertices of this set (they are "free" vertices).

Proof. By addition/subtraction we find

$$\bar{\chi}_{L,L\cup L_0} = 1 - \sum_{\substack{L_1,L_2\\ L_1\cup L_2=L\\ L_1\cap L_2=\emptyset\\ l_1\ge 1}} \chi_{L_1,L\cup L_0}\bar{\chi}_{L_2,L\cup L_0} = 1 - \sum_{\substack{L_1,L_2\\ L_1\cup L_2=L\\ L_1\cap L_2=\emptyset\\ l_1\ge 1}} \chi_{L_1,L_1\cup L_0}\bar{\chi}_{L_2,L_0\cup L_1\cup L_2} \ . \tag{3.2.12}$$

Note that $l_1 > 0$ and $\chi_{L_1,L\cup L_0} = \chi_{L_1,L_1\cup L_0}$, because any vertex in L_2 is not connected. Iterating once,

$$\bar{\chi}_{L,L\cup L_0} = 1 - \sum_{\substack{L_1\subset L\\ l_1\ge 1}} \chi_{L_1,L_1\cup L_0} + \sum_{\substack{L_1,L_2,L_3\\ L_1\cup L_2\cup L_3=L\\ L_i\cap L_j=\emptyset, i\ne j\\ l_1\ge 1, l_2\ge 1}} \chi_{L_1,L_0\cup L_1}\chi_{L_2,L_0\cup L_1\cup L_2}\bar{\chi}_{L_3,L_0\cup L_1\cup L_2\cup L_3} \ . \tag{3.2.13}$$

Then, successive iterations yield the following expansion:

$$\bar{\chi}_{L,L\cup L_0} = \sum_{r=0}^{|L|}(-1)^r \sum_{\substack{L_1,\cdots,L_r\\ \cup_i L_i\subset L\\ l_i\ge 1\\ L_i\cap L_j=\emptyset, i\ne j}} \chi_{L_1,L_0\cup L_1}\cdots\chi_{L_r,L_0\cup L_1\cdots\cup L_r} \ , \tag{3.2.14}$$

where the $r = 0$ term has to be interpreted as 1. I.e.

$$\bar{\chi}_{L,L\cup L_0} = \sum_{Q\subset L} R(Q, L_0) \ , \tag{3.2.15}$$

with

$$R(Q, L_0) := \sum_{r=1}^{q}(-1)^r \sum_{\substack{L_1,\cdots,L_r\\ \cup_i L_i=Q\\ l_i\ge 1\\ L_i\cap L_j=\emptyset, i\ne j}} \chi_{L_1,L_0\cup L_1}\cdots\chi_{L_r,L_0\cup L_1\cdots\cup L_r} \ , \tag{3.2.16}$$

and $R(\emptyset, L_0) = 1$.

From this expression it follows

$$\begin{aligned}
|R(Q,L_0)| &\leq \sum_{r=1}^{q} \sum_{\substack{L_1,\cdots,L_r \\ \cup_i L_i = Q \\ l_i \geq 1 \\ L_i \cap L_j = \emptyset, i \neq j}} \chi_{Q,Q\cup L_0} \\
&\leq \chi_{Q,Q\cup L_0} \sum_{r=1}^{q} \sum_{\substack{l_1,\cdots,l_r \\ l_i \geq 1}} \frac{q!}{l_1! \cdots l_r!} \\
&\leq \chi_{Q,Q\cup L_0}\, q! \sum_{r=1}^{q} (e-1)^r \\
&\leq \chi_{Q,Q\cup L_0}\, q!\, C^q \, . \qquad (3.2.17)
\end{aligned}$$

■

3.2 Proof of (3.4)

We start by rewriting the formula, introduced in Section 2.2, yielding the reduced correlation functions at time t, namely

$$f_J^{\varepsilon}(\mathbf{z}_J, t) = \sum_{\mathbf{n}_j, \mathbf{\Gamma}_j} \int d\Lambda \; \prod B^{\varepsilon} \; f_{0,S(J)}^{\varepsilon} \, , \qquad (3.3.1)$$

where we abbreviate

$$\int d\Lambda = \prod_{i=1}^{j} \int d\Lambda(\mathbf{t}_{n_i}^i, \boldsymbol{\omega}_{n_i}^i, \mathbf{v}_{1,n_i}^i) \qquad (3.3.2)$$

and

$$f_{0,S(J)}^{\varepsilon} = f_{0,|S(J)|}^{\varepsilon}(\boldsymbol{\zeta}_{S(J)}^{\varepsilon}(0)) \, . \qquad (3.3.3)$$

Remind that $\mathbf{\Gamma}_j = \{\Gamma_1 \ldots \Gamma_j\}$ denotes the set of j trees, $\mathbf{n}_j = \{n_1 \ldots n_j\}$ are the numbers of particles created by each tree, $(\mathbf{t}_{n_i}^i, \boldsymbol{\omega}_{n_i}^i, \mathbf{v}_{1,n_i}^i)$ are the collections of node variables in the tree Γ_i and $S(J)$ denotes the set of indices of the particles created in the backwards flow $\boldsymbol{\zeta}^{\varepsilon}$ at time 0 (Eq.s (2.2.18)–(2.2.22)). Clearly, $|S(J)| = j + \sum_i n_i$.

3.2.1 The recollision constraints

In this part of the proof we will focus only on the external recollisions. Let us fix a terminology that will play an important role in the future. *We say that two trees, say* Γ_i

and Γ_k, (briefly i and k from now on) recollide if there is a particle in $S(i)$ which recollides with a particle in $S(k)$.

Now we may introduce the characteristic function $\chi^{rec}_{i,K}$ defined by:

$$\chi^{rec}_{i,K} = 1$$

if and only if the tree i recollides with some tree in $K \subset J$. This depends of course on the IBF, $\boldsymbol{\zeta}^\varepsilon$. Also, we introduce

$$\chi^{rec}_K = \prod_{i \in K} \chi^{rec}_{i,K} \ ,$$

so that $\chi^{rec}_K = 1$ if and only if all the trees in K recollide with some other tree in K. Finally,

$$\bar{\chi}^{rec}_{i,K} = 1 - \chi^{rec}_{i,K}$$

and, for $H \subset J$,

$$\bar{\chi}^{rec}_{H,K} = \prod_{i \in H} \bar{\chi}^{rec}_{i,K} \ .$$

That is, $\bar{\chi}^{rec}_{H,K} = 1$ if and only if the trees in H do not recollide with any tree in K.

With these definitions, we have

$$1 = \sum_{L_0 \subset J} \chi^{rec}_{L_0} \bar{\chi}^{rec}_{J \setminus L_0, J} \ . \tag{3.3.4}$$

Observe that, if $L_0 \neq \emptyset$, $l_0 = |L_0| \geq 2$.

Inserting this expansion into (3.3.1), we find

$$f^\varepsilon_J(\mathbf{z}_J, t) = \sum_{\mathbf{n}_j, \mathbf{\Gamma}_j} \sum_{L_0 \subset J} \int d\Lambda \ \prod B^\varepsilon \ \chi^{rec}_{L_0} \bar{\chi}^{rec}_{J \setminus L_0, J} \, f^\varepsilon_{0,S(J)} \ . \tag{3.3.5}$$

3.2.2 The overlap constraints

In what follows (recall Section 2.2.3), we shall use a *mixed* backwards flow, where the particles of the trees in L_0 are evolved by taking into account all the recollisions among themselves, while the particles belonging to the trees in $J \setminus L_0$ are evolved via the flow $\tilde{\boldsymbol{\zeta}}^\varepsilon$ (i.e. by ignoring their external recollisions). We shall indicate such a flow

$$\hat{\boldsymbol{\zeta}}^\varepsilon = (\boldsymbol{\zeta}^{\varepsilon,(L_0)}, \tilde{\boldsymbol{\zeta}}^{\varepsilon,(J \setminus L_0)}) \ , \tag{3.3.6}$$

where $\boldsymbol{\zeta}^{\varepsilon,(L_0)}$ is the flow of particles of the trees in L_0 and $\tilde{\boldsymbol{\zeta}}^{\varepsilon,(J \setminus L_0)}$ is the flow of particles of the trees in $J \setminus L_0$. In particular, note that $\hat{\boldsymbol{\zeta}}^\varepsilon(0) = (\boldsymbol{\zeta}^{\varepsilon,(L_0)}_{S(L_0)}(0), \tilde{\boldsymbol{\zeta}}^\varepsilon_{S(J \setminus L_0)}(0))$.

If $i \in H \subset J \setminus L_0$ and $K \subset J$, we introduce the following characteristic functions:

$$\chi^{ov}_{i,K} = 1$$

if and only if the tree i *overlaps* with some tree in $K \subset J$ in the dynamics (3.3.6) (in the sense that some particle in $S(i)$ reaches a distance smaller than ε from some other particle in $S(K)$). Moreover we set

$$\chi^{ov}_{H,K} = \prod_{i \in H} \chi^{ov}_{i,K},$$
$$\bar{\chi}^{ov}_{i,K} = 1 - \chi^{ov}_{i,K},$$
$$\bar{\chi}^{ov}_{H,K} = \prod_{i \in H} \bar{\chi}^{ov}_{i,K}.$$

That is, $\chi^{ov}_{H,K} = 1$ if and only if all the trees in H overlap with some tree in K while $\bar{\chi}^{ov}_{H,K} = 1$ if and only if all the trees in H do not overlap with any tree in K.

With these definitions, the following trivial identity holds:

$$\Big(\chi^{rec}_{L_0} \bar{\chi}^{rec}_{J \setminus L_0, J}\, f^{\varepsilon}_{0,S(J)}\Big)(\boldsymbol{\zeta}^{\varepsilon}) = \Big(\chi^{rec}_{L_0} \bar{\chi}^{ov}_{J \setminus L_0, J}\, f^{\varepsilon}_{0,S(J)}\Big)(\hat{\boldsymbol{\zeta}}^{\varepsilon}) \;, \tag{3.3.7}$$

which inserted into (3.3.5) leads to

$$f^{\varepsilon}_J(\mathbf{z}_J, t) = \sum_{\mathbf{n}_j, \boldsymbol{\Gamma}_j} \sum_{L_0 \subset J} \int d\Lambda \, \prod B^{\varepsilon} \; \chi^{rec}_{L_0} \bar{\chi}^{ov}_{J \setminus L_0, J}\, f^{\varepsilon}_{0,S(J)} \;, \tag{3.3.8}$$

with the integrand function calculated via the flow (3.3.6).

3.2.3 Computation of E_K

We may apply Lemma 3 to the case $\mathcal{I} = J$, $\bar{\chi} = \bar{\chi}^{ov}$ and $L_0 = J \setminus L$, to obtain

$$\bar{\chi}^{ov}_{J \setminus L_0, J} = \bar{\chi}^{ov}_{L, L \cup L_0} = \sum_{Q \subset L} R^{ov}(Q, L_0), \tag{3.3.9}$$

where

$$|R^{ov}(Q, L_0)| \le C^q q! \chi^{ov}_{Q, Q \cup L_0} \tag{3.3.10}$$

for some $C > 0$.

Inserting the above expansion in (3.3.8), we find

$$f^{\varepsilon}_J(\mathbf{z}_J, t) = \sum_{L_0 \subset J} \sum_{Q \subset J \setminus L_0} \sum_{\mathbf{n}_j, \boldsymbol{\Gamma}_j} \int d\Lambda \, \prod B^{\varepsilon} \chi^{rec}_{L_0} R^{ov}(Q, L_0)\, f^{\varepsilon}_{0,S(J)}(\hat{\boldsymbol{\zeta}}^{\varepsilon}) \;. \tag{3.3.11}$$

By (3.3.11), we infer that each tree in Q must obey an overlap–condition in order that $R^{ov} \neq 0$. In contrast, the trees $J \setminus (L_0 \cup Q)$ are *free*, in the sense that there is no condition over them, so that they are not dynamically correlated. Nevertheless, they are still correlated through the initial data $f^{\varepsilon}_{0,S(J)}$.

To eliminate this correlation, we have to expand the initial data, as done in (2.3.10), with respect to the following partition of $S(J)$ in clusters of particles:

$$\mathcal{S}_J = \Big\{ S(i_1), \cdots, S(i_{j-l_0-q}), S(Q \cup L_0) \Big\} , \tag{3.3.12}$$

where $\{i_1, \cdots, i_{j-l_0-q}\} = J \setminus (L_0 \cup Q)$. In other words we are selecting the $j - l_0 - q$ free trees to eliminate the residual correlations. Namely, denoting

$$\mathcal{S}_H = \Big\{ S \in \mathcal{S}_J \;\Big|\; S \subset S(H) \Big\} \qquad (H \subset J) , \tag{3.3.13}$$

by (2.3.10) we write

$$f^{\varepsilon}_{0,S(J)} = \sum_{P \subset J \setminus (L_0 \cup Q)} \left(\prod_{i \in P} f^{\varepsilon}_{0,S(i)} \right) \left(\bar{E}^0_{\mathcal{S}_{J \setminus P}} + \bar{E}^0_{\mathcal{S}_{J \setminus (P \cup Q \cup L_0)}} f^{\varepsilon}_{0,S(Q \cup L_0)} \right) . \tag{3.3.14}$$

Unfortunately this is not enough. In fact, in the integral (3.3.11), $f^{\varepsilon}_{0,S(J)}$ is evaluated in

$$(\boldsymbol{\zeta}^{\varepsilon,(L_0)}_{S(L_0)}(0), \tilde{\boldsymbol{\zeta}}^{\varepsilon}_{S(J \setminus L_0)}(0)) . \tag{3.3.15}$$

Due to the dynamics $\tilde{\boldsymbol{\zeta}}^{\varepsilon}$, particles of different clusters of $\mathcal{S}_J$ *may* actually overlap at time zero (and when this happens, we have no available bounds on the errors $\bar{E}^0$ appearing in (3.3.14)). This is another source of correlation since $f^{\varepsilon}_{0,S(J)}$, as a function on the whole space, is of course proportional to

$$\bar{\chi}^0_{\mathcal{S}_J} := \prod_{\substack{S, S' \in \mathcal{S}_J \\ S \neq S'}} \bar{\chi}^0_{S,S'} , \tag{3.3.16}$$

where

$$\bar{\chi}^0_{S,S'} = \prod_{k \in S} \prod_{k' \in S'} \bar{\chi}^0_{k,k'} \tag{3.3.17}$$

and Definition (2.3.19) is used. Observe that this proportionality is not taken into account in formula (3.3.14).

Proceeding as above, we introduce characteristic functions of "time–zero overlaps between clusters of $\mathcal{S}_J$". That is, for $H, K \subset J$ and $S \in \mathcal{S}_J$,

$$\chi^0_{S,\mathcal{S}_K} = 1$$

if and only if the cluster of particles S *overlaps*, at time zero, with some cluster in $\mathcal{S}_K$ in (3.3.15) (in the sense that at least one particle belonging to S has a distance smaller than ε from some particle belonging to a cluster of $\mathcal{S}_K$). Moreover,

$$\chi^0_{\mathcal{S}_H,\mathcal{S}_K} = \prod_{S\in\mathcal{S}_H} \chi^0_{S,\mathcal{S}_K} \ ;$$

$$\bar{\chi}^0_{S,\mathcal{S}_K} = 1 - \chi^0_{S,\mathcal{S}_K} \ ;$$
$$\bar{\chi}^0_{\mathcal{S}_H,\mathcal{S}_K} = \prod_{S\in\mathcal{S}_H} \bar{\chi}^0_{S,\mathcal{S}_K} \ .$$

Before substituting (3.3.14) into (3.3.11), we add to the right hand side of (3.3.14) the factor (3.3.16), and use the following identities:

$$\begin{aligned}
f^\varepsilon_{0,S(J)} &= \bar{\chi}^0_{\mathcal{S}_J} \sum_{P\subset J\setminus(L_0\cup Q)} \left(\prod_{i\in P} f^\varepsilon_{0,S(i)}\right) \left(\bar{E}^0_{\mathcal{S}_{J\setminus P}} + \bar{E}^0_{\mathcal{S}_{J\setminus(P\cup Q\cup L_0)}} f^\varepsilon_{0,S(Q\cup L_0)}\right) \\
&= \sum_{P\subset J\setminus(L_0\cup Q)} \bar{\chi}^0_{\mathcal{S}_P,\mathcal{S}_J} \prod_{i\in P} f^\varepsilon_{0,S(i)} \\
&\qquad \cdot \bar{\chi}^0_{\mathcal{S}_{J\setminus P}} \left(\bar{E}^0_{\mathcal{S}_{J\setminus P}} + \bar{E}^0_{\mathcal{S}_{J\setminus(P\cup Q\cup L_0)}} f^\varepsilon_{0,S(Q\cup L_0)}\right) \\
&= \sum_{P\subset J\setminus(L_0\cup Q)} \sum_{Q'\subset P} R^0(\mathcal{S}_{Q'},\mathcal{S}_{J\setminus P}) \prod_{i\in P} f^\varepsilon_{0,S(i)} \\
&\qquad \cdot \bar{\chi}^0_{\mathcal{S}_{J\setminus P}} \left(\bar{E}^0_{\mathcal{S}_{J\setminus P}} + \bar{E}^0_{\mathcal{S}_{J\setminus(P\cup Q\cup L_0)}} f^\varepsilon_{0,S(Q\cup L_0)}\right) \\
&= \sum_{H\subset J\setminus(L_0\cup Q)} \sum_{Q'\subset J\setminus(L_0\cup Q\cup H)} R^0(\mathcal{S}_{Q'},\mathcal{S}_{J\setminus(H\cup Q')}) \prod_{i\in H\cup Q'} f^\varepsilon_{0,S(i)} \\
&\qquad \cdot \bar{\chi}^0_{\mathcal{S}_{J\setminus(H\cup Q')}} \left(\bar{E}^0_{\mathcal{S}_{J\setminus(H\cup Q')}} + \bar{E}^0_{\mathcal{S}_{J\setminus(H\cup Q'\cup Q\cup L_0)}} f^\varepsilon_{0,S(Q\cup L_0)}\right) \ .
\end{aligned} \tag{3.3.18}$$

In the third step, we applied Lemma 3 to expand $\bar{\chi}^0_{\mathcal{S}_P,\mathcal{S}_J}$ (i.e. to the case $\mathcal{I} = \mathcal{S}_J$, $\bar{\chi} = \bar{\chi}^0$ and $L = \mathcal{S}_P$, $L_0 = \mathcal{S}_{J\setminus P}$), while in the last equality we simply changed sum variables $P, Q' \to H, Q'$, where $H = P \setminus Q'$ is the set of "free" particles (there are no conditions involving $S(H)$). We remind that, in the above formula, $\mathcal{S}_{Q'} = \{S(i),\ i \in Q'\}$, and $\mathcal{S}_{J\setminus(H\cup Q')} = \{S(i),\ i \in J \setminus (H \cup Q' \cup Q \cup L_0)\} \cup S(Q \cup L_0)$.

Inserting (3.3.18) into (3.3.11), it follows that

$$f^{\varepsilon}_J(t) = \sum_{H\subset J} \sum_{\substack{L_0,Q,Q' \\ \subset J\setminus H \\ \text{disjoint}}} \sum_{\mathbf{n}_j,\boldsymbol{\Gamma}_j} \int d\Lambda \prod B^{\varepsilon} \chi^{rec}_{L_0} R^{ov}(Q,L_0)\, R^0(\mathcal{S}_{Q'},\mathcal{S}_{J\setminus(H\cup Q')})$$
$$\prod_{i\in H\cup Q'} f^{\varepsilon}_{0,S(i)}\ \bar{\chi}^0_{\mathcal{S}_{J\setminus(H\cup Q')}} \left(\bar{E}^0_{\mathcal{S}_{J\setminus(H\cup Q')}} + \bar{E}^0_{\mathcal{S}_{J\setminus(H\cup Q'\cup Q\cup L_0)}} f^{\varepsilon}_{0,S(Q\cup L_0)}\right)\ , \tag{3.3.19}$$

with the integrand calculated through the flow (3.3.6).

Observe that now the only dependence of the integrand function on particles of $S(H)$ is in $\left(\prod B^{\varepsilon} \prod_{i\in H} f^{\varepsilon}_{0,S(i)}\right)(\tilde{\boldsymbol{\zeta}}^{\varepsilon})$, which is a *completely factorized* function. Therefore, recalling Eq. (3.3.2), we may "extract" from the formula the product

$$\prod_{i\in H}\left(\sum_{n_i,\Gamma_i}\int d\Lambda(\mathbf{t}^i_{n_i},\boldsymbol{\omega}^i_{n_i},\mathbf{v}^i_{1,n_i}) \prod B^{\varepsilon} f^{\varepsilon}_{0,S(i)}\right) \equiv (f^{\varepsilon}_1(t))^{\otimes H}\ .$$

The result is

$$f^{\varepsilon}_J(t) = \sum_{H\subset J} (f^{\varepsilon}_1(t))^{\otimes H} E_{J\setminus H}(t)\ , \tag{3.3.20}$$

where

$$E_K(t) = \sum_{\substack{L_0,Q,Q' \\ \subset\ K \\ \text{disjoint}}} \sum_{\mathbf{n}_k,\boldsymbol{\Gamma}_k} \int d\Lambda \prod B^{\varepsilon} \chi^{rec}_{L_0} R^{ov}(Q,L_0)\, R^0(\mathcal{S}_{Q'},\mathcal{S}_{K\setminus Q'})$$
$$\cdot \prod_{i\in Q'} f^{\varepsilon}_{0,S(i)}\ \bar{\chi}^0_{\mathcal{S}_{K\setminus Q'}} \left(\bar{E}^0_{\mathcal{S}_{K\setminus Q'}} + \bar{E}^0_{\mathcal{S}_{K\setminus(Q'\cup Q\cup L_0)}} f^{\varepsilon}_{0,S(Q\cup L_0)}\right)\ . \tag{3.3.21}$$

Here, we remind that

$$\mathcal{S}_{Q'} = \{S(i),\ i\in Q'\}\ ,$$
$$\mathcal{S}_{K\setminus Q'} = \{S(i),\ i\in K\setminus(Q'\cup Q\cup L_0)\}\cup S(Q\cup L_0)\ ,$$
$$\mathcal{S}_{K\setminus(Q'\cup Q\cup L_0)} = \{S(i),\ i\in K\setminus(Q'\cup Q\cup L_0)\}\ .$$

Moreover, by Lemma 3, R^{ov} is bounded as in (3.3.10), and

$$\begin{aligned} |R^0(\mathcal{S}_{Q'},\mathcal{S}_{K\setminus Q'})| \quad &\leq C^{q'} q'!\, \chi^0_{\mathcal{S}_{Q'},\mathcal{S}_{K\setminus Q'}} \\ &\leq C^{q'} q'!\, \chi^{ov}_{Q',K}\ . \end{aligned} \tag{3.3.22}$$

3.2.4 Estimate (3.4) (I). Expansions and initial data

We proceed to give an estimate of $\int d\mathbf{v}_K |E_K(t)|$, uniformly in $\mathbf{x}_k \in \mathcal{M}_k^x(\delta)$. In what follows we assume $K = (1, 2, \cdots, k)$ and remind that $|Q| = q, |Q'| = q', |L_0| = l_0$.

Looking at (3.3.21), we first estimate

$$\Big| \prod_{i \in Q'} f^\varepsilon_{0,S(i)} \; \bar{\chi}^0_{\mathcal{S}_{K\setminus Q'}} \left(\bar{E}^0_{\mathcal{S}_{K\setminus Q'}} + \bar{E}^0_{\mathcal{S}_{K\setminus (Q'\cup Q\cup L_0)}} f^\varepsilon_{0,S(Q\cup L_0)} \right) \Big|$$
$$\leq \varepsilon^{\gamma_0''(k-q'-q-l_0)} \, z^{k+n} \, e^{-(\beta/2)\sum_{i\in S(K)} v_i^2} \; , \qquad k+n < \varepsilon^{-\alpha_0} \; . \tag{3.3.23}$$

Here $n = \sum_{i=1}^k n_i$. For $k + n \geq \varepsilon^{-\alpha_0}$, we have nothing else than

$$\Big| \prod_{i \in Q'} f^\varepsilon_{0,S(i)} \; \bar{\chi}^0_{\mathcal{S}_{K\setminus Q'}} \left(\bar{E}^0_{\mathcal{S}_{K\setminus Q'}} + \bar{E}^0_{\mathcal{S}_{K\setminus (Q'\cup Q\cup L_0)}} f^\varepsilon_{0,S(Q\cup L_0)} \right) \Big|$$
$$\leq 2^k z^{k+n} \, e^{-(\beta/2)\sum_{i\in S(K)} v_i^2} \; . \tag{3.3.24}$$

The two estimates above make use of (2.1.18) and of (2.3.11), (2.3.12).

Let θ_0 be an arbitrary positive constant and $\alpha < \alpha_0$. The condition $k + n < \varepsilon^{-\alpha_0}$ is ensured by $n \leq \log \varepsilon^{-\theta_0 k}$ and $k < \varepsilon^{-\alpha}$, as soon as ε is sufficiently small. By (3.3.21) and estimates (3.3.10), (3.3.22), we deduce then

$$\int d\mathbf{v}_K |E_K(t)|$$
$$\leq z^k \sum_{\substack{L_0,Q,Q' \\ \subset K \\ \text{disjoint}}} C^q q! C^{q'} q'! \sum_{n=0}^{\log \varepsilon^{-\theta_0 k}} z^n \sum_{\mathbf{n}_k, \mathbf{\Gamma}_k} \int d\mathbf{v}_k d\Lambda \prod |B^\varepsilon| \chi^{rec}_{L_0} \chi^{ov}_{Q,Q\cup L_0} \, \chi^{ov}_{Q',K}$$
$$\cdot \varepsilon^{\gamma_0''(k-q'-q-l_0)} e^{-(\beta/2)\sum_{i\in S(K)} v_i^2}$$
$$+ (2z)^k \sum_{\substack{L_0,Q,Q' \\ \subset K \\ \text{disjoint}}} C^q q! C^{q'} q'! \sum_{n \geq \log \varepsilon^{-\theta_0 k}} z^n \sum_{\mathbf{n}_k, \mathbf{\Gamma}_k} \int d\mathbf{v}_k d\Lambda \prod |B^\varepsilon| \chi^{rec}_{L_0} \chi^{ov}_{Q,Q\cup L_0} \, \chi^{ov}_{Q',K}$$
$$\cdot e^{-(\beta/2)\sum_{i\in S(K)} v_i^2} \; . \tag{3.3.25}$$

The second term is bounded by

$$(2^4 z C)^k (k!)^2 \sum_{n \geq \log \varepsilon^{-\theta_0 k}} z^n \sum_{\Gamma(k,n)} \int d\mathbf{v}_k d\Lambda \prod |B^\varepsilon| \; e^{-(\beta/2)\sum_{i\in S(K)} v_i^2} \; , \tag{3.3.26}$$

where we used also the factorization property pointed out by formula (2.2.18). Proceeding exactly as in the proof of Lemma 2 (case $a = 1$), this is in turn bounded by

$$
\begin{aligned}
&(2^4 C)^k (k!)^2 (2\pi/\beta)^{\frac{3}{2}k} (C(z,\beta)e)^k \sum_{n \geq \log \varepsilon^{-\theta_0 k}} (\bar{t}C(z,\beta)e)^n \\
&\leq C_1^k \varepsilon^{\theta_0 \log(\bar{t}C(z,\beta)e)^{-1} k - 2\alpha k} \\
&\leq \varepsilon^{\gamma k}/4 \, ,
\end{aligned}
\tag{3.3.27}
$$

for a suitable $C_1 = C_1(z, \beta, \bar{t}) > 0$, $t < \bar{t}$ and $k < \varepsilon^{-\alpha}$, having taken $\gamma < \theta_0 \log(\bar{t}C(z,\beta)e)^{-1} - 2\alpha$ and ε small enough.

Let us simplify the first term in (3.3.25) by estimating $\chi^{ov}_{Q,Q\cup L_0} \leq \chi^{ov}_{Q,K}$, and changing variable $Q \to Q \cup Q'$. It follows that

$$
\begin{aligned}
\int d\mathbf{v}_K |E_K(t)| \leq \frac{\varepsilon^{\gamma k}}{4} + (zC)^k (k!)^2 \sum_{\substack{L_0, Q \\ \subset K \\ \text{disjoint}}} \varepsilon^{\gamma_0''(k-q-l_0)} \\
\cdot \sum_{n=0}^{\log \varepsilon^{-\theta_0 k}} z^n \sum_{\substack{\mathbf{n}_k, \mathbf{\Gamma}_k \\ n = \sum_i n_i}} \int d\mathbf{v}_k d\Lambda \prod |B^\varepsilon| \chi^{rec}_{L_0} \chi^{ov}_{Q,K} \, e^{-(\beta/2)\sum_{i \in S(K)} v_i^2} \, .
\end{aligned}
\tag{3.3.28}
$$

3.2.5 Estimate (3.4) (II). High energies, factors B^ε

To estimate the integral over the constraints $\chi^{rec}_{L_0} \chi^{ov}_{Q,K}$, it will be convenient to truncate the domain to a compact set of velocities (more precisely, this truncation will be used in Lemma 5 below).

Setting

$$
\mathcal{H}_K = \sum_{i \in S(K)} v_i^2
$$

and given an arbitrary $\theta_2 > 0$, one finds

$$
\begin{aligned}
&z^k \sum_{n \geq 0} z^n \sum_{\substack{\mathbf{n}_k, \mathbf{\Gamma}_k \\ n = \sum_i n_i}} \int d\mathbf{v}_k d\Lambda \prod |B^\varepsilon| e^{-(\beta/2)\sum_{i \in S(K)} v_i^2} \mathbb{1}_{\mathcal{H}_K > \varepsilon^{-\theta_2}} \\
&\leq e^{-(\beta/4)\varepsilon^{-\theta_2}} z^k \sum_{n \geq 0} z^n \sum_{\substack{\mathbf{n}_k, \mathbf{\Gamma}_k \\ n = \sum_i n_i}} \int d\mathbf{v}_k d\Lambda \prod |B^\varepsilon| e^{-(\beta/4)\sum_{i \in S(K)} v_i^2} \\
&\leq e^{-(\beta/4)\varepsilon^{-\theta_2}} (\bar{C}'')^k
\end{aligned}
\tag{3.3.29}
$$

for suitable $\bar{C}'' > 0$ and t small enough. Note that in the last step we applied Lemma 2 with $a = 1$ and $\beta \to \beta/2$. In particular, we used $t < t^*$ for some arbitrary $t^* < (C(z, \beta/2)e)^{-1}$.

If $k < \varepsilon^{-\alpha}$ and we choose $\theta_2 > \alpha$, then $(const.)^k \varepsilon^{-2\alpha k} e^{-(\beta/4)\varepsilon^{-\theta_2}}$ can be made smaller than $\varepsilon^{\gamma k}/4$ by taking ε small enough. Using (3.3.28) it follows that

$$\int d\mathbf{v}_K |E_K(t)| \leq \frac{2\varepsilon^{\gamma k}}{4} + (zC)^k (k!)^2 \sum_{\substack{L_0,Q \\ \subset K \\ \text{disjoint}}} \varepsilon^{\gamma_0''(k-q-l_0)} \tag{3.3.30}$$

$$\cdot \sum_{n=0}^{\log \varepsilon^{-\theta_0 k}} z^n \sum_{\substack{\mathbf{n}_k, \mathbf{\Gamma}_k \\ n=\sum_i n_i}} \int d\mathbf{v}_k d\Lambda \prod |B^\varepsilon| \chi^{rec}_{L_0} \chi^{ov}_{Q,K}\, e^{-(\beta/2)\sum_{i\in S(K)} v_i^2}\, \mathbb{1}_{\mathcal{H}_K \leq \varepsilon^{-\theta_2}} \ .$$

Finally, we shall simplify further the last expression, by eliminating the cross–section factors $\prod |B^\varepsilon|$. To do this, we apply Corollary 1, i.e. Eq. (3.2.2). Note that, with respect to that result, the only difference here is that the integral is computed by using the flow (3.3.6) instead of the IBF. This causes of course no modification, except for the expression of the characteristic function in (3.2.2).

Set

$$\mathbb{1}_{L_0}$$

the characteristic function ensuring that the particles created in the trees $\mathbf{\Gamma}_{L_0}$ do not overlap among each other, at the moments of creation. Similarly, set

$$\tilde{\mathbb{1}}_{K\setminus L_0}$$

the characteristic function ensuring that the particles created in the trees $\mathbf{\Gamma}_{K\setminus L_0}$ do not overlap "internally" (i.e. with particles of the same tree) at the moments of creation.

Then we obtain

$$\int d\mathbf{v}_K |E_K(t)| \leq \frac{2\varepsilon^{\gamma k}}{4} + (k!)^2 (2^2 C\bar{C}')^k \varepsilon^{\theta_1 k} + (zC)^k (k!)^2 \varepsilon^{-\theta_1 k} \sum_{\substack{L_0,Q \\ \subset K \\ \text{disjoint}}} \varepsilon^{\gamma_0''(k-q-l_0)}$$

$$\cdot \sum_{n=0}^{\log \varepsilon^{-\theta_0 k}} z^n \sum_{\substack{\mathbf{n}_k, \mathbf{\Gamma}_k \\ n=\sum_i n_i}} \int d\mathbf{v}_k d\Lambda\, \mathbb{1}_{L_0}\, \tilde{\mathbb{1}}_{K\setminus L_0}\, \chi^{rec}_{L_0} \chi^{ov}_{Q,K}\, e^{-(\beta/2)\sum_{i\in S(K)} v_i^2}\, \mathbb{1}_{\mathcal{H}_K \leq \varepsilon^{-\theta_2}}$$

$$\leq \frac{3\varepsilon^{\gamma k}}{4} + (zC)^k (k!)^2 \varepsilon^{-\theta_1 k} \sum_{\substack{L_0,Q \\ \subset K \\ \text{disjoint}}} \varepsilon^{\gamma_0''(k-q-l_0)}$$

$$\cdot \sum_{n=0}^{\log \varepsilon^{-\theta_0 k}} z^n \sum_{\substack{\mathbf{n}_k, \mathbf{\Gamma}_k \\ n=\sum_i n_i}} \int d\mathbf{v}_k d\Lambda\, \mathbb{1}_{L_0}\, \tilde{\mathbb{1}}_{K\setminus L_0}\, \chi^{rec}_{L_0} \chi^{ov}_{Q,K}\, e^{-(\beta/2)\sum_{i\in S(K)} v_i^2}\, \mathbb{1}_{\mathcal{H}_K \leq \varepsilon^{-\theta_2}} \ , \tag{3.3.31}$$

where in the last inequality we used again $k < \varepsilon^{-\alpha}$, took $\gamma < \theta_1 - 2\alpha$ and ε small enough.

Note that, in the remaining integral, the function $\mathbb{1}_{L_0}\tilde{\mathbb{1}}_{K\setminus L_0}$ ensures well posedness of the flow $\hat{\boldsymbol{\zeta}}^\varepsilon$ (which is needed to evaluate $\chi^{rec}_{L_0}\chi^{ov}_{Q,K}$).

3.2.6 Estimate (3.4) (III). Conclusion

We are left with the estimate of the "many–recollision integral" in the last line of (3.3.31). In the next sections we shall prove that there exists $C_2 > 0$ such that, for all $n \le \log \varepsilon^{-\theta_0 k}$, $Q, L_0 \subset K, Q \cap L_0 = \emptyset$ and $\gamma_1 \in (0,1)$,

$$
\begin{aligned}
& z^n \sum_{\substack{\mathbf{n}_k,\boldsymbol{\Gamma}_k \\ n=\sum_i n_i}} \int d\mathbf{v}_k d\Lambda \, \mathbb{1}_{L_0}\, \tilde{\mathbb{1}}_{K\setminus L_0}\, \chi^{rec}_{L_0}\chi^{ov}_{Q,K}(\hat{\boldsymbol{\zeta}}^\varepsilon)\, e^{-(\beta/2)\sum_{i\in S(K)} v_i^2}\, \mathbb{1}_{\mathcal{H}_K \le \varepsilon^{-\theta_2}} \\
& \le C_2^k k^{(5/2)k} \varepsilon^{\gamma_1(\frac{q+l_0-1}{2})} (C_2 t)^n \ . \qquad (3.3.32)
\end{aligned}
$$

The latter bound allows to conclude the proof of (3.4). Indeed by (3.3.31), using that $k < \varepsilon^{-\alpha}$ and choosing $t^* < C_2^{-1}$, we deduce that, for $t < t^*$,

$$
\begin{aligned}
& \sup_{\mathcal{M}^x_k(\delta)} \int_{\mathbb{R}^{3k}} d\mathbf{v}_k |E_K(t)| \\
& \le \frac{3\,\varepsilon^{\gamma k}}{4} + (zCC_2)^k (k!)^2 k^{(5/2)k} \varepsilon^{-\theta_1 k} \sum_{\substack{L_0,Q \\ \subset K \\ \text{disjoint}}} \varepsilon^{\gamma_0''(k-q-l_0)} \varepsilon^{\gamma_1(\frac{q+l_0-1}{2})} \sum_{n\ge 0} (C_2 t^*)^n \\
& \le \frac{3\,\varepsilon^{\gamma k}}{4} + C_3^k\, \varepsilon^{-(9/2)\alpha k}\, \varepsilon^{-\theta_1 k}\, \varepsilon^{\gamma_2(k-1)} \qquad (3.3.33)
\end{aligned}
$$

for some suitable $C_3 > 0$ and $\gamma_2 = \min(\gamma_0'', \gamma_1/2)$.

Taking $\gamma < \gamma_2/2 - (9/2)\alpha - \theta_1$, we conclude that

$$
\sup_{\mathcal{M}^x_k(\delta)} \int_{\mathbb{R}^{3k}} d\mathbf{v}_k |E_K(t)| \le \varepsilon^{\gamma k}
$$

for ε small enough. ■

Remarks.

(1) It can be shown that the above computations are optimized by the choice $\theta_0 = (1/\log(\bar{t}C(z,\beta)e)^{-1})\theta_1$ and $\theta_1 = (-5\alpha + \gamma_2)/4$. The proof of Lemma 5 will require also $\theta_2 < 2 - 2\alpha_0$. It follows that, with our estimates, the result is true for any $\gamma < (\gamma_2 - 13\alpha)/4$ and $\alpha < \min(\alpha_0, \gamma_2/13, 2 - 2\alpha_0)$.

(2) Note that we did not optimize the time interval $(0, t^*)$ for which bound (3.4) holds. In particular, the time t^* introduced above (see (3.3.29) and (3.3.33)) will be possibly

smaller than the value $\bar{t}$, appearing in Proposition 3 and ensuring Lanford's validity result. Nevertheless, our estimate of the correlation error can be easily extended up to $\bar{t}$, paying the price of worst values of $\gamma, \alpha, \varepsilon_0$.

Indeed, remind that in the first step of the proof, Eq.s (3.3.25)–(3.3.27), we wrote

$$\sup_{\mathcal{M}_k^x(\delta)} \int_{\mathbb{R}^{3k}} d\mathbf{v}_k |E_K(t)| \leq \sum_{n=0}^{\log \varepsilon^{-\theta_0 k}} (\cdots) + \sum_{n \geq \log \varepsilon^{-\theta_0 k}} (\cdots)$$

and showed that the second term is $O(\varepsilon^{\gamma k})$ for $t < \bar{t}$. Subsequently, we focused on the first term. In particular, observe that a trivial modification of our estimates implies

$$\sup_{\mathcal{M}_k^x(\delta)} \int_{\mathbb{R}^{3k}} d\mathbf{v}_k |E_K(t)| \leq \varepsilon^{\gamma k} \sum_{n=0}^{\log \varepsilon^{-\theta_0 k}} (t/\, t^*)^n + \varepsilon^{\gamma k} \,, \tag{3.3.34}$$

where $(t^*)^{-1} > \max(C(z, \beta/2)e, C_2)$ (see (3.3.29) and (3.3.33)), $\gamma < \theta_0 \log(\bar{t}/\, t) - 2\alpha$ and ε is smaller than a suitable $\varepsilon_0 = \varepsilon_0(t)$. Note that, in this context, ε_0, γ and α flow to zero as t approaches $\bar{t}$.

If $t^* < \bar{t}$, the limiting series in (3.3.34) is not convergent for all $t < \bar{t}$. Nevertheless, it is easy to see that

$$\sup_{\mathcal{M}_k^x(\delta)} \int_{\mathbb{R}^{3k}} d\mathbf{v}_k |E_K(t)| \leq \varepsilon^{\tilde{\gamma} k}$$

if $\tilde{\gamma} < \min(\gamma - \theta_0 \log(t/t^*), \gamma)$, as soon as θ_0 and ε are small enough.

3.2.7 Control of the many–recollision events

In this section we prove estimate (3.3.32).

We recall that, by virtue of $\chi^{rec}_{L_0} \chi^{ov}_{Q,K}(\hat{\boldsymbol{\zeta}}^\varepsilon)$ (see Eq. (3.3.6)):

(i) particles belonging to the trees in L_0 evolve with the flow $\boldsymbol{\zeta}^{\varepsilon,(L_0)}$ (interacting dynamics) in such a way that each tree shows at least one external recollision;

(ii) particles belonging to the trees in $K \setminus L_0$ evolve with the dynamics $\tilde{\boldsymbol{\zeta}}^{\varepsilon,(K\setminus L_0)}$ (each tree ignores the others), and

(iii) the trees in $Q \subset K \setminus L_0$ show at least one (external) overlap.

We start by reordering in time this set of constraints.

Definition 2 (table of recollisions) *Let $L_0, Q \subset K$, $L_0 \cap Q = \emptyset$. A "table of recollisions" associated to (K, L_0, Q) is a set of couples $\mathcal{C} = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \cdots\}$ with $\alpha_i \in L_0 \cup Q$ and $\beta_i \in K$, such that:*

– $\cup_i \alpha_i = L_0 \cup Q$;

– $\alpha_i \neq \alpha_1, \cdots, \alpha_{i-1}, \beta_1, \cdots, \beta_{i-1}$. *We call "bullet" a particle of type α and "target" a particle of type β.*

Note that the bullets α and the targets β are indices of the particles generating the trees Γ_α, Γ_β. An external recollision and/or overlap between Γ_α and Γ_β indicates a recollision and/or overlap between a pair of particles of the two trees.

In Definition 2, the α_i are all distinct and

$$|\mathcal{C}| \geq (q + l_0)/2 \; . \tag{3.3.35}$$

Moreover, the number of all the tables of recollision associated to (K, L_0, Q) is bounded as

$$\sum_{\mathcal{C}} \leq (q + l_0)!\, k^{q+l_0} \leq k!\, k^k \; . \tag{3.3.36}$$

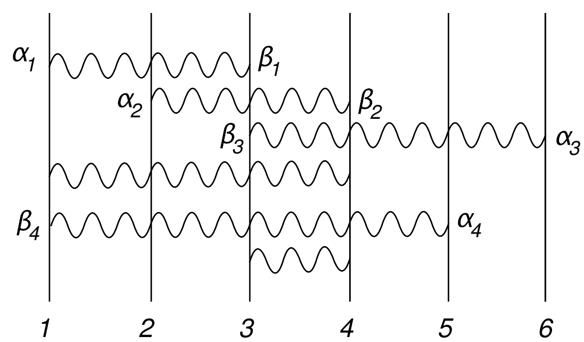

Figure 4: A scheme for a table of recollisions associated to $K = \{1, 2, \cdots, 6\}$, $Q \cup L_0 = \{1, 2, 5, 6\}$. Here $|\mathcal{C}| = 4$. The vertical lines can be associated to particles of a backwards flow and the wavy lines to their external recollisions / overlaps. In this case, the fourth and the last wavy lines represent recollisions (or overlaps) that do not appear in the table.

We introduce a new characteristic function:

$$\chi^{(\alpha,\beta)} = 1$$

if and only if the first overlap, in the dynamics $\tilde{\boldsymbol{\zeta}}^{\varepsilon,(\alpha)}$, of the tree Γ_α generated by the bullet α, occurs with the tree Γ_β generated by the target β.

Let $\hat{\boldsymbol{\zeta}}^{\varepsilon}_{\setminus\{S(\alpha_h),\, h \geq i\}}$ be constructed as the mixed flow $\hat{\boldsymbol{\zeta}}^{\varepsilon}$ but *ignoring* the particles of the bullet trees Γ_{α_h}, $h \geq i$. Note that, given a table of recollisions $\mathcal{C}$, the function $\chi^{(\alpha_i,\beta_i)}$ depends only on $(\hat{\boldsymbol{\zeta}}^{\varepsilon}_{\setminus\{S(\alpha_h),\, h \geq i\}}, \tilde{\boldsymbol{\zeta}}^{\varepsilon,(\alpha_i)})$.

Then we have the following:

Lemma 4 (ordering of events) *There holds*

$$\chi^{rec}_{L_0}\,\chi^{ov}_{Q,K}\big(\hat{\boldsymbol{\zeta}}^\varepsilon\big) \leq \sum_{\mathcal{C}} \prod_{(\alpha_i,\beta_i)\in\mathcal{C}} \chi^{(\alpha_i,\beta_i)}\left(\hat{\boldsymbol{\zeta}}^\varepsilon_{\backslash\{S(\alpha_h),\,h\geq i\}},\tilde{\boldsymbol{\zeta}}^{\varepsilon,(\alpha_i)}\right)\;, \tag{3.3.37}$$

where the sum is extended over all the table of recollisions associated to (K, L_0, Q).

Proof. For any value of $\hat{\boldsymbol{\zeta}}^\varepsilon$ such that $\chi^{rec}_{L_0}\,\chi^{ov}_{Q,K} = 1$, we may construct a table of recollisions as follows. The first backwards external recollision or overlap identifies the couple (α_1,β_1) (up to the exchange $\alpha_1 \leftrightarrow \beta_1$, if both particles are in $L_0\cup Q$). Going further backwards in time, we consider the first external recollision / overlap involving at least one tree in $L_0\cup Q$ and different from α_1,β_1. This identifies the couple (α_2,β_2), with the following constraint. If one (and only one) of the two trees involved is α_1 or β_1, we put such tree $=\ \beta_2$, and its partner $=\ \alpha_2$. We iterate this procedure until all the particles in $L_0\cup Q$ have received a name.

Once the table has been constructed observe that, if a bullet α_i recollides / overlaps with the target β_i in the dynamics $\hat{\boldsymbol{\zeta}}^\varepsilon$, then there must be also an overlap when we evolve the bullet with the dynamics $\tilde{\boldsymbol{\zeta}}^{\varepsilon,(\alpha_i)}$ and the target with the dynamics $\hat{\boldsymbol{\zeta}}^\varepsilon_{\backslash\{S(\alpha_h),\,h\geq i\}}$, since the considered recollision / overlap is the *first* (for the tree Γ_{α_i}) by going backwards in time. ■

(a) Reordering of the integrations in (3.3.32). Applying Lemma 4 to the left hand side of (3.3.32), we find

$$\begin{aligned} &z^n \sum_{\substack{\mathbf{n}_k,\boldsymbol{\Gamma}_k\\ n=\sum_i n_i}} \int d\mathbf{v}_k d\Lambda\, \mathbb{1}_{L_0}\, \tilde{\mathbb{1}}_{K\backslash L_0}\, \chi^{rec}_{L_0}\chi^{ov}_{Q,K}\big(\hat{\boldsymbol{\zeta}}^\varepsilon\big)\, e^{-(\beta/2)\sum_{i\in S(K)} v_i^2}\, \mathbb{1}_{\mathcal{H}_K\leq\varepsilon^{-\theta_2}} \\ &\leq z^n \sum_{\substack{\mathbf{n}_k,\boldsymbol{\Gamma}_k\\ n=\sum_i n_i}} \sum_{\mathcal{C}} \int d\mathbf{v}_k d\Lambda\, \mathbb{1}_{L_0}\, \tilde{\mathbb{1}}_{K\backslash L_0} \prod_{(\alpha_i,\beta_i)\in\mathcal{C}} \chi^{(\alpha_i,\beta_i)} e^{-(\beta/2)\sum_{i\in S(K)} v_i^2}\, \mathbb{1}_{\mathcal{H}_K\leq\varepsilon^{-\theta_2}}\;, \end{aligned} \tag{3.3.38}$$

where the $\chi^{(\alpha_i,\beta_i)}$ in the second line is evaluated via the flow

$$\left(\hat{\boldsymbol{\zeta}}^\varepsilon_{\backslash\{S(\alpha_h),\,h\geq i\}},\tilde{\boldsymbol{\zeta}}^{\varepsilon,(\alpha_i)}\right)\;. \tag{3.3.39}$$

To estimate the integral

$$z^n \sum_{\substack{\mathbf{n}_k,\boldsymbol{\Gamma}_k\\ n=\sum_i n_i}} \sum_{\mathcal{C}} \int d\mathbf{v}_k d\Lambda\, \mathbb{1}_{L_0}\, \tilde{\mathbb{1}}_{K\backslash L_0} \prod_{(\alpha_i,\beta_i)\in\mathcal{C}} \chi^{(\alpha_i,\beta_i)} e^{-(\beta/2)\sum_{i\in S(K)} v_i^2}\, \mathbb{1}_{\mathcal{H}_K\leq\varepsilon^{-\theta_2}}\;, \tag{3.3.40}$$

we will exploit the *first* bullet particles of the table $\mathcal{C}$. Then, putting

$$\mathcal{C} = \{(\alpha_1,\beta_1),(\alpha_2,\beta_2),\cdots\}$$

and recalling (3.3.35), we estimate simply with 1 the factors $\chi^{(\alpha_i,\beta_i)}$ with $i > (q+l_0)/2$.

Abbreviating

$$\ell = \lfloor (q+l_0)/2 \rfloor \,, \tag{3.3.41}$$

$$A = K \setminus \{\alpha_1, \alpha_2, \cdots, \alpha_\ell\} \,,$$

$$d\Lambda_A = \prod_{i\in A} \int d\Lambda(\mathbf{t}^i_{n_i}, \boldsymbol{\omega}^i_{n_i}, \mathbf{v}^i_{1,n_i}) \,,$$

$$\mathbb{1}^{(i)} = \mathbb{1}_{L_0\setminus\{\alpha_h, h\geq i\}}\, \tilde{\mathbb{1}}_{K\setminus(L_0\cup\{\alpha_h,h\geq i\})}$$

and ordering the integrals, we obtain

$$\begin{aligned}
(3.3.40) \leq z^n \sum_{\mathcal{C}} \Big\{ \sum_{\mathbf{n}_A, \boldsymbol{\Gamma}_A} \int d\mathbf{v}_A d\Lambda_A e^{-(\beta/2)\sum_{i\in S(A)} v_i^2}\, \mathbb{1}_{\mathcal{H}_A \leq \varepsilon^{-\theta_2}} \\
\cdot \sum_{n_{\alpha_1}, \Gamma_{\alpha_1}} \int dv_{\alpha_1} d\Lambda_{\alpha_1}\, \mathbb{1}^{(1)}\, \tilde{\mathbb{1}}_{\alpha_1}\, \chi^{(\alpha_1,\beta_1)}\, e^{-(\beta/2)\sum_{i\in S(\alpha_1)} v_i^2}\, \mathbb{1}_{\mathcal{H}_{\alpha_1} \leq \varepsilon^{-\theta_2}} \\
\cdot \sum_{n_{\alpha_2}, \Gamma_{\alpha_2}} \int dv_{\alpha_2} d\Lambda_{\alpha_2} \cdots \\
\cdots \\
\cdot \sum_{n_{\alpha_\ell}, \Gamma_{\alpha_\ell}} \mathbb{1}_{n=\sum_i n_i} \int dv_{\alpha_\ell} d\Lambda_{\alpha_\ell}\, \mathbb{1}^{(\ell)}\, \tilde{\mathbb{1}}_{\alpha_\ell}\, \chi^{(\alpha_\ell,\beta_\ell)} e^{-(\beta/2)\sum_{i\in S(\alpha_\ell)} v_i^2}\, \mathbb{1}_{\mathcal{H}_{\alpha_\ell} \leq \varepsilon^{-\theta_2}} \Big\} \,.
\end{aligned} \tag{3.3.42}$$

Note again that, in each integral, the indicator functions $\mathbb{1}^{(i)}\, \tilde{\mathbb{1}}_{\alpha_i}$ are needed only to ensure well posedness of the flow (3.3.39) appearing in the argument of $\chi^{(\alpha_i,\beta_i)}$. Moreover, the function $\mathbb{1}_{n=\sum_i n_i}$ in the last line takes into account that the total number of created particles has been fixed to n. Remind that $n \leq \log \varepsilon^{-\theta_0 k} < \varepsilon^{-\alpha_0}$.

(b) Estimate of a single external recollision. Our purpose is to estimate iteratively the above expression, starting from the last line and going upwards. At each step, we have a single tree with backwards flow overlapping an external, fixed (possibly complicated) trajectory. Indeed, with the above decomposition we have reduced the problem to the following bound:

Lemma 5 (estimate of the external recollision) *Let $s \to \eta(s)$ be a piecewise constant function in $\mathbb{R}^3$ with at most $\varepsilon^{-\alpha_0}$ jumps over $(0,t)$, with $\alpha_0 > 0$, and such that $|\eta| \leq \varepsilon^{-\theta_2/2}$ with $\theta_2 > 0$. Let $s \to \xi(s)$ be a piecewise–free trajectory in $\mathbb{R}^3$ with velocity*

$$\frac{d\xi}{ds} = \eta \tag{3.3.43}$$

except on the times of discontinuity of η, where jumps of entity ε may possibly occur ($|\xi(s^+) - \xi(s^-)| = \varepsilon$). Fix $x_1 \in \mathbb{R}^3$ and assume $|x_1 - \xi(t)| > \varepsilon^{\theta_2/2}$. Then there exists *a choice of θ_2 and a constant $D > 0$ such that, for any $\gamma_1 < 1$, $n_1 < \varepsilon^{-\alpha_0}$ and ε small enough,*

$$F^{\xi,n_1}(x_1,t) := \sum_{\Gamma(1,n_1)} \int dv_1 d\Lambda\, \tilde{\mathbb{1}}_1\, \chi^{ov}_\xi\, e^{-(\beta/2)\sum_{i\in S(1)} v_i^2}\, \mathbb{1}_{\mathcal{H}_1 \leq \varepsilon^{-\theta_2}} \leq \varepsilon^{\gamma_1}(Dt)^{n_1}\ , \tag{3.3.44}$$

where the integral in the left hand side is evaluated through the IBF $\boldsymbol{\zeta}^\varepsilon = (\zeta_1^\varepsilon, \cdots, \zeta_{1+n_1}^\varepsilon)$ associated to the 1–particle, n_1-collision tree $\Gamma(1,n_1)$, and $\chi^{ov}_\xi = \chi^{ov}_\xi(\boldsymbol{\zeta}^\varepsilon)$ is the indicator function of the event

$$\min_{i=1,\cdots,n_1+1} \inf_{s\in(0,t_{i-1})} |\xi(s) - \xi_i^\varepsilon(s)| < \varepsilon\ .$$

In the last formula we used the convention (2.2.14) (hence $(0, t_{i-1})$ is the time of existence of particle i in the IBF). Of course, since in Eq. (3.3.44) a single tree appears, $\boldsymbol{\zeta}^\varepsilon \equiv \tilde{\boldsymbol{\zeta}}^\varepsilon$. Observe also that here $\sum_{\Gamma(1,n_1)}$ is the sum over all 1−particle trees with n_1 (fixed) creations.

Before going through the proof, for which we refer to the following section 3.2.8, let us check the implications of Lemma 5. We shall deduce from it the final result of this section, i.e. (3.3.32).

To do that, it is convenient to introduce first the notion of virtual trajectory associated to a particle i. Following [19], this trajectory, denoted with upper indices, e.g. $\zeta^{\varepsilon,i}(s)$ in the IBF case, coincides with $\zeta_i^\varepsilon(s)$ for $s > 0$ and up to the time of creation of i. Then, it is extended by the trajectory of its ancestor up to its creation time, and so on. More precisely:

Definition 3 (virtual trajectory) *Consider particle i in the graph of a tree $\Gamma(k,n) = (k_1, \cdots, k_n)$. Let $\mathbf{t}_n = t_1, \cdots, t_n$ be the sequence of times associated to the nodes of the tree.*

(i) A polygonal path p_i is uniquely defined if we walk on the tree by going forward in time, starting from the time–zero endpoint of line i and going up to the root–point at time t (e.g. Figure 5).

(ii) Let $t_{i_1}, \cdots, t_{i_{n^i}}$ be the (decreasing) subsequence of $t_1, \cdots, t_n$, made of the times corresponding to the nodes met by following the path p_i (n^i being the number of such nodes, with the convention $i_0 = 0, t_{i_0} = t$). Then, for any backwards flow $\bar{\boldsymbol{\zeta}}$ which can be constructed from $\Gamma(k,n), \mathbf{t}_n$ [2]*, we call* **virtual trajectory associated to particle i in the flow***, and*

[2] In this definition, $\bar{\boldsymbol{\zeta}}$ can be either the IBF $\boldsymbol{\zeta}^\varepsilon$, the uncorrelated flow $\tilde{\boldsymbol{\zeta}}^\varepsilon$, the EBF $\boldsymbol{\zeta}^{\mathscr{E}}$ or a mixed flow as $\hat{\boldsymbol{\zeta}}^\varepsilon$. We shall use it in different contexts.

indicate it with upper indices $\bar{\zeta}^i(s) = (\bar{\xi}^i(s), \bar{\eta}^i(s)) \in \mathbb{R}^6$, $s \in [0,t]$, *the trajectory given by:*

$$\bar{\zeta}^i(s) = \begin{cases} \bar{\zeta}_i(s) & \textit{for } s \in [0, t_{i_{n^i}}) \\ \bar{\zeta}_{k_{i_r}}(s) & \textit{for } s \in [t_{i_r}, t_{i_{r-1}}), \quad 0 < r \leq n^i \end{cases} . \tag{3.3.45}$$

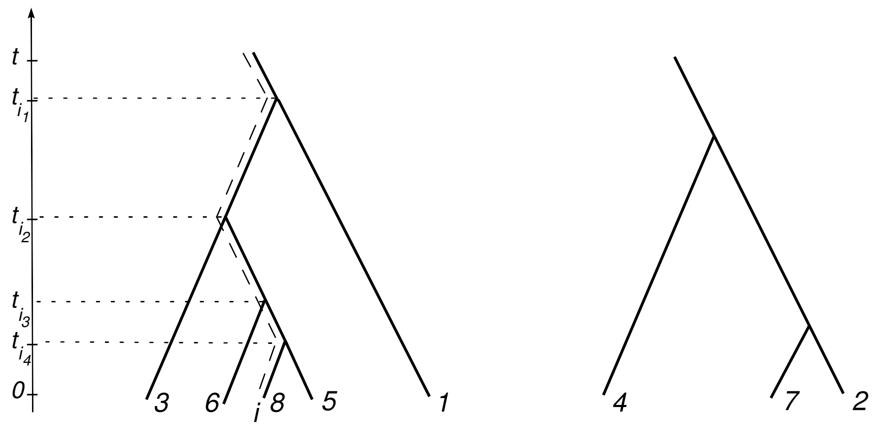

Figure 5: The line closest to the dashed line is the path p_i in the tree $\Gamma(2,6)$, with $i = 8$. The states of the particle associated to it via the flow $\bar{\boldsymbol{\zeta}}$ form the"virtual trajectory of i".

Note that the virtual trajectory is piecewise–free, and built up with pieces of trajectories of (different) particles of $\bar{\boldsymbol{\zeta}}$. Instantaneous jumps of entity ε occur at creation times, when the name of the particle in the flow $\bar{\boldsymbol{\zeta}}$ changes (e.g. t_{i_1}, t_{i_2} and t_{i_4} in Figure 5). Only during the time of existence of particle i in the flow, $\bar{\zeta}^i(s) = \bar{\zeta}_i(s)$.

(c) Iterative estimate of (3.3.42) **and conclusion.** The functions $\chi^{(\alpha_i,\beta_i)}$ appearing in (3.3.42) do not specify which particles in the trees Γ_α and Γ_β realize the condition of first external overlap. Therefore, we start with $\chi^{(\alpha_\ell,\beta_\ell)} \leq \sum_{i' \in S(\beta_\ell)} \chi_{i'}^{(\alpha_\ell,\beta_\ell)}$, where the index i' specifies which particle in Γ_{β_ℓ} is exactly the "target".

Clearly if the bullet tree Γ_{α_ℓ} (evolved with the uncorrelated dynamics $\tilde{\boldsymbol{\zeta}}^{\varepsilon,(\alpha_\ell)}$) overlaps with particle i' (evolved with the flow $\hat{\zeta}^{\varepsilon}_{\setminus\{S(\alpha_h),\,h\geq i\},i}$), then it must also overlap with the *virtual* trajectory associated to i' (i.e. $\hat{\zeta}^{\varepsilon,i'}_{\setminus\{S(\alpha_h),\,h\geq\ell\}}$). It follows that

$$\chi^{(\alpha_\ell,\beta_\ell)}\left(\hat{\boldsymbol{\zeta}}^{\varepsilon}_{\setminus\{S(\alpha_h),\,h\geq\ell\}}, \tilde{\boldsymbol{\zeta}}^{\varepsilon,(\alpha_\ell)}\right) \leq \sum_{i' \in S(\beta_\ell)} \chi^{ov}_{\hat{\xi}^{\varepsilon,i'}_{\setminus\{S(\alpha_h),\,h\geq\ell\}}}(\tilde{\boldsymbol{\zeta}}^{\varepsilon,(\alpha_\ell)}) ,$$

where χ^{ov}_{ξ} has been defined in Lemma 5, and we used Definition 3 (applying it to the flow $\hat{\zeta}^{\varepsilon}_{\setminus\{S(\alpha_h),\,h\geq i\},i}$).

Hence, using the notation of (3.3.44) (with particle 1 replaced by particle α_ℓ), the last line in (3.3.42) is bounded by

$$\sum_{n_{\alpha_\ell},\Gamma_{\alpha_\ell}} \mathbb{1}_{n=\sum_i n_i} \int dv_{\alpha_\ell} d\Lambda_{\alpha_\ell}\, \mathbb{1}^{(\ell)}\, \tilde{\mathbb{1}}_{\alpha_\ell}\, \chi^{(\alpha_\ell,\beta_\ell)} e^{-(\beta/2)\sum_{i\in S(\alpha_\ell)} v_i^2}\, \mathbb{1}_{\mathcal{H}_{\alpha_\ell}\le\varepsilon^{-\theta_2}}$$
$$\le \sum_{n_{\alpha_\ell}} \mathbb{1}_{n=\sum_i n_i}\, \mathbb{1}^{(\ell)} \sum_{i'\in S(\beta_\ell)} F^{\hat{\xi}^{\varepsilon,i'}_{\setminus\{S(\alpha_h),\,h\ge\ell\}},n_{\alpha_\ell}}(x_{\alpha_\ell},t) \tag{3.3.46}$$

(where $\mathbb{1}^{(\ell)}$ ensures existence of the flow $\hat{\zeta}^{\varepsilon,i'}_{\setminus\{S(\alpha_h),\,h\ge\ell\}}$).

Recall that we are assuming $\mathbf{x}_k \in \mathcal{M}^x_k(\delta)$ (see (3.1)). Taking $\delta = \varepsilon^{\theta_2/2}$, we may apply Lemma 5:

$$\sum_{n_{\alpha_\ell}} \mathbb{1}_{n=\sum_i n_i}\, \mathbb{1}^{(\ell)} \sum_{i'\in S(\beta_\ell)} F^{\hat{\xi}^{\varepsilon,i'}_{\setminus\{S(\alpha_h),\,h\ge\ell\}},n_{\alpha_\ell}}(x_{\alpha_\ell},t) \le \varepsilon^{\gamma_1}(1+n)\sum_{n_{\alpha_\ell}} \mathbb{1}_{n=\sum_i n_i}\,(Dt)^{n_{\alpha_\ell}}\ . \tag{3.3.47}$$

Inserting into Eq. (3.3.42) and using

$$\chi^{(\alpha_{\ell-1},\beta_{\ell-1})}\left(\hat{\boldsymbol{\zeta}}^{\varepsilon}_{\setminus\{S(\alpha_h),\,h\ge\ell-1\}}, \tilde{\boldsymbol{\zeta}}^{\varepsilon,(\alpha_{\ell-1})}\right) \le \sum_{i'\in S(\beta_{\ell-1})} \chi^{ov}_{\hat{\xi}^{\varepsilon,i'}_{\setminus\{S(\alpha_h),\,h\ge\ell-1\}}}(\tilde{\boldsymbol{\zeta}}^{\varepsilon,(\alpha_{\ell-1})})\ ,$$

we obtain

$$(3.3.40) \le \left[\varepsilon^{\gamma_1}(1+n)z^n\right]\sum_{n_{\alpha_\ell}}(Dt)^{n_{\alpha_\ell}} \sum_{\mathcal{C}}\sum_{\mathbf{n}_A,\boldsymbol{\Gamma}_A}\cdots\sum_{n_{\alpha_1},\Gamma_{\alpha_1}}\cdots\sum_{\substack{n_{\alpha_{\ell-2}}\\ \Gamma_{\alpha_{\ell-2}}}}\cdots$$
$$\cdot\sum_{n_{\alpha_{\ell-1}}} \mathbb{1}_{n=\sum_i n_i}\, \mathbb{1}^{(\ell-1)} \sum_{i'\in S(\beta_{\ell-1})} F^{\hat{\xi}^{\varepsilon,i'}_{\setminus\{S(\alpha_h),\,h\ge\ell-1\}},n_{\alpha_{\ell-1}}}(x_{\alpha_{\ell-1}},t)\ . \tag{3.3.48}$$

We apply again Lemma 5 to the last line, gaining a factor

$$\varepsilon^{\gamma_1}(1+n)\sum_{n_{\alpha_{\ell-1}}} \mathbb{1}_{n=\sum_i n_i}\,(Dt)^{n_{\alpha_{\ell-1}}}\ .$$

We iterate this procedure until all the sums over the bullet trees have disappeared. After $\ell = \lfloor (q+l_0)/2 \rfloor \le k/2$ steps, we get

$$(3.3.40) \le \left[\varepsilon^{\gamma_1(\frac{q+l_0-1}{2})}(1+n)^{k/2}z^n\right]\sum_{n_{\alpha_1}}(Dt)^{n_{\alpha_1}}\cdots\sum_{n_{\alpha_\ell}}(Dt)^{n_{\alpha_\ell}} \tag{3.3.49}$$
$$\cdot\sum_{\mathcal{C}}\sum_{\mathbf{n}_A,\boldsymbol{\Gamma}_A} \mathbb{1}_{n=\sum_i n_i}\int d\mathbf{v}_A d\Lambda_A\, e^{-(\beta/2)\sum_{i\in S(A)} v_i^2}\ .$$

Using (3.3.36) and taking D large enough (so that the last line above is bounded by $k^{2k} D^{|A|} \sum_{\mathbf{n}_A} \mathbb{1}_{n=\sum_i n_i} (Dt)^{|\mathbf{n}_A|}$) we finally have

$$(3.3.40) \leq \varepsilon^{\gamma_1 (\frac{q+l_0-1}{2})} (1+n)^{k/2} (zDt)^n k^{2k} D^k \sum_{\substack{n_1, \cdots, n_k \\ n=\sum_i n_i}} 1 \ . \tag{3.3.50}$$

Since the last sum can be bounded by e^{k+n} (see e.g. (3.2.8)) and since

$$(eD)^k (1+n)^{k/2} (ezD)^n \leq k^{k/2} C_2^n$$

for a suitable large $C_2 > 0$, we conclude that

$$(3.3.40) \leq C_2^k k^{(5/2)k} \varepsilon^{\gamma_1 (\frac{q+l_0-1}{2})} (C_2 t)^n \ . \tag{3.3.51}$$

Estimate (3.3.32) is proved. ■

3.2.8 Proof of Lemma 5

(i) Substitution of the IBF with the EBF. The integral $F^{\xi, n_1}(x_1, t)$ involves an IBF associated to a 1−particle tree, in which some particle needs to "overlap" with a given external trajectory, namely $\xi(s)$. The IBF involves of course also internal recollisions, which is convenient to eliminate first. Let

$$\chi^{i.r.} = \chi^{i.r.}(\boldsymbol{\zeta}^\varepsilon) = 1 \tag{3.3.52}$$

if and only if the IBF associated to the 1−particle tree delivers an internal recollision.

Lemma 6 (estimate of the internal recollision) *There exists a constant $D > 0$ such that, for any $\gamma_1 < 1$ and ε small enough,*

$$\sum_{\Gamma(1,n_1)} \int dv_1 d\Lambda \, \tilde{\mathbb{1}}_1 \, \chi^{i.r.} \, e^{-(\beta/2) \sum_{i \in S(1)} v_i^2} \leq \frac{\varepsilon^{\gamma_1}}{2} (Dt)^{n_1} \ . \tag{3.3.53}$$

Proof. It is convenient to use the Enskog backwards flow $\boldsymbol{\zeta}^{\mathscr{E}}$ introduced in Section 2.2.4. For any given value of the variables $(x_1, \Gamma(1,n_1), \mathbf{v}_{n_1+1}, \boldsymbol{\omega}_{n_1}, \mathbf{t}_{n_1})$, if the IBF $\boldsymbol{\zeta}^\varepsilon$ delivers an internal recollision, then the EBF $\boldsymbol{\zeta}^{\mathscr{E}}$ delivers an *overlap* (two particles of the flow having a distance smaller than ε). That is,

$$\left(\tilde{\mathbb{1}}_1 \, \chi^{i.r.}\right) (\boldsymbol{\zeta}^\varepsilon) \leq \chi^{i.o.}(\boldsymbol{\zeta}^{\mathscr{E}}) \ , \tag{3.3.54}$$

where

$$\chi^{i.o.} = \chi^{i.o.}(\boldsymbol{\zeta}^{\mathscr{E}}) = 1 \tag{3.3.55}$$

if and only if the EBF associated to the 1−particle tree shows at least one overlap between two particles. In what follows we shall prove the estimate

$$\sum_{\Gamma(1,n_1)} \int dv_1 d\Lambda\, \chi^{i.o.}\, e^{-(\beta/2)\sum_{i\in S(1)} v_i^2} \leq \frac{\varepsilon^{\gamma_1}}{2}(Dt)^{n_1}\ . \tag{3.3.56}$$

We start with

$$\chi^{i.o.} \leq \sum_{s=0}^{n_1} \sum_{\substack{i,h=1\\ i\neq h}}^{1+s} \chi^{i.o.}_{(i,h),s}\ , \tag{3.3.57}$$

where $\chi^{i.o.}_{(i,h),s} = \chi^{i.o.}_{(i,h),s}(\boldsymbol{\zeta}^{\mathscr{E}}) = 1$ if and only if:
(i) after (going backwards) time t_s in the EBF, free evolution leads particles i and h to overlap:

$$\inf_{\tau\in(0,t_s)} \Big| \big(\xi_i^{\mathscr{E}}(t_s) - \xi_h^{\mathscr{E}}(t_s)\big) - (\eta_i^{\mathscr{E}}(t_s^-) - \eta_h^{\mathscr{E}}(t_s^-))(t_s-\tau)\Big| < \varepsilon\ ;$$

(ii) at time t_s, the virtual trajectory of particle h undergoes a deviation in velocity:

$$\eta^{\mathscr{E},h}(t_s^-) \neq \eta^{\mathscr{E},h}(t_s^+)\ .$$

Note that we are using virtual trajectories (Definition 3, applied to $\bar{\boldsymbol{\zeta}} = \boldsymbol{\zeta}^{\mathscr{E}}$). Clearly, condition (ii) implies that particle h is involved in the creation process at time t_s. Furthermore, we are excluding situations as the one in case 2 of Figure 6 below with an incoming collision configuration at the creation time t_s.

From (3.3.56)–(3.3.57) one gets

$$\begin{aligned} &\sum_{\Gamma(1,n_1)} \int dv_1 d\Lambda\, \chi^{i.o.}\, e^{-(\beta/2)\sum_{i\in S(1)} v_i^2} \\ &\leq \sum_{s=0}^{n_1} \sum_{\substack{i,h=1\\ i\neq h}}^{1+s} \sum_{\Gamma(1,n_1)} \int dv_1 d\Lambda\, \chi^{i.o.}_{(i,h),s}\, e^{-(\beta/2)\sum_{i\in S(1)} v_i^2}\ . \end{aligned} \tag{3.3.58}$$

Remind that $d\Lambda = d\Lambda(\mathbf{t}_{n_1}, \boldsymbol{\omega}_{n_1}, \mathbf{v}_{1,1+n_1})$ and $\Gamma(1,n_1) = (k_1,\cdots,k_{n_1})$. Observe that $\chi^{i.o.}_{(i,h),s}$ depends actually only on $\boldsymbol{\zeta}^{\mathscr{E}}_{1+s}$. Integrating in the node variables

$$t_{s+1},\cdots,t_{n_1},\omega_{s+1}\cdots,\omega_{n_1},v_{s+2},\cdots,v_{1+n_1}$$

and summing over the tree variables $k_{s+1},\cdots,k_{n_1}$ (as in the proof of Lemma 2), we infer that for some suitable $D'>0$

$$\begin{aligned} &\sum_{\Gamma(1,n_1)} \int dv_1 d\Lambda\, \chi^{i.o.}\, e^{-(\beta/2)\sum_{i\in S(1)} v_i^2} \leq \sum_{s=0}^{n_1} \sum_{\substack{i,h=1\\ i\neq h}}^{1+s} (D't)^{n_1-s} \\ &\cdot \sum_{\Gamma(1,s)} \int dv_1 d\Lambda(\mathbf{t}_s, \boldsymbol{\omega}_s, \mathbf{v}_{1,1+s})\, \chi^{i.o.}_{(i,h),s}\, e^{-(\beta/2)\sum_{i=1}^{1+s} v_i^2}\ , \end{aligned} \tag{3.3.59}$$

where in the last line we are left with integrals associated to 1−particle, s−collision trees.

If $\chi^{i.o.}_{(i,h),s} = 1$, then there are two possibilities: either $h = s+1$ (h is created at t_s) or $k_s = h$ (h is the progenitor of $s+1$); see Figure 6. Let us resort again to the notation of

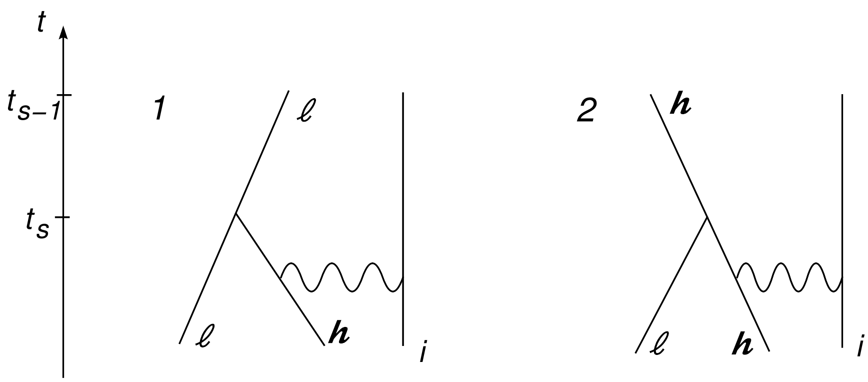

Figure 6: Case 1: $h = s+1$, $k_s = \ell$. Case 2: $h = k_s$, $\ell = s+1$.

virtual trajectories, to deal with both cases simultaneously. We set

$$W = \eta^{\mathscr{E}}_h(t_s^-) - \eta^{\mathscr{E}}_i(t_s) \, , \qquad W_0 = \eta^{\mathscr{E},h}(t_s^+) - \eta^{\mathscr{E}}_i(t_s)$$

and

$$Y = \xi^{\mathscr{E}}_h(t_s) - \xi^{\mathscr{E}}_i(t_s) \, , \qquad Y_0 = \xi^{\mathscr{E},h}(t_{s-1}) - \xi^{\mathscr{E}}_i(t_{s-1}) \, .$$

We remind that t^+, t^- denote the limit from the future (post–collision) or from the past (pre–collision) respectively.

The overlap–condition implies

$$\inf_{\tau \in (0,t_s)} |Y - W\tau| \leq \varepsilon \, . \tag{3.3.60}$$

Putting $\hat{W} = \frac{W}{|W|}$, (3.3.60) implies in turn

$$|Y \wedge \hat{W}| \leq \varepsilon \, ,$$

i.e.

$$|(Y_0 - W_0 t_{s-1}) \wedge \hat{W} + (W_0 \wedge \hat{W}) t_s| \leq \varepsilon \, . \tag{3.3.61}$$

Therefore, we may bound the last line in (3.3.59), by replacing $\chi^{i.o.}_{(i,h),s}$ with the indicator function of the events (3.3.61) and $W \neq W_0$ (which takes into account condition (ii) above).

By definition of the Enskog flow, Y_0 and W_0 do not depend on t_s (since they concern the previous history). Moreover, the velocities in (t_{s+1}, t_s), which we indicate

$$(\eta_1^-, \cdots, \eta_{s+1}^-) = (\eta^{\mathscr{E}}_1(t_s^-), \cdots, \eta^{\mathscr{E}}_{s+1}(t_s^-)) \, , \tag{3.3.62}$$

are also independent on the times $\mathbf{t}_s$. They depend only on previous velocities and impact vectors. In particular, W does not depend on t_s, so that in (3.3.61) a *linear* relation in t_s appears. On the other hand, the integral in t_s over the condition (3.3.61) is bounded by $\min(t, 2\varepsilon|W_0 \wedge \hat{W}|^{-1})$.

Hence there holds, for $b \in (0, 1)$ arbitrary,

$$\begin{aligned}&\sum_{\Gamma(1,s)} \int dv_1 d\Lambda(\mathbf{t}_s, \boldsymbol{\omega}_s, \mathbf{v}_{1,1+s})\ \chi^{i.o.}_{(i,h),s}\, e^{-(\beta/2)\sum_{i=1}^{1+s} v_i^2} \\ &\leq (2/t)^b t\, \varepsilon^b \sum_{\Gamma(1,s)} \int dv_1 d\Lambda'(\mathbf{t}_{s-1}, \boldsymbol{\omega}_s, \mathbf{v}_{1,1+s})\, \frac{1}{|W_0 \wedge \hat{W}|^b}\, e^{-(\beta/2)\sum_{i=1}^{1+s} v_i^2}\ ,\end{aligned} \tag{3.3.63}$$

where $d\Lambda'(\mathbf{t}_{s-1}, \boldsymbol{\omega}_s, \mathbf{v}_{1,1+s})$ is the measure $d\Lambda(\mathbf{t}_s, \boldsymbol{\omega}_s, \mathbf{v}_{1,1+s})$ deprived of dt_s and multiplied, in case 2 of Figure 6, by the characteristic function of $\omega_s \cdot (v_{1+s} - \eta^{\mathscr{E},h}(t_s^+)) > 0$.

It remains to bound the integral of the singular function $|W_0 \wedge \hat{W}|^{-b}$. To do so, let us first express W_0 in terms of the pre–collisional variables (3.3.62). Applying the elastic collision rule, Eq. (2.1.3), one readily finds

$$\begin{aligned} W_0 \quad &= \left(\eta^{\mathscr{E},h}(t_s^+) - \eta^{\mathscr{E},h}(t_s^-)\right) + W \\ &= P_s W_\ell + W\ ,\end{aligned}$$

where

$$W_\ell = \eta_\ell^- - \eta_h^-$$

and

$$P_s X := \begin{cases} P^\perp_{\omega_s} X := X - \omega_s(\omega_s \cdot X) & \text{case 1, outgoing collision} \\ X & \text{case 1, incoming collision} \\ P^\parallel_{\omega_s} X := \omega_s(\omega_s \cdot X) & \text{case 2, outgoing collision} \\ 0 & \text{case 2, incoming collision} \end{cases}\ , \tag{3.3.64}$$

where cases $1, 2$ are those in Figure 6, while the incoming / outgoing collisions are depicted in Figure 2 on page 18 (here corresponding respectively to the negative / positive sign of the scalar product $\omega_s \cdot (v_{1+s} - \eta^{\mathscr{E},h}(t_s^+))$).

Observe that the "case" depends only on the structure of the chosen tree $\Gamma(1, s)$. Fixed one such a tree, it follows that

$$\int dv_1 d\Lambda'(\mathbf{t}_{s-1}, \boldsymbol{\omega}_s, \mathbf{v}_{1,1+s})\, \frac{e^{-(\beta/2)\sum_{i=1}^{1+s} v_i^2}}{|W_0 \wedge \hat{W}|^b} \leq \frac{t^{s-1}}{(s-1)!} \int d\boldsymbol{\omega}_s \int d\mathbf{v}_{s+1} \frac{e^{-(\beta/2)\sum_{i=1}^{1+s} v_i^2}}{|\tilde{P}_s W_\ell \wedge \hat{W}|^b}\ , \tag{3.3.65}$$

where $\tilde{P}_s = P_s$ except for "case 2, incoming collision", for which we put for simplicity $\tilde{P}_s X = X$.

We change variables according to $v_1, v_2, \cdots, v_{s+1} \to \eta_1^-, \cdots, \eta_{s+1}^-$. This is an invertible and measure–preserving transformation, for any fixed value of $\omega_1, \cdots, \omega_s$, (since the single hard–sphere collision rule (2.1.3) is so). Using the conservation of energy at collisions, we obtain

$$\begin{aligned}\int d\mathbf{v}_{s+1} \frac{e^{-(\beta/2)\sum_{i=1}^{1+s} v_i^2}}{|\tilde{P}_s W_\ell \wedge \hat{W}|^b} &= \int d\boldsymbol{\eta}_{s+1}^- \frac{e^{-(\beta/2)\sum_{i=1}^{1+s}(\eta_i^-)^2}}{|\tilde{P}_s W_\ell \wedge \hat{W}|^b} . \\ &\leq \int d\boldsymbol{\eta}_{s+1}^- e^{-(\beta/2)\sum_{i\neq \ell,h}(\eta_i^-)^2} \frac{e^{-(\beta/8)(W_\ell^2+W^2)}}{|\tilde{P}_s W_\ell \wedge \hat{W}|^b} ,\end{aligned} \tag{3.3.66}$$

where in the last step we also used that $W^2 + W_\ell^2 \leq 4(\eta_\ell^2 + \eta_h^2)$.

Applying the translations $(\eta_\ell^-, \eta_h^-) \to (W_\ell = \eta_\ell^- - \eta_h^-, W = \eta_h^- - \eta_i^-)$,

$$\int d\mathbf{v}_{s+1} \frac{e^{-(\beta/2)\sum_{i=1}^{1+s} v_i^2}}{|\tilde{P}_s W_\ell \wedge \hat{W}|^b} = \int d\boldsymbol{\eta}_{s+1}'^- e^{-(\beta/2)\sum_{i\neq \ell,h}(\eta_i^-)^2} \int dW_\ell dW \frac{e^{-(\beta/8)(W_\ell^2+W^2)}}{|\tilde{P}_s W_\ell \wedge \hat{W}|^b} ,$$

where $\boldsymbol{\eta}_{s+1}'^-$ is the collection $\boldsymbol{\eta}_{s+1}^-$ deprived of η_ℓ, η_h.

The last integral can be easily computed. It is

$$\int dW_\ell dW \frac{e^{-(\beta/8)(W_\ell^2+W^2)}}{|\tilde{P}_s W_\ell \wedge \hat{W}|^b} = C_\beta \int dW_\ell \frac{e^{-(\beta/8)W_\ell^2}}{|P_s W_\ell|^b} = C_{\beta,b} , \tag{3.3.67}$$

for suitable constants $C_\beta, C_{\beta,b} > 0$ (possibly diverging as $b \to 1$).

From the two last equations we conclude that the integral over velocities in (3.3.65) is bounded by $(C'_{\beta,b})^s$ for some constant $C'_{\beta,b} > 0$. Thus, from (3.3.63),

$$\sum_{\Gamma(1,s)} \int dv_1 d\Lambda(\mathbf{t}_s, \boldsymbol{\omega}_s, \mathbf{v}_{1,1+s})\ \chi_{(i,h),s}^{i.o.}\, e^{-(\beta/2)\sum_{i=1}^{1+s} v_i^2} \leq (D''t)^s \varepsilon^b \tag{3.3.68}$$

for some $D'' > 0$.

Inserting this into Eq. (3.3.59) and performing the sums, we obtain the final result. ■

We come back to the proof of Lemma 5. By Lemma 6, we can bound $F^{\xi,n_1}(x_1,t)$ by a much simpler expression, i.e.

$$\begin{aligned}F^{\xi,n_1}(x_1,t) &\leq \frac{\varepsilon^{\gamma_1}}{2}(Dt)^{n_1} + \sum_{\Gamma(1,n_1)} \int dv_1 d\Lambda\, \tilde{\mathbb{1}}_1 \left(1-\chi^{i.r.}\right) \chi_\xi^{ov}(\boldsymbol{\zeta}^\varepsilon)\, e^{-(\beta/2)\sum_{i\in S(1)} v_i^2}\, \mathbb{1}_{\mathcal{H}_1 \leq \varepsilon^{-\theta_2}} \\ &\equiv \frac{\varepsilon^{\gamma_1}}{2}(Dt)^{n_1} + \sum_{\Gamma(1,n_1)} \int dv_1 d\Lambda\, \tilde{\mathbb{1}}_1 \left(1-\chi^{i.r.}\right) \chi_\xi^{ov}(\boldsymbol{\zeta}^{\mathscr{E}})\, e^{-(\beta/2)\sum_{i\in S(1)} v_i^2}\, \mathbb{1}_{\mathcal{H}_1 \leq \varepsilon^{-\theta_2}} \\ &\leq \frac{\varepsilon^{\gamma_1}}{2}(Dt)^{n_1} + \sum_{\Gamma(1,n_1)} \int dv_1 d\Lambda\, \chi_\xi^{ov}(\boldsymbol{\zeta}^{\mathscr{E}})\, e^{-(\beta/2)\sum_{i\in S(1)} v_i^2}\, \mathbb{1}_{\mathcal{H}_1 \leq \varepsilon^{-\theta_2}} ,\end{aligned}$$

(3.3.69)

where in the last step we added all the trajectories of the EBF that show up some overlap among the particles of the tree $\Gamma(1, n_1)$.

(ii) Integration over virtual trajectories. We shall reduce the problem to the estimate of an integral spanning a single virtual trajectory.

We start by specifying which particle of the tree $\Gamma(1, n_1)$ is exactly the bullet:

$$\int dv_1 d\Lambda\, \chi^{ov}_{\xi}(\boldsymbol{\zeta}^{\mathscr{E}})\, e^{-(\beta/2)\sum_{i\in S(1)} v_i^2}\, \mathbb{1}_{\mathcal{H}_1 \le \varepsilon^{-\theta_2}}$$
$$\le \sum_{i=1}^{n_1+1} \int dv_1 d\Lambda\, \chi^{ov}_{\xi,i}(\boldsymbol{\zeta}^{\mathscr{E}})\, e^{-(\beta/2)\sum_{i\in S(1)} v_i^2}\, \mathbb{1}_{\mathcal{H}_1 \le \varepsilon^{-\theta_2}}\ , \tag{3.3.70}$$

where $\chi^{ov}_{\xi,i}(\boldsymbol{\zeta}^{\mathscr{E}}) = 1$ if and only if

$$\inf_{s\in(0,t_{i-1})} |\xi(s) - \xi^{\mathscr{E}}_i(s)| < \varepsilon\ .$$

Note now that $\chi^{ov}_{\xi,i}$ depends actually not on the full EBF, but *only* on the virtual trajectory associated to particle i:

$$\chi^{ov}_{\xi,i}(\boldsymbol{\zeta}^{\mathscr{E}}) = \chi^{ov}_{\xi,i}(\zeta^{\mathscr{E},i})\ .$$

Consequently, we may integrate out easily from (3.3.70) all the variables which do *not* enter in the description of $\zeta^{\mathscr{E},i}(s)$.

More precisely, according to Definition 3, for any given $\Gamma(1, n_1), i$, we call n^i the number of nodes met by $\zeta^{\mathscr{E},i}$, and $i_1, i_2, \cdots, i_{n^i}$ their names (ordered as an increasing sequence): then the integration variables describing completely the virtual trajectory are

$$v_1, t_{i_1}, \cdots, t_{i_{n^i}}, \omega_{i_1}, \cdots, \omega_{i_{n^i}}, v_{i_1}, \cdots, v_{i_{n^i}} \longrightarrow \zeta^{\mathscr{E},i}\ ,$$

which we rename here for convenience

$$v_1, t^1, \cdots, t^{n^i}, \omega^1, \cdots, \omega^{n^i}, v^1, \cdots, v^{n^i} \longrightarrow \zeta^{\mathscr{E},i}\ .$$

With this notation and

$$\mathcal{H}^i_1 = v_1^2 + \sum_{k=1}^{n^i} (v^k)^2\ ,$$

we get from (3.3.70):

$$\int dv_1 d\Lambda\, \chi^{ov}_{\xi}(\boldsymbol{\zeta}^{\mathscr{E}})\, e^{-(\beta/2)\sum_{i\in S(1)} v_i^2}\, \mathbb{1}_{\mathcal{H}_1 \le \varepsilon^{-\theta_2}} \tag{3.3.71}$$
$$\le \sum_{i=1}^{n_1+1} \frac{(D't)^{n_1-n^i}}{(n_1-n^i)!} \int_0^t dt^1 \int_0^{t^1} dt^2 \cdots \int_0^{t^{n^i-1}} dt^{n^i} \int d\omega^1 \cdots d\omega^{n^i}$$
$$\cdot \int dv_1 dv^1 \cdots dv^{n^i}\, \mathbb{1}_{\{\inf_{s\in(0,t)} |\xi(s) - \xi^{\mathscr{E},i}(s)| < \varepsilon\}}\, e^{-(\beta/2)\mathcal{H}^i_1}\, \mathbb{1}_{\mathcal{H}^i_1 \le \varepsilon^{-\theta_2}}\ ,$$

for suitable $D' > 0$.

(iii) Change to relative velocities. To integrate over the above characteristic function, it is convenient to use the variable v_1, together with the relative velocities at the creation times, which we introduce in what follows.

As already explained, the virtual trajectory has piecewise constant velocity. Furthermore, in the EBF a change of velocity may occur only at a creation time. Let us call

$$\eta^1, \eta^2, \cdots, \eta^{n^i+1}$$

the values assumed by the velocity in $\eta^{\mathscr{E},i}$, namely

$$\eta^1 = v_1 \ ,$$

$$\eta^k \equiv \eta^{\mathscr{E},i}((t^{k-1})^-) \equiv \eta^{\mathscr{E},i}(s) \qquad \text{for} \qquad s \in (t^k, t^{k-1}) \ .$$

The relative velocities at creations are then

$$\begin{aligned} & V_1 = v^1 - \eta^1 \ , \\ & V_2 = v^2 - \eta^2 \ , \\ & \cdots \\ & V_{n^i} = v^{n^i} - \eta^{n^i} \ . \end{aligned}$$

Since η^k is independent of v^k, the previous relations are simple translations. Thus, we may rewrite (3.3.71) as

$$\begin{aligned} & \int dv_1 d\Lambda \, \chi_\xi^{ov}(\boldsymbol{\zeta}^{\mathscr{E}}) \, e^{-(\beta/2)\sum_{i\in S(1)} v_i^2} \, \mathbb{1}_{\mathcal{H}_1 \le \varepsilon^{-\theta_2}} \\ & \le \sum_{i=1}^{n_1+1} \frac{(D't)^{n_1-n^i}}{(n_1-n^i)!} \int_0^t dt^1 \int_0^{t^1} dt^2 \cdots \int_0^{t^{n^i-1}} dt^{n^i} \int d\omega^1 \cdots d\omega^{n^i} \\ & \quad \cdot \int dv_1 \, dV_1 \cdots dV_{n^i} \, \mathbb{1}_{\{\inf_{s\in(0,t)} |\xi(s) - \xi^{\mathscr{E},i}(s)| < \varepsilon\}} \, e^{-(\beta/2)\mathcal{H}_1^i} \, \mathbb{1}_{\mathcal{H}_1^i \le \varepsilon^{-\theta_2}} \ , \end{aligned} \tag{3.3.72}$$

where now $\zeta^{\mathscr{E},i}(s)$ and $\mathcal{H}_1^i$ are computed by using $V_1, \cdots, V_{n^i}$. In particular,

$$\mathcal{H}_1^i = v_1^2 + \sum_{k=1}^{n^i} (V_k + \eta^k)^2 \ . \tag{3.3.73}$$

In what follows, we find the explicit expression of this function in terms of the new integration variables.

First, observe that each jump of velocity in the virtual trajectory $\zeta^{\mathscr{E},i}$, i.e.

$$\eta^k = \eta^{\mathscr{E},i}((t^k)^+) \to \eta^{k+1} = \eta^{\mathscr{E},i}((t^k)^-) ,$$

can be of two types, determined uniquely by the structure of the tree $\Gamma(1, n_1)$ (and corresponding again to the two cases shown in Figure 6). That is:

- **Type** 1. The position jumps according to $\xi^{\mathscr{E},i}((t^k)^+) \to \xi^{\mathscr{E},i}((t^k)^-) = \xi^{\mathscr{E},i}((t^k)^+) + \varepsilon\omega^k$; the velocity jumps according to

$$\eta^{k+1} - \eta^k = \begin{cases} P^{\perp}_{\omega^k} V_k = V_k - \omega^k(\omega^k \cdot V_k) & (\omega^k \cdot V_k) \geq 0 \quad \text{(outgoing collision)} \\ V_k & (\omega^k \cdot V_k) < 0 \quad \text{(incoming collision)} \end{cases} .$$

- **Type** 2. The position does not jump: $\xi^{\mathscr{E},i}((t^k)^+) = \xi^{\mathscr{E},i}((t^k)^-)$; the velocity jumps according to

$$\eta^{k+1} - \eta^k = \begin{cases} P^{\parallel}_{\omega^k} V_k = \omega^k(\omega^k \cdot V_k) & (\omega^k \cdot V_k) \geq 0 \quad \text{(outgoing collision)} \\ 0 & (\omega^k \cdot V_k) < 0 \quad \text{(incoming collision)} \end{cases} .$$

To have a compact notation, we write the above transformation as

$$\eta^{k+1} - \eta^k = P^k V_k . \tag{3.3.74}$$

The energy (3.3.73) can be now written as

$$\mathcal{H}^i_1 = \sum_{k=0}^{n^i} \left(V_k + \sum_{h=1}^{k-1} P^h V_h + v_1 \right)^2 , \tag{3.3.75}$$

with the convention $V_0 = 0$.

(iv) The overlap constraint as an integral over "tubes". With the notations introduced, we are ready to write more explicitly the constraint

$$\inf_{s\in(0,t)} |\xi(s) - \xi^{\mathscr{E},i}(s)| < \varepsilon . \tag{3.3.76}$$

Given $r \in \{0, 1, \cdots, n^i\}$, the virtual trajectory at time $s \in (t^{r+1}, t^r)$ (with $t^0 = t$) reads

$$\begin{aligned} \xi^{\mathscr{E},i}(s) \quad &= x_1 - v_1(t - t^1) - \eta^2(t^1 - t^2) - \cdots - \eta^{r+1}(t^r - s) + \sum_{k\leq r}^{*} \varepsilon\, \omega^k \\ &= x_1 - v_1(t - s) - (\eta^2 - \eta^1)(t^1 - s) - \cdots - (\eta^{r+1} - \eta^r)(t^r - s) + \sum_{k\leq r}^{*} \varepsilon\, \omega^k \\ &= x_1 - v_1(t - s) - \sum_{k=1}^{r} P^k V_k (t^k - s) + \sum_{k\leq r}^{*} \varepsilon\, \omega^k , \end{aligned} \tag{3.3.77}$$

where the sum $\sum^*$ runs over all the nodes of type 1. The condition (3.3.76) is then

$$\inf_{s\in(0,t)} \Big|(\xi(s)-\xi(t))+(\xi(t)-x_1)+v_1(t-s)+\sum_{k=1}^{r} P^k V_k(t^k-s)-\sum_{k\leq r}^{*} \varepsilon\,\omega^k\Big| < \varepsilon\ . \tag{3.3.78}$$

A couple of remarks on the above condition follow.

Remark 1. In our hypotheses on the trajectory $s\to\xi(s)$ (see the statement of Lemma 5) and by virtue of $\mathcal{H}_1^i \leq \varepsilon^{-\theta_2}$, $n_1 < \varepsilon^{-\alpha_0}$ and the condition on the initial distance $|\xi(t)-x_1| \geq \varepsilon^{\theta_2/2}$, one can argue that the time s realizing the overlap constraint can *not* be too close to t. In fact, the displacement of the target is $|\xi(s)-\xi(t)| \leq (\varepsilon^{-\theta_2/2}(t-s)+\varepsilon^{1-\alpha_0})$. Recalling that $r\leq n^i \leq n_1 < \varepsilon^{-\alpha_0}$ and observing that, by conservation of energy, $|\eta^k|^2 \leq \mathcal{H}_1^i \leq \varepsilon^{-\theta_2}$, it follows

$$\begin{aligned} |\xi(s)-\xi^{\mathscr{E},i}(s)| \quad &\geq |\xi(t)-x_1|-|\xi(s)-\xi(t)|-\sup_k|\eta^k|(t-s)-\varepsilon^{1-\alpha_0} \\ &\geq \varepsilon^{\theta_2/2}-(\varepsilon^{-\theta_2/2}(t-s)+\varepsilon^{1-\alpha_0})-(\varepsilon^{-\theta_2/2}(t-s)+\varepsilon^{1-\alpha_0})\ . \end{aligned}$$

Thus, choosing $\theta_2/2 < 1-\alpha_0$, the constraint $|\xi(s)-\xi^{\mathscr{E},i}(s)|<\varepsilon$ implies

$$(t-s) > \varepsilon^{\theta_2}/4 \tag{3.3.79}$$

for ε small enough.

Remark 2. A similar computation shows that (3.3.76) implies

$$|\xi(t)-x_1| \leq G\varepsilon^{-\theta_2/2}\ , \tag{3.3.80}$$

for a sufficiently large constant G. Namely, the initial distance between the compared backwards trajectories can not be too large.

Now, using the position variable

$$X := -v_1 t\ , \tag{3.3.81}$$

simple algebra leads to rewrite (3.3.78) as

$$\inf_{s\in(0,t)} \Big|X-\Delta(s)\Big| < \frac{\varepsilon\, t}{t-s}\ , \tag{3.3.82}$$

where we introduced the trajectory

$$\Delta(s) := \frac{t}{t-s}\Big[(\xi(s)-\xi(t))+(\xi(t)-x_1)+\sum_{k=1}^{r} P^k V_k(t^k-s)-\sum_{k}^{*}\varepsilon\,\omega^k\Big]\ . \tag{3.3.83}$$

Using (3.3.79), it follows that

$$\inf_{s\in(0,t-\varepsilon^{\theta_2}/4)}\Big|X-\Delta(s)\Big|<\varepsilon^{1-\theta_2}\,4t\,. \tag{3.3.84}$$

Therefore, we are allowed to rewrite

$$\begin{aligned}&\int dv_1\,dV_1\cdots dV_{n^i}\,\mathbb{1}_{\{\inf_{s\in(0,t)}|\xi(s)-\xi^{\mathscr{E},i}(s)|<\varepsilon\}}\,e^{-(\beta/2)\mathcal{H}_1^i}\,\mathbb{1}_{\mathcal{H}_1^i\leq\varepsilon^{-\theta_2}}\\&\leq\int dV_1\cdots dV_{n^i}\int_{\mathbb{R}^3}dX\,\mathbb{1}_{\{\inf_{s\in(0,t-\varepsilon^{\theta_2}/4)}|X-\Delta(s)|<\varepsilon^{1-\theta_2}4t\}}\,e^{-(\beta/2)\mathcal{H}_1^i}\,,\end{aligned} \tag{3.3.85}$$

where the energy assumes now the expression

$$\mathcal{H}_1^i=\sum_{k=0}^{n^i}\left(V_k+\sum_{h=1}^{k-1}P^hV_h-\frac{X}{t}\right)^2\,. \tag{3.3.86}$$

The important feature, in (3.3.85), is that $\Delta(s)$ does *not* depend on X. This makes simple the estimate of the $\int dX$. Namely X is restricted to a "tube" with axis given by the curve of parametric equation $\Delta(s)$ and radius $\varepsilon^{1-\theta_2}4t$.

(v) Volume of the tube and conclusion. Indicating

$$a_k=V_k+\sum_{h=1}^{k-1}P^hV_h\,, \tag{3.3.87}$$

which is an $X-$independent quantity, it is easy to see that

$$\begin{aligned}\inf_X\mathcal{H}_1^i\qquad&=\inf_X\sum_{k=0}^{n^i}\left(a_k-\frac{X}{t}\right)^2\\&\geq\sum_{k=0}^{n^i}a_k^2-\frac{\left(\sum_{k=0}^{n^i}a_k\right)^2}{n^i+1}\,.\end{aligned} \tag{3.3.88}$$

Using this to bound the integrand in (3.3.85), one gets

$$\begin{aligned}&\int dv_1\,dV_1\cdots dV_{n^i}\,\mathbb{1}_{\{\inf_{s\in(0,t)}|\xi(s)-\xi^{\mathscr{E},i}(s)|<\varepsilon\}}\,e^{-(\beta/2)\mathcal{H}_1^i}\,\mathbb{1}_{\mathcal{H}_1^i\leq\varepsilon^{-\theta_2}}\\&\leq\int dV_1\cdots dV_{n^i}\,e^{-(\beta/2)\sum_k a_k^2+\frac{(\beta/2)}{n^i+1}(\sum_k a_k)^2}\\&\quad\cdot\int_{\mathbb{R}^3}dX\,\mathbb{1}_{\{\inf_{s\in(0,t-\varepsilon^{\theta_2}/4)}|X-\Delta(s)|<\varepsilon^{1-\theta_2}4t\}}\,.\end{aligned} \tag{3.3.89}$$

It remains to compute the volume of the region defined by the last line of the previous formula.

Remind that, as shown above, the displacements $|\xi(s)-\xi(t)|$ and $|\xi^{\mathscr{E},i}(s)-\xi^{\mathscr{E},i}(t)|$ are bounded by $\varepsilon^{-\theta_2/2}t+\varepsilon^{1-\alpha_0}$ Using this, (3.3.80) and (3.3.79), a straightforward calculation leads to show that, for ε small enough, the length of the curve $\Delta(s)$ is bounded by $C_t\varepsilon^{-(5/2)\theta_2}$, where $C_t>0$ is a suitable constant depending only on t.

Hence,

$$\int_{\mathbb{R}^3} dX\, \mathbb{1}_{\{\inf_{s\in(0,t-\varepsilon^{\theta_2}/4)}|X-\Delta(s)|<\varepsilon^{1-\theta_2}4t\}} \leq C_t'\varepsilon^{2-(9/2)\theta_2} \ . \tag{3.3.90}$$

for a suitable $C_t'>0$. Choosing $\theta_2<2/9$, the above volume can be bounded by ε.

Thus we obtain

$$\begin{aligned}&\int dv_1\, dV_1\cdots dV_{n^i}\, \mathbb{1}_{\{\inf_{s\in(0,t)}|\xi(s)-\xi^{\mathscr{E},i}(s)|<\varepsilon\}}\, e^{-(\beta/2)\mathcal{H}_1^i}\, \mathbb{1}_{\mathcal{H}_1^i\leq\varepsilon^{-\theta_2}}\\ &\leq \varepsilon\int dV_1\cdots dV_{n^i}\, e^{-(\beta/2)\sum_k a_k^2+\frac{(\beta/2)}{n^i+1}(\sum_k a_k)^2} \ .\end{aligned} \tag{3.3.91}$$

The Gaussian integral is conveniently computed in the variables a_k. Note that $V_k\to a_k$, defined by (3.3.87), is a further translation. There holds

$$\begin{aligned}&\int dV_1\cdots dV_{n^i}\, e^{-(\beta/2)\sum_k a_k^2+\frac{(\beta/2)}{n^i+1}(\sum_k a_k)^2}\\ &=\int da_1\cdots da_{n^i}\, e^{-(\beta/2)\sum_k a_k^2+\frac{(\beta/2)}{n^i+1}(\sum_k a_k)^2}\\ &=\left(\frac{2\pi}{\beta}\right)^{\frac{3}{2}n^i}(n^i+1)^{3/2} \ ,\end{aligned} \tag{3.3.92}$$

as can be checked for instance by induction.

We put together (3.3.92), (3.3.91) (3.3.72) and (3.3.69). Performing the remaining integrals and summing as usual over the trees, the proof of Lemma 5 is concluded. ■

3.3 Proof of (3.6)

We start from the expansion (3.3), and insert in it the solution to the Enskog equation $g^\varepsilon(t)$:

$$\begin{aligned}f_J^\varepsilon(t) &= \sum_{H\subset J}(f_1^\varepsilon(t))^{\otimes H}E_{J\setminus H}(t)\\ &= \sum_{H\subset J}(f_1^\varepsilon(t)-g^\varepsilon(t)+g^\varepsilon(t))^{\otimes H}E_{J\setminus H}(t)\\ &= \sum_{H\subset J}(g^\varepsilon(t))^{\otimes H}E_{J\setminus H}^{\mathscr{E}}(t) \ ,\end{aligned} \tag{3.3.1}$$

where

$$E_K^{\mathscr{E}}(t) = \sum_{Q\subset K} (f_1^\varepsilon(t) - g^\varepsilon(t))^{\otimes Q} E_{K\setminus Q}(t)$$
$$= \sum_{Q\subset K} (E_1^{\mathscr{E}}(t))^{\otimes Q} E_{K\setminus Q}(t)\ . \tag{3.3.2}$$

The correlation error $E_{K\setminus Q}(t)$ is already controlled by (3.4). It is enough to estimate the one–point error

$$E_1^{\mathscr{E}}(t) = f_1^\varepsilon(t) - g^\varepsilon(t)\ . \tag{3.3.3}$$

Resorting to the respective BBGKY and Enskog tree expansions, Eq.s (2.2.18) and (2.2.26), we have

$$\begin{aligned} E_1^{\mathscr{E}}(t) &= \sum_{n=0}^{\infty}\sum_{\Gamma(1,n)}\int d\Lambda \left(\prod B^\varepsilon f^\varepsilon_{0,1+n}(\boldsymbol{\zeta}^\varepsilon(0)) - \prod B^{\mathscr{E}} g^\varepsilon_{0,1+n}(\boldsymbol{\zeta}^{\mathscr{E}}(0))\right) \\ &= \sum_{n=0}^{\infty}\sum_{\Gamma(1,n)}\int d\Lambda \prod B^\varepsilon \left(f^\varepsilon_{0,1+n}(\boldsymbol{\zeta}^\varepsilon(0)) - g^\varepsilon_{0,1+n}(\boldsymbol{\zeta}^\varepsilon(0))\right) \\ &+\sum_{n=0}^{\infty}\sum_{\Gamma(1,n)}\int d\Lambda \left(\prod B^\varepsilon g^\varepsilon_{0,1+n}(\boldsymbol{\zeta}^\varepsilon(0)) - \prod B^{\mathscr{E}} g^\varepsilon_{0,1+n}(\boldsymbol{\zeta}^{\mathscr{E}}(0))\right)\ , \end{aligned} \tag{3.3.4}$$

where $g^\varepsilon_{0,1+n} = f_0^{\otimes(1+n)}$.

The first term in the right hand side is the error due to the difference in the initial data. This is bounded by Hypothesis 2 (and the bound (2.1.18)); see for instance (2.3.6)–(2.3.7), which imply immediately

$$\begin{aligned} &\Big|\sum_{n=0}^{\infty}\sum_{\Gamma(1,n)}\int d\Lambda \prod B^\varepsilon \left(f^\varepsilon_{0,1+n}(\boldsymbol{\zeta}^\varepsilon(0)) - g^\varepsilon_{0,1+n}(\boldsymbol{\zeta}^\varepsilon(0))\right)\Big| \\ &\leq \varepsilon^{\gamma_0'}\sum_{n=0}^{\infty}\sum_{\Gamma(1,n)}\int d\Lambda \prod |B^\varepsilon| C^{1+n}\ e^{-(\beta/2)\sum_{i\in S(1)}(\eta_1^\varepsilon(0))^2} \\ &\leq \varepsilon^{\gamma_0'} C e^{-(\beta/4)v_1^2} \end{aligned} \tag{3.3.5}$$

for suitable constants $C > 0$ and $t < \bar{t}$, having used Lemma 2 for the convergence of the series.

The last term in (3.3.4) is due to the differences among the IBF and the EBF. Since, in absence of recollisions of the IBF and of overlaps of the EBF, the two flows coincide, there holds

$$\sum_{n=0}^{\infty}\sum_{\Gamma(1,n)}\int d\Lambda\left(\prod B^{\varepsilon}g^{\varepsilon}_{0,1+n}(\boldsymbol{\zeta}^{\varepsilon}(0))-\prod B^{\mathscr{E}}g^{\varepsilon}_{0,1+n}(\boldsymbol{\zeta}^{\mathscr{E}}(0))\right)\ ,$$

$$=\sum_{n=0}^{\infty}\sum_{\Gamma(1,n)}\int d\Lambda\left(\prod B^{\varepsilon}\chi^{i.r.}g^{\varepsilon}_{0,1+n}(\boldsymbol{\zeta}^{\varepsilon}(0))-\prod B^{\mathscr{E}}\chi^{i.o.}g^{\varepsilon}_{0,1+n}(\boldsymbol{\zeta}^{\mathscr{E}}(0))\right) \tag{3.3.6}$$

where $\chi^{i.r.} = \chi^{i.r.}(\boldsymbol{\zeta}^{\varepsilon})$ and $\chi^{i.o.} = \chi^{i.o.}(\boldsymbol{\zeta}^{\mathscr{E}})$ were defined when discussing Lemma 6 (see (3.3.52) and (3.3.55)).

Using $|g^{\varepsilon}_{0,1+n}| \leq C^{n+1}e^{-(\beta/2)\sum_{i=1}^{n+1}v_i^2}$, Corollary 1 with $\theta_1 = \gamma_1/2$, Lemma 6 and Eq. (3.3.56), we deduce:

$$\sum_{n=0}^{\infty}\sum_{\Gamma(1,n)}\int dv_1 d\Lambda\prod|B^{\varepsilon}|\chi^{i.r.}g^{\varepsilon}_{0,1+n}(\boldsymbol{\zeta}^{\varepsilon}(0))+\prod|B^{\mathscr{E}}|\chi^{i.o.}g^{\varepsilon}_{0,1+n}(\boldsymbol{\zeta}^{\mathscr{E}}(0))=O(\varepsilon^{\gamma_1/2})$$

for any $\gamma_1 \in (0,1)$, $t < t^*$ and ε small enough. Remind that γ has been chosen smaller than $\gamma_1/2$ (see Section 3.2.6).

The final result follows then from the above estimates together with (3.3.4), (3.3.2) and (3.4), by taking $\gamma' < \min(\gamma,\gamma'_0)$. ■

3.4 Proof of (3.8)

Given the estimates derived in the previous section, we are left with the comparison of the solutions of the Boltzmann and the Enskog equations, Eq.s (2.2.26) and (2.2.29)–(2.2.30). In fact, proceeding as previously, one gets

$$\begin{aligned} f^{\varepsilon}_J(t) &= \sum_{H\subset J}(f(t))^{\otimes H}E^{\mathcal{B}}_{J\setminus H}(t)\ ,\\ E^{\mathcal{B}}_K(t) &= \sum_{Q\subset K}(g^{\varepsilon}(t)-f(t))^{\otimes Q}E^{\mathscr{E}}_{K\setminus Q}(t)\ , \end{aligned} \tag{3.4.1}$$

where the correlation errors $E^{\mathscr{E}}_{K\setminus Q}$ are bounded by (3.6).

By $g^{\varepsilon}_{0,1+n} = f_0^{\otimes(1+n)} = f_{0,1+n}$ and (2.2.33), it follows that

$$\begin{aligned} g^{\varepsilon}(t)-f(t) &= \sum_{n=0}^{\infty}\sum_{\Gamma(1,n)}\int d\Lambda\left(\prod B^{\mathscr{E}}g^{\varepsilon}_{0,1+n}(\boldsymbol{\zeta}^{\mathscr{E}}(0))-\prod Bf_{0,1+n}(\boldsymbol{\zeta}(0))\right)\\ &= \sum_{n=0}^{\infty}\sum_{\Gamma(1,n)}\int d\Lambda\prod B\Big[f_{0,1+n}(\boldsymbol{\zeta}^{\mathscr{E}}(0))-f_{0,1+n}(\boldsymbol{\zeta}(0))\Big]\ . \end{aligned} \tag{3.4.2}$$

Using the Lipschitz–regularity assumption on f_0 and (2.2.32), one finds

$$\left|f_{0,1+n}(\boldsymbol{\zeta}^{\mathscr{E}}(0)) - f_{0,1+n}(\boldsymbol{\zeta}(0))\right| \leq C^n\, e^{-(\beta/2)\sum_{i=1}^{n+1}\eta_i^2}\, L \sum_{i=1}^{n+1} |\xi_i^{\mathscr{E}}(0) - \xi_i(0)| \tag{3.4.3}$$

for suitable $C > 0$. But

$$|\xi_i^{\mathscr{E}}(0) - \xi_i(0)| \leq n\varepsilon\ . \tag{3.4.4}$$

Hence, applying Lemma 2, we prove

$$\int dv |g^{\varepsilon}(t) - f(t)| = O(\varepsilon)\ . \tag{3.4.5}$$

The estimate (3.8) follows finally from (3.4.5), (3.4.1) and the bound (3.6). ■

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