Renormalization group and divergences

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Abstract: *Application of asymptotic freedom to the ultraviolet stability in Euclidean quantum field theories is revisited and illustrated through the hierarchical model making also use of a few technical developments that followed the original works of Wilson on the renormalization group.*

Key words: *Renormalization group, Coupling renormalization, Hierarchical model, Quantum field theory*

1 Euclidean quantum fields

The first examples that Wilson worked out about the constructions of the renormalization flow are in two quite similar works, [15, 16]. The second is slightly simpler because it deals with a system consisting entirely of spins (*i.e.* described by bounded operators). It essentially introduces the renormalization group method via the consideration of hierarchical models as a tool to understand the essence of renormalization theory. The hierarchical models also appeared explicitly essentially at the same time in the work of [2] devoted to the theory of phase transitions (in 1 dimension) but not in relation to renormalization theory: the intimate relation between the two domains (statistical mechanics and quantum field theory) was a consequent development.

The work [16] ideally foreshadows the theory of the Kondo effect developed shortly afterwards and presented in detail in [20]; the papers [17, 18] reduce to the theory of a dynamical system the study of the critical point in the Ising model: a breakthrough making possible, for the first time, a computer aided approach to the calculation of critical exponents in dimension < 4. At the same time it gave a solution, via the same recursion, to the ultraviolet stability in QFT of dimension < 4, a classical renormalization problem studied until then by rather different methods, [12, 6].

The work of Wilson that most influenced *constructive theory* of quantum fields has been the analysis of the hierarchical model performed applying his view of renormalization, [17],[19, Eq.(23)] to scalar field theory: it made crystal clear that the divergences removal (already known since the early days of renormalization theory to be a "multiscale problem", [9]) was reducible to controlling a dynamical system governing the evolution, as the "scale" changed up from the ultraviolet (from short distances to distances of $O(1)$) or up in the infrared (from distances of $O(1)$ to large distances), of a few "running couplings" with a technique that unified conceptually the quantum field theory renormalization and the classical critical point theory via the new concept of asymptotic freedom, to the emergence of which his work gave an important contribution, [8, 14].

The hierarchical model analysis for the scalar $\lambda\varphi^4$ field in space-time dimensions ≤ 3, performed following Wilson's renormalization methods, teaches how to treat functional integrals (at least in the asymptotically free theories) as chains of "naive" sums. In the end it shows, for instance in scalar QFT at low dimension, that there is no divergence problem if the analysis is properly set up: because the physically interesting quantities (like the "Schwinger functions") are expressed as power series in the running couplings with no divergences at all.

This is an important result, although the model is a simplified version of a theory, the "$\lambda\varphi^4$ field theory", which at the time "had no obvious application anywhere in elementary particle physics", [14].

Divergences arise if the running couplings are expanded in power series of the constants in the Lagrangian function; the point being the lack of analyticity of the running couplings in terms of the parameters present in the Lagrangian, called "bare constants". Attempting an (*unnecessary*) expansion of the running couplings in terms of the parameters present in the Lagrangian, called bare constants, results in divergent expressions.

In Wilson's approach bare constants will never appear (and therefore the accompanying divergences will never arise): the theory will be described by the sequence of the running couplings which are related to their values on the physical scale [1] by a map, called the beta function. At least not in theories which are asymptotically free: in the others, which represent many physically relevant problems, like the critical point theory, the question is still very hard as it relies on the possible existence of non trivial fixed points for the map describing the running constants flow through the different scales.

Implicitly the hierarchical model was introduced already in [16] (related to "meson theory") and it was preceded by an even simpler version (related to the "Lee model") [15]. In its simplest version it is a model for the Euclidean φ^4-theory in the ultraviolet region. This is a theory which in space-time dimension ≤ 3 is asymptotically free in the ultraviolet region and

[1] The observer's length and time scales are by definition of $O(1)$.

asymptotically non trivial in the infrared region: and the basic mathematical problem is to give a meaning to the functional integral

$$Z = \int e^{-\int_\Lambda (\lambda\varphi_x^4 + \mu\varphi_x^2 + \nu)dx} \Big[C\, e^{-\frac12 \int ((\partial_x\varphi_x)^2 + \varphi_x^2)dx} \prod_x d\varphi_x\Big] \qquad (1.1)$$

where φ_x is a function on Λ and, in the "ultraviolet problem", the integral in the exponent is over a finite volume Λ, *e.g.* a cube for simplicity (if $d = 3$) or a square (if $d = 2$). The easy case $\lambda = \mu = \nu = 0$, "free field", corresponds to interpreting the quantity in square brackets in Eq.(1.1) as a Gaussian probability distribution assigning average value $\langle\, \varphi_x\varphi_y\, \rangle$ to the product $\varphi_x\varphi_y$ as:

$$\Gamma(x,y) \stackrel{def}{=} \langle\, \varphi_x\varphi_y\, \rangle = \frac{1}{(2\pi)^d} \int \frac{e^{ip(x-y)}}{1+p^2} d^dp \simeq const \frac{e^{-|x-y|}}{|x-y|^{d-2}} \qquad (1.2)$$

which, through the rules for Gaussian integrals ("Wick's rules"), defines all the averages $\langle\, \varphi_{x_1}\varphi_{x_2}\cdots\varphi_{x_{2n}}\, \rangle$.

The basic difficulties can be seen from the fact that if $d \ge 2$ then $\langle\, \varphi_x^2\, \rangle = +\infty$: with the consequent failure of any attempt to evaluate Z through an expansion in powers of λ, μ, ν, for instance, the integral in Eq.(1.1) or

$$\frac{1}{Z} \int \varphi_x\varphi_y\, e^{-\int_\Lambda (\lambda\varphi_x^4 + \mu\varphi_x^2 + \nu)dx} \Big[C e^{-\frac12 \int ((\partial_x\varphi_x)^2 + \varphi_x^2)dx} \prod_x d\varphi_x\Big] \qquad (1.3)$$

Yet it is well known that the founding fathers devised a resummation scheme, the "renormalization", of the series so that divergences would disappear.

In the work [15] Wilson undertook to define an algorithm that would produce the resummation of the formal series (with divergent coefficients) transforming it into a power series of a new sequence of *finite* constants related to each other as subsequent elements of a trajectory of a map in a finite dimensional space (*very low dimensional*, actually one dimensional in the quoted paper) with initial data suitably restricted.

The simple but new idea was that the functional integral had to be thought of as a sequence of almost identical integrals each of which simple enough to be computable naively. The hierarchical model realizes a paradigmatic case.

2 The hierarchical model

Before discussing in detail the model it is interesting to quote what appears to be its birth moment:

"*In this approximation the free-meson field has been replaced by independent harmonic oscillators for each phase space cell, with a frequency depending only on the mean momentum of the cell. The interaction of the meson field with the source has been replaced by an interaction of those oscillators located at the origin (where the source is) with the source. The remaining terms of the original Hamiltonian are to be considered as a perturbation*" [15, p.455].

As will be seen below this viewpoint, very clearly presented again in [18, p.3184], where the following heuristic remark summarizes another key idea:

This means that $s_L(\mathbf{x})$ does not vary enormously within a block of size L and for qualitative purposes one can think of $s_L(\mathbf{x})$ within a block as if it were a single block variable,

and in [19, Eqs.(23),(33)], opens the way to a totally new conception of renormalization theory through functional integrals: I allows himself to remember here a talk by Wilson at the University of Roma in the early '70's. There I was amazed to see the way and ease he was using to compute functional integrals: it was in sharp contrast to what I was used to after learning the mathematical theory of Brownian motion (no functional spaces in sight, no Banach spaces, no subtle almost everywhere statements, ...), and the procedure seemed to me far from mathematical rigor. I raised hand and signified my disappointment: the lapidary reply was just "you do not understand functional integration". *Therefore* I tried to understand why and shortly afterwards I was working intensely on the renormalization group in scalar quantum fields, using the methods that he had described, and I kept doing so for the next two decades.

Imagine Λ of side L and paved by cubes or squares Δ of side $2^{-n}L$, $n = 0, 1, \ldots$; the pavements $\mathcal{Q}_n$ will be said to have "scale n". To each Δ associate a normal Gaussian random variable z_Δ with distribution $P(dz_\Delta)$ and define

$$\varphi_x \stackrel{def}{=} \sum_{n=0}^{\infty} \sum_{x\in\Delta\in\mathcal{Q}_n} 2^{\frac{d-2}{2}n} z_\Delta, \qquad P(dz_\Delta) \stackrel{def}{=} \frac{e^{-\frac12 z_\Delta^2}}{\sqrt{2\pi}} dz_\Delta \tag{2.1}$$

The distribution of the φ_x's thus constructed is "quite close" to the Gaussian process defined by Eq.(1.2). Let $d_h(x,y)$ denote $2^{-n(x,y)}$ with $n(x,y)-1$ being the scale of the smallest Δ that contains both x and y; then $d_h(x,y)$, called *dyadic distance of* x, y, will often enough be close to the actual distance between x, y: in the sense that the average $\langle\, \varphi_x \varphi_y \,\rangle$ of the product of two φ's as defined by Eq.(2.1) is

$$C(x,y)\stackrel{def}{=}\langle\,\varphi_x\varphi_y\,\rangle=\begin{cases}-\log_2 d_h(x,y) & \text{if } d=2\\ \frac{1}{d_h(x,y)^{d-2}}\frac{1-d_h(x,y)^{d-2}}{2^{d-2}-1}\simeq\frac{1}{d_h(x,y)^{d-2}} & \text{if } d>2\end{cases}\tag{2.2}$$

Certainly the value of the field φ_x is infinite for every x: nevertheless $C(x,y)<\infty$ if $x\neq y$. A precise meaning of Eq.(1.1),(1.2) can be defined via a "regularization procedure": define $\varphi_x^{[\le N]}$ as

$$\varphi_x^{[\le N]}\stackrel{def}{=}\sum_{n=0}^{N}\sum_{x\in\Delta\in\mathcal{Q}_n}2^{\frac{d-2}{2}n}z_\Delta\tag{2.3}$$

which is a well defined finite sum and therefore

$$Z_N\stackrel{def}{=}\int e^{-\int_\Lambda(\lambda(\varphi_x^{[\le N]})^4+\mu(\varphi_x^{[\le N]})^2+\nu)dx}P(d\varphi)\tag{2.4}$$

is well defined if $P(d\varphi)=\prod_\Delta P(dz_\Delta)$ denotes integration with respect to the z_Δ variables introduced in Eq.(2.1).

The plan is then to integrate the z_Δ variables for Δ on a given scale and prodeed to integrate the other z-variables "one scale at a time": the correct question to pose is whether the parameters λ,μ,ν can be so chosen *as functions of* N in such a way that the limit as $N\to\infty$, called *ultraviolet limit*, of

$$S_N(x_1,\ldots x_{2s})\stackrel{def}{=}\int\varphi_{x_1}\varphi_{x_2}\cdots\varphi_{x_{2s}}\,\frac{e^{-\int_\Lambda(\lambda(\varphi_x^{[\le N]})^4+\mu(\varphi_x^{[\le N]})^2+\nu)\,dx}P(d\varphi)}{Z_N}\tag{2.5}$$

is not only well defined for all pairwise distinct $x_1,\ldots,x_{2s}$ and all s. but it is also "non trivial", *i.e.* it is not computable via Wick's rule from $S_\infty(x_1,x_2)$ (which means that after removing the cut-off, $N\to\infty$, the theory is not a free theory).

In applications the physically relevant quantities are expressed in terms of the *Schwinger functions*, $S_\infty(x_1,\ldots x_{2s})$: so on the one hand the bare constants disappear and, on the other hand, one is left with the problem of checking that the $S_\infty(x_1,\ldots x_{2s})$ have the properties needed to describe a theory that agrees with the basic laws of dynamics: which essentially amount at suitable analyticity properties of the Schwinger functions, [13].

The point of the hierarchical model is that the construction of its Schwinger functions as limits of regularized probability distributions of the fields φ_x presents the same difficulties, *in dimension* 2 *and* 3, that are encountered in the study of the integrals like Eq.(1.3).

Namely attempting an expansion in powers of the couplings leads to divergent quantities which can be eliminated through suitable resummations. Its study via Wilson's renormalization group method *simply avoids introducing divergences.*

3 Effective potentials and running couplings

A first key remark is that if in the integral Eq.(2.4) the integration is performed only with respect to the z_Δ with $\Delta \in \mathcal{Q}_N$ then the computation can be performed via perturbation theory and *with complete control of the remainders.* The argument of the exponential should be appropriately regarded as a function of the "ultraviolet z_Δ's"; let for $\Delta \in \mathcal{Q}_N$

$$X_\Delta^{[\le N]} \stackrel{def}{=} \frac{\varphi_\Delta^{[\le N]}}{\sqrt{\langle\, (\varphi_\Delta^{[\le N]})^2\,\rangle}} = \alpha_N z_\Delta + \beta_N X_{\Delta'}^{[<N]} \tag{3.1}$$

where $\Delta \subset \Delta' \in \mathcal{Q}_{N-1}$, and $\alpha_N^2 = \frac{2^{(d-2)N}}{\sum_{k=0}^N 2^{(d-2)k}}$, $\beta_N^2 = 1-\alpha_N^2$, so that

$$\begin{aligned} \alpha_N^2 =& \frac{1}{N+1}, \quad \beta_N^2 = \frac{N}{N+1}, \qquad \text{if} \quad d=2\\ \alpha_N^2,\beta_N^2 =& \frac{1}{2} + O(2^{-(d-2)N}), \qquad \text{if} \quad d=3 \end{aligned} \tag{3.2}$$

In the following the $O(2^{-(d-2)N})$ will be neglected (for the purpose of simplified notations).

Since the volume of Δ is 2^{-dN} the integrals in the exponential are

$$\begin{aligned} \mathcal{L}(X^{[\le N]}) =& \sum_{\Delta\in\mathcal{Q}_N} (\lambda 2^{-dN} C_N^2 (\alpha_N z_\Delta + \beta_N X_{\Delta'}^{[<N]})^4\\ &\qquad + \mu 2^{-dN} C_N (\alpha_N z_\Delta + \beta_N X_{\Delta'}^{[<N]})^2 + 2^{-dN}\nu)\\ =& \sum_{\Delta\in\mathcal{Q}_N} V_N(\alpha_N z_\Delta + \beta_N X_\Delta) \end{aligned} \tag{3.3}$$

where $C_N \stackrel{def}{=} \langle\, (\varphi_\Delta^{[\le N]})^2\,\rangle$, *i.e.* $C_N = 1+N$ if $d=2$ and in general $2^{(d-2)N}(1+O(2^{-(d-2)N}))$ if $d>2$.

Therefore in performing the integral over z_Δ the variable z_Δ appears multiplied by a factor $2^{-dN} C_N^2 \sim 2^{-(4-d)N}$ or $2^{-dN} C_N \sim 2^{-2N}$. [2]

[2]Here $\sim$ means that the equalities are true in dimension $d=2$ up to a factor N^2 or N or up to a factor $O(1+2^{-(d-2)N})$ in dimension $d=3$.

For definiteness suppose hereafter that $d = 3$ (the case $d = 2$ is actually much simpler) and, to simplify notations, take α_k^2, β_k^2 to be $\alpha_k^2 = \beta_k^2 = \frac{1}{2}$ (thus neglecting the mentioned correction of $O(2^{-N})$).

Call λ_N, μ_N, ν_N the "bare coupling constants" in $\mathcal{L}_N$: the *ultraviolet stability problem* is to show that the couplings can be determined so that the Z_N, Eq.(2.4), as well as all Schwinger functions, Eq.2.5, are bounded above and below uniformly in N and *cannot be evaluated by a Wick rule starting from* $S(x_1, x_2)$.

The idea is to define the "effective potential" V_k on scale $k < N$ as

$$e^{\sum_{\Delta' \in \mathcal{Q}_k} V_k(X'_\Delta)} = \int \prod_{\Delta' \in \mathcal{Q}_k} \Big(\prod_{\Delta \subset \Delta'} e^{V_{k+1}(\frac{z_\Delta + X_{\Delta'}}{\sqrt{2}})} \frac{e^{-\frac{1}{2} z_\Delta^2}}{\sqrt{2\pi}} dz_\Delta \Big) \tag{3.4}$$

The hierarchical structure reduces the study to the recursion

$$e^{V'(X)} = \Big(\int e^{V(\frac{X+z}{\sqrt{2}})} P(dz) \Big)^{2^3} \tag{3.5}$$

and it has to be shown that starting with a polynomial of degree 4 in X, of the form $V_N(X) = \lambda_{0,N} + \lambda_{1,N} : X^2 : + \lambda_{2,N} : X^4 :$, and *fixed* $p > 3$ the recursion defines a sequence of effective potentials $V_k(X)$ which, up to a remainder $\eta_k = O(\lambda^p 2^{-(p-3)k})$ with $\lambda \stackrel{def}{=} \lambda_{2,N}$, is a polynomial $\mathcal{L}_k(X)$ of degree $2p$:[3]

$$\mathcal{L}_k(X) = -\sum_{n=0}^{p} \lambda_{k,n} : X^{2n} : \tag{3.6}$$

and $\lambda_{k,n}$ are called *running couplings* on scale k.

In other words the effective potential V_k on scale k is a polynomial of degree $2p$ within a remainder, of order λ^p, *summable over* k *uniformly in* N.

The recursion is therefore reduced to a polynomial map in p dimensions, if the analysis has to be performed up to a remainder λ^p. In the present work the theory of the recursion, *i.e. of the beta function*, will be presented and reduced to the iteration of a map involving finitely many "running couplings" in dimenson $d = 3$: a point of view which was not literally followed in the earlier works on the hierarchical model, [3, 1].

[3]Rather than in terms of the monomials X^n it will be expressed in terms of Wick's monomials $: X^n :$, because this simplifies the algebra (if the calculation of several needed Gaussian integrals is performed via Feynman's graphs, reducing substantially their number). Recall that Wick's monomials of a Gaussian variable X are defined in terms of the Hermite polynomials $H_n(X)$ (with leading coefficient 2^n) as $: X^n : \stackrel{def}{=} \big(\frac{C}{2}\big)^{\frac{n}{2}} H_n(\frac{X}{\sqrt{2C}})$, with $C = \langle X^2 \rangle$, [7, 8.950.2].

Remarks: (1) The ultraviolet problem is essentially reduced to prove that the "trivial fixed point", $V = 0$, of the recursion Eq.(3.5) is unstable and therefore, if after N iterations a N–independent non trivial result is desired, it is possible to start with a V close enough to 0 so that after the N steps it evolves into a $O(1)$ final V_0.
(2) In other words in the ultraviolet problem the "bare couplings" tend to 0 as the cut-off $N \to \infty$ and the problem can be studied via perturbation theory *if the large values of the fields can be controlled* (note that no matter how small is λ_N there will always be fields so large that V is large).
(3) The infrared problem, directly related to the critical point theory, cannot be studied by simply reducing it to the analysis of a polynomial map. Since the recursion is the same in the ultraviolet and infrared problems, what makes the analysis easy in the ultraviolet problem makes it difficult in the infrared problem, where the role of the trivial fixed point has to be played by another fixed point V^* which is non trivial and unstable so that by starting close enough to it it is possible to stay close to it until the infrared cut-off is reached.
(4) Wilson used a computer aided approach to show the existence of the non trivial fixed point in dimension $d = 2, 3$. This was an important result also because it made clear, in a concrete case, that the idea of the fixed point was a generalization of the Gell-Mann-Low eigenvalue condition for the bare coupling constant of quantum electrodynamics, [16], and opened the way to the understanding of the critical point scaling properties. A rigorous determination of the existence and of several analytic properties of V^* have been later studied in the remarkable works [10, 11].

4 The beta function

In superrenormalizable theories, like φ^4 in dimension (2 or) 3, the beta function is a polynomial transformation mapping the coupling constants on a scale $k + 1$ into the couplings on scale k. Its definition is based on the formal integration with respect to the Gaussian $P(dz) \stackrel{def}{=} \frac{e^{-\frac{1}{2}z^2} dz}{\sqrt{2\pi}}$

$$\Big(\int e^{\mathcal{L}(\frac{X+z}{\sqrt{2}})} P(dz)\Big)^{2^3} = \exp 2^3 \sum_{n=0}^{\infty} \frac{1}{n!} \langle\, \mathcal{L}(\frac{X+z}{\sqrt{2}})^n \,\rangle^T \tag{4.1}$$

where ^{T} indicates that the $\langle \mathcal{L}^n \rangle$ is the order n truncated expectation with respect to the Gaussian variable z.[4]

The heuristic reason behind the procedure is in the comment following Eq.(3.3): once reduced the field $\varphi_x^{[\le N]}$ to $2^{\frac{d-2}{2}N} X_x^{[\le N]}$, *i.e.* to a quantity of order 1 times its (average) size $2^{\frac{d-2}{2}N}$ and after extracting the size 2^{-dN} of the volume element over which the field of scale $\le N$ is constant, it remains to integrate over z_Δ the exponential of a sum of *very small* quantities, of $O(\lambda 2^{(d-4)N})$, functions of the z_Δ; therefore it looks possible (and even apparently easy) to use explicit perturbation methods (*i.e.* evaluate the integrals via Taylor's expansions).

A perturbation method [5] will stop at some order and the remainder will have to be carefully estimated. It is clear that the best that it is possible to hope is that if perturbation calculations are pushed to order $p-1$ the remainder will be at least of the p-th power of the small parameter, *i.e.* $O((\lambda 2^{(d-4)N})^p)$.

The error will be repeated once per each of the 2^{dN} boxes $\Delta \in \mathcal{Q}_N$ and this will add up to $O((\lambda 2^{(d-4)N})^p\, 2^{dN})$: *therefore the calculation of the integral has to be performed up to order* $p-1$ *such that* $(d-4)p+d<0$ *which means* $p \ge 2$ *if* $d=2$*, i.e. a calculation to first order is sufficient (which makes the problem a bit too easy), and* $p \ge 4$ *if* $d=3$: where an exact calculation is necessary at least to order 3.

Of course after the first integration the effective potential on scale $N-1$ will be quite different from the initial $\mathcal{L}(X)$: therefore the parameters initially in $\mathcal{L}$ will have to be adjusted so that the form of the new $\mathcal{L}'$ is as close as possible to that of $\mathcal{L}$ and the procedure can be iterated.

This puts a severe constraint on the initial parameters: it imposes that upon integration they change according to a precise rule, called the *beta function* constraint.

Let $\mathcal{L}(X)$ be a polynomial of degree $2p$ as in Eq.(3.6). Given p the beta function is obtained by replacing the *r.h.s.* series in Eq.(4.1) (which at best is asymptotic) by its "approximation"

[4]The n-th truncated expectation of a random variable Y, *with any distribution*, also called the n-th "cumulant", is defined as $\langle Y^n \rangle^T = \partial_\varepsilon^n \log \langle \varepsilon Y \rangle \Big|_{\varepsilon=0}$.

[5]Usually called in this context "exact" as it is not merely a formal expansion but provides exact results once the tolerance of the approximation is, arbitrarily, prefixed and if the physical couplings of the theory are small enough (but neither infinitesimal nor of size depending on the approximation order $p-1$).

$$\mathcal{L}'(X) = 2^3 \Big(\sum_{n=0}^{p-1} \frac{1}{n!} \langle\, \mathcal{L}(\frac{X+z}{\sqrt{2}})^n \,\rangle_p^T \Big) \tag{4.2}$$

where the $\mathcal{L}'$ in the *r.h.s.* is calculated by

(1) first compute the truncated expectations $\widetilde{\mathcal{L}}'(X) = 2^3 \sum_{n=1}^{p-1} \langle\, \mathcal{L}(\frac{X+z}{\sqrt{2}})^n \,\rangle^T$, for instance using Wick's rule. The result will be a polynomial in the constants λ_k, $k \neq 2$ in $\mathcal{L}(X)$, see Eq.(3.6), with coefficients depending on X.
(2) assign degree 1 to the coefficient[6] λ_2 of $: X^4 :$ and degree ≥ 2 to the other constants λ_k, $k \neq 2$ and then truncate the polynomials in the λ_k by retaining only their monomials of degree $< p$.
(3) Express the even polynomial of degree $2p$, thus obtained, again on the Wick's monomials basis and call it $\mathcal{L}'(X)$: it will have the form Eq.(3.6) with suitable coefficients λ'_k.

Therefore the transformation $\mathcal{L} \to \mathcal{L}'$ maps $\{\lambda_n\}_{n<p}$ into $\{\lambda'_n\}_{n<p}$. For instance:

$$\begin{aligned} &p = 1 \to \mathcal{L}'(X) = 0, \\ &p = 2 \to \mathcal{L}'(X) = 2\lambda_2 : X^4 : \\ &p = 3 \to \mathcal{L}'(X) = 2^3\lambda_0 + 2^2\lambda_1 : X^2 : +2\lambda_2 : X^4 : +\lambda_3 : X^6 : \\ &\qquad\qquad + \lambda_2^2(a_6 : X^6 : +a_4 : X^4 : +a_2 : X^2 : +a_0) \end{aligned} \tag{4.3}$$

and for $p = 4$, calling $\lambda_0 \equiv \nu, \lambda_1 \equiv \mu, \lambda_2 = \lambda, \lambda_3 \equiv \sigma, \lambda_4 \equiv \vartheta$

$$\begin{aligned} \nu' =& 2^3\nu + a_0\lambda^2 + (d_0\lambda^3) \\ \mu' =& 2^2\mu + a_2\lambda^2 + (b_2\lambda\mu + c_2\lambda^3 + d_2\lambda\sigma) \\ \lambda' =& 2\lambda + a_4\lambda^2 + (b_4\lambda\mu + c_4\lambda^3 + d_4\lambda\sigma + e_4\lambda\vartheta) \\ \sigma' =& \sigma + a_6\lambda^2 + (c_6\lambda^3 + d_6\lambda\sigma + e_6\lambda\vartheta) \\ \vartheta' =& 2^{-1}\vartheta + (a_8\lambda^3 + d_8\lambda\sigma + e_8\lambda\vartheta) \end{aligned} \tag{4.4}$$

The first three constants are called *relevant couplings*, the fourth is called *marginal* and the fifth *irrelevant*. The coefficients a_j, b_j, c_j, d_j, e_j can be computed exactly via elementary integrations: they have a combinatorial nature and are expressible in terms of Feynman graphs.

Needless to say the qualification "irrelevant" is not supposed to convey an implication of "negligible"; on the contrary the irrelevant terms are very

[6] As the φ^4 model is being studied.

important and needed in the applications of the theory. The whole problem is to control them and their contributions to the remainders. For larger p similar relations hold and more "irrelevant" terms arise.

From now on $p = 4$ will be fixed, once understood this case it should be clear how to treat the cases $p > 4$ and no new problems will arise: *by the above comments* (about the errors, see the two paragraphs preceding Eq.(4.2)) *this is the lowest possible choice of* p.

The Eq.(4.4) maps $(\lambda, \mu, \nu, \sigma, \vartheta) \to (\lambda', \mu', \nu', \sigma', \vartheta')$: since the origin is an unstable fixed point (in three directions and marginal in one) there will be a trajectory which starting close to 0 in N steps reaches a point at finite distance from the origin; one checks (by substitution) that for $k = 0, \ldots, N$:

$$\begin{aligned}\lambda_k =&\lambda 2^{-k}, \qquad \mu_k = -2^{-2k} k a_2 \lambda^2, \qquad \nu_k = 2^{-2k-1}\, a_2\, \lambda^2\\ \sigma_k =&2^{-2k}\lambda^2 s_{6,N}, \qquad \vartheta_k = 2^{-3k}\lambda^3 t_{8,N}\end{aligned} \tag{4.5}$$

with $s_{6,N} = a_6 \sum_{n=1}^{N-1} 2^{-2n}$, $t_{8,N} = d_8 \sum_{n=1}^{N-1} \frac{2^{-3n}}{2^n}$ and a_2, a_6, d_8 suitably chosen, is a trajectory of the map for $k = 0, \ldots, N-1$ if $\sigma_N = \vartheta_N = 0$ *up to corrections amounting at factors* $(1 + const\, k 2^{-k})$ *in each term*: for instance a correction to ν_k is $-2^{-3k}\, k\, d_0\, \lambda^3$, for a suitable (precise) choice of d_0 and to σ_k a correction is $\sum_{n=1}^{N-1} d_6 \lambda^3 2^{-3(k+n)}$, and there are other similar corrections to the trajectory in Eq.(4.5); here empty sums mean 0.

In the next section it will be shown that the existence of a trajectory with the properties Eq.(4.5) with $\lambda > 0$ (a quite elementary fact) is all what is needed for a complete analysis.

5 The renormalization group

Given a polynomial $\mathcal{L}(X)$ with the property that there are constants $m > 0$ and $B > B' > \overline{B}$ such that $B > \frac{B' \pm \overline{B}}{\sqrt{2}} > \frac{1}{2} B$ (*e.g.* $B' = B(1 - \frac{1}{8}), \overline{B} = \frac{B}{8}$), $B > 1$, and

$$\begin{aligned}&\mathcal{L}(X) < 0, \quad B > |X| > \frac{B}{2}, \quad \text{and}\\ &\mathcal{L}(X) < m, \quad |X| < B\,.\end{aligned} \tag{5.1}$$

A concrete case to keep in mind could be $\lambda : X^4 : + \mu : X^2 : + \nu$ with $: X^{2k} := 2^{-\frac{k}{2}} H_{2k}(\frac{X}{\sqrt{2}})$ with $\lambda > |\mu|, |\nu|$ and B large enough.

Then, for $Y \stackrel{def}{=} \frac{X+z}{\sqrt{2}}$ and $\chi(condition) \stackrel{def}{=} 1$ if conditon is true, $\stackrel{def}{=} 0$ otherwise:

$$\begin{aligned}&\int e^{\mathcal{L}(Y)}P(dz)\le\int e^{\mathcal{L}(Y)\chi(|Y|<B)}dP\\&\quad\le\chi(|X|>B')\int e^{\mathcal{L}(\frac{X+z}{\sqrt{2}})\chi(|Y|<B)}\Big(\chi(|z|>\overline{B})+\chi(|z|<\overline{B})\Big)dP\\&\qquad+\chi(|X|<B')\int e^{\mathcal{L}(\frac{X+z}{\sqrt{2}})\chi(|Y|<B)}\Big(\chi(|z|>\overline{B})+\chi(|z|<\overline{B})\Big)dP\end{aligned}\tag{5.2}$$

If $||\mathcal{L}||=\max_{|Y|<B}|\mathcal{L}(Y)|$ then (making use of $|\frac{X+z}{\sqrt{2}}|\ge\frac{B'-\overline{B}}{\sqrt{2}}>\frac{1}{2}B$ for $|X|\ge B',|z|<\overline{B}$, of Eq.(5.1) and of Taylor's remainder estimate)

$$\begin{aligned}&\chi(|X|>B')\Big(e^{m}e^{-\frac{1}{2}\overline{B}^2}+1\Big)\\&\chi(|X|<B')\Big(e^{m}e^{-\frac{1}{2}\overline{B}^2}+\exp\Big(\sum_{n=1}^{p-1}\frac{\langle\,\chi\mathcal{L}^n\,\rangle^T}{n!}+c'_p||\mathcal{L}||^p\Big)\Big)\end{aligned}\tag{5.3}$$

where $\chi\equiv\chi(|z|<\overline{B})$ and c'_p is a constant depending only on p. Hence $|\langle\,\chi\mathcal{L}^n\,\rangle^T-\langle\,\mathcal{L}^n\,\rangle^T|\le||\mathcal{L}||^n e^{-\frac{\overline{B}^2}{2}}$, there is c_p such that

$$\begin{aligned}&\sum_{n=1}^{p-1}\frac{\langle\,\chi\mathcal{L}^n\,\rangle^T}{n!}\le\sum_{n=1}^{p-1}\frac{\langle\,\mathcal{L}^n\,\rangle^T}{n!}+e^{-\frac{1}{2}\overline{B}^2}\sum_{n=1}^{p-1}\frac{||\mathcal{L}||^n}{n!}\\&|\langle\,\mathcal{L}^n\,\rangle^T-\langle\,\mathcal{L}^n\,\rangle^T_p|\le\Lambda^pB^{2p}c_p,\qquad\Lambda=\max_{0\le n\le 2p}|\lambda_n|\end{aligned}\tag{5.4}$$

Therefore

$$\int e^{\mathcal{L}(Y)\chi(|Y|<B)}dP\le(1+e^me^{-\frac{1}{2}\overline{B}^2})^{2^3}e^{\mathcal{L}'(X)\chi(|X|<B')}\tag{5.5}$$

Introduce sequences $B_k,\overline{B}_k$ are such that for all $k\ge0$

$$B_k=(k+2)^4b,\quad B'_k=B_{k-1},\quad\overline{B}_k=(k+2)^2b\tag{5.6}$$

for a constant $b>0$ to be fixed later (as $b=12$).

Let $1>\lambda_k>0,\mu_k,\nu_k,\sigma_k,\vartheta_k$ be a trajectory of the flow generated by the beta function with $|\mu_k|,|\nu_k|,|\sigma_k|,|\vartheta_k|<\lambda_k^2(1+k)$ satisfying Eq.(4.5). Notice that

$$\begin{aligned}&:X^4:\equiv X^4-6X^2+3\ge-6,\ \text{and}\\&:X^4:>\frac{1}{2}X^4,\ |X|\ge12=b\end{aligned}\tag{5.7}$$

therefore if $2^{-k}\lambda > 2^{-2k}k\lambda^2 c$ for suitable c, m it will be

$$\begin{aligned}\mathcal{L}_k(X) <& 0, \qquad \text{for} \quad B_k > |X| > \frac{B_k}{2},\\ \mathcal{L}_k(X) <& m \qquad \text{for} \quad |X| < B_k,\end{aligned} \tag{5.8}$$

and Eq.(5.1) hold with $B' = B_{k-1}$. It follows

$$\begin{aligned}V_k(X) \le& \sum_{\Delta'\in\mathcal{Q}_k}\sum_{\Delta\subset\Delta'}\Big(\sum_{n=1}^{p-1}\frac{1}{n!}2^3\langle\,\mathcal{L}(\frac{X_{\Delta'}+z_\Delta}{\sqrt{2}})\,\rangle_p^T\Big)\\ &+\sum_{j=k}^{N}2^{3(j+1)}\log(1+e^m e^{-\frac12\overline{B}_j^2})\stackrel{def}{=}V_k^0(X)+\varepsilon_k^+\end{aligned} \tag{5.9}$$

and $\varepsilon_k^+ \le \varepsilon_0^+ \stackrel{def}{=} \sum_{k=0}^\infty 2^{3(k+1)}\log(1+e^m e^{-\frac12\overline{B}_k^2})$ is an estimate of the total error on V_k for all j.

In other words the value of $\log Z$ is determined via an asymptotic expansion with finite coefficients, *provided* a lower bound coinciding with the upper bound up to order $p-1$ and with an error estimate of the same size as that on the upper bound.

A lower bound can be easily constructed simply by restricting the integration domain:

$$\int e^{-\mathcal{L}_N(X)}P(dz) \ge \int e^{-\mathcal{L}_N(X)}\prod_{k=0}^{N}\prod_{\Delta\in\mathcal{Q}_k}\chi(|z_\Delta|<\overline{B}_k)\,g(dz_\Delta) \tag{5.10}$$

Since $|z_\Delta| < \overline{B}_k$, for $\Delta\in\mathcal{Q}_k, \forall k$ implies $|X_\Delta| < B_k$ it appears that the estimate is essentially the same as the one used to find the upper bound to the last of the integrals in Eq.(5.2): the result is similar to Eq.(5.9) the integral yields

$$V_k(X) \ge V_k^0(X) - \varepsilon_k^- \tag{5.11}$$

where $\varepsilon_k^- = -\sum_{j=k}^N 2^{3(j+1)}\log(1-e^m e^{-\frac12\overline{B}_j^2}) \le \varepsilon_0^-$.

This is iterated leading to a lower bound $e^{\nu_0-\sum_{k=0}^\infty \varepsilon_k^- 2^{3k}|\Lambda|}$, proceeding as in the upper bound.

Finally the errors $\varepsilon_k^\pm$ sum up to a quantity that is $o(\lambda^3)$ provided the constants $B_k, \overline{B}_k$ have the form $B(\lambda)(k+1)^a, \overline{B}(\lambda)(k+1)^b$ and $B(\lambda)$ is chosen so large that the error due to the truncation of the z integrals which contain $e^{-\frac12\overline{B}_k^2}$ become more infinitesimal than any power (hence not affecting corrections of any order in λ): this can be achieved simply by $B(\lambda) = B(\log(1+\frac{1}{\lambda}))^2$ and $B > 1$, [3, 1].

Remarks: **(1)**: It must be stressed that the possibility of the iteration with controlled remainders relies on the possibility of eliminating the "*large fields*" at the first integration (*i.e.* on scale N) and replacing $\mathcal{L}_N$ with $\mathcal{L}_N\chi$ controlling the error: which could only be done because $\sigma_N, \vartheta_N = 0$; as a consequence they will never grow enough to affect the positivity of $\mathcal{L}_k$ which remains controlled by $: X^4 :$ as long as $|X|$ is bounded by a power of k, because the coefficients of the other terms of $\mathcal{L}$ will be exponentially small relative to the coefficient of $: X^4 :$.
(2) Analysing the proof it is seen that the $\mathcal{L}(X)$ could have been kept a polynomial of degree 4: namely $\mathcal{L}(X) = \nu_N + \mu_N : X^2 : +\lambda_N : X^4 :$ defining the beta function by Eq.(4.4) with $\sigma, \vartheta = 0$. The upper and lower bounds would have been obtained in the same way (including the contributions with σ, ϑ in the error). The procedure followed has been chosen because it can be extended to all $p \geq 4$ to prove that the perturbatiion theory yields upper and lower bounds correct to any prefixed order. It can also be extended to obtain bounds on the Schwinger functions.
(3) A natural question is whether the $d = 4$ case can be studied in a similar way. In this case $d - 4 = 0$ and the only small parameter can be found among the bare couplings. No power of 2^{-N} helps, thus spoiling the main tool which consisted in taking advantage of the $2^{(d-4)N}$ dimesionless size of the interaction coupling. Nevertheless a formal theory of the resummation is possible, see [4] for a beta function analysis, in the case of φ^4 model *on* R^4: but *not in the hierarchical case*. The hierarchical case could be studied if the recursion

$$e^{V'(X)} = \Big(\int e^{V(\sqrt{\frac{3}{4}}z + \sqrt{\frac{1}{4}}X)} P(dz)\Big)^{2^4} \tag{5.12}$$

which is the $d = 4$ version of the $d = 3$ Eq.(3.5), had an unstable fixed point. However, *as Wilson pointed out*, [19, endnote 8], no such fixed point could be found, neither by theoretical investigations nor by computer assisted search. The latter all indicate that, on the contrary, no matter which choice of the bare couplings was made the only possiblity for the final Schwinger functions would be that they were the free field functions.
(4) The non hierarchical case is very different but, although a formal resummation is possible the beta function that drives it can only be defined as a formal power series. In spite of several results supporting the conjecture that it is impossible to obtain obtain nontrivial Schwinger functions in a scalar quantum field theory in dimension 4 is still (wide) open, [19, endnote8],[5].
(5) The models φ^6 in $d = 3$ is only superficially similar to the φ^4 in $d = 4$: in the hierarchical case it still appears to lead to a trivial result or possibly,

if $\lambda_N = \lambda 2^{-\frac{N}{2}}$, back to the φ^4 case. However in dimension 3 it was a major discovery by Wilson, [19], that (in the hierarchical case) it admits a non trivial theory different from the φ^4 one: *i.e.* a non trivial fixed point V^* which is unstable in only one direction (in the space of the V's). Its stable manifold is crossed by the family of V's of the form $rX^2 + \lambda X^6$ as r varies reaching a critical value $r_c(\lambda_0)$. Therefore the stable manifold of V^* can play the same role of the trivial fixed point for the φ^4 model discussed above. Starting $V_N = r_N X^2 - \lambda_0 X^6$ with r_N close enough to the critical $r_c(\lambda_0)$ the V_k are exponentially repelled by the stable manifold of V^* and reach a finite distance from V^* on scale 1. The V^* can also be used to obtain a nontrivial infrared behavior: if $r = r_c(\lambda_0)$ the V_k for $k < 0$ will approach V^*, and a scale invariant long distance family of Schwinger functions describing a critical point of a model in which $r - r_c(\lambda_0)$ plays the role of $T - T_c$. Changing λ_0 (Wilson fixes $\lambda_0 = 0.1$) only changes the critical value $r_c(\lambda_0)$ and has no influence on V^*. A rigorous proof of the existence of V^* in dimensions $2, 3$ is, as mentioned above, in [10, 11].
(6) In dimension $d = 2$ it is possible with the renormalization group method (whether hierarchical, very easy, or in the non hierarchical model) to check that φ^{2n} can be defined for all n: this was the first case in which ultraviolet stability was established, [12], via an alternative approach that, however, could not be extended to $d = 3$, not even in the φ^4 model. In dimension 3 only the φ^4 can be treated, essentially along the lines of the above hierarchical analysis.

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