

# Mean-field Evolution of Fermionic Mixed States

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## Abstract

In this paper we study the dynamics of fermionic mixed states in the mean-field regime. We consider initial states which are close to quasi-free states and prove that, under suitable assumptions on the initial data and on the many-body interaction, the quantum evolution of such initial data is well approximated by a suitable quasi-free state. In particular we prove that the evolution of the reduced one-particle density matrix converges, as the number of particles goes to infinity, to the solution of the time-dependent Hartree-Fock equation. Our result holds for all times, and gives effective estimates on the rate of convergence of the many-body dynamics towards the Hartree-Fock one.

## 1 Introduction

The time-evolution of many-body quantum systems in the so called mean-field regime has received a lot of attention in the recent years. For bosonic systems, the mean-field dynamics can be generated by an  $N$ -particle Hamiltonian

$$H_N = \sum_{j=1}^N -\Delta_{x_j} + \frac{1}{N} \sum_{i<j}^N V(x_i - x_j) \quad (1.1)$$

acting on the bosonic Hilbert space  $L_s^2(\mathbb{R}^{3N}, dx_1 \dots dx_N)$  in the limit of large  $N$ . The coupling constant  $N^{-1}$  in front of the potential energy guarantees that both terms in the Hamiltonian are typically of the same size which is a necessary condition for the existence of a non-trivial limiting evolution.

The physical picture one has in mind is that particles are initially trapped in a volume of order one and are in equilibrium. When the traps are switched off, the system starts to evolve, following the evolution generated by (1.1). Thus, the object of interest is the dynamics  $\psi_{N,t} = e^{-iH_N t} \psi_N$  where the initial data  $\psi_N$  describes trapped particles at equilibrium. At low temperature, the system exhibits complete condensation and it is natural to consider

approximately factorized initial data  $\psi_N \simeq \varphi^{\otimes N}$  for an appropriate  $\varphi \in L^2(\mathbb{R}^3)$ . It turns out that in the limit of large  $N$  the evolution preserves the approximate factorization. The relation  $\psi_{N,t} \simeq \varphi_t^{\otimes N}$  then easily leads to the self-consistent nonlinear Hartree equation

$$i\partial_t \varphi_t = -\Delta \varphi_t + (V * |\varphi_t|^2) \varphi_t$$

for the evolution of the condensate wave function.

The convergence of the many-body Schrödinger evolution towards the Hartree dynamics has to be understood in the sense of reduced density matrices. The one-particle reduced density  $\gamma_{N,t}^{(1)}$  associated with the solution  $\psi_{N,t}$  of the Schrödinger equation is given by the partial trace

$$\gamma_{N,t}^{(1)} = N \operatorname{tr}_{2,\dots,N} |\psi_{N,t}\rangle \langle \psi_{N,t}|.$$

In other words,  $\gamma_{N,t}^{(1)}$  is the non-negative trace class operator on  $L^2(\mathbb{R}^3)$  with the integral kernel

$$\gamma_{N,t}^{(1)}(x, y) = N \int dx_2 \dots dx_N \psi_{N,t}(x, x_2, \dots, x_N) \bar{\psi}_{N,t}(y, x_2, \dots, x_N).$$

Analogously, we can define the  $k$ -particle reduced density  $\gamma_{N,t}^{(k)}$  associated with  $\psi_{N,t}$  by

$$\gamma_{N,t}^{(k)} = \binom{N}{k} \operatorname{tr}_{k+1,\dots,N} |\psi_{N,t}\rangle \langle \psi_{N,t}|. \quad (1.2)$$

Notice that we choose the normalization

$$\operatorname{tr} \gamma_{N,t}^{(k)} = \binom{N}{k}.$$

Of course, if  $k < N$ , the  $k$ -particle reduced density  $\gamma_{N,t}^{(k)}$  does not contain the full information about the state  $\psi_{N,t}$ ; however, knowledge of  $\gamma_{N,t}^{(k)}$  is sufficient to compute the expectation of any  $k$ -particle observable. In fact, if  $J^{(k)}$  is an operator acting on  $L^2(\mathbb{R}^{3k})$ , and if we denote by  $J_{i_1,\dots,i_k}^{(k)}$  the operator acting as  $J^{(k)}$  on the  $k$  particles  $i_1, \dots, i_k$  and as the identity on the other  $(N - k)$  particles, we have

$$\sum_{(i_1,\dots,i_k)} \langle \psi_{N,t}, J_{i_1,\dots,i_k}^{(k)} \psi_{N,t} \rangle = \operatorname{tr} J^{(k)} \gamma_{N,t}^{(k)},$$

where, on the left hand side, we are summing over all clusters of  $k$  particles.

Reduced density matrices provide the appropriate language to express the convergence of the many-body Schrödinger dynamics towards the limiting Hartree equation. Notice that the  $k$ -particle reduced density matrix associated with the product state  $\varphi^{\otimes N}$  is given by

$$\binom{N}{k} |\varphi\rangle \langle \varphi|^{\otimes k} = \binom{N}{k} |\varphi^{\otimes k}\rangle \langle \varphi^{\otimes k}|.$$

Under appropriate conditions on the interaction potential  $V$ , and assuming that, at time  $t = 0$ ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \gamma_{N,0}^{(1)} = |\varphi\rangle \langle \varphi|$$

one can show that, for every  $t \in \mathbb{R}$ ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \gamma_{N,t}^{(1)} = |\varphi_t\rangle \langle \varphi_t|. \quad (1.3)$$

Similarly, one also gets convergence of the reduced densities  $\gamma_{N,t}^{(k)}$ , for any fixed  $k \in \mathbb{N}$ . Eq. (1.3) states the stability of condensation, understood as approximate factorization of the wave function, with respect to the time evolution.

The first results in the direction of (1.3) were obtained in [26, 24, 34]. More recently, the case of a Coulomb interaction has been studied in [20, 3]. Different points of view on the bosonic mean-field limit have been proposed in [23, 22] (where the convergence towards the Hartree equation has been interpreted as an Egorov-type theorem) and in [1] (focusing on the mean-field propagation of Wigner measures). Bounds on the rate of convergence towards the Hartree equation have been proven in [32, 27, 13] and fluctuations around the Hartree dynamics have been analyzed in [25, 5, 9, 28]. The more singular Gross-Pitaevskii regime has been considered in [17, 18, 19, 31, 6, 12]. For one-dimensional systems, the focusing nonlinear Schrödinger equation has been derived in [11]. In the case of three-body interactions, the convergence towards nonlinear Schrödinger equation with a quintic nonlinearity has been established in [14, 10].

For fermionic systems, the mean-field limit is more delicate. Consider a gas of  $N$  fermions described by the Hamiltonian

$$H_N = \sum_{j=1}^N -\Delta_{x_j} + \lambda \sum_{i<j}^N V(x_i - x_j)$$

acting on the fermionic Hilbert space  $L_a^2(\mathbb{R}^{3N}, dx_1 \dots dx_N)$  (the subspace of  $L^2(\mathbb{R}^{3N})$  consisting of permutation antisymmetric functions satisfying

$$\psi_N(x_{\pi(1)}, \dots, x_{\pi(N)}) = \sigma_\pi \psi_N(x_1, \dots, x_N)$$

for every permutation  $\pi \in S_N$ , where  $\sigma_\pi \in \{\pm 1\}$  denotes the sign of the permutation  $\pi$ ). Because of the Pauli principle (reflected in the antisymmetry of the wave functions), if the  $N$  particles are trapped in a volume of order one, their kinetic energy is (at least) of the order  $N^{5/3}$ . To make sure that the potential energy is of the same order, one has to take  $\lambda = N^{-1/3}$ , which is much larger than  $\lambda = N^{-1}$  in the bosonic case. Moreover, since the typical velocity of the particles is very large, of the order  $N^{1/3}$ , one can only hope to follow the time evolution for short times, of the order  $N^{-1/3}$ . After rescaling time, we are led to the  $N$ -particle Schrödinger equation

$$iN^{1/3} \partial_t \psi_{N,t} = \left[ \sum_{j=1}^N -\Delta_{x_j} + \frac{1}{N^{1/3}} \sum_{i<j}^N V(x_i - x_j) \right] \psi_{N,t}. \quad (1.4)$$

In these variables, we are interested in the solution  $\psi_{N,t}$  for times  $t$  of order one. Setting  $\varepsilon = N^{-1/3}$  and multiplying (1.4) with  $\varepsilon^2$ , we conclude that

$$i\varepsilon \partial_t \psi_{N,t} = \left[ \sum_{j=1}^N -\varepsilon^2 \Delta_{x_j} + \frac{1}{N} \sum_{i<j}^N V(x_i - x_j) \right] \psi_{N,t}. \quad (1.5)$$

Hence, we recover the  $N^{-1}$  coupling constant characterizing the mean-field limit in the bosonic setting. Here, however, the mean-field regime is naturally linked with a semiclassical limit, with  $\varepsilon = N^{-1/3}$  playing the role of Planck's constant. Let us remark, however, that different scaling limits in the fermionic setting have been considered in [4, 21, 30].

Similarly to the bosonic case, we are interested in the solution of (1.5) for initial data  $\psi_{N,0}$  describing fermions confined to a volume of order one and in thermal equilibrium.

At zero temperature, the ground state of the system of trapped fermions is expected (and in certain cases, it is known) to be approximated by a Slater determinant

$$\psi_{\text{slater}}(x_1, \dots, x_N) = \det(f_i(x_j))$$

for an appropriate set of  $N$  orthonormal orbitals  $f_1, \dots, f_N \in L^2(\mathbb{R}^3)$ . The Slater determinant is completely characterized by its one-particle reduced density

$$\omega_N = N \operatorname{tr}_{2, \dots, N} |\psi_{\text{slater}}\rangle\langle\psi_{\text{slater}}| = \sum_{j=1}^N |f_j\rangle\langle f_j|,$$

given by the orthogonal projection onto the  $N$ -dimensional space spanned by  $f_1, \dots, f_N$ . In the presence of an external potential  $V_{\text{ext}}$ , the energy of the Slater determinant with reduced density  $\omega_N$  is given by the Hartree-Fock functional

$$\mathcal{E}_{\text{HF}}(\omega_N) = \operatorname{tr}(-\varepsilon^2 \Delta + V_{\text{ext}}(x)) \omega_N + \frac{1}{2N} \int dx dy V(x-y) [\omega_N(x, x) \omega_N(y, y) - |\omega_N(x, y)|^2]. \quad (1.6)$$

Semiclassical analysis suggests that, in the limit of large  $N$ , the minimizer of (1.6) can be approximated by the Weyl quantization

$$\operatorname{Op}_M^W(x, y) = \frac{1}{(2\pi)^3} \int dp M\left(\frac{x+y}{2}, p\right) e^{ip \cdot \frac{x-y}{\varepsilon}} \quad (1.7)$$

of the phase-space density  $M(x, p) = \chi(|p| \leq c\rho_{\text{TF}}^{1/3}(x))$ , where  $\rho_{\text{TF}}$  is the minimizer of the Thomas-Fermi functional

$$\mathcal{E}_{\text{TF}}(\rho) = \frac{3}{5} \varepsilon^2 c_{\text{TF}} \int \rho^{5/3}(x) dx + \int V_{\text{ext}}(x) \rho(x) dx + \frac{1}{2N} \int dx dy V(x-y) \rho(x) \rho(y) \quad (1.8)$$

under the condition  $\int \rho(x) dx = N$ . Eq. (1.7) implies in particular that the integral kernel of the Hartree-Fock minimizer  $\omega_N$  should vary on scales of order one in the  $x+y$  direction and on much shorter scales, of order  $\varepsilon$ , in the  $x-y$  direction. Accordingly, we expect that

$$\begin{aligned} \operatorname{tr} |[x, \omega_N]| &\leq CN\varepsilon, \\ \operatorname{tr} |[\varepsilon \nabla, \omega_N]| &\leq CN\varepsilon. \end{aligned} \quad (1.9)$$

In fact, it is easy to show that (1.9) holds true for the Weyl quantization (1.7) under reasonable assumptions on the Thomas-Fermi density  $\rho_{\text{TF}}$ .

It is therefore interesting to study the solution  $\psi_{N,t}$  of the many-body Schrödinger equation (1.5) for initial data close to a Slater determinant with one-particle reduced density satisfying (1.9). This was done in [16], for analytic interaction and for short times, and, more recently, in [7] for arbitrary times and for a much larger class of regular interactions (a similar analysis for semi-relativistic fermions can be found in [8]). In [7], the one-particle reduced density  $\gamma_{N,t}^{(1)}$  associated with  $\psi_{N,t}$  was proven to satisfy

$$\|\gamma_{N,t}^{(1)} - \omega_{N,t}\|_{\text{HS}} \leq C \exp(c \exp(c|t|)) \quad (1.10)$$

uniformly in  $N$ . Here  $\omega_{N,t}$  is the solution of the time-dependent Hartree-Fock equation

$$i\varepsilon \partial_t \omega_{N,t} = [-\varepsilon^2 \Delta + V * \rho_t + X_t, \omega_{N,t}] \quad (1.11)$$

with initial data  $\omega_{N,0} = \omega_N$ . Here we used the notation  $\rho_t(x) = N^{-1}\omega_{N,t}(x,x)$  for the normalized density associated with  $\omega_{N,t}$  and  $X_t$  to denote the exchange operator defined by the integral kernel

$$X_t(x,y) = N^{-1}V(x-y)\omega_{N,t}(x,y)$$

The right hand side of (1.10) should be compared with the Hilbert-Schmidt norm of  $\gamma_{N,t}^{(1)}$  and of  $\omega_{N,t}$ , which is of the order  $N^{1/2}$  (it is also possible to get bounds for the trace norm of the difference  $\gamma_{N,t}^{(1)} - \omega_{N,t}$ , see [7]). In other words, (1.10) states that the evolution of an initial Slater determinant satisfying the semiclassical bounds (1.9) remains close to a Slater determinant, whose reduced one-particle density evolves according to (1.11). Notice that the assumption (1.9) for the initial data plays a crucial role in the analysis. Physically, it is motivated by the fact that we are interested in initial data describing systems at equilibrium.

In contrast with the bosonic case, the limiting Hartree-Fock equation (1.11) still depends on  $N$ . As  $N \rightarrow \infty$ , the Hartree-Fock dynamics converges, at least formally, towards the Vlasov dynamics. More precisely, let

$$W_{N,t}(x,v) = \frac{\varepsilon^3}{(2\pi)^3} \int dy \gamma_{N,t}^{(1)}\left(x + \frac{\varepsilon y}{2}, x - \frac{\varepsilon y}{2}\right) e^{iy \cdot v}$$

be the Wigner transform of the one-particle reduced density  $\gamma_{N,t}^{(1)}$  associated with the solution of the Schrödinger equation (1.5). Then a simple computation suggests that, as  $N \rightarrow \infty$ ,  $W_{N,t} \rightarrow W_{\infty,t}$ , where  $W_{\infty,t}$  satisfies the Vlasov equation

$$\partial_t W_{\infty,t} + v \cdot \nabla_x W_{\infty,t} + \nabla \cdot (V * \rho_t) \cdot \nabla_v W_{\infty,t} = 0 \quad (1.12)$$

with the density  $\rho_t(x) = \int dv W_{\infty,t}(x,v)$ . Convergence of the many-body evolution towards the Vlasov equation (1.12) was first established, for analytic interaction potentials, in [29]. Later, this result was extended to a larger class of potentials in [35]. Also (1.10) implies convergence (in a weak sense) towards the Vlasov dynamics. It should be noted, however, that establishing the convergence (1.10) is more complicated than comparing the solution of the Schrödinger equation with the Vlasov equation, and requires a deeper understanding of the many-body dynamics.

To establish the convergence (1.10) and also to generalize the analysis to positive temperatures, it is useful to switch to a Fock space representation of the fermionic system. The fermionic Fock space over  $L^2(\mathbb{R}^3)$  is defined as the direct sum

$$\mathcal{F} = \bigoplus_{n \geq 0} L_a^2(\mathbb{R}^{3n}, dx_1 \dots dx_n).$$

In particular, states with exactly  $N$  particles are described on the Fock space by sequences with only one non-vanishing component. On  $\mathcal{F}$ , Slater determinants can be conveniently obtained by applying Bogoliubov transformations to the Fock vacuum  $\Omega = \{1, 0, \dots\}$ . In fact, for every orthogonal projection  $\omega_N$  on  $L^2(\mathbb{R}^3)$  with  $\text{tr} \omega_N = N$  there exists a unitary operator  $R_{\omega_N} : \mathcal{F} \rightarrow \mathcal{F}$  preserving the canonical anticommutation relations such that  $R_{\omega_N} \Omega$  is exactly the  $N$ -particle Slater determinant with reduced one-particle density  $\omega_N$ .

In the present paper, we are interested in the dynamics at positive temperature. More precisely, we are interested in initial data describing interacting fermions trapped in a volume of order one, in equilibrium at a certain temperature  $T > 0$  and with an appropriate chemical potential  $\mu$ , so that the average number of particles is  $N$ , the same parameter appearing in the Hamiltonian generating the many-body evolution (the precise definition of the Hamiltonian

will be given in the next section; notice however, that the Hamiltonian is the same as the one considered at zero temperature).

In our setting it is expected that equilibrium states at positive temperature can be approximated by quasi-free states on  $\mathcal{F}$ . Quasi-free states are completely characterized by their one-particle reduced density; all higher order correlations can be expressed in terms of the one-particle reduced density using Wick's theorem (in general, one also needs the pairing density, but here we will assume it to vanish). In fact, every operator  $\omega_N$  with  $0 \leq \omega_N \leq 1$  and  $\text{tr} \omega_N = N$  is the one-particle reduced density of a unique quasi-free state on  $\mathcal{F}$ . A simple example of quasi-free states are Slater determinants; in this case,  $\omega_N$  is additionally an orthogonal projection. However, there is an important difference between Slater determinants (which are important at zero temperature) and the quasi-free states which are relevant at positive temperature. Slater determinants are *pure* quasi-free states. At positive temperature, on the other hand, we will have to consider *mixed* quasi-free states on  $\mathcal{F}$ . This difference makes the analysis at positive temperature more involved, compared with [7].

Mixed states are described by density matrices (and not simply by vectors) on the fermionic Fock space  $\mathcal{F}$ . For our purposes, it will be useful to represent mixed states on  $\mathcal{F}$  through vectors on the double Fock space  $\mathcal{F} \otimes \mathcal{F} \simeq \mathcal{F}(L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3))$ . In this way, we will be able to obtain any mixed quasi-free state by applying a Bogoliubov transformation, this time acting on the vacuum of the double Fock space  $\mathcal{F} \otimes \mathcal{F} \simeq \mathcal{F}(L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3))$  (this representation of mixed quasi-free states is a special case of the Araki-Wyss construction, see [2, 15]).

Similarly to (1.9), we will need the (approximate) initial quasi-free state to satisfy certain estimates, which are justified by the fact that we are starting from an equilibrium state and that we observe its evolution resulting from a change of the external field. Semiclassical analysis suggests that equilibrium states at positive temperature  $T > 0$  can be approximated by the Weyl quantization

$$\text{Op}_M^W(x, y) = \frac{1}{(2\pi)^3} \int dp M\left(\frac{x+y}{2}, p\right) e^{ip \cdot \frac{x-y}{\varepsilon}} \quad (1.13)$$

of a phase-space density

$$M(x, p) = g_{T, \mu}\left(p^2 - c\rho_{\text{TF}}^{2/3}(x)\right) \quad (1.14)$$

for the smooth function

$$g_{T, \mu}(E) = \frac{1}{1 + e^{(E-\mu)/T}}$$

depending on the temperature  $T > 0$  and on the chemical potential  $\mu \geq 0$ . In (1.14),  $\rho_{\text{TF}}$  denotes the minimizer of the Thomas-Fermi functional (1.8) and the normalization constant  $c > 0$  has to be chosen so that

$$\int M(x, v) dx dv = 1$$

Similarly to (1.7), the kernel (1.13) vanishes, unless  $|x - y| \lesssim \varepsilon$ . Since the function  $g_{T, \mu}$  for  $T > 0$  is smoother than at zero temperature ( $g_{T, \mu}$  is a characteristic function for  $T = 0$ ), we observe that (1.13) satisfies even stronger semiclassical estimates, compared with (1.9). We will be interested in approximately quasi-free states, with one-particle reduced density

matrix  $0 \leq \omega_N \leq 1$  with  $\text{tr } \omega_N = N$  and satisfying the bounds

$$\begin{aligned} \|\llbracket \sqrt{\omega_N}, e^{ip \cdot x} \rrbracket\|_{\text{HS}} &\leq C\varepsilon N^{1/2}(1 + |p|), \\ \|\llbracket \sqrt{1 - \omega_N}, e^{ip \cdot x} \rrbracket\|_{\text{HS}} &\leq C\varepsilon N^{1/2}(1 + |p|), \\ \|\llbracket \sqrt{\omega_N}, \varepsilon \nabla \rrbracket\|_{\text{HS}} &\leq C\varepsilon N^{1/2}, \\ \|\llbracket \sqrt{1 - \omega_N}, \varepsilon \nabla \rrbracket\|_{\text{HS}} &\leq C\varepsilon N^{1/2}. \end{aligned} \tag{1.15}$$

It is then easy to check that the Weyl quantization (1.13) satisfies the estimates (1.15), under reasonable assumptions on the Thomas-Fermi density  $\rho_{\text{TF}}$ . Our main result is that for such initial data the time evolution remains close to a quasi-free state, with reduced one-particle density  $\omega_{N,t}$  satisfying the time-dependent Hartree-Fock equation (1.11).

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## 2 Fock space representation, quasi-free states and main result

*Fock space.* Let  $\mathfrak{h} = L^2(\mathbb{R}^3)$ . We define the fermionic Fock space  $\mathcal{F}(\mathfrak{h})$  over  $\mathfrak{h}$  as the direct sum

$$\mathcal{F}(\mathfrak{h}) = \bigoplus_{n \geq 0} \mathfrak{h}^{\wedge n} \tag{2.1}$$

where  $\mathfrak{h}^{\wedge n} := \mathfrak{h} \wedge \mathfrak{h} \cdots \wedge \mathfrak{h}$  is the  $n$ -fold antisymmetric tensor product of  $\mathfrak{h}$ . Vectors in  $\mathcal{F}(\mathfrak{h})$  are sequences

$$\mathcal{F}(\mathfrak{h}) \ni \xi = (\xi^{(0)}, \xi^{(1)}, \dots, \xi^{(n)}, \dots) \tag{2.2}$$

with  $\xi^{(n)} \in \mathfrak{h}^{\wedge n}$ . In particular, the vacuum  $\Omega_{\mathcal{F}(\mathfrak{h})} = (1, 0, \dots)$  describes a state with no particles.  $\mathcal{F}(\mathfrak{h})$  is a Hilbert space, with respect to the inner product

$$\langle \xi, \chi \rangle = \sum_{n \geq 0} \langle \xi^{(n)}, \chi^{(n)} \rangle_{\mathfrak{h}^{\wedge n}}$$

The creation and annihilation operators on  $\mathcal{F}(\mathfrak{h})$  are defined as follows: for  $f \in \mathfrak{h}$ ,

$$\begin{aligned} (a^*(f)\psi)^{(n)}(x_1, \dots, x_n) &= \frac{1}{\sqrt{n}} \sum_{j=1}^n (-1)^j f(x_j) \psi^{(n-1)}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n), \\ (a(f)\psi)^{(n)}(x_1, \dots, x_n) &= \sqrt{n+1} \int dx \overline{f(x)} \psi^{(n+1)}(x, x_1, \dots, x_n) \end{aligned}$$

As the notation suggests, the creation operator  $a^*(f)$  is the adjoint of the annihilation operator  $a(f)$ . Observe that  $a^*(f)$  is linear, while  $a(f)$  is antilinear in the argument  $f \in \mathfrak{h}$ . Creation and annihilation operators satisfy canonical anticommutation relations

$$\{a(f), a^*(g)\} = \langle f, g \rangle_{\mathfrak{h}}, \quad \{a(f), a(g)\} = \{a^*(f), a^*(g)\} = 0 \tag{2.3}$$

for all  $f, g \in \mathfrak{h}$ . Here  $\{A, B\} = AB + BA$  denotes the anticommutator of the operators  $A, B$ . Notice that (in contrast to the bosonic operators) fermionic creation and annihilation operators are *bounded*, with

$$\|a^\sharp(f)\| = \|f\| \tag{2.4}$$

where  $a^\sharp$  is either  $a$  or  $a^*$ .

It is also useful to introduce the operator-valued distributions  $a_x^*$  and  $a_x$ , which formally create and annihilate, respectively, a particle at point  $x \in \mathbb{R}^3$ . They satisfy

$$a(f) = \int dx a_x \overline{f(x)}, \quad a^*(f) = \int dx a_x^* f(x)$$

for all  $f \in \mathfrak{h}$ .

Given a one-particle observable  $O : \mathfrak{h} \rightarrow \mathfrak{h}$ , we define its second quantization  $d\Gamma(O) : \mathcal{F} \rightarrow \mathcal{F}$  by

$$(d\Gamma(O)\psi)^{(n)} = \sum_{j \leq n} O^{(j)} \psi^{(n)}, \quad (2.5)$$

where  $O^{(j)} = 1 \otimes \dots \otimes 1 \otimes O \otimes 1 \otimes \dots \otimes 1$  is the operator acting as  $O$  on the  $j$ -th particle and as the identity on the other  $(n-1)$  particles. An important example is the particle number operator  $\mathcal{N} = d\Gamma(1)$ , which counts the number of particles in each sector of the Fock space:  $(\mathcal{N}\psi)^{(n)} = n\psi^{(n)}$ . If the one-particle operator  $O$  has the integral kernel  $O(x; y)$ , then  $d\Gamma(O)$  can be expressed in terms of operator-valued distributions as

$$d\Gamma(O) = \int dx dy O(x; y) a_x^* a_y.$$

In particular, the number operator can be written as

$$\mathcal{N} = \int dx a_x^* a_x.$$

*Mixed states.* A fermionic mixed state is represented by a density matrix on  $\mathcal{F}(\mathfrak{h})$ . A density matrix is a non-negative trace class operator  $\rho : \mathcal{F}(\mathfrak{h}) \rightarrow \mathcal{F}(\mathfrak{h})$  with  $\text{tr } \rho = 1$ . The mixed state with density matrix  $\rho$  is pure if and only if  $\rho = |\psi\rangle\langle\psi|$  is an orthogonal projection onto a  $\psi \in \mathcal{F}(\mathfrak{h})$ . In general,

$$\rho = \sum_n \lambda_n |\psi_n\rangle\langle\psi_n| \quad (2.6)$$

where  $\lambda_n \geq 0$ ,  $\{\psi_n\}$  is an orthonormal set in  $\mathcal{F}(\mathfrak{h})$ , and  $\sum_n \lambda_n = 1$ . Physically,  $\rho$  describes an incoherent superposition of pure states and  $\lambda_n$  is the probability that the system is in the state  $\psi_n$ . Obviously, the expectation of an arbitrary operator  $A$  on  $\mathcal{F}(\mathfrak{h})$  with respect to the mixed state with density matrix  $\rho$  is

$$\text{tr } A\rho = \sum_n \lambda_n \langle\psi_n, A\psi_n\rangle.$$

Given the density matrix (2.6) we define the operator  $\tilde{\kappa} : \mathcal{F}(\mathfrak{h}) \rightarrow \mathcal{F}(\mathfrak{h})$  by

$$\tilde{\kappa} = \sum_n \varepsilon_n |\psi_n\rangle\langle\phi_n|$$

where  $\varepsilon_n \in \mathbb{C}$  is a sequence satisfying  $|\varepsilon_n|^2 = \lambda_n$  and  $\{\phi_n\}$  is an orthonormal set in  $\mathcal{F}(\mathfrak{h})$ . Clearly,

$$\tilde{\kappa}\tilde{\kappa}^* = \rho$$

Of course, such a decomposition of  $\rho$  is far from being unique and later we shall choose a convenient one. Since  $\rho$  is trace class,  $\tilde{\kappa} \in \mathcal{L}^2(\mathcal{F}(\mathfrak{h}))$ , the set of Hilbert-Schmidt operators on  $\mathcal{F}(\mathfrak{h})$ .



Next, we observe that  $\mathcal{L}^2(\mathcal{F}(\mathfrak{h})) \simeq \mathcal{F}(\mathfrak{h}) \otimes \mathcal{F}(\mathfrak{h})$ . The isomorphism is induced by the map  $|\psi\rangle\langle\phi| \rightarrow \psi \otimes \bar{\phi}$ , extended by linearity to the whole space  $\mathcal{L}^2(\mathcal{F}(\mathfrak{h}))$ . The mixed state with density matrix (2.6) is described on  $\mathcal{F}(\mathfrak{h}) \otimes \mathcal{F}(\mathfrak{h})$  by the vector

$$\kappa = \sum_n \varepsilon_n \psi_n \otimes \bar{\phi}_n.$$

The expectation of the operator  $A$  on  $\mathcal{F}(\mathfrak{h})$  in the state described by  $\kappa \in \mathcal{F}(\mathfrak{h}) \otimes \mathcal{F}(\mathfrak{h})$  is then given by

$$\text{tr } A\rho = \text{tr } A\tilde{\kappa}\tilde{\kappa}^* = \langle \kappa, (A \otimes 1)\kappa \rangle_{\mathcal{F}(\mathfrak{h}) \otimes \mathcal{F}(\mathfrak{h})}. \quad (2.7)$$

It is well-known that the doubled Fock space  $\mathcal{F}(\mathfrak{h}) \otimes \mathcal{F}(\mathfrak{h})$  is isomorphic to the Fock space  $\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$  (see [15] or any book on mathematical quantum field theory). The unitary map  $U$  that implements this isomorphism is known as the *exponential law* and it is defined by the relations

$$U(\Omega_{\mathcal{F}(\mathfrak{h})} \otimes \Omega_{\mathcal{F}(\mathfrak{h})}) = \Omega_{\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})}$$

and

$$\begin{aligned} U[a(f) \otimes 1]U^* &= a(f \oplus 0) =: a_l(f) \\ U[(-1)^{\mathcal{N}} \otimes a(f)]U^* &= a(0 \oplus f) =: a_r(f) \end{aligned} \quad (2.8)$$

for all  $f \in \mathfrak{h}$ , where  $a_\sigma(f)$ ,  $\sigma = l, r$ , are the *left and right representations* of  $a(f)$ , respectively. By hermitian conjugation, we also find

$$\begin{aligned} U[a^*(f) \otimes 1]U^* &= a^*(f \oplus 0) =: a_l^*(f) \\ U[(-1)^{\mathcal{N}} \otimes a^*(f)]U^* &= a^*(0 \oplus f) =: a_r^*(f) \end{aligned} \quad (2.9)$$

where  $a_\sigma^*(f)$ ,  $\sigma = l, r$ , are the left and right representations of  $a^*(f)$ . Notice that the presence of the operator  $(-1)^{\mathcal{N}}$  on the second line of (2.8) and (2.9) guarantees that creation operators on the space  $\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$  satisfy the canonical anticommutation relations (and in particular that  $a_l^\sharp(f)$  anticommutes with  $a_r^\sharp(g)$ , for all  $f, g \in \mathfrak{h}$ ).

It is convenient to introduce the left and right representations of the operator-valued distributions  $a_x, a_x^*$  by the relations

$$\begin{aligned} a_l(f) &= \int dx a_{x,l} \overline{f(x)}, & a_r(f) &= \int dx a_{x,r} \overline{f(x)}, \\ a_l^*(f) &= \int dx a_{x,l}^* f(x), & a_r^*(f) &= \int dx a_{x,r}^* f(x), \end{aligned} \quad (2.10)$$

for all  $f \in \mathfrak{h}$ .

We also define the left and right representations of the second quantization of operators on  $\mathfrak{h}$  by

$$\begin{aligned} U[d\Gamma(O) \otimes 1]U^* &= d\Gamma(O \oplus 0) =: d\Gamma_l(O) \\ U[1 \otimes d\Gamma(O)]U^* &= d\Gamma(0 \oplus O) =: d\Gamma_r(O). \end{aligned}$$

The left and right representations of  $d\Gamma(O)$  can be written in terms of the operator-valued distributions as

$$d\Gamma_l(O) = \int dx dy O(x; y) a_{x,l}^* a_{y,l}, \quad d\Gamma_r(O) = \int dx dy O(x; y) a_{x,r}^* a_{y,r}. \quad (2.11)$$

*Reduced densities.* According to (2.7), the expectation of the observable  $d\Gamma(O)$  in the mixed state described by  $\psi \in \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$  is given by

$$\langle \psi, d\Gamma_l(O)\psi \rangle = \int dx dy O(x, y) \langle \psi, a_{x,l}^* a_{y,l} \psi \rangle = \text{tr } O \gamma_\psi^{(1)},$$

where  $\gamma_\psi^{(1)}$  is the one-particle reduced density associated with  $\psi$ , defined as the non-negative trace-class operator on  $\mathfrak{h}$  with the integral kernel

$$\gamma_\psi^{(1)}(x; y) = \langle \psi, a_{y,l}^* a_{x,l} \psi \rangle. \quad (2.12)$$

It is also useful to introduce the pairing density  $\alpha_\psi$  associated with  $\psi$ . It is defined as the Hilbert-Schmidt operator on  $\mathfrak{h}$  with integral kernel

$$\alpha_\psi(x; y) = \langle \psi, a_{y,l} a_{x,l} \psi \rangle.$$

It is convenient to combine the reduced one-particle density  $\gamma_\psi^{(1)}$  and the pairing density  $\alpha_\psi$  in a single non-negative operator  $\Gamma_\psi : \mathfrak{h} \oplus \mathfrak{h} \rightarrow \mathfrak{h} \oplus \mathfrak{h}$  acting as

$$\Gamma_\psi = \begin{pmatrix} \gamma_\psi^{(1)} & \alpha_\psi \\ \alpha_\psi^* & 1 - \bar{\gamma}_\psi^{(1)} \end{pmatrix}. \quad (2.13)$$

$\Gamma_\psi$  is known as the generalized one-particle reduced density associated with the mixed state described by  $\psi \in \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$ . It allows us to compute the expectation of arbitrary quadratic expressions in creation and annihilation operators on  $\mathcal{F}(\mathfrak{h})$ . In fact, for any  $f_1, f_2, g_1, g_2 \in \mathfrak{h}$ , we find

$$\langle \psi, (a_l^*(f_1) + a_l(\bar{g}_1)) (a_l(f_2) + a_l^*(\bar{g}_2)) \psi \rangle = \langle (f_1, g_1), \Gamma_\psi(f_2, g_2) \rangle \quad (2.14)$$

where, on the r.h.s.,  $\langle \cdot, \cdot \rangle$  denotes the inner product on  $\mathfrak{h} \oplus \mathfrak{h}$ . Using (2.14), it is easy to check that  $0 \leq \Gamma_\psi \leq 1$  for any  $\psi \in \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$ .

To compute the expectation of higher order observables (given by the product of more than two creation and annihilation operators), we need to define higher order correlations. We define the  $k$ -particle reduced density  $\gamma_\psi^{(k)}$  associated with  $\psi \in \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$  as the non-negative trace-class operator on  $\mathfrak{h}^{\wedge k}$  with the integral kernel

$$\gamma_\psi^{(k)}(x_1, \dots, x_k; y_1, \dots, y_k) = \frac{1}{k!} \langle \psi, a_{y_1,l}^* \dots a_{y_k,l}^* a_{x_k,l} \dots a_{x_1,l} \psi \rangle. \quad (2.15)$$

Notice that we use the normalization

$$\text{tr } \gamma_\psi^{(k)} = \frac{1}{k!} \langle \psi, \mathcal{N}_l(\mathcal{N}_l - 1) \dots (\mathcal{N}_l - k + 1) \psi \rangle$$

with  $\mathcal{N}_l = d\Gamma_l(1)$ . Hence, for states with average number of particles  $N$ , we expect  $\text{tr } \gamma_\psi^{(k)}$  to be of the order  $N^k$  (for states with exactly  $N$  particles we have  $\text{tr } \gamma_\psi^{(k)} = \binom{N}{k}$ , consistently with the definition (1.2)).

*Time evolution of mixed states.* On  $\mathcal{F}$ , we define the Hamilton operator  $\mathcal{H}_N$  by

$$(\mathcal{H}_N \Psi)^{(n)} = \mathcal{H}_N^{(n)} \Psi^{(n)}$$

with

$$\mathcal{H}_N^{(n)} = \sum_{j=1}^n -\varepsilon^2 \Delta_{x_j} + \frac{1}{N} \sum_{i < j}^n V(x_i - x_j).$$

By definition,  $\mathcal{H}_N$  commutes with the particle number operator, and, when restricted to the  $N$ -particle sector, coincides with the Hamiltonian generating the evolution (1.5). It is sometimes useful to express the Hamiltonian  $\mathcal{H}_N$  in terms of the operator-valued distributions  $a_x, a_x^*$ . We find

$$\mathcal{H}_N = \varepsilon^2 \int dx \nabla_x a_x^* \nabla_x a_x + \frac{1}{2N} \int dx dy V(x-y) a_x^* a_y^* a_y a_x. \quad (2.16)$$

The time evolution of the density matrix  $\rho$  is given by

$$\rho_t = e^{-i\mathcal{H}_N t/\varepsilon} \rho e^{i\mathcal{H}_N t/\varepsilon}.$$

Thus, we define the time evolution of  $\kappa \in \mathcal{F}(\mathfrak{h}) \otimes \mathcal{F}(\mathfrak{h})$  by  $\kappa_t = e^{-i\mathcal{H}_N t/\varepsilon} \otimes e^{i\mathcal{H}_N t/\varepsilon} \kappa$ . We denote by  $\psi_t = U \kappa_t$  the vector in  $\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$  describing the mixed state with density matrix  $\rho_t$ . Then

$$\psi_t = U \kappa_t = U e^{-i(\mathcal{H}_N \otimes 1 - 1 \otimes \mathcal{H}_N)t/\varepsilon} \kappa = e^{-i\mathcal{L}_N t/\varepsilon} \psi$$

where the *Liouvillian*  $\mathcal{L}_N$  is defined by

$$\mathcal{L}_N = U(\mathcal{H}_N \otimes 1 - 1 \otimes \mathcal{H}_N)U^*.$$

A more explicit expression follows from (2.8) and (2.10):

$$\begin{aligned} \mathcal{L}_N &= \varepsilon^2 \int dx \nabla_x a_{x,l}^* \nabla_x a_{x,l} + \frac{1}{2N} \int dx dy V(x-y) a_{x,l}^* a_{y,l}^* a_{y,l} a_{x,l} \\ &\quad - \varepsilon^2 \int dx \nabla_x a_{x,r}^* \nabla_x a_{x,r} - \frac{1}{2N} \int dx dy V(x-y) a_{x,r}^* a_{y,r}^* a_{y,r} a_{x,r}. \end{aligned} \quad (2.17)$$

Hence, the expectation of an arbitrary operator  $A$  on  $\mathcal{F}(\mathfrak{h})$  in the evolved mixed state is given by

$$\text{tr } A \rho_t = \langle \psi_t, (A \otimes 1) \psi_t \rangle = \langle \psi, e^{i\mathcal{L}_N t/\varepsilon} (A \otimes 1) e^{-i\mathcal{L}_N t/\varepsilon} \psi \rangle.$$

*Bogoliubov transformations.* We will be interested in the time evolution of quasi-free (or approximately quasi-free) initial data. It turns out that mixed quasi-free states can be represented in  $\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$  by applying Bogoliubov transformations on the vacuum of  $\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$ .

Let us briefly review the concept of Bogoliubov transformation. For  $f, g \in \mathfrak{h} \oplus \mathfrak{h}$  we define the field operators

$$A(f, g) = a(f) + a^*(\bar{g}), \quad A^*(f, g) = (A(f, g))^* = a^*(f) + a(\bar{g}).$$

We observe that

$$A^*(f, g) = A(Jg, Jf) = A(\mathcal{J}(f, g)) \quad (2.18)$$

where  $J : \mathfrak{h} \oplus \mathfrak{h} \rightarrow \mathfrak{h} \oplus \mathfrak{h}$  is the antilinear operator defined by  $Jf = \bar{f}$  and  $\mathcal{J} : (\mathfrak{h} \oplus \mathfrak{h}) \oplus (\mathfrak{h} \oplus \mathfrak{h}) \rightarrow (\mathfrak{h} \oplus \mathfrak{h}) \oplus (\mathfrak{h} \oplus \mathfrak{h})$  is given by

$$\mathcal{J} = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}.$$

The anticommutation relations (2.3) can be rewritten as

$$\{A(f_1, g_1), A^*(f_2, g_2)\} = \langle (f_1, g_1), (f_2, g_2) \rangle \quad (2.19)$$

for all  $f_1, f_2, g_1, g_2 \in \mathfrak{h} \oplus \mathfrak{h}$ . Here  $\langle \cdot, \cdot \rangle$  denotes the inner product on  $(\mathfrak{h} \oplus \mathfrak{h}) \oplus (\mathfrak{h} \oplus \mathfrak{h})$ .

A Bogoliubov transformation is a linear map  $\nu : (\mathfrak{h} \oplus \mathfrak{h}) \oplus (\mathfrak{h} \oplus \mathfrak{h}) \rightarrow (\mathfrak{h} \oplus \mathfrak{h}) \oplus (\mathfrak{h} \oplus \mathfrak{h})$  such that  $\nu^* \nu = 1$  and

$$\nu \mathcal{J} = \mathcal{J} \nu.$$

These two conditions guarantee that the rotated field operators  $B(f, g) := A(\nu(f, g))$  satisfy the same relations (2.18) and (2.19) as the original operators  $A(f, g)$ , for all  $f, g \in \mathfrak{h} \oplus \mathfrak{h}$ .

It is easy to check that a map  $\nu$  is a Bogoliubov transformation if and only if it is of the form

$$\nu = \begin{pmatrix} U & \overline{V} \\ V & \overline{U} \end{pmatrix} \quad (2.20)$$

where  $U, V : \mathfrak{h} \oplus \mathfrak{h} \rightarrow \mathfrak{h} \oplus \mathfrak{h}$  are linear maps with

$$U^*U + V^*V = 1, \quad U^*\overline{V} + V^*\overline{U} = 0 \quad (2.21)$$

A Bogoliubov transformation  $\nu : (\mathfrak{h} \oplus \mathfrak{h}) \oplus (\mathfrak{h} \oplus \mathfrak{h})$  is said to be implementable if there exists a unitary operator  $R_\nu : \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h}) \mapsto \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$  such that

$$A(\nu(f, g)) = R_\nu^* A(f, g) R_\nu \quad (2.22)$$

for all  $f, g \in \mathfrak{h} \oplus \mathfrak{h}$ . The *Shale-Stinespring condition* states that the Bogoliubov transformation (2.20) is implementable if and only if  $V$  is a Hilbert-Schmidt operator. The proof of this theorem and more information about Bogoliubov transformations can be found in the lecture notes [33].

*Quasi-free states.* A vector  $\psi \in \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$  is said to describe a quasi-free state if for all  $f_1, \dots, f_{2n+1} \in \mathfrak{h}$ ,

$$\langle \psi, a_l^{\#1}(f_1) \dots a_l^{\#2n+1}(f_{2n+1}) \psi \rangle = 0,$$

and for all  $f_1, \dots, f_{2n} \in \mathfrak{h}$ , the Wick rule

$$\langle \psi, a_l^{\#1}(f_1) \dots a_l^{\#2n}(f_{2n}) \psi \rangle = \sum_{\pi} \sigma_{\pi} \prod_{j=1}^n \langle \psi, a_l^{\#i_j}(f_{i_j}) a_l^{\#\ell_j}(f_{\ell_j}) \psi \rangle \quad (2.23)$$

holds. The sum on the r.h.s. runs over all pairings  $\pi$ , mapping the  $n$  indices  $i_1, \dots, i_n$  into  $\ell_1, \dots, \ell_n$ .

Hence, as long as we are interested in expectations of the physically relevant left - observables (i.e. observables expressed through left creation and annihilation operators), quasi-free states are completely characterized by their generalized one-particle reduced density (2.13). Notice that for every  $\Gamma : \mathfrak{h} \oplus \mathfrak{h} \rightarrow \mathfrak{h} \oplus \mathfrak{h}$  satisfying  $0 \leq \Gamma \leq 1$ , there exists a quasi-free state on  $\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$  with generalized one-particle density matrix  $\Gamma$ ; see [33].

It turns out that quasi-free states can be conveniently written in terms of Bogoliubov transformations. More precisely, the vector  $\psi \in \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$  describing a quasi-free state can always be written as  $\psi = R_\nu \Omega$ , for a suitable Bogoliubov transformation  $\nu$  on  $(\mathfrak{h} \oplus \mathfrak{h}) \oplus (\mathfrak{h} \oplus \mathfrak{h})$ ; see [33]. This is a particular example of a famous construction in quantum statistical mechanics called Araki-Wyss representation [2] (see also [15] for additional information and references).

In the following, we will be interested in quasi-free states  $\psi \in \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$  with average number of particles equal to  $N$  and with vanishing pairing density  $\alpha_\psi = 0$ . The justification for this restriction comes from physics; equilibrium states for systems of weakly interacting fermions (described in the grand canonical ensemble by the Hamiltonian (2.16)) trapped in a volume of order one can be approximated by quasi-free states with vanishing pairing density. In fact, the pairing density is only expected to play an important role when the

potential varies on very short scales, of the order  $\varepsilon$  (in this limit, the Bardeen-Cooper-Schrieffer (BCS) theory is believed to be relevant and  $\alpha$  can be significantly different from zero). The generalized one-particle density matrix of such quasi-free states has the form

$$\Gamma^{(1)} = \begin{pmatrix} \omega_N & 0 \\ 0 & 1 - \bar{\omega}_N \end{pmatrix} \quad (2.24)$$

for a non-negative trace class operator  $\omega_N$  on  $\mathfrak{h}$  satisfying  $0 \leq \omega_N \leq 1$  and  $\text{tr} \omega_N = N$ .

It is then easy to check that the quasi-free state with generalized reduced density (2.24) can be written as  $\psi = R_{\nu_N} \Omega$ , with the Bogoliubov transformation  $\nu_N : (\mathfrak{h} \oplus \mathfrak{h}) \oplus (\mathfrak{h} \oplus \mathfrak{h}) \rightarrow (\mathfrak{h} \oplus \mathfrak{h}) \oplus (\mathfrak{h} \oplus \mathfrak{h})$  defined by

$$\nu_N = \begin{pmatrix} U_N & \bar{V}_N \\ V_N & \bar{U}_N \end{pmatrix}, \quad (2.25)$$

where

$$U_N = \begin{pmatrix} u_N & 0 \\ 0 & \bar{u}_N \end{pmatrix}, \quad V = \begin{pmatrix} 0 & \bar{v}_N \\ -v_N & 0 \end{pmatrix}, \quad (2.26)$$

and  $u_N = \sqrt{1 - \omega_N}$ ,  $v_N = \sqrt{\omega_N}$  (notice that  $U_N, V_N$  satisfy the relations (2.21)). Since  $\text{tr} V_N^* V_N = 2 \text{tr} \omega_N = 2N < \infty$ , the Bogoliubov transformation  $\nu_N$  is implementable.

To check that the  $\nu_N$  defined by (2.26) is indeed the correct Bogoliubov transformation, we note first that, for any  $f \in \mathfrak{h}$ ,

$$\begin{aligned} R_{\nu_N}^* a_l(f) R_{\nu_N} &= a_l(u_N f) - a_r^*(\bar{v}_N \bar{f}), \\ R_{\nu_N}^* a_r(f) R_{\nu_N} &= a_r(\bar{u}_N f) + a_l^*(v_N \bar{f}). \end{aligned} \quad (2.27)$$

Equivalently,

$$\begin{aligned} R_{\nu_N}^* a_{x,l} R_{\nu_N} &= a_l(u_x) - a_r^*(\bar{v}_x), \\ R_{\nu_N}^* a_{x,r} R_{\nu_N} &= a_r(\bar{u}_x) + a_l^*(v_x), \end{aligned} \quad (2.28)$$

where we used the notation  $u_x(y) = u_N(y, x)$ ,  $v_x(y) = v_N(y, x)$ , and where  $u_N(y, x)$  and  $v_N(y, x)$  denote the kernels of the operators  $u_N, v_N$ .

Using (2.28) and the unitarity of  $R_{\nu_N}$  we find

$$\begin{aligned} \gamma_\psi(x; y) &= \langle R_{\nu_N} \Omega, a_{y,l}^* a_{x,l} R_{\nu_N} \Omega \rangle \\ &= \langle \Omega, R_{\nu_N}^* a_{y,l}^* R_{\nu_N} R_{\nu_N}^* a_{x,l} R_{\nu_N} \Omega \rangle \\ &= \langle \Omega, (a_l^*(u_y) - a_r(\bar{v}_y))(a_l(u_x) - a_r^*(\bar{v}_x)) \Omega \rangle \\ &= \langle \Omega, a_r(\bar{v}_y) a_r^*(\bar{v}_x) \Omega \rangle \\ &= (v_N^* v_N)(x; y) \\ &= \omega_N(x; y) \end{aligned} \quad (2.29)$$

where we used the anticommutation relations and the fact that  $a_l, a_r$  annihilate the vacuum. Also,

$$\begin{aligned} \alpha_\psi(x; y) &= \langle R_{\nu_N} \Omega, a_{y,l} a_{x,l} R_{\nu_N} \Omega \rangle \\ &= \langle \Omega, (a_l(u_y) - a_r^*(\bar{v}_y))(a_l(u_x) - a_r^*(\bar{v}_x)) \Omega \rangle \\ &= \langle \Omega, a_l(u_y) a_r^*(\bar{v}_x) \Omega \rangle \\ &= 0 \end{aligned} \quad (2.30)$$

where we used that  $\{a_l(u_y), a_r^*(\bar{v}_x)\} = 0$ . Finally, notice that the vector  $\psi = R_{\nu_N}\Omega \in \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$  corresponds to a specific choice of  $\tilde{\kappa} \in \mathcal{L}^2(\mathcal{F}(L^2))$  such that  $\tilde{\kappa}\tilde{\kappa}^* = \rho$ , where  $\rho$  is the quasi-free state with reduced one-particle density matrix given by  $\omega_N$ . In fact,  $\tilde{\kappa} \simeq U^*R_{\nu_N}\Omega \in \mathcal{F}(\mathfrak{h}) \otimes \mathcal{F}(\mathfrak{h})$ ; the isomorphism is the one induced by the map  $\psi \otimes \bar{\phi} \rightarrow |\psi\rangle\langle\phi|$ , extended by linearity to the whole space  $\mathcal{F}(\mathfrak{h}) \otimes \mathcal{F}(\mathfrak{h})$ .

*Main result.* In our main theorem, we consider the time evolution of initial mixed states which are close to quasi-free states satisfying certain semiclassical estimates. For such initial data, we show that the evolution remains close to a (mixed) quasi-free state, with reduced density evolving according to the Hartree-Fock equation (1.11).

**Theorem 2.1.** *Let  $V \in L^1(\mathbb{R}^3)$ , and assume that*

$$\int dp (1 + |p|)^2 |\widehat{V}(p)| < \infty. \quad (2.31)$$

*Let  $\omega_N$  be a sequence of operators on  $\mathfrak{h} = L^2(\mathbb{R}^3)$  such that  $0 \leq \omega_N \leq 1$ ,  $\text{tr } \omega_N = N$ , and*

$$\begin{aligned} \| [v_N, x] \|_{HS} &\leq CN^{1/2}\varepsilon, & \| [v_N, \varepsilon \nabla] \|_{HS} &\leq CN^{1/2}\varepsilon, \\ \| [u_N, x] \|_{HS} &\leq CN^{1/2}\varepsilon, & \| [u_N, \varepsilon \nabla] \|_{HS} &\leq CN^{1/2}\varepsilon, \end{aligned} \quad (2.32)$$

*with  $v_N = \sqrt{\omega_N}$ ,  $u_N = \sqrt{1 - \omega_N}$ , for a suitable constant  $C > 0$ . Let  $\nu_N$  denote the Bogoliubov transformations (2.25), such that  $R_{\nu_N}\Omega_{\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})}$  is the quasi-free state on  $\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$  with the generalized one-particle density matrix*

$$\Gamma_{\nu_N} = \begin{pmatrix} \omega_N & 0 \\ 0 & 1 - \bar{\omega}_N \end{pmatrix}. \quad (2.33)$$

*Let  $\xi_N \in \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$  be a sequence with  $\langle \xi_N, \mathcal{N}^{10} \xi_N \rangle \leq C$  uniformly in  $N$ , and such that  $\xi_N = \chi(\mathcal{N} \leq CN)\xi_N$ , where  $\chi(\cdot)$  is the characteristic function and  $C \geq 0$  is independent of  $N$ . Here  $\mathcal{N} = d\Gamma_l(1) + d\Gamma_r(1)$  is the particle number operator in the extended Fock-space  $\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$ .*

*Let  $\gamma_{N,t}^{(1)}$  be the one-particle reduced density matrix associated with the evolved state*

$$\psi_{N,t} = e^{-i\mathcal{L}_N t/\varepsilon} R_{\nu_N} \xi_N \quad (2.34)$$

*where the Liouillian  $\mathcal{L}_N$  has been defined in (2.17). Let  $\omega_{N,t}$  be the solution of the time-dependent Hartree-Fock equation*

$$i\varepsilon \partial_t \omega_{N,t} = [-\varepsilon^2 \Delta + V * \rho_t - X_t, \omega_{N,t}], \quad (2.35)$$

*with the initial data  $\omega_{N,0} = \omega_N$ . Then, there exist constants  $K, c_1, c_2$ , such that*

$$\| \gamma_{N,t}^{(1)} - \omega_{N,t} \|_{HS} \leq K \exp(c_2 \exp(c_1 |t|)) \quad (2.36)$$

*and*

$$\text{tr} | \gamma_{N,t}^{(1)} - \omega_{N,t} | \leq KN^{1/2} \exp(c_2 \exp(c_1 |t|)). \quad (2.37)$$

We can also prove convergence of higher order densities. The  $k$ -particle reduced density associated with the quasi-free state having one-particle reduced density  $\omega_{N,t}$  is given by the  $k$ -fold wedge product

$$\omega_{N,t}^{\wedge k}(x_1, \dots, x_k; y_1, \dots, y_k) = \sum_{\pi \in S_k} \sigma_\pi \prod_{j=1}^k \omega_{N,t}(x_j; y_{\pi(j)}).$$

**Theorem 2.2.** *Let  $V \in L^1(\mathbb{R}^3)$  satisfy (2.31). Let  $\omega_N$  be a sequence of trace-class operators on  $\mathfrak{h}$  with  $0 \leq \omega_N \leq 1$ ,  $\text{tr} \omega_N = N$  and satisfying (2.32). Fix  $k \in \mathbb{N}$  and let  $\xi_N$  be a sequence in  $\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$  such that  $\langle \xi_N, \mathcal{N}^{10k} \xi_N \rangle \leq C$  uniformly in  $N$ , and such that  $\xi_N = \chi(\mathcal{N} \leq CN) \xi_N$  for  $C \geq 0$  independent of  $N$ . Again,  $\mathcal{N} = d\Gamma_l(1) + d\Gamma_r(1)$ .*

*We denote by  $\gamma_{N,t}^{(k)}$  the reduced  $k$ -particle density matrix (defined as in (2.15)) associated with  $\psi_{N,t} = e^{-i\mathcal{L}_{Nt}/\varepsilon} R_{\nu_N} \xi_N$ . Then there are constants  $K, c_1, c_2$  depending also on  $k \in \mathbb{N}$ , such that*

$$\|\gamma_{N,t}^{(k)} - \omega_{N,t}^{\wedge k}\|_{HS} \leq KN^{(k-1)/2} \exp(c_2 \exp(c_1|t|)) \quad (2.38)$$

and

$$\text{tr} |\gamma_{N,t}^{(k)} - \omega_{N,t}^{\wedge k}| \leq KN^{k-1/2} \exp(c_2 \exp(c_1|t|)). \quad (2.39)$$

The general strategy of the proof is similar to the one in [7] for pure quasi-free states. The main new ingredient is the representation of fermionic mixed states on  $\mathcal{F}(\mathfrak{h})$  as pure states on  $\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$ . As explained above, on  $\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$  time-dependent quasi-free states can be represented using Bogoliubov transformations. With  $\omega_{N,t}$  denoting the solution of the Hartree-Fock equation (1.11), we define the corresponding Bogoliubov transformation  $\nu_{N,t} : (\mathfrak{h} \oplus \mathfrak{h}) \oplus (\mathfrak{h} \oplus \mathfrak{h}) \rightarrow (\mathfrak{h} \oplus \mathfrak{h}) \oplus (\mathfrak{h} \oplus \mathfrak{h})$  by

$$\nu_{N,t} = \begin{pmatrix} U_{N,t} & \bar{V}_{N,t} \\ V_{N,t} & \bar{U}_{N,t} \end{pmatrix}$$

where the linear maps  $U_{N,t}, V_{N,t} : \mathfrak{h} \oplus \mathfrak{h} \rightarrow \mathfrak{h} \oplus \mathfrak{h}$  have the form

$$U_{N,t} = \begin{pmatrix} u_{N,t} & 0 \\ 0 & \bar{u}_{N,t} \end{pmatrix}, \quad V_{N,t} = \begin{pmatrix} 0 & \bar{v}_{N,t} \\ -v_{N,t} & 0 \end{pmatrix} \quad (2.40)$$

with  $u_{N,t} = \sqrt{1 - \omega_{N,t}}$  and  $v_{N,t} = \sqrt{\omega_{N,t}}$ . Since  $\text{tr} V_{N,t}^* V_{N,t} = 2N < \infty$ , this Bogoliubov transformation is implementable, and we shall denote by  $R_t \equiv R_{\nu_{N,t}}$  its unitary implementor on  $\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$ .

To show that  $\psi_{N,t} = e^{-i\mathcal{L}_{Nt}/\varepsilon} R_0 \xi_N$  is close to the quasi-free state  $R_t \Omega$ , we define the fluctuation vectors  $\xi_{N,t}$  by

$$\psi_{N,t} = e^{-i\mathcal{L}_{Nt}/\varepsilon} R_0 \xi_N = R_t \xi_{N,t}.$$

Equivalently  $\xi_{N,t} = \mathcal{U}_N(t; 0) \xi_N$ , with the fluctuation dynamics

$$\mathcal{U}_N(t; s) = R_t^* e^{-i\mathcal{L}_N(t-s)/\varepsilon} R_s \quad (2.41)$$

Similarly to [7], the problem of proving the convergence (2.36) of the one-particle reduced density reduces to controlling the growth of the expectation of the particle number operator  $\mathcal{N} = d\Gamma_l(1) + d\Gamma_r(1)$  with respect to the fluctuation dynamics. Analogously, the convergence (2.38) of the  $k$ -particle reduced density follows from an estimate for the  $k$ -th moment of the particle number operator with respect to  $\mathcal{U}_N$ .

The paper is organized as follows. In Section 3 we compute the generator of  $\mathcal{U}_N$ . In Section 4 we control the growth of fluctuations with respect to the dynamics  $\mathcal{U}_N$ . To do so we first control the growth of fluctuations with respect to a suitable auxiliary dynamics  $\tilde{\mathcal{U}}_N$ , and then show that  $\tilde{\mathcal{U}}_N$  is close in norm to  $\mathcal{U}_N$ . Then, in Section 5 we use the bounds on the growth of fluctuations to prove Theorems 2.1 and 2.2. Finally, in Appendix A we prove the propagation of the semiclassical structure of the initial data.

### 3 The generator of the fluctuation dynamics

The fluctuation dynamics  $\mathcal{U}_N(t; s)$  defined in (2.41) satisfies

$$i\varepsilon \frac{d}{dt} \mathcal{U}_N(t; s) = \mathcal{G}_N(t) \mathcal{U}_N(t; s), \quad (3.1)$$

with  $\mathcal{U}_N(s; s) = 1$  for all  $s \in \mathbb{R}$  and where

$$\mathcal{G}_N(t) = (i\varepsilon \partial_t R_t^*) R_t + R_t^* \mathcal{L}_N R_t. \quad (3.2)$$

In the next proposition, we compute explicitly the generator  $\mathcal{G}_N(t)$ .

**Proposition 3.1.** *The generator (3.2) of the fluctuation dynamics is given by*

$$\mathcal{G}_N(t) = d\Gamma_l(h_{HF}(t)) - d\Gamma_r(\overline{h_{HF}(t)}) + \mathcal{C}_N + \mathcal{Q}_N \quad (3.3)$$

where

$$\begin{aligned} \mathcal{C}_N = \frac{1}{2N} \int dx dy V(x-y) & \left( a_l^*(u_x) a_l^*(u_y) a_l(u_y) a_l(u_x) + 2a_l^*(u_x) a_r^*(v_x) a_r(v_y) a_l(u_y) \right. \\ & - 2a_l^*(u_x) a_r^*(\overline{v_y}) a_r(\overline{v_y}) a_l(u_y) + a_r^*(\overline{v_y}) a_r^*(\overline{v_x}) a_r(\overline{v_x}) a_r(\overline{v_y}) \\ & - a_r^*(\overline{u_x}) a_r^*(\overline{u_y}) a_r(\overline{u_y}) a_r(\overline{u_x}) - 2a_r^*(\overline{u_x}) a_l^*(v_x) a_l(v_y) a_r(\overline{u_y}) \\ & \left. + 2a_r^*(\overline{u_x}) a_l^*(v_y) a_l(v_y) a_r(\overline{u_y}) - a_l^*(v_y) a_l^*(v_x) a_l(v_x) a_l(v_y) \right) \end{aligned} \quad (3.4)$$

contains the quartic terms that commute with  $\mathcal{N} = d\Gamma_l(1) + d\Gamma_r(1)$ , and

$$\begin{aligned} \mathcal{Q}_N = \frac{1}{2N} \int dx dy V(x-y) & \left( a_l^*(u_x) a_l^*(u_y) a_r^*(\overline{v_y}) a_r^*(\overline{v_x}) + \right. \\ & + 2a_l^*(u_x) a_l^*(u_y) a_r^*(\overline{v_x}) a_l(u_y) - 2a_l^*(u_x) a_r^*(\overline{v_y}) a_r^*(\overline{v_x}) a_r(\overline{v_y}) - \\ & - a_r^*(\overline{u_x}) a_r^*(\overline{u_y}) a_l^*(v_y) a_l^*(v_x) + 2a_r^*(\overline{u_x}) a_r^*(\overline{u_y}) a_l^*(v_x) a_r(\overline{u_y}) - \\ & \left. - 2a_r^*(\overline{u_x}) a_l^*(v_y) a_l^*(v_x) a_l(v_y) + h.c. \right) \end{aligned} \quad (3.5)$$

are the quartic terms that do not commute with  $\mathcal{N}$ . In (3.4) and (3.5), we introduced the notation  $u_x(y) := u_{N,t}(y, x)$ ,  $v_x(y) := v_{N,t}(y, x)$  and similarly for the complex conjugate kernels.

*Proof.* Computation of  $(i\varepsilon \partial_t R_t^*) R_t$ . To evaluate the first contribution in (3.2), we proceed as follows. By definition

$$R_t^* A(f, g) R_t = A(\nu_{N,t}(f, g)) \quad (3.6)$$

for all  $f, g \in \mathfrak{h} \oplus \mathfrak{h}$ . Here  $\nu_{N,t}$  is the Bogoliubov transformation

$$\nu_{N,t} = \begin{pmatrix} U_{N,t} & \overline{V}_{N,t} \\ V_{N,t} & \overline{U}_{N,t} \end{pmatrix} \quad (3.7)$$

with  $U_{N,t}, V_{N,t}$  given by

$$U_{N,t} = \begin{pmatrix} u_{N,t} & 0 \\ 0 & \overline{u_{N,t}} \end{pmatrix}, \quad V_{N,t} = \begin{pmatrix} 0 & \overline{v_{N,t}} \\ -v_{N,t} & 0 \end{pmatrix} \quad (3.8)$$



and  $u_{N,t} = \sqrt{1 - \omega_{N,t}}$ ,  $v_{N,t} = \sqrt{\omega_{N,t}}$ . Differentiating the left hand side of (3.6) we get

$$\begin{aligned}
(i\varepsilon\partial_t(R_t^*A(f,g)R_t)) &= (i\varepsilon\partial_t R_t^*)A(f,g)R_t + R_t^*A(f,g)i\varepsilon\partial_t R_t \\
&= (i\varepsilon\partial_t R_t^*)R_t R_t^*A(f,g)R_t + R_t^*A(f,g)R_t R_t^*i\varepsilon\partial_t R_t \\
&= (i\varepsilon\partial_t R_t^*)R_t A(\nu_{N,t}(f,g)) + A(\nu_{N,t}(f,g))R_t^*i\varepsilon\partial_t R_t \\
&= (i\varepsilon\partial_t R_t^*)R_t A(\nu_{N,t}(f,g)) - A(\nu_{N,t}(f,g))(i\varepsilon\partial_t R_t^*)R_t \\
&= \left[ (i\varepsilon\partial_t R_t^*)R_t, A(\nu_{N,t}(f,g)) \right]
\end{aligned} \tag{3.9}$$

where in the fourth line we used that  $0 = \partial_t(R_t^*R_t) = (\partial_t R_t^*)R_t + R_t^*\partial_t R_t$ . Therefore, from (3.6), (3.9) we find

$$\left[ (i\varepsilon\partial_t R_t^*)R_t, A(\nu_{N,t}(f,g)) \right] = i\varepsilon A(\partial_t \nu_{N,t}(f,g)). \tag{3.10}$$

Since (3.10) holds true for all  $f, g \in \mathfrak{h} \oplus \mathfrak{h}$ ,  $(i\varepsilon\partial_t R_t^*)R_t$  must be quadratic in the fermionic operators,

$$\begin{aligned}
(i\varepsilon\partial_t R_t^*)R_t &= \sum_{\sigma, \sigma' = l, r} \left[ \int dx dy C_t(x, \sigma; y, \sigma') a_{x, \sigma}^* a_{y, \sigma'} \right. \\
&\quad \left. + \frac{1}{2} \left( \int dx dy D_t(x, \sigma; y, \sigma') a_{x, \sigma}^* a_{y, \sigma'}^* + h.c. \right) \right] \\
&\equiv \int dx dy C_t(\mathbf{x}; \mathbf{y}) a_{\mathbf{x}}^* a_{\mathbf{y}} + \frac{1}{2} \left( \int dx dy D_t(\mathbf{x}; \mathbf{y}) a_{\mathbf{x}}^* a_{\mathbf{y}}^* + h.c. \right)
\end{aligned} \tag{3.11}$$

where we used the shorthand notations  $\mathbf{x} = (x, \sigma)$ ,  $\int d\mathbf{x} = \sum_{\sigma=l,r} \int dx$ . Eq. (3.11) holds for suitable kernels  $C_t, D_t$ , to be determined later, such that

$$\overline{C_t(\mathbf{x}; \mathbf{y})} = C_t(\mathbf{y}; \mathbf{x}), \quad D_t(\mathbf{x}; \mathbf{y}) = -D_t(\mathbf{y}; \mathbf{x}). \tag{3.12}$$

Eq. (3.10) is the equation that determines  $C_t, D_t$ . Taking  $f(x) = (\delta(x-w)\delta_{l,\sigma''}, \delta(x-w)\delta_{r,\sigma''})$ ,  $g(x) = (0, 0)$ , and setting  $\mathbf{w} = (w, \sigma'')$ , we find

$$A(\nu_{N,t}(f,g)) = A(U_{N,t}f, V_{N,t}f) = a(U_{N,t}f) + a^*(\overline{V_{N,t}f}) = a(U_{t,\mathbf{w}}) + a^*(\overline{V_{t,\mathbf{w}}})$$

where we introduced the notation

$$a(U_{t,\mathbf{w}}) := \sum_{\sigma} \int dz a_{z,\sigma} \overline{U_{N,t}(z, \sigma; w, \sigma'')}, \quad a^*(\overline{V_{t,\mathbf{w}}}) := \sum_{\sigma} \int dz a_{z,\sigma}^* \overline{V_{N,t}(z, \sigma; w, \sigma'')}.$$

Thus, the left hand side of (3.10) can be rewritten as:

$$\begin{aligned}
&\left[ (i\varepsilon\partial_t R_t^*)R_t, a(U_{t,\mathbf{w}}) + a^*(\overline{V_{t,\mathbf{w}}}) \right] \\
&= \int dx dy C_t(\mathbf{x}; \mathbf{y}) \left[ a_{\mathbf{x}}^* a_{\mathbf{y}}, a(U_{t,\mathbf{w}}) + a^*(\overline{V_{t,\mathbf{w}}}) \right] \\
&\quad + \frac{1}{2} \int dx dy D_t(\mathbf{x}; \mathbf{y}) \left[ a_{\mathbf{x}}^* a_{\mathbf{y}}^*, a(U_{t,\mathbf{w}}) + a^*(\overline{V_{t,\mathbf{w}}}) \right] \\
&\quad + \frac{1}{2} \int dx dy \overline{D_t(\mathbf{x}; \mathbf{y})} \left[ a_{\mathbf{y}} a_{\mathbf{x}}, a(U_{t,\mathbf{w}}) + a^*(\overline{V_{t,\mathbf{w}}}) \right].
\end{aligned} \tag{3.13}$$

Let us compute the first term on the r.h.s. of (3.13). We get

$$\begin{aligned}
&\int dx dy C_t(\mathbf{x}; \mathbf{y}) \left[ a_{\mathbf{x}}^* a_{\mathbf{y}}, a(U_{t,\mathbf{w}}) + a^*(\overline{V_{t,\mathbf{w}}}) \right] \\
&= \int dx dy C_t(\mathbf{x}; \mathbf{y}) (-a_{\mathbf{y}} \overline{U_t(\mathbf{x}; \mathbf{w})} + a_{\mathbf{x}}^* \overline{V_t(\mathbf{y}; \mathbf{w})}) \\
&= -a(C_t U_{t,\mathbf{w}}) + a^*(C_t \overline{V_{t,\mathbf{w}}}).
\end{aligned} \tag{3.14}$$

Similarly, we find that

$$\begin{aligned}
& \frac{1}{2} \int d\mathbf{x}d\mathbf{y} D_t(\mathbf{x}; \mathbf{y}) \left[ a_{\mathbf{x}}^* a_{\mathbf{y}}^*, a(U_{t,\mathbf{w}}) + a^*(\bar{V}_{t,\mathbf{w}}) \right] \\
&= \int d\mathbf{x}d\mathbf{y} D_t(\mathbf{x}; \mathbf{y}) (a_{\mathbf{x}}^* \bar{U}_t(\mathbf{y}; \mathbf{w}) - a_{\mathbf{y}}^* \bar{U}_t(\mathbf{x}; \mathbf{w})) \\
&= a^*(D_t \bar{U}_{t,\mathbf{w}})
\end{aligned} \tag{3.15}$$

and

$$\frac{1}{2} \int d\mathbf{x}d\mathbf{y} \overline{D_t(\mathbf{x}; \mathbf{y})} \left[ a_{\mathbf{y}} a_{\mathbf{x}}, a(U_{t,\mathbf{w}}) + a^*(\bar{V}_{t,\mathbf{w}}) \right] = -a(D_t V_{t,\mathbf{w}}). \tag{3.16}$$

Therefore, summing up (3.14) – (3.16) and substituting them into (3.10), we obtain

$$a^*(C_t \bar{V}_{t,\mathbf{w}}) + a^*(D_t \bar{U}_{t,\mathbf{w}}) - a(C_t U_{t,\mathbf{w}}) - a(D_t V_{t,\mathbf{w}}) = a^*(i\varepsilon \partial_t \bar{V}_{t,\mathbf{w}}) - a(i\varepsilon \partial_t U_{t,\mathbf{w}}). \tag{3.17}$$

This translates into a system of equations for the kernels  $C_t$  and  $D_t$ , namely:

$$\begin{aligned}
C_t \bar{V}_t + D_t \bar{U}_t &= i\varepsilon \partial_t \bar{V}_t, \\
C_t U_t + D_t V_t &= i\varepsilon \partial_t U_t,
\end{aligned} \tag{3.18}$$

which can be rewritten more compactly as

$$(C_t \ D_t) \begin{pmatrix} U_t & \bar{V}_t \\ V_t & \bar{U}_t \end{pmatrix} = (i\varepsilon \partial_t U_t \quad i\varepsilon \partial_t \bar{V}_t). \tag{3.19}$$

This gives

$$(C_t \ D_t) = (i\varepsilon \partial_t U_t \quad i\varepsilon \partial_t \bar{V}_t) \begin{pmatrix} U_t & V_t^* \\ \bar{V}_t^* & \bar{U}_t \end{pmatrix}. \tag{3.20}$$

Let

$$\begin{aligned}
H_{HF}(t) &:= \begin{pmatrix} h_{HF}(t) & 0 \\ 0 & -h_{HF}(t) \end{pmatrix}, \\
h_{HF}(t) &:= -\varepsilon^2 \Delta + \rho_t * V - X_t.
\end{aligned}$$

The time-dependent Hartree-Fock equation  $i\varepsilon \partial_t \omega_t = [h_{HF}(t), \omega_t]$  implies

$$\begin{aligned}
i\varepsilon \partial_t U_t &= [H_{HF}(t), U_t], \\
i\varepsilon \partial_t \bar{V}_t &= H_{HF}(t) \bar{V}_t + \overline{V_t H_{HF}(t)}.
\end{aligned} \tag{3.21}$$

Inserting these equations into (3.20) we have

$$\begin{aligned}
C_t &= H_{HF}(t) - U_t H_{HF}(t) U_t + \overline{V_t H_{HF}(t) \bar{V}_t^*} \\
&= \begin{pmatrix} h_{HF}(t) - u_t h_{HF}(t) u_t - v_t h_{HF}(t) v_t & 0 \\ 0 & -\overline{h_{HF}(t)} + \overline{u_t h_{HF}(t) \bar{u}_t} + \overline{v_t h_{HF}(t) \bar{v}_t} \end{pmatrix}, \\
D_t &= -U_t H_{HF}(t) V_t^* + \overline{V_t H_{HF}(t) \bar{U}_t} \\
&= \begin{pmatrix} 0 & u_t h_{HF}(t) v_t - v_t h_{HF}(t) u_t \\ \overline{u_t h_{HF}(t) \bar{v}_t} - \overline{v_t h_{HF}(t) \bar{u}_t} & 0 \end{pmatrix}.
\end{aligned} \tag{3.22}$$

Therefore, we conclude that

$$\begin{aligned}
(i\varepsilon \partial_t R_t^*) R_t &= d\Gamma_l \left( h_{HF}(t) - u_t h_{HF}(t) u_t - v_t h_{HF}(t) v_t \right) \\
&\quad - d\Gamma_r \left( \overline{h_{HF}(t)} - \overline{u_t h_{HF}(t) \bar{u}_t} - \overline{v_t h_{HF}(t) \bar{v}_t} \right) \\
&\quad + \left( \int dxdy a_{x,l}^* a_{y,r}^* (u_t h_{HF}(t) v_t - v_t h_{HF}(t) u_t)(x, y) + h.c. \right).
\end{aligned} \tag{3.23}$$

Computation of  $R_t^* \mathcal{L}_N R_t$ . Let us first compute the contribution of the kinetic energy, that is

$$R_t^* d\Gamma_l(-\varepsilon^2 \Delta) R_t - R_t^* d\Gamma_r(-\varepsilon^2 \Delta) R_t. \quad (3.24)$$

Using the definition of Bogoliubov transformation, after a straightforward computation we have

$$\begin{aligned} R_t^* d\Gamma_l(-\varepsilon^2 \Delta) R_t &= \text{tr}(-\varepsilon^2 \Delta \omega) + d\Gamma_l(u_t(-\varepsilon^2 \Delta) u_t) - d\Gamma_r(\bar{v}_t(-\varepsilon^2 \Delta) \bar{v}_t) \\ &\quad - \left( \int dx dy a_{x,l}^* a_{y,r}^* (u_t(-\varepsilon^2 \Delta) v_t)(x, y) + h.c. \right) \end{aligned} \quad (3.25)$$

and, analogously,

$$\begin{aligned} R_t^* d\Gamma_r(-\varepsilon^2 \Delta) R_t &= \text{tr}(-\varepsilon^2 \Delta \bar{\omega}) - d\Gamma_l(v_t(-\varepsilon^2 \Delta) v_t) + d\Gamma_r(\bar{u}_t(-\varepsilon^2 \Delta) \bar{u}_t) \\ &\quad + \left( \int dx dy a_{x,r}^* a_{y,l}^* (\bar{u}_t(-\varepsilon^2 \Delta) \bar{v}_t)(x, y) + h.c. \right). \end{aligned} \quad (3.26)$$

In other words,

$$\begin{aligned} R_t^* [d\Gamma_l(-\varepsilon^2 \Delta) - d\Gamma_r(-\varepsilon^2 \Delta)] R_t &= \\ &= d\Gamma_l(u_t(-\varepsilon^2 \Delta) u_t + v_t(-\varepsilon^2 \Delta) v_t) \\ &\quad - d\Gamma_r(\bar{u}_t(-\varepsilon^2 \Delta) \bar{u}_t + \bar{v}_t(-\varepsilon^2 \Delta) \bar{v}_t) \\ &\quad - \left( \int dx dy a_{x,l}^* a_{y,r}^* (u_t(-\varepsilon^2 \Delta) v_t - v_t(-\varepsilon^2 \Delta) u_t)(x, y) + h.c. \right). \end{aligned} \quad (3.27)$$

Finally, we compute the contribution coming from the many-body interaction, namely

$$R_t^* \left( \frac{1}{2N} \int dx dy V(x-y) (a_{x,l}^* a_{y,l}^* a_{y,l} a_{x,l} - a_{x,r}^* a_{y,r}^* a_{y,r} a_{x,r}) \right) R_t. \quad (3.28)$$

Consider first the ‘‘left’’ contribution to (3.28). Putting all terms in normal order and recalling the shorthand notation  $u_x(y) := u_{N,t}(y, x)$  and  $v_x(y) := v_{N,t}(y, x)$  we find

$$\begin{aligned} R_t^* a_{x,l}^* a_{y,l}^* a_{y,l} a_{x,l} R_t &= (a_l^*(u_x) - a_r^*(\bar{v}_x))(a_l^*(u_y) - a_r^*(\bar{v}_y))(a_l(u_y) - a_r^*(\bar{v}_y))(a_l(u_x) - a_r^*(\bar{v}_x)) \\ &= a_l^*(u_x) a_l^*(u_y) a_r^*(\bar{v}_y) a_r^*(\bar{v}_x) + 2a_l^*(u_x) a_l^*(u_y) a_r^*(\bar{v}_x) a_l(u_y) \\ &\quad - 2a_l^*(u_x) a_r^*(\bar{v}_y) a_r^*(\bar{v}_x) a_r(\bar{v}_y) + h.c. \\ &\quad + a_l^*(u_x) a_l^*(u_y) a_l(u_y) a_l(u_x) + 2a_l^*(u_x) a_r^*(\bar{v}_x) a_r(\bar{v}_y) a_l(u_y) \\ &\quad - 2a_l^*(u_x) a_r^*(\bar{v}_y) a_r(\bar{v}_y) a_l(u_y) + a_r^*(\bar{v}_y) a_r^*(\bar{v}_x) a_r(\bar{v}_x) a_r(\bar{v}_y) \\ &\quad + Q + S \end{aligned} \quad (3.29)$$

where

$$\begin{aligned} Q &= -2a_l^*(u_x) a_r^*(\bar{v}_x) \omega(y, y) + 2a_l^*(u_x) a_r^*(\bar{v}_y) \omega(x, y) + h.c. \\ &\quad + 2a_l^*(u_y) a_l(u_y) \omega(x, x) - 2a_l^*(u_y) a_l(u_x) \omega(y, x) - 2a_r^*(\bar{v}_y) a_r(\bar{v}_y) \omega(x, x) \\ &\quad + 2a_r^*(\bar{v}_y) a_r(\bar{v}_x) \omega(x, y) \end{aligned} \quad (3.30)$$

are the quadratic terms and

$$S = \omega(x, x) \omega(y, y) - \omega(y, x) \omega(x, y) \quad (3.31)$$

are the scalar terms (both the quadratic and the scalar terms arise from the normal ordering procedure). To compute the “right” contribution to (3.28), one just has to perform the replacement  $l \leftrightarrow r$ ,  $u_x \rightarrow \overline{u_x}$ ,  $\overline{v_x} \rightarrow -v_x$  in the above computation (see Eq. (2.28)).

Summing all together and combining the quadratic terms in (3.27), (3.30) and in its “right” counterpart with (3.23) we conclude that

$$\begin{aligned}\mathcal{G}_N(t) &= (i\varepsilon\partial_t R_t^*)R_t + R_t^*\mathcal{L}_N R_t \\ &= d\Gamma_l(h_{HF}(t)) - d\Gamma_r(\overline{h_{HF}(t)}) + \mathcal{C}_N + \mathcal{Q}_N\end{aligned}\quad (3.32)$$

where  $\mathcal{C}_N$  and  $\mathcal{Q}_N$  are given by (3.4) and (3.5).  $\square$

## 4 Bounds on the growth of fluctuations

The goal of this section is to control the growth of the particle number operator with respect to the fluctuation dynamics.

**Theorem 4.1.** *Assume  $V \in L^1(\mathbb{R}^3)$  satisfies (2.31). Let  $\omega_N$  be a sequence of trace class operators on  $\mathfrak{h} = L^2(\mathbb{R}^3)$  with  $0 \leq \omega_N \leq 1$ ,  $\text{tr} \omega_N = N$ , and satisfying the semiclassical bounds (2.32). Let  $\xi \in \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$  be such that  $\xi = \chi(\mathcal{N} \leq CN)\xi$ , where  $C \geq 0$  is independent of  $N$ . Let  $\mathcal{U}_N(t; s)$  be the fluctuation dynamics, as defined in (2.41). Let  $k \in \mathbb{N}$ . Then there exists a constant  $c_1$  depending on  $V$  and on  $k$  and a constant  $c_2$  depending only on  $V$  such that*

$$\left\langle \mathcal{U}_N(t; 0)\xi, (\mathcal{N} + 1)^k \mathcal{U}_N(t; 0)\xi \right\rangle \leq \exp(c_1 \exp(c_2|t|)) \left\langle \xi, (\mathcal{N} + 1)^{10k}\xi \right\rangle \quad (4.1)$$

for all  $t \in \mathbb{R}$ .

The proof of this theorem relies on Propositions 4.3 and 4.5 that will be proven in the next two sections. In turn, these results rely on the estimates for operators on the extended Fock space  $\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$  contained in the following lemma (which is the analog of Lemma 3.1 in [7]).

**Lemma 4.2.** *For every bounded operator  $O$  on  $\mathfrak{h} = L^2(\mathbb{R}^3)$ , we have*

$$\|d\Gamma_\sigma(O)\psi\| \leq \|O\| \|\mathcal{N}\psi\| \quad (4.2)$$

for every  $\psi \in \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$  and every  $\sigma \in \{l, r\}$ . Here  $\mathcal{N} = d\Gamma_l(1) + d\Gamma_r(1) = d\Gamma(1 \oplus 1)$  denotes the particle number operator on  $\mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$ .

If  $O$  is a Hilbert-Schmidt operator, we also have

$$\begin{aligned}\|d\Gamma_\sigma(O)\psi\| &\leq \|O\|_{HS} \|\mathcal{N}^{1/2}\psi\|, \\ \left\| \int dx dx' O(x; x') a_{x,\sigma} a_{x',\sigma'} \psi \right\| &\leq \|O\|_{HS} \|\mathcal{N}^{1/2}\psi\|, \\ \left\| \int dx dx' O(x; x') a_{x,\sigma}^* a_{x',\sigma'}^* \psi \right\| &\leq 2\|O\|_{HS} \|(\mathcal{N} + 1)^{1/2}\psi\|\end{aligned}\quad (4.3)$$

for all  $\psi \in \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$ .

*Proof.* To prove (4.2) notice that, for bounded operators  $A, B$  on  $\mathfrak{h}$ ,

$$\begin{aligned}\|d\Gamma(A \oplus B)\psi\|^2 &= \sum_{n \geq 0} \sum_{i,j \leq n} \langle \psi^{(n)}, (A \oplus B)^{(i)*} (A \oplus B)^{(j)} \psi^{(n)} \rangle \\ &\leq \|A \oplus B\|_{\mathfrak{h} \oplus \mathfrak{h}}^2 \sum_{n \geq 0} n^2 \|\psi^{(n)}\|^2 \\ &\leq \|A \oplus B\|_{\mathfrak{h} \oplus \mathfrak{h}}^2 \|\mathcal{N}\psi\|^2.\end{aligned}\quad (4.4)$$

The proof of (4.2) follows from (4.4) and the inequality  $\|A \oplus B\|_{\mathfrak{h} \oplus \mathfrak{h}} \leq \|A\|_{\mathfrak{h}} + \|B\|_{\mathfrak{h}}$ . Let us now prove the bounds (4.3). To prove the first two, notice that, denoting by  $a^\sharp$  either  $a$  or  $a^*$  and by  $O^\sharp$  either  $\bar{O}$  or  $O$ :

$$\begin{aligned} \left\| \int dx dx' O(x; x') a_{x, \sigma}^\sharp a_{x', \sigma'} \psi \right\| &= \left\| \int dx' a_\sigma^\sharp (O^\sharp(\cdot; x')) a_{x', \sigma'} \psi \right\| \\ &\leq \int dx' \|O(\cdot; x')\|_2 \|a_{x', \sigma'} \psi\| \\ &\leq \left( \int dx' \|O(\cdot; x')\|_2^2 \right)^{1/2} \left( \int dx' \|a_{x', \sigma'} \psi\|^2 \right)^{1/2} \\ &\leq \|O\|_{HS} \|\mathcal{N}_\sigma^{1/2} \psi\| \leq \|O\|_{HS} \|\mathcal{N}^{1/2} \psi\| \end{aligned} \quad (4.5)$$

where in the second line we used the boundedness of the fermionic operators, Eqs. (2.4). This proves in particular that

$$\left\| \int dx dx' O(x; x') a_{x, \sigma} a_{x', \sigma'} (\mathcal{N} + 1)^{-1/2} \right\| \leq \|O\|_{HS}.$$

Since the adjoint of a bounded operator is again bounded and has the same operator norm, we find

$$\begin{aligned} \left\| \int dx dx' O(x; x') (\mathcal{N} + 1)^{-1/2} a_{x, \sigma}^* a_{x', \sigma'}^* \right\| &= \left\| \int dx dx' O(x; x') a_{x, \sigma}^* a_{x', \sigma'}^* (\mathcal{N} + 3)^{-1/2} \right\| \\ &\leq \|O\|_{HS}. \end{aligned}$$

This concludes the proof, since  $\|(\mathcal{N} + 3)^{1/2} \psi\| \leq 2\|(\mathcal{N} + 1)^{1/2} \psi\|$  for all  $\psi \in \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$ .  $\square$

#### 4.1 The auxiliary dynamics

We start by studying the growth of the number of fluctuations with respect to an *auxiliary dynamics* that we shall denote by  $\tilde{\mathcal{U}}_N(t; s)$ . Later we will prove that  $\tilde{\mathcal{U}}_N(t; s)$  remains close (in norm) to the true fluctuation dynamics  $\mathcal{U}_N(t; s)$ .

The auxiliary dynamics  $\tilde{\mathcal{U}}_N(t; s)$  is defined through the differential equation

$$i\varepsilon \frac{d}{dt} \tilde{\mathcal{U}}_N(t; s) = \tilde{\mathcal{G}}_N(t) \tilde{\mathcal{U}}_N(t; s), \quad \tilde{\mathcal{U}}_N(s; s) = 1, \quad (4.6)$$

with the generator

$$\tilde{\mathcal{G}}_N(t) = d\Gamma_l(h_{HF}(t)) - d\Gamma_r(\overline{h_{HF}(t)}) + \mathcal{C}_N + \tilde{\mathcal{Q}}_N \quad (4.7)$$

where

$$\begin{aligned} \tilde{\mathcal{Q}}_N &= \frac{1}{2N} \int dx dy V(x - y) \\ &\times \left( a_l^*(u_x) a_l^*(u_y) a_r^*(\bar{v}_y) a_r^*(\bar{v}_x) - a_r^*(\bar{u}_x) a_r^*(\bar{u}_y) a_l^*(v_y) a_l^*(v_x) + h.c. \right). \end{aligned} \quad (4.8)$$

In (4.7), the operator  $\mathcal{C}_N$  is defined as in (3.4) and contains quartic terms commuting with the particle number operator  $\mathcal{N} = d\Gamma_l(1) + d\Gamma_r(1)$ .

In other words,  $\tilde{\mathcal{G}}_N(t)$  is obtained from  $\mathcal{G}_N(t)$  by dropping the second, third, fifth and sixth terms in (3.5) (together with their Hermitian conjugates). Next, we prove a bound for the growth of the fluctuations with respect to  $\tilde{\mathcal{U}}_N(t; s)$ .

**Proposition 4.3.** *Let  $V \in L^1(\mathbb{R}^3)$  and suppose that (2.31) holds. Let  $\omega_N$  be a sequence of non-negative trace class operators on  $\mathfrak{h} = L^2(\mathbb{R}^3)$ , with  $\text{tr} \omega_N = N$ ,  $\|\omega_N\| \leq 1$ , satisfying (2.32). Let  $\tilde{\mathcal{U}}_N(t; s)$  be the auxiliary dynamics defined in (4.6) and  $k \in \mathbb{N}$ . Then there exist a constant  $c_1$ , depending only on  $V$ , and a constant  $c_2$ , depending on  $V$  and  $k$ , such that*

$$\left\langle \tilde{\mathcal{U}}_N(t; s)\xi, (\mathcal{N} + 1)^k \tilde{\mathcal{U}}_N(t; s)\xi \right\rangle \leq \exp(c_2 \exp(c_1|t - s|)) \left\langle \xi, (\mathcal{N} + 1)^k \xi \right\rangle. \quad (4.9)$$

As in [7, 8], the proof of the theorem makes use of the propagation of the commutator estimates (2.32) along the solution of the Hartree-Fock equation. We defer the proof of the following Lemma to Appendix A.

**Lemma 4.4.** *Let  $V \in L^1(\mathbb{R}^3)$ , and suppose that (2.31) holds. Let  $\omega_N$  be a sequence of non-negative trace class operators on  $\mathfrak{h} = L^2(\mathbb{R}^3)$ , with  $\text{tr} \omega_N = N$ ,  $0 \leq \omega_N \leq 1$ , and such that*

$$\begin{aligned} \| [v_N, x] \|_{HS} &\leq CN^{1/2}\varepsilon, & \| [v_N, \varepsilon \nabla] \|_{HS} &\leq CN^{1/2}\varepsilon, \\ \| [u_N, x] \|_{HS} &\leq CN^{1/2}\varepsilon, & \| [u_N, \varepsilon \nabla] \|_{HS} &\leq CN^{1/2}\varepsilon, \end{aligned} \quad (4.10)$$

where  $v_N = \sqrt{\omega_N}$ ,  $u_N = \sqrt{1 - \omega_N}$ . Let  $v_{N,t} = \sqrt{\omega_{N,t}}$  and  $u_{N,t} = \sqrt{1 - \omega_{N,t}}$ , where  $\omega_{N,t}$  denotes the solution of the Hartree-Fock equation with initial data  $\omega_{N,0} = \omega_N$ .

Then, there exist constants  $K, c > 0$  only depending on the potential  $V$  such that

$$\begin{aligned} \| [v_{N,t}, x] \|_{HS} &\leq K \exp(ct) N^{1/2} \varepsilon, & \| [v_{N,t}, \varepsilon \nabla] \|_{HS} &\leq K \exp(ct) N^{1/2} \varepsilon, \\ \| [u_{N,t}, x] \|_{HS} &\leq K \exp(ct) N^{1/2} \varepsilon, & \| [u_{N,t}, \varepsilon \nabla] \|_{HS} &\leq K \exp(ct) N^{1/2} \varepsilon, \end{aligned} \quad (4.11)$$

for all  $t \in \mathbb{R}$ .

We are now ready to prove Proposition 4.3.

*Proof of Proposition 4.3.* For simplicity, we set  $s = 0$ ; the case  $s \neq 0$  can be handled in a completely analogous way. We compute

$$\begin{aligned} &i\varepsilon \frac{d}{dt} \left\langle \xi, \tilde{\mathcal{U}}_N^*(t; 0) (\mathcal{N} + 1)^k \tilde{\mathcal{U}}_N(t; 0) \xi \right\rangle \\ &= \left\langle \xi, \tilde{\mathcal{U}}_N^*(t; 0) \left[ (\mathcal{N} + 1)^k, \tilde{\mathcal{G}}_N(t) \right] \tilde{\mathcal{U}}_N(t; 0) \xi \right\rangle \\ &= \sum_{j=1}^k \left\langle \xi, \tilde{\mathcal{U}}_N^*(t; 0) (\mathcal{N} + 1)^{j-1} \left[ \mathcal{N}, \tilde{\mathcal{G}}_N(t) \right] (\mathcal{N} + 1)^{k-j} \tilde{\mathcal{U}}_N(t; 0) \xi \right\rangle. \end{aligned} \quad (4.12)$$

We have

$$\begin{aligned} \left[ \mathcal{N}, \tilde{\mathcal{G}}_N(t) \right] &= -\frac{4i}{N} \text{Im} \int dx dy V(x - y) \\ &\quad \times \left\{ a_l^*(u_x) a_l^*(u_y) a_r^*(\bar{v}_y) a_r^*(\bar{v}_x) - a_r^*(\bar{u}_x) a_r^*(\bar{u}_y) a_l^*(v_y) a_l^*(v_x) \right\}. \end{aligned}$$

It is important to recognize a cancellation arising from the two terms in the parenthesis. We

rewrite the commutator as

$$\begin{aligned}
[\mathcal{N}, \tilde{\mathcal{G}}_N(t)] &= -\frac{4i}{N} \text{Im} \int dp \hat{V}(p) \int dz_1 dz_2 dz_3 dz_4 \\
&\quad \times \left[ a_{z_1, l}^* a_{z_2, l}^* a_{z_3, r}^* a_{z_4, r}^* (u_t e^{ipx} v_t)(z_1, z_4) (u_t e^{-ipx} v_t)(z_2, z_3) \right. \\
&\quad \left. - a_{z_1, r}^* a_{z_2, r}^* a_{z_3, l}^* a_{z_4, l}^* (\bar{u}_t e^{ipx} \bar{v}_t)(z_1, z_4) (\bar{u}_t e^{-ipx} \bar{v}_t)(z_2, z_3) \right] \\
&= -\frac{4i}{N} \text{Im} \int dp \hat{V}(p) \int dz_1 dz_2 dz_3 dz_4 a_{z_1, l}^* a_{z_2, l}^* a_{z_3, r}^* a_{z_4, r}^* \\
&\quad \times \left[ (u_t e^{ipx} v_t)(z_1, z_4) (u_t e^{-ipx} v_t)(z_2, z_3) - (\bar{u}_t e^{ipx} \bar{v}_t)(z_4, z_1) (\bar{u}_t e^{-ipx} \bar{v}_t)(z_3, z_2) \right] \\
&= -\frac{4i}{N} \text{Im} \int dp \hat{V}(p) \int dz_1 dz_2 dz_3 dz_4 a_{z_1, l}^* a_{z_2, l}^* a_{z_3, r}^* a_{z_4, r}^* \\
&\quad \times \left[ (u_t e^{ipx} v_t)(z_1, z_4) (u_t e^{-ipx} v_t)(z_2, z_3) - (v_t e^{ipx} u_t)(z_1, z_4) (v_t e^{-ipx} u_t)(z_2, z_3) \right].
\end{aligned} \tag{4.13}$$

Using the identity

$$u_t e^{ipx} v_t = v_t e^{ipx} u_t + u_t [e^{ipx}, v_t] + v_t [u_t, e^{ipx}] \tag{4.14}$$

we can rewrite (4.13) as

$$\begin{aligned}
[\mathcal{N}, \tilde{\mathcal{G}}_N(t)] &= -\frac{4i}{N} \text{Im} \int dp \hat{V}(p) \int dz_1 dz_2 dz_3 dz_4 a_{z_1, l}^* a_{z_2, l}^* a_{z_3, r}^* a_{z_4, r}^* \\
&\quad \times \left[ (v_t e^{ipx} u_t)(z_1, z_4) \left( (u_t [e^{-ipx}, v_t])(z_2, z_3) + (v_t [u_t, e^{-ipx}])(z_2, z_3) \right) \right. \\
&\quad \left. + \left( (u_t [e^{ipx}, v_t])(z_1, z_4) + (v_t [u_t, e^{ipx}])(z_1, z_4) \right) (u_t e^{-ipx} v_t)(z_2, z_3) \right].
\end{aligned} \tag{4.15}$$

Using Lemma 4.2 we find, for any  $\varphi \in \mathcal{F}$ ,

$$\begin{aligned}
&\left\| \int dz_1 dz_2 a_{z_1, l}^* a_{z_2, r}^* (u_t [e^{ipx}, v_t])(z_1, z_2) \int dz_3 dz_4 a_{z_3, l}^* a_{z_4, r}^* (u_t e^{-ipx} v_t)(z_3, z_4) \varphi \right\| \\
&\leq C \|u_t [e^{ipx}, v_t]\|_{HS} \|u_t e^{-ipx} v_t\|_{HS} \|(\mathcal{N} + 1)\varphi\| \\
&\leq CN^{1/2} \|[e^{ipx}, v_t]\|_{HS} \|(\mathcal{N} + 1)\varphi\|.
\end{aligned} \tag{4.16}$$

Hence, substituting (4.15) into (4.12), we find, after some simple manipulations,

$$\begin{aligned}
&\varepsilon \left| \frac{d}{dt} \left\langle \xi, \tilde{\mathcal{U}}_N^*(t; 0) (\mathcal{N} + 1)^k \tilde{\mathcal{U}}_N(t; 0) \xi \right\rangle \right| \\
&\leq \frac{C}{N^{1/2}} \int dp |\hat{V}(p)| \left( \|[e^{ipx}, v_t]\|_{HS} + \|[e^{ipx}, u_t]\|_{HS} \right) \left\langle \xi, \tilde{\mathcal{U}}_N^*(t; 0) (\mathcal{N} + 1)^k \tilde{\mathcal{U}}_N(t; 0) \xi \right\rangle
\end{aligned}$$

where the constant  $C$  depends on  $k$ . Since

$$\|[e^{ipx}, v_t]\|_{HS} \leq |p| \|[x, v_t]\|_{HS}, \quad \|[e^{ipx}, u_t]\|_{HS} \leq |p| \|[x, u_t]\|_{HS} \tag{4.17}$$

we find, applying Proposition 4.4,

$$\left| \frac{d}{dt} \left\langle \xi, \tilde{\mathcal{U}}_N^*(t; 0) (\mathcal{N} + 1)^k \tilde{\mathcal{U}}_N(t; 0) \xi \right\rangle \right| \leq C \exp(c|t|) \left\langle \xi, \tilde{\mathcal{U}}_N^*(t; 0) (\mathcal{N} + 1)^k \tilde{\mathcal{U}}_N(t; 0) \xi \right\rangle. \tag{4.18}$$

Finally, Gronwall's Lemma implies that

$$\left\langle \xi, \tilde{\mathcal{U}}_N^*(t; 0) (\mathcal{N} + 1)^k \tilde{\mathcal{U}}_N(t; 0) \xi \right\rangle \leq \exp(c_2 \exp(c_1 |t|)) \left\langle \xi, (\mathcal{N} + 1)^k \xi \right\rangle \tag{4.19}$$

where  $c_2$  depends on  $k \in \mathbb{N}$  and on the potential  $V$ , and  $c_1$  depends only on  $V$ .  $\square$

## 4.2 Comparison of fluctuation and auxiliary dynamics

In this section we prove that the auxiliary dynamics  $\tilde{\mathcal{U}}_N$  is a good approximation of the fluctuation dynamics  $\mathcal{U}_N$ , in the sense that  $\|\mathcal{U}_N(t; s)\xi - \tilde{\mathcal{U}}_N(t; s)\xi\|$  goes to zero as  $N$  goes to infinity. This is the content of the following proposition.

**Proposition 4.5.** *Let  $V \in L^1(\mathbb{R}^3)$  and suppose that (2.31) holds. Let  $\omega_N$  be a sequence of trace-class operators on  $\mathfrak{h} = L^2(\mathbb{R}^3)$  with  $\text{tr} \omega_N = N$ ,  $0 \leq \omega_N \leq 1$ , and satisfying (2.32). Let  $\tilde{\mathcal{U}}_N(t; s)$  be the auxiliary dynamics defined in (4.6). Then there exist constants  $c_1 > 0$ ,  $c_2 > 0$ , only depending on  $V$ , such that*

$$\|(\mathcal{U}_N(t; s) - \tilde{\mathcal{U}}_N(t; s))\xi\| \leq N^{-1/6} \exp(c_2 \exp(c_1|t - s|)) \|(\mathcal{N} + 1)^{3/2}\xi\|. \quad (4.20)$$

*Proof.* We write

$$\begin{aligned} i\varepsilon \partial_t \|(\mathcal{U}_N(t; s) - \tilde{\mathcal{U}}_N(t; s))\xi\|^2 &= i\varepsilon \partial_t \langle \Omega, (\mathcal{U}_N(t; s)^* - \tilde{\mathcal{U}}_N(t; s)^*)(\mathcal{U}_N(t; s) - \tilde{\mathcal{U}}_N(t; s))\Omega \rangle \\ &= i\varepsilon \partial_t \langle \Omega, (2 - \mathcal{U}_N(t; s)^* \tilde{\mathcal{U}}_N(t; s) - \tilde{\mathcal{U}}_N(t; s)^* \mathcal{U}_N(t; s))\Omega \rangle \\ &= -2i\varepsilon \text{Im} \partial_t \langle \Omega, \mathcal{U}_N(t; s)^* \tilde{\mathcal{U}}_N(t; s)\Omega \rangle \\ &= 2\text{Im} \langle \mathcal{U}_N(t; s)\xi, (\mathcal{G}_N(t) - \tilde{\mathcal{G}}_N(t))\tilde{\mathcal{U}}_N(t; s)\xi \rangle \\ &= 2\text{Im} \langle \mathcal{U}_N(t; s) - \tilde{\mathcal{U}}_N(t; s)\xi, (\mathcal{G}_N(t) - \tilde{\mathcal{G}}_N(t))\tilde{\mathcal{U}}_N(t; s)\xi \rangle \end{aligned}$$

where in the last step we used that  $\langle \tilde{\mathcal{U}}_N(t; s)\xi, (\mathcal{G}_N(t) - \tilde{\mathcal{G}}_N(t))\tilde{\mathcal{U}}_N(t; s)\xi \rangle$  is real. This gives

$$\varepsilon \left| \partial_t \|(\mathcal{U}_N(t; s) - \tilde{\mathcal{U}}_N(t; s))\xi\| \right| \leq \|(\mathcal{G}_N(t) - \tilde{\mathcal{G}}_N(t))\tilde{\mathcal{U}}_N(t; s)\xi\|. \quad (4.21)$$

Recall that

$$\begin{aligned} \mathcal{G}_N(t) - \tilde{\mathcal{G}}_N(t) &= \frac{1}{N} \int dx dy V(x - y) \left( a_l^*(u_x) a_l^*(u_y) a_r^*(\bar{v}_x) a_l(u_y) - a_l^*(u_x) a_r^*(\bar{v}_y) a_r^*(\bar{v}_x) a_l(\bar{v}_y) \right. \\ &\quad \left. + a_r^*(\bar{u}_x) a_r^*(\bar{u}_y) a_l^*(v_x) a_r(\bar{u}_y) - a_r^*(\bar{u}_x) a_l^*(v_y) a_l^*(v_x) a_l(v_y) + h.c. \right). \end{aligned} \quad (4.22)$$

The contribution from the first term on the r.h.s. of (4.22) can be bounded by using Lemma 4.2. We find

$$\begin{aligned} &\left\| \frac{1}{N} \int dx dy V(x - y) a_l^*(u_x) a_l^*(u_y) a_r^*(\bar{v}_x) a_l(u_y) \tilde{\mathcal{U}}_N(t; s)\xi \right\| \\ &\leq \frac{1}{N} \int dp |\hat{V}(p)| \left\| \int dx a_l^*(u_x) a_r^*(\bar{v}_x) e^{ipx} \int dy e^{-ipy} a_l^*(u_y) a_l(u_y) \tilde{\mathcal{U}}_N(t; s)\Omega \right\| \\ &\leq \frac{1}{N} \int dp |\hat{V}(p)| \|ue^{ipx}v\|_{HS} \|ue^{-ipx}u\|_{op} \|(\mathcal{N} + 1)^{3/2}\tilde{\mathcal{U}}_N(t; s)\xi\| \\ &\leq \frac{C}{N^{1/2}} \|(\mathcal{N} + 1)^{3/2}\tilde{\mathcal{U}}_N(t; s)\xi\|. \end{aligned} \quad (4.23)$$

It is easy to see that the same estimate holds true for all the other contributions arising from (4.22). Therefore, we have:

$$\|(\mathcal{G}_N(t) - \tilde{\mathcal{G}}_N(t))\tilde{\mathcal{U}}_N(t; s)\xi\| \leq \frac{C}{N^{1/2}} \|(\mathcal{N} + 1)^{3/2}\tilde{\mathcal{U}}_N(t; s)\xi\|. \quad (4.24)$$

Substituting this estimate in (4.21) and using Proposition 4.3, we get

$$\left| \partial_t \|\mathcal{U}_N(t; s)\xi - \tilde{\mathcal{U}}_N(t; s)\xi\| \right| \leq N^{-1/6} \exp(c_2 \exp(c_1|t - s|)) \|(\mathcal{N} + 1)^{3/2}\xi\|.$$

The integration yields the claim.  $\square$



### 4.3 Proof of Theorem 4.1

We claim that there exists a constant  $c_1 > 0$  and, for every  $k \in \mathbb{N}$ , a constant  $c_2 > 0$  such that

$$\begin{aligned} \langle \mathcal{U}_N(t; s)\xi_N, (\mathcal{N} + 1)^k \mathcal{U}_N(t; s)\xi_N \rangle \\ \leq N^{k-n/3} \exp(2(n+1)c_2 \exp(c_1|t-s|)) \langle \xi_N, (\mathcal{N} + 1)^{k+3n}\xi_N \rangle \end{aligned} \quad (4.25)$$

for every  $n \in \mathbb{N}$  with  $n \leq 3k$ . We prove the claim (4.25) by induction over  $n$ . For  $n = 0$ , we observe from (2.41) that

$$\langle \mathcal{U}_N(t; s)\xi_N, (\mathcal{N} + 1)^k \mathcal{U}_N(t; s)\xi_N \rangle = \langle R_t^* e^{-i\mathcal{L}_N(t-s)} R_s \xi_N, (\mathcal{N} + 1)^k R_t^* e^{-i\mathcal{L}_N(t-s)} R_s \xi_N \rangle. \quad (4.26)$$

Since

$$R_t^* \mathcal{N} R_t = \mathcal{N} + 2N - 2d\Gamma(\omega_{N,t} \oplus \bar{\omega}_{N,t}) + 2 \int dx dy [a_{x,r}^* a_{y,t}^* \bar{u}_{N,t} \bar{v}_{N,t}(x, y) + \text{h.c.}]$$

we easily get

$$R_t^*(\mathcal{N} + c)R_t \leq C(\mathcal{N} + N + c)$$

for a universal constant  $C > 0$  and any  $c \geq 0$ . Also, iterating this bound  $k$  times (and always shifting the factors of  $\mathcal{N}$  to the left and to the right), we obtain

$$R_t^*(\mathcal{N} + c)^k R_t \leq C(\mathcal{N} + N + c)^k \quad (4.27)$$

for a  $k$ -dependent constant  $C > 0$ . Hence, applying (4.27) twice (once for  $R_t$ , once for  $R_s$ ) and using the fact that  $\mathcal{N}$  commutes with the Liouville operator  $\mathcal{L}_N$ , we find from (4.26) that

$$\begin{aligned} \langle \mathcal{U}_N(t; s)\xi_N, (\mathcal{N} + 1)^k \mathcal{U}_N(t; s)\xi_N \rangle &\leq CN^k \langle \xi_N, (\mathcal{N} + 1)^k \xi_N \rangle \\ &\leq N^k \exp(2c_2 \exp(c_1|t-s|)) \langle \xi_N, (\mathcal{N} + 1)^k \xi_N \rangle \end{aligned}$$

if  $c_2 > 0$  is large enough, depending on  $k$ . This proves (4.25) in the case  $n = 0$ .

Next we assume (4.25) to hold, and we prove it with  $n$  replaced by  $(n+1)$ , assuming  $n+1 \leq 3k$ . To this end, we write

$$\begin{aligned} \langle \xi_N, \mathcal{U}_N^*(t; s)(\mathcal{N} + 1)^k \mathcal{U}_N(t; s)\xi_N \rangle \\ = \langle \xi_N, \tilde{\mathcal{U}}_N^*(t; s)(\mathcal{N} + 1)^k \tilde{\mathcal{U}}_N(t; s)\xi_N \rangle \\ + 2 \operatorname{Re} \langle \xi_N, (\mathcal{U}_N^*(t; s) - \tilde{\mathcal{U}}_N^*(t; s))(\mathcal{N} + 1)^k \tilde{\mathcal{U}}_N(t; s)\xi_N \rangle \\ + \langle \xi_N, (\mathcal{U}_N^*(t; s) - \tilde{\mathcal{U}}_N^*(t; s))(\mathcal{N} + 1)^k (\mathcal{U}_N(t; s) - \tilde{\mathcal{U}}_N(t; s))\xi_N \rangle \\ = \text{I} + \text{II} + \text{III}. \end{aligned} \quad (4.28)$$

Using Proposition 4.3, we find

$$\begin{aligned} \text{I} &\leq \exp(c_2 \exp(c_1|t-s|)) \langle \xi_N, (\mathcal{N} + 1)^k \xi_N \rangle \\ &\leq N^{k-(n+1)/3} \exp((2n+3)c_2 \exp(c_1|t-s|)) \langle \xi_N, (\mathcal{N} + 1)^{k+3(n+1)} \xi_N \rangle. \end{aligned} \quad (4.29)$$

If  $k = 0$ , the second term on the r.h.s of (4.28) can be bounded trivially. For  $k \geq 1$ , on the other hand, we note that, for a vector  $\xi_N = \chi(\mathcal{N} \leq CN)\xi_N$ :

$$\begin{aligned}
|\text{III}| &\leq 2\|(\mathcal{U}_N(t; s) - \tilde{\mathcal{U}}_N(t; s))\xi_N\| \|(\mathcal{N} + 1)^k \tilde{\mathcal{U}}_N(t; s)\xi_N\| \\
&\leq 2N^{-1/6} \exp(c_2 \exp(c_1|t - s|)) \langle \xi_N, (\mathcal{N} + 1)^3 \xi_N \rangle^{1/2} \langle \xi_N, (\mathcal{N} + 1)^{2k} \xi_N \rangle^{1/2} \\
&\leq 2N^{-1/6} \exp(c_2 \exp(c_1|t - s|)) \langle \xi_N, (\mathcal{N} + 1)^{2k+1} \xi_N \rangle \\
&\leq 2(CN + 1)^{k-(n+1)/3} \exp(c_2 \exp(c_1|t - s|)) \langle \xi_N, (\mathcal{N} + 1)^{k+(n+1)/3+1} \xi_N \rangle \\
&\leq \tilde{C} N^{k-(n+1)/3} \exp((2n+3)c_2 \exp(c_1|t - s|)) \langle \xi_N, (\mathcal{N} + 1)^{k+3(n+1)} \xi_N \rangle
\end{aligned} \tag{4.30}$$

where  $\tilde{C} > 0$  is a  $k$ -dependent constant. In (4.30) we used: Propositions 4.3 and 4.5 in the second inequality; the fact that  $\max\{2k, 3\} \leq 2k + 1$  for  $k \geq 1$  in the third inequality; the assumption  $\xi_N = \chi(\mathcal{N} \leq CN)\xi_N$  in the fourth inequality; and the fact that  $n/3 + 1 \leq 3n$  for all  $n \geq 1$  in the last estimate.

Finally, we consider the last term in (4.28). To control it, we use the Duhamel formula

$$\mathcal{U}_N(t; s) - \tilde{\mathcal{U}}_N(t; s) = \frac{1}{i\varepsilon} \int_s^t ds' \mathcal{U}_N(t; s') (\mathcal{G}_N(s') - \tilde{\mathcal{G}}_N(s')) \tilde{\mathcal{U}}_N(s'; s).$$

Setting  $\Delta\mathcal{G}_N(s') = \mathcal{G}_N(s') - \tilde{\mathcal{G}}_N(s')$ , we rewrite III as

$$\frac{1}{\varepsilon^2} \int_s^t ds' \int_s^t ds'' \langle \mathcal{U}_N(t; s') \Delta\mathcal{G}_N(s') \tilde{\mathcal{U}}_N(s'; s) \xi_N, (\mathcal{N} + 1)^k \mathcal{U}_N(t; s'') \Delta\mathcal{G}_N(s'') \tilde{\mathcal{U}}_N(s''; s) \xi_N \rangle. \tag{4.31}$$

By Cauchy-Schwarz we find

$$|\text{III}| \leq \frac{2|t - s|}{\varepsilon^2} \int_s^t ds' \langle \mathcal{U}_N(t; s') \Delta\mathcal{G}_N(s') \tilde{\mathcal{U}}_N(s'; s) \xi_N, (\mathcal{N} + 1)^k \mathcal{U}_N(t; s') \Delta\mathcal{G}_N(s') \tilde{\mathcal{U}}_N(s'; s) \xi_N \rangle, \tag{4.32}$$

and the induction assumption yields

$$\begin{aligned}
|\text{III}| &\leq \frac{2|t - s| N^{k-n/3}}{\varepsilon^2} \int_s^t ds' \exp(2(n+1)c_2 \exp(c_1|t - s'|)) \\
&\quad \times \langle \Delta\mathcal{G}_N(s') \tilde{\mathcal{U}}_N(s'; s) \xi_N, (\mathcal{N} + 1)^{k+3n} \Delta\mathcal{G}_N(s') \tilde{\mathcal{U}}_N(s'; s) \xi_N \rangle.
\end{aligned}$$

Writing  $\Delta\mathcal{G}_N(s') = A(s') + A^*(s')$ , with

$$\begin{aligned}
A(s') &= \frac{1}{N} \int dx dy V(x - y) \left( a_l^*(u_x) a_l^*(u_y) a_r^*(\bar{v}_x) a_l(u_y) - a_l^*(u_x) a_r^*(\bar{v}_y) a_r^*(\bar{v}_x) a_r(\bar{v}_y) \right. \\
&\quad \left. + a_r^*(\bar{u}_x) a_r^*(\bar{u}_y) a_l^*(v_x) a_r(\bar{u}_y) - a_r^*(\bar{u}_x) a_l^*(v_y) a_l^*(v_x) a_l(v_y) \right)
\end{aligned}$$

we conclude that, using  $A(s')\mathcal{N} = (\mathcal{N} - 2)A(s')$  and Cauchy-Schwarz inequality:

$$\begin{aligned}
|\text{III}| &\leq \frac{C|t - s| N^{k-n/3} \exp(2(n+1)c_2 \exp(c_1|t - s|))}{\varepsilon^2} \\
&\quad \times \int_s^t ds' \left\{ \|A(s')(\mathcal{N} + 1)^{(k+3n)/2} \tilde{\mathcal{U}}_N(s'; s) \xi_N\|^2 + \|A^*(s')(\mathcal{N} + 1)^{(k+3n)/2} \tilde{\mathcal{U}}_N(s'; s) \xi_N\|^2 \right\}
\end{aligned}$$

for a  $k$ -dependent constant  $C > 0$ . Proceeding as in the proof of Proposition 4.5 (see (4.24)) and applying Proposition 4.3, we find

$$|\text{III}| \leq C|t - s|^2 N^{k-(n+1)/3} \exp((2n+3)c_2 \exp(c_1|t - s|)) \|(\mathcal{N} + 1)^{(k+3(n+1))/2} \xi_N\|^2 \tag{4.33}$$

for an appropriate  $k$ -dependent constant  $C > 0$ . Combining the last bound with (4.29) and (4.30) we can estimate  $|\text{I} + \text{II} + \text{III}|$  as follows:

$$(1 + \tilde{C} + C|t - s|^2)N^{k-(n+1)/3} \exp((2n+3)c_2 \exp(c_1|t-s|)) \|(\mathcal{N}+1)^{(k+3(n+1))/2} \xi_N\|^2. \quad (4.34)$$

If  $c_2 > 0$  is large enough, we have

$$(1 + \tilde{C} + C|t - s|^2) \leq \exp(c_2 \exp(c_1|t-s|))$$

for all  $t, s \in \mathbb{R}$ . Hence,

$$|\text{I} + \text{II} + \text{III}| \leq N^{k-(n+1)/3} \exp(2(n+2)c_2 \exp(c_1|t-s|)) \|(\mathcal{N}+1)^{(k+3(n+1))/2} \xi_N\|^2$$

which concludes the proof of (4.25) with  $n$  replaced by  $(n+1)$ . Finally, putting  $n = 3k$  and  $s = 0$  in (4.25), we obtain

$$\langle \xi_N, \mathcal{U}_N^*(t; 0) (\mathcal{N}+1)^k \mathcal{U}_N(t; 0) \xi_N \rangle \leq \exp(c_2 \exp(c_1|t|)) \langle \xi_N, (\mathcal{N}+1)^{10k} \xi_N \rangle$$

for an appropriate newly defined constant  $c_2$  depending on  $k$ . This completes the proof.  $\square$

## 5 Proof of main results

In this section we prove Theorems 2.1 and 2.2.

*Proof of Theorem 2.1.* We start from the expression

$$\begin{aligned} \gamma_{N,t}^{(1)}(x; y) &= \langle \psi_{N,t}, a_{y,l}^* a_{x,l} \psi_{N,t} \rangle \\ &= \langle e^{-i\mathcal{L}_N t/\varepsilon} R_0 \xi_N, a_{y,l}^* a_{x,l} e^{-i\mathcal{L}_N t/\varepsilon} R_0 \xi_N \rangle \\ &= \langle \xi_N, R_0^* e^{i\mathcal{L}_N t/\varepsilon} a_{y,l}^* a_{x,l} e^{-i\mathcal{L}_N t/\varepsilon} R_0 \xi_N \rangle. \end{aligned} \quad (5.1)$$

Introducing the fluctuation dynamics  $\mathcal{U}_N(t; 0) = R_t^* e^{-i\mathcal{L}_N t/\varepsilon} R_0$ , we can rewrite (5.1) as

$$\begin{aligned} \gamma_{N,t}^{(1)}(x; y) &= \langle \xi_N, \mathcal{U}_N(t; 0)^* R_t^* a_{y,l}^* R_t a_{x,l} R_t \mathcal{U}_N(t; 0) \xi_N \rangle \\ &= \langle \xi_N, \mathcal{U}_N(t; 0)^* (a_l^*(u_y) - a_r(\bar{v}_y)) (a_l(u_x) - a_r^*(\bar{v}_x)) \mathcal{U}_N(t; 0) \xi_N \rangle \\ &= \langle \xi_N, \mathcal{U}_N(t; 0)^* \{ a_l^*(u_y) a_l(u_x) - a_l^*(u_y) a_r^*(\bar{v}_x) - a_r(\bar{v}_y) a_l(u_x) - \\ &\quad - a_r^*(\bar{v}_x) a_r(\bar{v}_y) + \langle v_x, v_y \rangle \} \mathcal{U}_N(t; 0) \xi_N \rangle. \end{aligned}$$

Observe that

$$\langle v_x, v_y \rangle = \int dz \overline{v_{N,t}(z, x)} v_{N,t}(z, y) = \omega_{N,t}(x; y). \quad (5.2)$$

Therefore, we can write

$$\begin{aligned} \gamma_{N,t}^{(1)}(x; y) - \omega_{N,t}(x; y) &= \langle \xi_N, \mathcal{U}_N(t; 0)^* \{ a_l^*(u_y) a_l(u_x) - a_l^*(u_y) a_r^*(\bar{v}_x) - \\ &\quad - a_r(\bar{v}_y) a_l(u_x) - a_r^*(\bar{v}_x) a_r(\bar{v}_y) \} \mathcal{U}_N(t; 0) \xi_N \rangle. \end{aligned}$$

Integrating against the kernel of a one-particle Hilbert-Schmidt observable  $O$ , we find

$$\begin{aligned} \text{tr } O(\gamma_{N,t}^{(1)} - \omega_{N,t}) &= \langle \xi_N, \mathcal{U}_N^*(t; 0) (d\Gamma_l(u_t O u_t) - d\Gamma_r(\bar{v}_t \bar{O}^* v_t)) \mathcal{U}_N(t; 0) \xi_N \rangle \\ &\quad - 2 \text{Re} \langle \xi_N, \mathcal{U}_N^*(t; 0) \int dz_1 dz_2 a_{r,z_1} a_{l,z_2} (v_t O u_t)(z_1; z_2) \mathcal{U}_N(t; 0) \xi_N \rangle. \end{aligned} \quad (5.3)$$

Combining the estimates of Lemma 4.2 with the bounds  $\|u_t\| \leq 1$ ,  $\|v_t\| \leq 1$ , we conclude that

$$\begin{aligned} \left| \operatorname{tr} O(\gamma_{N,t}^{(1)} - \omega_{N,t}) \right| &\leq (\|u_t O u_t\| + \|\bar{v}_t \bar{O}^* v_t\|) \langle \xi_N, \mathcal{U}_N^*(t; 0) \mathcal{N} \mathcal{U}_N(t; 0) \xi_N \rangle + \\ &\quad + 2 \|v_t O u_t\|_{HS} \|\mathcal{N}^{1/2} \mathcal{U}_N(t; 0) \xi_N\| \|\xi_N\| \\ &\leq C \|O\|_{HS} \langle \xi_N, \mathcal{U}_N^*(t; 0) \mathcal{N} \mathcal{U}_N(t; 0) \xi_N \rangle. \end{aligned} \quad (5.4)$$

Finally, from Theorem 4.1 we find

$$\|\gamma_{N,t}^{(1)} - \omega_{N,t}\|_{HS} \leq \exp(c_2 \exp(c_1 |t|)) \langle \xi_N, (\mathcal{N} + 1)^{10} \xi_N \rangle,$$

which concludes the proof of convergence in the Hilbert-Schmidt norm. To prove convergence in trace norm, we proceed as follows. Let  $O$  be a compact operator on  $\mathfrak{h} = L^2(\mathbb{R}^3)$ . Using the bound  $\|u_t O v_t\|_{HS} \leq \|O\| \|v_t\|_{HS} \leq N^{1/2} \|O\|$ , one notices that (5.3) can be alternatively estimated by

$$\left| \operatorname{tr} O(\gamma_{N,t}^{(1)} - \omega_{N,t}) \right| \leq C \|O\| N^{1/2} \langle \xi_N, \mathcal{U}_N^*(t; 0) \mathcal{N} \mathcal{U}_N(t; 0) \xi_N \rangle,$$

and the proof is concluded by applying again Theorem 4.1.  $\square$

Next, we show the convergence of the  $k$ -particle reduced density, as stated in Theorem 2.2.

*Proof of Theorem 2.2.* We start from the expression

$$\begin{aligned} &\gamma_{N,t}^{(k)}(x_1, \dots, x_k; x'_1, \dots, x'_k) \\ &= \langle e^{-i\mathcal{L}_N t/\varepsilon} R_0 \xi_N, a_{x'_k, l}^* \dots a_{x'_1, l}^* a_{x_1, l} \dots a_{x_k, l} e^{-i\mathcal{L}_N t/\varepsilon} R_0 \xi_N \rangle \\ &= \langle \mathcal{U}_N(t; 0) \xi_N, R_t^* a_{x'_k, l}^* \dots a_{x'_1, l}^* a_{x_1, l} \dots a_{x_k, l} R_t \mathcal{U}_N(t; 0) \xi_N \rangle \\ &= \langle \mathcal{U}_N(t; 0) \xi_N, (a_l^*(u_{t, x'_k}) - a_r(\bar{v}_{t, x'_k})) \dots (a_l^*(u_{t, x'_1}) - a_r(\bar{v}_{t, x'_1})) \\ &\quad \times (a_l(u_{t, x_1}) - a_r^*(\bar{v}_{t, x_1})) \dots (a_l(u_{t, x_k}) - a_r^*(\bar{v}_{t, x_k})) \mathcal{U}_N(t; 0) \xi_N \rangle. \end{aligned} \quad (5.5)$$

The strategy of the proof is to expand the last product into a sum of  $2^k$  terms. Each summand can be put into normal order, using Wick's theorem. The fully contracted term will give rise to the kernel of  $\omega_{N,t}^{(k)}$ , while all the others will be proven to be of smaller order in  $N$ . The proofs of these facts follow the proof of Theorem 2.2 in [7]. Notice that, in contrast to [7], for a mixed state the operators  $u_{N,t}$  and  $\bar{v}_{N,t}$  are *not* orthogonal. However, the arguments of the fermionic operators  $u_{N,t} \oplus 0$  and  $0 \oplus \bar{v}_{N,t}$  are orthogonal operators in  $\mathfrak{h} \oplus \mathfrak{h}$ , and that suffices for our purposes. In order to make the proof of Theorem 2.1 self-contained, we sketch the main steps.

Proceeding as in the proof of Theorem 2.2 in [7], we expand the products in Eq. (5.5) and we put the result into normal order. The end result is a sum of contributions of the following form:

$$\begin{aligned} &\pm \left\langle \mathcal{U}_N(t; 0) \xi_N, : a_{\alpha_1}^\sharp(w_1(\cdot; x'_{\sigma(1)})) \dots a_{\alpha_{k-j}}^\sharp(w_{k-j}(\cdot; x'_{\sigma(k-j)})) \right. \\ &\quad \times a_{\alpha_1}^\sharp(\eta_1(\cdot; x_{\pi(1)})) \dots a_{\alpha_{k-j}}^\sharp(\eta_{k-j}(\cdot; x_{\pi(k-j)})) : \mathcal{U}_N(t; 0) \xi_N \rangle \\ &\quad \times \omega_{N,t}(x_{\pi(k-j+1)}; x'_{\sigma(k-j+1)}) \dots \omega_{N,t}(x_{\pi(k)}; x'_{\sigma(k)}) \end{aligned} \quad (5.6)$$

where the symbol  $\dots :$  denotes normal ordering;  $j \leq k$  denotes the number of contractions;  $\pi, \sigma \in S_k$  are two appropriate permutations; for every  $s = 1, \dots, k-j$ ,  $w_s, \eta_s : \mathfrak{h} \rightarrow \mathfrak{h}$  are

either the operator  $u_t$  or the operator  $\bar{v}_t$  (the operators are identified with their integral kernels); the  $\alpha, \alpha'$  labels are either  $l$  or  $r$ .

*Fully contracted term.* For  $j = k$ , the contractions produce the kernel of the  $k$ -particle density matrix  $\omega_{N,t}^{(k)}$ :

$$\sum_{\pi \in S_k} \sigma_\pi \omega_{N,t}(x_1; x'_{\pi(1)}) \cdots \omega_{N,t}(x_k; x'_{\pi(k)}) = \omega_{N,t}^{(k)}(\mathbf{x}_k; \mathbf{x}'_k)$$

where  $\mathbf{x}_k = (x_1, \dots, x_k)$ ,  $\mathbf{x}'_k = (x'_1, \dots, x'_k)$ .

*Error terms.* Consider now the contributions corresponding to  $j < k$ . Let  $O$  be a Hilbert-Schmidt operator on  $L^2(\mathbb{R}^{3k})$ , with integral kernel  $O(\mathbf{x}_k; \mathbf{x}'_k)$ . Integrating (5.6) against  $O(\mathbf{x}_k; \mathbf{x}'_k)$ , we get

$$\begin{aligned} \text{I} := & \left| \int d\mathbf{x}_k d\mathbf{x}'_k O(\mathbf{x}_k; \mathbf{x}'_k) \langle \mathcal{U}_N(t; 0) \xi_N, : a_{\alpha'_1}^\#(w_1(\cdot; x'_{\sigma(1)})) \cdots a_{\alpha'_{k-j}}^\#(w_{k-j}(\cdot; x'_{\sigma(k-j)})) \right. \\ & \times a_{\alpha_1}^\#(\eta_1(\cdot; x_{\pi(1)})) \cdots a_{\alpha_{k-j}}^\#(\eta_{k-j}(\cdot; x_{\pi(k-j)})) : \mathcal{U}_N(t; 0) \xi_N \rangle \\ & \left. \times \omega_{N,t}(x_{\pi(k-j+1)}; x'_{\sigma(k-j+1)}) \cdots \omega_{N,t}(x_{\pi(k)}; x'_{\sigma(k)}) \right|. \end{aligned} \quad (5.7)$$

This contribution can be rewritten as

$$\begin{aligned} \text{I} = & \left| \int d\mathbf{x}_k d\mathbf{x}'_k \left[ \eta_1^{(\pi(1))} \cdots \eta_{k-j}^{(\pi(k-j))} O w_1^{(\sigma(1))} \cdots w_{k-j}^{(\sigma(k-j))} \right] (\mathbf{x}_k; \mathbf{x}'_k) \right. \\ & \times \langle \mathcal{U}_N(t; 0) \xi_N, : a_{x'_{\sigma(1)}, \alpha'_1}^\# \cdots a_{x'_{\sigma(k-j)}, \alpha'_{k-j}}^\# a_{x_{\pi(1)}, \alpha_1}^\# \cdots a_{x_{\pi(k-j)}, \alpha_{k-j}}^\# : \mathcal{U}_N(t; 0) \xi_N \rangle \\ & \left. \times \omega_{N,t}(x_{\pi(k-j+1)}; x'_{\sigma(k-j+1)}) \cdots \omega_{N,t}(x_{\pi(k)}; x'_{\sigma(k)}) \right| \end{aligned}$$

where  $\eta_\ell^{(\pi(\ell))}$  and  $w_\ell^{(\sigma(\ell))}$  denote the one-particle operators  $\eta_\ell$  and  $w_\ell$  acting only on particle  $\pi(\ell)$  and particle  $\sigma(\ell)$ , respectively. Notice that some of the operators  $\eta_\ell^{(\pi(\ell))}$  and  $w_\ell^{(\sigma(\ell))}$  should actually be replaced by their transpose or their complex conjugate. This change however does not affect our analysis, since we will only need the bounds  $\|\eta_j\|, \|w_j\| \leq 1$  for the operator norms. Applying Hölder's inequality, we get

$$\begin{aligned} \text{I} & \leq \left\| \eta_1^{(\pi(1))} \cdots \eta_{k-j}^{(\pi(k-j))} O w_1^{(\sigma(1))} \cdots w_{k-j}^{(\sigma(k-j))} \right\|_{\text{HS}} \times \\ & \left( \int d\mathbf{x}_k d\mathbf{x}'_k \left| \langle \mathcal{U}_N(t; 0) \xi_N, : a_{x'_{\sigma(1)}, \alpha'_1}^\# \cdots a_{x'_{\sigma(k-j)}, \alpha'_{k-j}}^\# a_{x_{\pi(1)}, \alpha_1}^\# \cdots a_{x_{\pi(k-j)}, \alpha_{k-j}}^\# : \mathcal{U}_N(t; 0) \xi_N \rangle \right|^2 \right. \\ & \quad \left. \times \left| \omega_{N,t}(x_{\pi(k-j+1)}; x'_{\sigma(k-j+1)}) \right|^2 \cdots \left| \omega_{N,t}(x_{\pi(k)}; x'_{\sigma(k)}) \right|^2 \right)^{1/2} \\ & \leq \|O\|_{\text{HS}} \|\omega_{N,t}\|_{\text{HS}}^j \\ & \quad \times \left( \int dx_{\pi(1)} \cdots dx_{\pi(k-j)} dx'_{\sigma(1)} \cdots dx'_{\sigma(k-j)} \right. \\ & \quad \left. \times \left| \langle \mathcal{U}_N(t; 0) \xi_N, : a_{x'_{\sigma(1)}, \alpha'_1}^\# \cdots a_{x'_{\sigma(k-j)}, \alpha'_{k-j}}^\# a_{x_{\pi(1)}, \alpha_1}^\# \cdots a_{x_{\pi(k-j)}, \alpha_{k-j}}^\# : \mathcal{U}_N(t; 0) \xi_N \rangle \right|^2 \right)^{1/2}. \end{aligned}$$

Since  $\|\omega_{N,t}\|_{\text{HS}} \leq N^{1/2}$  and since the operators in the inner product are normal ordered, we obtain (possibly by moving some factors of  $(\mathcal{N} + 1)$  and  $(\mathcal{N} + 1)^{-1}$  around, to re-equilibrate the number of creation and annihilation operators left):

$$\text{I} \leq C \|O\|_{\text{HS}} N^{j/2} \langle \mathcal{U}_N(t; 0) \xi_N, (\mathcal{N} + 1)^{k-j} \mathcal{U}_N(t; 0) \xi_N \rangle.$$

Therefore, the contributions of the contractions with  $j < k$  arising from (5.5) after applying Wick's theorem and integrating against a Hilbert-Schmidt operator  $O$  can be bounded by

$$C\|O\|_{\text{HS}} N^{(k-1)/2} \langle \mathcal{U}_N(t; 0)\xi_N, (\mathcal{N} + 1)^k \mathcal{U}_N(t; 0)\xi_N \rangle. \quad (5.8)$$

This concludes the proof of convergence of many-body quantum dynamics to Hartree-Fock dynamics in Hilbert-Schmidt norm.

To prove convergence in trace norm, we proceed as follows. The goal is to prove that, for every compact operator  $O$  on  $L^2(\mathbb{R}^{3k})$ , the contribution (5.7) is estimated as follows:

$$\text{I} \leq C\|O\| N^{\frac{k+j}{2}} \exp(c_1 \exp(c_2|t|)) \quad (5.9)$$

for all  $\xi_N \in \mathcal{F}(\mathfrak{h} \oplus \mathfrak{h})$  with  $\langle \xi_N, \mathcal{N}^{10k} \xi_N \rangle < \infty$ , and the number of contractions  $0 \leq j < k$ . In fact, because of the fermionic symmetry of  $\gamma_{N,t}^{(k)}$  and  $\omega_{N,t}^{(k)}$ , it is enough to establish (5.9) for all bounded  $O$  with the symmetry

$$O(x_{\pi(1)}, \dots, x_{\pi(k)}; x'_{\sigma(1)}, \dots, x'_{\sigma(k)}) = \text{sgn}(\pi)\text{sgn}(\sigma) O(x_1, \dots, x_k; x'_1, \dots, x'_k)$$

for any permutations  $\pi, \sigma \in S_k$ . For such observables, (5.7) can be rewritten as

$$\begin{aligned} \text{I} &= \left| \int d\mathbf{x}_k d\mathbf{x}'_k O(\mathbf{x}_k, \mathbf{x}'_k) \langle \mathcal{U}_N(t; 0)\xi_N, : a_{\alpha'_1}^\#(w_1(\cdot, x'_1)) \cdots a_{\alpha'_{k-j}}^\#(w_{k-j}(\cdot, x'_{k-j})) \right. \\ &\quad \times a_{\alpha_1}^\#(\eta_1(\cdot, x_1)) \cdots a_{\alpha_{k-j}}^\#(\eta_{k-j}(\cdot, x_{k-j})) : \mathcal{U}_N(t; 0)\xi_N \rangle \\ &\quad \left. \times \omega_{N,t}(x_{k-j+1}, x'_{k-j+1}) \cdots \omega_{N,t}(x_k, x'_k) \right| \\ &= \left| \int d\mathbf{x}_{k-j} d\mathbf{x}'_{k-j} \left[ \eta_1^{(1)} \cdots \eta_{k-j}^{(k-j)} \left( \text{tr}_{k-j+1, \dots, k} O(1 \otimes \omega_{N,t}^{\otimes j}) \right) w_1^{(1)} \cdots w_{k-j}^{(k-j)} \right] (\mathbf{x}_{k-j}; \mathbf{x}'_{k-j}) \right. \\ &\quad \left. \times \langle \mathcal{U}_N(t; 0)\xi_N, : a_{\alpha'_1}^\# \cdots a_{\alpha'_{k-j}}^\# a_{\alpha_1}^\# \cdots a_{\alpha_{k-j}}^\# : \mathcal{U}_N(t; 0)\xi_N \rangle \right| \end{aligned}$$

where

$$\begin{aligned} &\left( \text{tr}_{k-j+1, \dots, k} O(1 \otimes \omega_{N,t}^{\otimes j}) \right) (\mathbf{x}_{k-j}; \mathbf{x}'_{k-j}) \\ &= \int dx_{k-j+1} dx'_{k-j+1} \cdots dx_k dx'_k O(\mathbf{x}_k; \mathbf{x}'_k) \prod_{\ell=k-j+1}^k \omega_{N,t}(x_\ell; x'_\ell) \end{aligned}$$

denotes the partial trace over the last  $j$  particles. Using Cauchy-Schwarz, we obtain

$$\begin{aligned} \text{I} &\leq \left\| \eta_1^{(1)} \cdots \eta_{k-j}^{(k-j)} \left( \text{tr}_{k-j+1, \dots, k} O(1 \otimes \omega_{N,t}^{\otimes j}) \right) w_1^{(1)} \cdots w_{k-j}^{(k-j)} \right\|_{\text{HS}} \left\| \mathcal{N}^{\frac{k-j}{2}} \mathcal{U}_N(t; 0)\xi_N \right\|^2 \\ &\leq \left\| \eta_1^{(1)} \cdots \eta_{k-j}^{(k-j)} \right\|_{\text{HS}} \left\| \text{tr}_{k-j+1, \dots, k} O(1 \otimes \omega_{N,t}^{\otimes j}) \right\| \left\| \mathcal{N}^{\frac{k-j}{2}} \mathcal{U}_N(t; 0)\xi_N \right\|^2 \\ &\leq N^{\frac{k-j}{2}} \left\| \text{tr}_{k-j+1, \dots, k} O(1 \otimes \omega_{N,t}^{\otimes j}) \right\| \left\| \mathcal{N}^{\frac{k-j}{2}} \mathcal{U}_N(t; 0)\xi_N \right\|^2 \end{aligned}$$

where in the second line we used that  $\|w_m^{(m)}\| = 1$  for all  $m = 1, \dots, k-j$ . Since

$$\begin{aligned} \left\| \text{tr}_{k-j+1, \dots, k} O(1 \otimes \omega_{N,t}^{\otimes j}) \right\| &= \sup_{\substack{\phi, \varphi \in L^2(\mathbb{R}^{3(k-j)}) \\ \|\phi\| = \|\varphi\| \leq 1}} \left| \langle \phi, \left( \text{tr}_{k-j+1, \dots, k} O(1 \otimes \omega_{N,t}^{\otimes j}) \right) \varphi \rangle \right| \\ &= \sup_{\substack{\phi, \varphi \in L^2(\mathbb{R}^{3(k-j)}) \\ \|\phi\| = \|\varphi\| \leq 1}} \left| \text{tr} O(|\varphi\rangle\langle\phi| \otimes \omega_{N,t}^{\otimes j}) \right| \\ &\leq (\text{tr}|\omega_{N,t}|)^j \|O\| \leq N^j \|O\|, \end{aligned} \quad (5.10)$$

we get

$$I \leq N^{\frac{k+j}{2}} \|O\| \left\| \mathcal{N}^{\frac{k-j}{2}} \mathcal{U}_N(t; 0) \xi \right\|^2,$$

which, by Theorem 4.1, proves (5.9). This concludes the proof of Theorem 2.2.  $\square$

## A Propagation of semiclassical structure

This appendix is devoted to the proof of Proposition 4.4. The argument follows [7] with some minor modifications.

*Proof of Proposition 4.4.* Let  $h_{HF}(t) = -\varepsilon^2 \Delta + \rho_t * V - X_t$  be the Hartree-Fock Hamiltonian, and let  $\omega_{N,t}$  denote the solution of the Hartree-Fock equation

$$i\varepsilon \partial_t \omega_{N,t} = [h_{HF}(t), \omega_{N,t}] \quad (\text{A.11})$$

with  $\omega_{N,0} = \omega_N$ . Since  $u_{N,t} = \sqrt{1 - \omega_{N,t}}$ ,  $v_{N,t} = \sqrt{\omega_{N,t}}$ , we have

$$i\varepsilon \partial_t v_{N,t} = [h_{HF}(t), v_{N,t}], \quad i\varepsilon \partial_t u_{N,t} = [h_{HF}(t), u_{N,t}] \quad (\text{A.12})$$

with  $u_{N,0} = u_N = \sqrt{1 - \omega_N}$  and  $v_{N,0} = v_N = \sqrt{\omega_N}$ . We first prove the propagation of the estimates for  $v_{N,t}$ . We compute

$$i\varepsilon \frac{d}{dt} [x, v_{N,t}] = [x, [h_{HF}(t), v_{N,t}]] = [v_{N,t}, [h_{HF}(t), x]] + [h_{HF}(t), [x, v_{N,t}]].$$

One eliminates the last term by conjugating  $[x, v_{N,t}]$  with the two-parameter group  $W(t; s)$  generated by the self-adjoint operator  $h_{HF}(t)$  satisfying

$$i\varepsilon \frac{d}{dt} W(t; s) = h_{HF}(t) W(t; s) \quad \text{with } W(s; s) = 1 \text{ for all } s \in \mathbb{R}.$$

We have

$$\begin{aligned} i\varepsilon \frac{d}{dt} W^*(t; 0) [x, v_{N,t}] W(t; 0) &= W^*(t; 0) [v_{N,t}, [h_{HF}(t), x]] W(t; 0) \\ &= W^*(t; 0) \left( [v_{N,t}, -2\varepsilon^2 \nabla] - [v_{N,t}, [X_t, x]] \right) W(t; 0) \end{aligned}$$

where we used that  $[-\varepsilon^2 \Delta, x] = -2\varepsilon^2 \nabla$  and that  $[\rho_t * V, x] = 0$ . Therefore,

$$\begin{aligned} W^*(t; 0) [x, v_{N,t}] W(t; 0) &= [x, v_{N,0}] + \frac{1}{i\varepsilon} \int_0^t ds i\varepsilon \frac{d}{ds} W^*(s; 0) [x, v_{N,s}] W(s; 0) \\ &= [x, v_{N,0}] - \frac{1}{i\varepsilon} \int_0^t ds W^*(s; 0) \left( [v_{N,s}, 2\varepsilon^2 \nabla] + [v_{N,s}, [X_s, x]] \right) W(s; 0). \end{aligned}$$

This implies that

$$\| [x, v_{N,t}] \|_{HS} \leq \| [x, v_{N,0}] \|_{HS} + \frac{1}{\varepsilon} \int_0^t ds \left( \| [v_{N,s}, 2\varepsilon^2 \nabla] \|_{HS} + \| [v_{N,s}, [X_s, x]] \|_{HS} \right). \quad (\text{A.13})$$

To control the second term, notice that

$$X_s = \frac{1}{N} \int dq \hat{V}(q) e^{ip \cdot x} \omega_{N,s} e^{-ip \cdot x}. \quad (\text{A.14})$$

Using that  $\|v_{N,s}\|_{op} \leq 1$ , we find:

$$\begin{aligned}
\| [v_{N,s}, [X_s, x]] \|_{HS} &\leq \frac{1}{N} \int dq |\hat{V}(q)| \| [v_{N,s}, [e^{ip \cdot x} \omega_{N,s} e^{-ip \cdot x}, x]] \|_{HS} \\
&\leq \frac{1}{N} \int dq |\hat{V}(q)| \| [v_{N,s}, e^{ip \cdot x} [\omega_{N,s}, x] e^{-ip \cdot x}] \|_{HS} \\
&\leq \frac{2}{N} \int dq |\hat{V}(q)| \| [\omega_{N,s}, x] \|_{HS} \\
&\leq \frac{C}{N} \| [v_{N,s}, x] \|_{HS}
\end{aligned} \tag{A.15}$$

where in the last step we used the identity

$$[\omega_{N,s}, x] = [v_{N,s} v_{N,s}, x] = v_{N,s} [v_{N,s}, x] + [v_{N,s}, x] v_{N,s} \tag{A.16}$$

and the triangle inequality. Substituting the estimate (A.15) in (A.13), we derive

$$\| [x, v_{N,t}] \|_{HS} \leq \| [x, v_{N,0}] \|_{HS} + C \int_0^t ds \left( \| [v_{N,s}, \varepsilon \nabla] \|_{HS} + N^{-2/3} \| [v_{N,s}, x] \|_{HS} \right). \tag{A.17}$$

To control  $[v_{N,s}, \varepsilon \nabla]$ , we start by writing

$$\begin{aligned}
i\varepsilon \frac{d}{dt} [\varepsilon \nabla, v_{N,t}] &= [\varepsilon \nabla, [h_{HF}(t), v_{N,t}]] = [v_{N,t}, [h_{HF}(t), \varepsilon \nabla]] + [h_{HF}(t), [\varepsilon \nabla, v_{N,t}]] \\
&= [h_{HF}(t), [\varepsilon \nabla, v_{N,t}]] + [v_{N,t}, [\rho_t * V, \varepsilon \nabla]] - [v_{N,t}, [X_t, \varepsilon \nabla]].
\end{aligned}$$

As before, the first term can be eliminated after conjugating with the unitary operator  $W(t; 0)$ . Hence, we find

$$\| [\varepsilon \nabla, v_{N,t}] \|_{HS} \leq \| [\varepsilon \nabla, v_{N,0}] \|_{HS} + \frac{1}{\varepsilon} \int_0^t ds \left( \| [v_{N,s}, [\rho_s * V, \varepsilon \nabla]] \|_{HS} + \| [v_{N,s}, [X_s, \varepsilon \nabla]] \|_{HS} \right). \tag{A.18}$$

The second term can be controlled by writing

$$\begin{aligned}
\| [v_{N,s}, [V * \rho_s, \varepsilon \nabla]] \|_{HS} &= \varepsilon \| [v_{N,s}, \nabla V * \rho_s] \|_{HS} \\
&\leq \varepsilon \int dq |\hat{V}(q)| |q| |\hat{\rho}_s(q)| \| [v_{N,s}, e^{iq \cdot x}] \|_{HS} \\
&\leq \varepsilon \int dq |\hat{V}(q)| |q|^2 |\hat{\rho}_s(q)| \| [v_{N,s}, x] \|_{HS} \\
&\leq C\varepsilon \| [v_{N,s}, x] \|_{HS}
\end{aligned} \tag{A.19}$$

where we used the bound  $\|\hat{\rho}_s\|_\infty \leq \|\rho_s\|_1 = 1$  and the assumption on the interaction potential. Let us now focus on the last term in (A.18). Rewriting  $X_s$  as in (A.14), we have

$$\begin{aligned}
\| [v_{N,s}, [X_s, \varepsilon \nabla]] \|_{HS} &\leq \frac{1}{N} \int dq |\hat{V}(q)| \| [v_{N,s}, [e^{iq \cdot x} \omega_{N,s} e^{-iq \cdot x}, \varepsilon \nabla]] \|_{HS} \\
&\leq \frac{2}{N} \int dq |\hat{V}(q)| \| [e^{iq \cdot x} \omega_{N,s} e^{-iq \cdot x}, \varepsilon \nabla] \|_{HS} \\
&\leq \frac{2}{N} \int dq |\hat{V}(q)| \| [\omega_{N,s}, \varepsilon \nabla] \|_{HS} \\
&\leq \frac{4\|\hat{V}\|_1}{N} \| [v_{N,s}, \varepsilon \nabla] \|_{HS}
\end{aligned} \tag{A.20}$$



where to get the third inequality we used that

$$[e^{iq \cdot x} \omega_{N,s} e^{-iq \cdot x}, \varepsilon \nabla] = e^{iq \cdot x} [\omega_{N,s}, \varepsilon(\nabla + iq)] e^{-iq \cdot x} = e^{iq \cdot x} [\omega_{N,s}, \varepsilon \nabla] e^{-iq \cdot x}$$

and to get the last inequality we used the analog of identity (A.16) with  $x$  replaced by  $\varepsilon \nabla$ . Substituting the estimates (A.19), (A.20) into (A.18), we get

$$\|[\varepsilon \nabla, v_{N,t}]\|_{HS} \leq \|[\varepsilon \nabla, v_{N,0}]\|_{HS} + C \int_0^t ds \left( \|v_{N,s}, x\|_{HS} + N^{-2/3} \|v_{N,s}, \varepsilon \nabla\|_{HS} \right).$$

Summing up this inequality with (A.17), using the assumption on the initial data and applying Gronwall's lemma, we obtain

$$\|x, v_{N,t}\|_{HS} \leq K \exp(c|t|) N^{1/2} \varepsilon, \quad \|[\varepsilon \nabla, v_{N,t}]\|_{HS} \leq K \exp(c|t|) N^{1/2} \varepsilon \quad (\text{A.21})$$

for suitable positive constants  $K, c$  only depending on  $V$ . This proves the propagation of the semiclassical estimates for  $v_{N,t}$ . To conclude the proof, we have to prove the propagation of the semiclassical estimates for  $u_{N,t}$ . The argument proceeds in the same way, with the following modifications. In the estimate for  $\|u_{N,s}, x\|_{HS}$ , in the step analogous to (A.15) one has to use that

$$\|\omega_{N,s}, x\|_{HS} = \|1 - \omega_{N,s}, x\|_{HS} \leq 2 \|u_{N,t}, x\|_{HS}$$

where we used that

$$[1 - \omega_{N,s}, x] = [u_{N,s} u_{N,s}, x] = u_{N,s} [u_{N,s}, x] + [u_{N,s}, x] u_{N,s}. \quad (\text{A.22})$$

Similarly, in the estimate for  $\|u_{N,s}, \varepsilon \nabla\|_{HS}$ , in the step analogous to (A.20) one has to use that

$$\|\omega_{N,s}, \varepsilon \nabla\|_{HS} = \|1 - \omega_{N,s}, \varepsilon \nabla\|_{HS} \leq 2 \|u_{N,s}, \varepsilon \nabla\|_{HS}$$

where we used the analog of (A.22) with  $x$  replaced by  $\varepsilon \nabla$ . Taking into account these small modifications, one derives

$$\|x, u_{N,t}\|_{HS} \leq K \exp(c|t|) N^{1/2} \varepsilon, \quad \|[\varepsilon \nabla, u_{N,t}]\|_{HS} \leq K \exp(c|t|) N^{1/2} \varepsilon \quad (\text{A.23})$$

for suitable constants  $K, c$  only depending on  $V$ . This proves the propagation of the semiclassical structure for  $u_{N,t}$ , and concludes the proof of Proposition 4.4.  $\square$

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