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Kondo effect in the hierarchical $s-d$ model

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Abstract The $s-d$ model describes a chain of spin-1/2 electrons interacting magnetically with a two-level impurity. It was introduced to study the Kondo effect, in which the magnetic susceptibility of the impurity remains finite in the 0-temperature limit as long as the interaction of the impurity with the electrons is anti-ferromagnetic. A variant of this model was introduced by Andrei, which he proved was exactly solvable via Bethe Ansatz. A hierarchical version of Andrei's model was studied by Benfatto and the authors. In the present letter, that discussion is extended to a hierarchical version of the $s-d$ model. The resulting analysis is very similar to the hierarchical Andrei model, though the result is slightly simpler.

Keywords Renormalization group · Non-perturbative renormalization · Kondo effect · Fermionic hierarchical model · Quantum field theory

The $s-d$ model was introduced by Anderson [1] and used by Kondo [4] to study what would subsequently be called the *Kondo effect*. It describes a chain of electrons interacting with a fixed spin-1/2 magnetic impurity. One of the manifestations of the effect is that when the coupling is anti-ferrmoagnetic, the magnetic susceptibility of the impurity remains finite in the 0-temperature limit, whereas it diverges for ferromagnetic and for vanishing interactions.

A modified version of the $s-d$ model was introduced by Andrei [2], which was shown to be exactly solvable by Bethe Ansatz. In [3], a hierarchical version of Andrei's model was introduced and shown to exhibit a Kondo effect. In the present letter, we show how the argument can be adapted to the $s-d$ model.

We will show that in the hierarchical $s-d$ model, the computation of the susceptibility reduces to iterating an *explicit* map relating 6 *running coupling constants* (rccs), and that this map can be obtained by restricting the flow equation for the hierarchical Andrei model [3] to one of its invariant manifolds. The physics of both models are therefore very closely related, as had already been argued in [3]. This is particularly noteworthy since, at 0-field, the flow in the hierarchical Andrei model is relevant, whereas it is marginal in the hierarchical $s-d$ model, which shows that the relevant direction carries little to no physical significance.

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The $s-d$ model [4] represents a chain of non-interacting spin-1/2 fermions, called *electrons*, which interact with an isolated spin-1/2 *impurity* located at site 0. The Hilbert space of the system is $\mathcal{F}_L\otimes\mathbb{C}^2$ in which $\mathcal{F}_L$ is the Fock space of a length-L chain of spin-1/2 fermions (the electrons) and $\mathbb{C}^2$ is the state space for the two-level impurity. The Hamiltonian, in the presence of a magnetic field of amplitude h in the direction $\boldsymbol{\omega}\equiv(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\boldsymbol{\omega}_3)$, is

$$\begin{aligned}
H_K &= H_0+V_0+V_h\stackrel{def}{=}H_0+V\\
H_0 &= \sum_{\alpha\in\{\uparrow,\downarrow\}}\Big(\sum_{x=-L/2}^{L/2-1}c^+_\alpha(x)\left(-\frac{\Delta}{2}-1\right)c^-_\alpha(x)\Big)\\
V_0 &= -\lambda_0\sum_{\substack{j=1,2,3\\ \alpha_1,\alpha_2}}c^+_{\alpha_1}(0)\sigma^j_{\alpha_1,\alpha_2}c^-_{\alpha_2}(0)\,\tau^j\\
V_h &= -h\sum_{j=1,2,3}\boldsymbol{\omega}_j\tau^j
\end{aligned}\tag{1}$$

where λ_0 is the interaction strength, Δ is the discrete Laplacian $c^\pm_\alpha(x)$, $\alpha=\uparrow,\downarrow$ are creation and annihilation operators acting on *electrons*, and $\sigma^j=\tau^j$, $j=1,2,3$, are Pauli matrices. The operators τ^j act on the *impurity*. The boundary conditions are taken to be periodic.

In the *Andrei model* [2], the impurity is represented by a fermion instead of a two-level system, that is the Hilbert space is replaced by $\mathcal{F}_L\otimes\mathcal{F}_1$, and the Hamiltonian is defined by replacing τ^j in Eq.(1) by $d^+\tau^jd^-$ in which $d^\pm_\alpha(x)$, $\alpha=\uparrow,\downarrow$ are creation and annihilation operators acting on the impurity.

The partition function $Z=\mathrm{Tr}\,e^{-\beta H_K}$ can be expressed formally as a functional integral:

$$Z=\mathrm{Tr}\int P(d\psi)\sum_{n=0}^{\infty}(-1)^n\int_{0<t_1<\cdots<t_n<\beta}dt_1\cdots dt_n\,\mathcal{V}(t_1)\cdots\mathcal{V}(t_n)\tag{2}$$

in which $\mathcal{V}(t)$ is obtained from V by replacing $c^\pm_\alpha(0)$ in Eq.(1) by a *Grassmann* field $\psi^\pm_\alpha(0,t)$, $P(d\psi)$ is a *Gaussian Grassmann measure* over the fields $\{\psi^\pm_\alpha(0,t)\}_{t,\alpha}$ whose *propagator* (*i.e. covariance*) is, in the $L\to\infty$ limit,

$$g(t,t')=\frac{1}{(2\pi)^2}\int dkdk_0\frac{e^{ik_0(t-t')}}{ik_0-\cos k},$$

and the trace is over the state-space of the spin-1/2 impurity, that is a trace over $\mathbb{C}^2$.

We will consider a *hierarchical* version of the $s-d$ model. The hierarchical model defined below is *inspired* by the $s-d$ model in the same way as the hierarchical model defined in [3] was inspired by the Andrei model. We will not give any details on the justification of the definition, as such considerations are entirely analogous to the discussion in [3].

The model is defined by introducing a family of *hierarchical fields* and specifying a *propagator* for each pair of fields. The average of any monomial of fields is then computed using the Wick rule.

Assuming $\beta=2^{N_\beta}$ with $N_\beta=\log_2\beta\in\mathbb{N}$, the time axis $[0,\beta)$ is paved with boxes (*i.e.* intervals) of size 2^{-m} for every $m\in\{0,-1,\dots,-N_\beta\}$: let

$$\mathcal{Q}_m\stackrel{def}{=}\left\{[i2^{|m|},(i+1)2^{|m|})\right\}_{\substack{i=0,1,\cdots 2^{N_\beta-|m|}-1,\\ m=0,-1,\dots}}.\tag{3}$$

Given a box $\Delta\in\mathcal{Q}_m$, let t_Δ denote the center of Δ, and given a point $t\in R$, let $\Delta^{[m]}(t)$ be the (unique) box on scale m that contains t. We further decompose each box $\Delta\in\mathcal{Q}_m$ into two *half boxes*: for $\eta\in\{-,+\}$, let

$$\Delta_\eta\stackrel{def}{=}\Delta^{[m+1]}(t_\Delta+\eta2^{-m-2})\tag{4}$$

for $m\le 0$. Thus Δ_- can be called the "lower half" of Δ and Δ_+ the "upper half".

The elementary fields used to define the hierarchical $s-d$ model will be *constant on each half-box* and will be denoted by $\psi^{[m]\pm}_\alpha(\Delta_\eta)$ for $m\in\{0,-1,\cdots,-N_\beta\}$, $\Delta\in\mathcal{Q}_m$, $\eta\in\{-,+\}$, $\alpha\in\{\uparrow,\downarrow\}$.

The propagator of the hierarchical $s-d$ model is defined as

$$\left\langle \psi_\alpha^{[m]-}(\Delta_{-\eta})\psi_\alpha^{[m]+}(\Delta_\eta)\right\rangle \stackrel{def}{=} \eta \tag{5}$$

for $m\in\{0,-1,\cdots,-N_\beta\}$, $\Delta\in\mathcal{Q}_m$, $\eta\in\{-,+\}$, $\alpha\in\{\uparrow,\downarrow\}$. The propagator of any other pair of fields is set to 0.

Finally, we define

$$\psi_\alpha^\pm(t)\stackrel{def}{=}\sum_{m=0}^{-N_\beta} 2^{\frac{m}{2}}\psi_\alpha^{[m]\pm}(\Delta^{[m+1]}(t)). \tag{6}$$

The partition function for the hierarchical $s-d$ model is

$$Z = \mathrm{Tr}\left\langle \sum_{n=0}^{\infty}(-1)^n\int_{0<t_1<\cdots<t_n<\beta} dt_1\cdots dt_n\,\mathcal{V}(t_1)\cdots\mathcal{V}(t_n)\right\rangle \tag{7}$$

in which the $\psi_\alpha^\pm(0,t)$ in $\mathcal{V}(t)$ have been replaced by the $\psi_\alpha^\pm(t)$ defined in Eq.(6):

$$\mathcal{V}(t)\stackrel{def}{=}-\lambda_0\sum_{\substack{j=1,2,3\\ \alpha_1,\alpha_2}}\psi_{\alpha_1}^+(t)\sigma_{\alpha_1,\alpha_2}^j\psi_{\alpha_2}^-(t)\,\tau^j - h\sum_{j=1,2,3}\boldsymbol{\omega}_j\tau^j. \tag{8}$$

This concludes the definition of the hierarchical $s-d$ model.

We will now show how to compute the partition function Eq.(7) using a renormalization group iteration. We first rewrite

$$\sum_{n=0}^{\infty}(-1)^n\int_{0<t_1<\cdots<t_n<\beta} dt_1\cdots dt_n\,\mathcal{V}(t_1)\cdots\mathcal{V}(t_n) = \prod_{\Delta\in\mathcal{Q}_0}\prod_{\eta=\pm}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\mathcal{V}(t_{\Delta_\eta})^n\right) \tag{9}$$

and find that

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\mathcal{V}(t_{\Delta_\eta^{[0]}})^n = C\left(1+\sum_p \ell_p^{[0]}O_{p,\eta}^{[\leq 0]}(\Delta^{[0]})\right) \tag{10}$$

with

$$\begin{aligned} O_{0,\eta}^{[\leq 0]}(\Delta)&\stackrel{def}{=}\frac{1}{2}\mathbf{A}_\eta^{[\leq 0]}(\Delta)\cdot\boldsymbol{\tau}, & O_{1,\eta}^{[\leq 0]}(\Delta)&\stackrel{def}{=}\frac{1}{2}\mathbf{A}_\eta^{[\leq 0]}(\Delta)^2,\\ O_{4,\eta}^{[\leq 0]}(\Delta)&\stackrel{def}{=}\frac{1}{2}\mathbf{A}_\eta^{[\leq 0]}(\Delta)\cdot\boldsymbol{\omega}, & O_{5,\eta}^{[\leq 0]}(\Delta)&\stackrel{def}{=}\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{\omega},\\ O_{6,\eta}^{[\leq 0]}(\Delta)&\stackrel{def}{=}\frac{1}{2}(\mathbf{A}_\eta^{[\leq 0]}(\Delta)\cdot\boldsymbol{\omega})(\boldsymbol{\tau}\cdot\boldsymbol{\omega}), & O_{7,\eta}^{[\leq 0]}(\Delta)&\stackrel{def}{=}\frac{1}{2}(\mathbf{A}_\eta^{[\leq 0]}(\Delta)^2)(\boldsymbol{\tau}\cdot\boldsymbol{\omega}) \end{aligned} \tag{11}$$

(the numbering is meant to recall that in [3]) in which $\boldsymbol{\tau}=(\tau^1,\tau^2,\tau^3)$ and $\mathbf{A}_\eta^{[\leq 0]}(\Delta)$ is a vector of polynomials in the fields whose j-th component for $j\in\{1,2,3\}$ is

$$A_\eta^{[\leq 0]j}(\Delta)\stackrel{def}{=}\sum_{(\alpha,\alpha')\in\{\uparrow,\downarrow\}^2}\psi_\alpha^{[\leq 0]+}(\Delta_\eta)\sigma_{\alpha,\alpha'}^j\psi_{\alpha'}^{[\leq 0]-}(\Delta_\eta) \tag{12}$$

$\psi_\alpha^{[\leq 0]\pm}:=\sum_{m\leq 0}2^{\frac{m}{2}}\psi_\alpha^{[m]\pm}$, and

$$\begin{aligned} &C=\cosh(\tilde h),\quad \ell_0^{[0]}=\frac{1}{C}\frac{\lambda_0}{\tilde h}\sinh(\tilde h)\\ &\ell_1^{[0]}=\frac{1}{C}\frac{\lambda_0^2}{12\tilde h}(\tilde h\cosh(\tilde h)+2\sinh(\tilde h))\\ &\ell_4^{[0]}=\frac{1}{C}\lambda_0\sinh(\tilde h),\quad \ell_5^{[0]}=\frac{2}{C}\sinh(\tilde h)\\ &\ell_6^{[0]}=\frac{1}{C}\frac{\lambda_0}{\tilde h}(\tilde h\cosh(\tilde h)-\sinh(\tilde h))\\ &\ell_7^{[0]}=\frac{1}{C}\frac{\lambda_0^2}{12\tilde h^2}(\tilde h^2\sinh(\tilde h)+2\tilde h\cosh(\tilde h)-2\sinh(\tilde h)) \end{aligned} \tag{13}$$

in which $\tilde{h} := h/2$.

By a straightforward induction, we find that the partition function Eq.(7) can be computed by defining

$$C^{[m]}\mathcal{W}^{[m-1]}(\Delta^{[m]})\stackrel{def}{=}\left\langle\prod_{\eta}\left(\mathcal{W}^{[m]}(\Delta_{\eta}^{[m]})\right)\right\rangle_{m} \tag{14}$$

in which $\langle\cdot\rangle_m$ denotes the average over $\psi^{[m]}$, $C^{[m]} > 0$ and

$$\mathcal{W}^{[m-1]}(\Delta^{[m]}) = 1 + \sum_{p} \ell_p^{[m]} O_p^{[\leq m]}(\Delta^{[m]}) \tag{15}$$

in terms of which

$$Z = C^{2|\mathcal{Q}_0|} \prod_{m=-N(\beta)+1}^{0} (C^{[m]})^{|\mathcal{Q}_{m-1}|} \tag{16}$$

in which $|\mathcal{Q}_m| = 2^{N(\beta)-|m|}$ is the cardinality of $\mathcal{Q}_m$. In addition, similarly to [3], the map relating $\ell_p^{[m]}$ to $\ell_p^{[m-1]}$ and $C^{[m]}$ can be computed explicitly from Eq.(14):

$$\begin{aligned}
C^{[m]} &= 1 + \frac{3}{2}\ell_0^2 + \ell_0\ell_6 + 9\ell_1^2 + \frac{\ell_4^2}{2} + \frac{\ell_5^2}{4} + \frac{\ell_6^2}{2} + 9\ell_7^2 \\
\ell_0^{[m-1]} &= \frac{1}{C}\Big(\ell_0 - \ell_0^2 + 3\ell_0\ell_1 - \ell_0\ell_6\Big) \\
\ell_1^{[m-1]} &= \frac{1}{C}\Big(\frac{\ell_1}{2} + \frac{\ell_0^2}{8} + \frac{\ell_0\ell_6}{12} + \frac{\ell_4^2}{24} + \frac{\ell_5\ell_7}{4} + \frac{\ell_6^2}{24}\Big) \\
\ell_4^{[m-1]} &= \frac{1}{C}\Big(\ell_4 + \frac{\ell_0\ell_5}{2} + 3\ell_0\ell_7 + 3\ell_1\ell_4 + \frac{\ell_5\ell_6}{2} + 3\ell_6\ell_7\Big) \\
\ell_5^{[m-1]} &= \frac{1}{C}\Big(2\ell_5 + 2\ell_0\ell_4 + 36\ell_1\ell_7 + 2\ell_4\ell_6\Big) \\
\ell_6^{[m-1]} &= \frac{1}{C}\Big(\ell_6 + \ell_0\ell_6 + 3\ell_1\ell_6 + \frac{\ell_4\ell_5}{2} + 3\ell_4\ell_7\Big) \\
\ell_7^{[m-1]} &= \frac{1}{C}\Big(\frac{\ell_7}{2} + \frac{\ell_0\ell_4}{12} + \frac{\ell_1\ell_5}{4} + \frac{\ell_4\ell_6}{12}\Big)
\end{aligned} \tag{17}$$

in which the $^{[m]}$ have been dropped from the right hand side.

The flow equation Eq.(17) can be recovered from that of the hierarchical Andrei model studied in [3] (see in particular [3, Eq.(C1)]) by restricting the flow to the invariant submanifold defined by

$$\ell_2^{[m]} = \frac{1}{3}, \quad \ell_3^{[m]} = \frac{1}{6}\ell_1^{[m]}, \quad \ell_8^{[m]} = \frac{1}{6}\ell_4^{[m]}. \tag{18}$$

This is of particular interest since $\ell_2^{[m]}$ is a relevant coupling and the fact that it plays no role in the $s-d$ model indicates that it has little to no physical relevance.

The qualitative behavior of the flow is therefore the same as that described in [3] for the hierarchical Andrei model. In particular the susceptibility, which can be computed by deriving $-\beta^{-1}\log Z$ with respect to h, remains finite in the 0-temperature limit as long as $\lambda_0 < 0$, that is as long as the interaction is anti-ferromagnetic.

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References

1. Anderson, P.: Local magnetized states in metals. Physical Review **124**, 41–53 (1961)
2. Andrei, N.: Diagonalization of the Kondo Hamiltonian. Physical Review Letters **45**, 379–382 (1980)
3. Benfatto, G., Gallavotti, G., Jauslin, I.: Kondo effect in a fermionic hierarchical model. arXiv: 1506.04381 (2015)
4. Kondo, J.: Resistance Minimum in Dilute Magnetic Alloys. Progress of Theoretical Physics **32**, 37–49 (1964)