Localization in interacting fermionic chains with quasi-random disorder

Vieri Mastropietro

Universitá di Milano, Via C. Saldini 50, 20133, Milano, Italy

We consider a system of fermions with a quasi-random almost-Mathieu disorder interacting through a many-body short range potential. We establish exponential decay of the zero temperature correlations, indicating localization of the interacting ground state, for weak hopping and interaction and almost everywhere in the frequency and phase; this extends the analysis in [17] to chemical potentials outside spectral gaps. The proof is based on Renormalization Group and it is inspired by techniques developed to deal with KAM Lindstedt series.

1. INTRODUCTION AND MAIN RESULTS

A. Introduction

It is due to Anderson [1] the discovery that disorder can produce *localization* of independent quantum particles, consisting in the exponential decay from some point of the eigenfunctions of the one-body Schroedinger operator. The mathematical understanding of Anderson localization required the development of powerful techniques and it was finally rigorously established in the case of random [2], [3] and quasi-random (or quasi-periodic) disorder [4],[5], [6],[7].

A natural question is what happens to localization in presence of a many-body interaction, which is always present in real systems. The interplay of disorder and interaction is believed to have deep consequences on the ground state and low temperature properties [8], [9], [10] and in the non equilibrium dynamics, like lack of thermalization and memory of initial state [11], [12],[13],[14],[15]. Mathematical results on localization for interacting systems are still very few [16],[17].

In this paper we consider a system of spinless fermions on a one dimensional lattice with a quasi-random disorder described by a quasi-periodic almost-Mathieu potential $\phi_x = u\cos 2\pi(\omega x + \theta)$, ω irrational, and interacting via a short range potential with coupling

U. Such model is known as the *interacting Aubry-André* model [14],[18] or the *Heisenberg quasi-periodic spin chain*, and it has been recently experimentally realized in cold atoms experiments [18].

In the absence of interaction the N-particle eigenstates can be constructed from the single particle eigenstates of the Schroedinger energy operator with *almost-Mathieu* potential, for which a rather detailed mathematical knowledge exists; in particular such system shows a metal-insulator transition, with an Anderson localized insulating phase with strong disorder and a metallic extended phase at weak disorder, similar to what happens in a random three dimensional situation. The *exponential* decay of the single particle eigenstates of the *almost-Mathieu* operator, almost everywhere in ω, θ, was proved in [5] and [6] , for ε small enough, ε being the hopping, and later up to ε/u equal to $\frac{1}{2}$ in [7]. In the opposite regime $\varepsilon/u > \frac{1}{2}$ the almost Mathieu has *extended* states [20],[21],[22],[23],[24]; in particular in [20] a Diophantine condition is assumed on the phase excluding values close to $2\theta = \omega k$, k integer, corresponding to gaps [24]. In both regimes and for all irrationals the spectrum is a Cantor set [19]. The non interacting Aubry-André model has ground state correlations with a power law decay for large $\frac{\varepsilon}{u}$ [25], even in presence of interaction [26], and an exponential decay for small $\frac{\varepsilon}{u}$ [27].

In this paper we prove localization of the ground state of interacting fermions with a strong quasi-random disorder, by establishing the exponential decay of the zero temperature grand-canonical truncated correlations of local operators. Our main results can be informally stated as follows.

Almost everywhere in ω, θ, *for small* $\frac{\varepsilon}{u}$, $\frac{U}{u}$, *with chemical potential* $\mu = \phi_{\widehat{x}}$, $\widehat{x} \in \mathbb{N}$ *the zero temperature grand canonical infinite volume truncated correlations of local operators decay exponentially for large distances*

The proof is based on a combination of constructive renormalization Group methods for fermions, see for instance [28] , with techniques developed for proving convergence of Lindstedt series for Kolmogorov-Arnold-Moser (KAM) invariant tori [29],[30]. Persistence of localization in the ground state is therefore established for almost all values of the chemical potentials (or the particle density), extending a previous result [17] in which the chemical potential was assumed in the middle of one of the infinitely many gaps of the non interacting spectrum, that is $2\theta/\omega \in \mathbb{N}$.

B. The model

If Λ is a one dimensional lattice $\Lambda=\{x\in\mathbb{Z}, 1\le x\le L\}$, we introduce fermionic creation and annihilation operators a^+_x, a^-_x, $x\in\Lambda$ on the Fock space verifying $\{a^+_x, a^-_y\}=\delta_{x,y}$, $\{a^+_x, a^+_y\}=\{a^-_x, a^-_y\}=0$. The Fock space Hamiltonian is

$$H=-\varepsilon(\sum_x a^+_{x+1}a_x+\sum_x a^+_{x-1}a^-_x)+\sum_x \phi_x a^+_x a^-_x+U\sum_{x,y} v(x-y)(a^+_x a^-_x-\frac{1}{2})(a^+_y a^-_y-\frac{1}{2}) \quad (1)$$

with $v(x-y)=\delta_{y-x,1}+\delta_{x-y,1}$, and $\phi_x=u\cos(2\pi(\omega x+\theta))$, ω irrational, $a^\pm_{L+1}$ and $a^\pm_0$ must be interpreted as zero and $u=1$ for definiteness. If $a^\pm_{\mathbf{x}}=e^{(H-\mu N)x_0}a^\pm_x e^{-(H-\mu N)x_0}$, $\mathbf{x}=(x,x_0)$, $N=\sum_x a^+_x a^-_x$ and μ the chemical potential, the Grand-Canonical imaginary time 2-point correlation is

$$<\mathbf{T}a^-_{\mathbf{x}}a^+_{\mathbf{y}}>|_T=\frac{Tre^{-\beta(H-\mu N)}\mathbf{T}\{a^-_{\mathbf{x}}a^+_{\mathbf{y}}\}}{Tre^{-\beta(H-\mu N)}} \quad (2)$$

where $\mathbf{T}$ is the time-order product, T denotes truncation and μ is the chemical potential. In the $\varepsilon=U=0$ the spectrum is given by $\sum_x \phi_x n_x$ with $n_x=0,1$ and the correlations are given by the Wick rule in terms of the fermionic 2-point function $<\mathbf{T}a^-_{\mathbf{x}}a^+_{\mathbf{y}}>|_{U=\varepsilon=0}=g(\mathbf{x},\mathbf{y})$ with

$$g(\mathbf{x},\mathbf{y})=\delta_{x,y}\frac{1}{\beta}\sum_{k_0=\frac{2\pi}{\beta}(n_0+\frac{1}{2})}\frac{e^{-ik_0(x_0-y_0)}}{-ik_0+\cos 2\pi(\omega x+\theta)-\mu}=\delta_{x,y}\bar{g}(x,x_0-y_0) \quad (3)$$

If $\mu=\cos 2\pi(\omega\widehat{x}+\theta)$, $\widehat{x}\in\Lambda$, the occupation number, defined as $\bar{g}(x,0^-)$, is at zero temperature $\chi(\cos 2\pi(\omega x+\theta)\le\mu)$, that is the ground state is obtained by filling all the one particle states with energy $\cos 2\pi(\omega x+\theta)$ up to the level $\cos 2\pi(\omega\widehat{x}+\theta)$.

The location of the singularity of the temporal Fourier transform of the 2-point function is expected to depend on the interaction, and this of course causes problems in a perturbative analysis, resulting in a lack of convergence of a naive power series expansion. It is therefore convenient to write the chemical potential as a function of the interaction, and to tune it so that the singularity in the free or interacting case are the same; this is done by writing $\mu=\cos 2\pi(\omega\widehat{x}+\theta)+\nu$ and choosing properly the counterterm ν as a function of ε, U.

The starting point of the Renormalization Group analysis is the representation of the correlations (2) in terms of *Grassmann integrals*. Let $M\in\mathbb{N}$ and $\bar{\chi}(t)$ a smooth compact support function that is 1 for $t\le 1$ and 0 for $t\ge\gamma$, with $\gamma>1$. Let $\mathcal{D}_\beta=D_\beta\cap\{k_0\ :$

$\bar\chi(\gamma^{-M}|k_0|)>0\}$, where $D_\beta=\{k_0=\frac{2\pi}{\beta}(n_0+\frac{1}{2}), n_0\in\mathbb{Z}\}$. If $x_0-y_0\neq n\beta$, we can write

$$g(\mathbf{x},\mathbf{y})=\lim_{M\to\infty}\delta_{x,y}\frac{1}{\beta}\sum_{k_0\in\mathcal{D}_\beta}\bar\chi(\gamma^{-M}|k_0|)\frac{e^{-ik_0(x_0-y_0)}}{-ik_0+\cos 2\pi(\omega x+\theta)-\mu}\equiv\lim_{M\to\infty}g^{(\leq M)}(\mathbf{x},\mathbf{y}) \quad (4)$$

Because of the jump discontinuities, $g^{(\leq M)}(\mathbf{x},\mathbf{y})$ is not absolutely convergent but is pointwise convergent and the limit is given by $g(\mathbf{x},\mathbf{y})$ at the continuity points, while at the discontinuities it is given by the mean of the right and left limits. If $\mathcal{B}_{\beta,L}=\{\Lambda\otimes\mathcal{D}_\beta\}$, we consider the Grassmann algebra generated by the Grassmannian variables $\{\widehat\psi^\pm_{x,k_0}\}_{x,k_0\in\mathcal{B}_{\beta,L}}$ and a Grassmann integration $\int\big[\prod_{x,k_0\in\mathcal{B}_{\beta,L}}d\widehat\psi^+_{x,k_0}d\widehat\psi^-_{x,k_0}\big]$ defined as the linear operator on the Grassmann algebra such that, given a monomial $Q(\widehat\psi^-,\widehat\psi^+)$ in the variables $\widehat\psi^\pm_{x,k_0}$, its action on $Q(\widehat\psi^-,\widehat\psi^+)$ is 0 except in the case $Q(\widehat\psi^-,\widehat\psi^+)=\prod_{x,k_0\in\mathcal{B}_{\beta,L}}\widehat\psi^-_{x,k_0}\widehat\psi^+_{x,k_0}$, up to a permutation of the variables. In this case the value of the integral is determined, by using the anticommuting properties of the variables, by the condition

$$\int\Big[\prod_{x,k_0\in\mathcal{B}_{\beta,L}}d\widehat\psi^+_{x,k_0}d\widehat\psi^-_{x,k_0}\Big]\prod_{x,k_0\in\mathcal{B}_{\beta,L}}\widehat\psi^-_{x,k_0}\widehat\psi^+_{x,k_0}=1 \quad (5)$$

We define also Grassmanian field as $\psi^\pm_{\mathbf{x}}=\frac{1}{\beta}\sum_{k_0\in\mathcal{B}_{\beta,L}}e^{\pm ik_0x_0}\widehat\psi^\pm_{x,k_0}$ with $x_0=m_0\frac{\beta}{\gamma^M}$ and $m_0\in(0,1,...,\gamma^M-1)$. The "Gaussian Grassmann measure" (also called integration) is defined as

$$P(d\psi)=[\prod_{x,k_0\in\mathcal{B}_{\beta,L}}\beta d\widehat\psi^-_{x,k_0}d\widehat\psi^+_{x,k_0}\widehat g^{(\leq M)}(x,k_0)]\exp\{-\frac{1}{\beta}\sum_{x,k_0}(\widehat g^{(\leq M)}(x,k_0))^{-1}\widehat\psi^+_{x,k_0}\widehat\psi^-_{x,k_0}\} \quad (6)$$

with

$$\widehat g^{(\leq M)}(x,k_0)=\frac{\bar\chi(\gamma^{-M}|k_0|)}{-ik_0+\cos 2\pi(\omega x+\theta)-\cos 2\pi(\omega\widehat x+\theta)} \quad (7)$$

We introduce the generating functional $W(\eta)$ defined in terms of the following Grassmann integral ($\psi^\pm_{L+1}$ and $\psi^\pm_{-1}$ must be interpreted as zero)

$$e^{W(\eta)}=\int P(d\psi)e^{-\mathcal{V}(\psi)-\mathcal{B}(\psi,\eta)} \quad (8)$$

with

$$\mathcal{V}(\psi)=U\int d\mathbf{x}\sum_{\alpha=\pm}\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}\psi^+_{\mathbf{x}+\alpha\mathbf{e_1}}\psi^-_{\mathbf{x}+\alpha\mathbf{e_1}}+\nu\int d\mathbf{x}\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}} \quad (9)$$

where $\int d\mathbf{x}=\sum_{x\in\Lambda}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}}dx_0$ and $\psi^\pm_{\mathbf{x}}$ is vanishing for $x=L/2+1$ and $x=-L/2-1$. Finally

$$\mathcal{B}(\psi,\eta)=\int d\mathbf{x}(\eta^+_{\mathbf{x}}\psi^-_{\mathbf{x}}+\psi^+_{\mathbf{x}}\eta^-_{\mathbf{x}}) \quad (10)$$

The 2-point function is given by

$$S_2^{L,\beta}(\mathbf{x},\mathbf{y}) = \frac{\partial^2}{\partial\eta^+_{\mathbf{x}}\partial\eta^-_{\mathbf{y}}} W|_0 \tag{11}$$

It is easy to check, see §1.C of [17], that the expansions in ε, U, ν of (2) with $\mu = \widehat{p}hi_{\widehat{x}} + \nu$ and of (11) coincide in the limit $M \to \infty$.

C. Main results

Our main result is the following.

Theorem 1.1 *Let us consider the 2-point function* $S_2^{L,\beta}(\mathbf{x},\mathbf{y})$ *(11) with* $\mu = \cos 2\pi(\omega\widehat{x}+\theta)$, $\widehat{x} \in \Lambda$, $\widehat{x}, \theta$ *non vanishing and assume that, for some* $C_0, \tau > 1$

$$||\omega x|| \geq C_0|x|^{-\tau}, \quad ||\omega x \pm 2\theta|| \geq C_0|x|^{-\tau} \quad \forall x \in \mathbb{Z}/\{0\} \tag{12}$$

with $||.||$ *is the norm on the one dimensional torus of period* 1*. There exists an* ε_0 *such that, for* $|\varepsilon|, |U| \leq \varepsilon_0$ *(*$u = 1$*),it is possible to choose* $\nu = \nu(\varepsilon, U)$ *so that the limit* $\lim_{\beta\to\infty}\lim_{L\to\infty}\lim_{M\to\infty} S_2^{L,\beta}(\mathbf{x},\mathbf{y}) = S_2(\mathbf{x},\mathbf{y})$ *exists and for any* $N \in \mathbb{N}$

$$|S_2(\mathbf{x},\mathbf{y})| \leq Ce^{-\xi|x-y|}\log(1+\min(|x||y|))^\tau \frac{1}{1+(\Delta|x_0-y_0)|)^N} \tag{13}$$

with $\Delta = (1+\min(|x|,|y|))^{-\tau}$, $\xi = |\log(\max(|\varepsilon|,|U|))|$ *and* C *is a suitable constant.*

Remarks

- The exponential decay in the coordinates of the zero temperature truncated correlations (and the much slower decay in the temporal direction) is a signature of localization of the many body ground state. One has to restrict to a full measure set of frequencies and phases satisfying two Diophantine conditions: one on the frequency of the almost-Mathieu potential, and the second excluding phases around integer values of $\frac{2\theta}{\omega}$; such conditions are often assumed in the analysis of the almost Mathieu equation [20], [6]. This above theorem extends a previous result [17] in which exponential decay was proven assuming ω diophantine and $\frac{2\theta}{\omega}$ half-integer, and it was announced in [31].

- A simple consequence of the theorem proof is a localization result formulated fixing the phase θ and varying the chemical potential; namely if we choose $\theta = 0$ and define $\mu = \cos 2\pi\omega\bar{x}$, $\bar{x} \in \mathbb{R}$, than the exponential decay of correlation (13) holds provided that the chemical potential is chosen in correspondence of a point of the non interacting spectrum, namely assuming a Diophantine condition on $\bar{x}$, $||\omega x \pm 2\omega\bar{x}|| \geq C|x|^{-\tau}$, $x \neq 0$. In [17] it was instead considered that case of the chemical potential in the middle of one of the infinitely many gaps, that is $\bar{x}$ half-integer; in such a case (13) still holds, provided that Δ in (13) is replaced by the gap size.

- The proof of Theorem 1.1 can be extended to more general form of quasi-periodic potential; one simply needs that $\phi_x = \bar{\phi}(2\pi(\omega x + \theta))$ with $\bar{\phi} \in C^1$, even $\bar{\phi}(t) = \bar{\phi}(-t)$ and periodic $\bar{\phi}(t) = \bar{\phi}(t+1)$; moreover one needs $\partial\bar{\phi}_{\omega\hat{x}+\theta} \neq 0$. Other classes of potentials were discussed in the non interacting case in [27] and one could easily extend the proof of the above theorem to such cases.

- Eq.(13) is in agreement with the proposed phase diagram of the interacting Aubry-André model obtained by numerical simulations [14], in which a *many body localized* phase is expected for small $\frac{\varepsilon}{u}$, $\frac{U}{u}$. Many body localization is however a stronger property, requiring exponential decay of truncated correlations not only on the ground states corresponding to different densities, but on each eigenstate of the many body Hamiltonian; if such correlations can be analyzed by an extension of the methods developed here is an important open problem.

- The assumption of spinless fermions plays an important role in controlling the contribution of the *resonant* terms. The methods developed in the present paper can be extended to spinning fermions at the cost of introducing a marginal running coupling constant quartic in the fields.

D. Feynman Graphs expansion and small divisors

Before starting the proof of Theorem 1.1 it is useful to figure out the main difficulties of the problem, related to the presence of small divisors. Let us consider the *effective potential* defined by

$$e^{-V(\eta)} = \int P(d\psi) e^{-\mathcal{V}(\psi+\eta)} \tag{14}$$

with $\mathcal{V}(\psi)$ given by (9). We can write

$$V(\eta) = -\log \int P(d\psi)\, e^{-\mathcal{V}(\psi+\eta)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathcal{E}^T(\mathcal{V}; n) \tag{15}$$

where $\mathcal{E}^T$ are the *fermionic truncated expectations*, that is, if $X(\psi)$ is a monomial

$$\mathcal{E}^T(X; n) \equiv \mathcal{E}^T(X; ...; X) = \frac{\partial^n}{\partial \alpha^n} \log \int P(d\psi) e^{\alpha X(\psi)})|_{\alpha=0} \tag{16}$$

It is well known that the truncated expectations can be computed using the *anticommutative Wick rule* defined in the following way, see for instance [28]. Given a set of indexes P and defining $\widetilde{\psi}(P) = \prod_{f\in P} \psi^{\varepsilon(f)}_{\mathbf{x}(f)}$ with $\varepsilon(f) = \pm$, we can represent each field $\psi^\varepsilon_{\mathbf{x}}$ as an oriented half line emerging from a point $\mathbf{x}$ and carrying an arrow, pointing towards the point if $\varepsilon = -$ and in the opposite direction if $\varepsilon = +$. We can enclose the points $\mathbf{x}(f)$, $f \in P$ in a box, and, if we have $P_1, .., P_s$ sets, we can associate a set of diagrams Γ obtained by joining pairwise the half-lines with consistent orientation, in such a way that all the boxes are connected; a line obtained by joining two half-lines is denoted by ℓ. If a line ℓ is contained in a diagram Γ we say $\ell \in \Gamma$, and the two fields are said *contracted*. Then

$$\mathcal{E}^T(\widetilde{\psi}(P_1); ...; \widetilde{\psi}(P_s)) = \sum_{\Gamma} \varepsilon_\Gamma \prod_{\ell\in\Gamma} g_\ell \tag{17}$$

where $g_\ell = g^{(\leq M)}(\mathbf{x}(f), \mathbf{y}(f))$ defined in (4) and ε_Γ is the sign of the permutation required to move every ψ^+ to the immediate right of the ψ^- operator it is paired with. If we use the graphical representation of the Wick rule described above in the truncated expectations in (15), we see that the effective potential V can be written as a series of graphs, called *Feynman graphs*, obtained taking n elements represented as in fig.1 and contracting the lines with consistent orientation so that all the n vertices are connected; the contribution of each Feynman graph is expressed by the sum over coordinates of product of propagators $g^{(\leq M)}(\mathbf{x}, \mathbf{y})$.

In absence of many body interaction, *i.e.*$U = 0$, the graphs have the simple form of chain graphs. The value of the graphs obtained contracting only ε vertices and bilinear in the external fields η, after summing over all the $n!$ choices of vertex labels and taking into account the $n!^{-1}$ in (15) , is given by, see Fig. 2 ($\alpha_i = \pm$),

$$\varepsilon^\nu \int \prod_{1=1}^{n} d\mathbf{x}_i \eta_{\mathbf{x}_1} [\prod_{i=1}^{n} \delta_{x_i+\alpha_i, x_{i+1}} \bar{g}(x_i + \alpha_i, x_{0,i} - x_{0,i+1})] \eta_{\mathbf{x}_{n+1}} = \tag{18}$$

$$\varepsilon^{\nu_\varepsilon} \sum_{x_1} \int dk_0 \widehat{\eta}_{x_1,k_0} [\prod_{k=1}^{n} \widehat{g}(x_1 + \sum_{i\leq k} \alpha_i, k_0)] \widehat{\eta}_{x_1+\sum_{i\leq k}\alpha_i,k_0} = \varepsilon^\nu \sum_{x_1} \int dk_0 \widehat{\eta}_{x_1,k_0} \widehat{\eta}_{x_1+\sum_{i\leq k}\alpha_i,k_0} H_n(k_0, x_1)$$

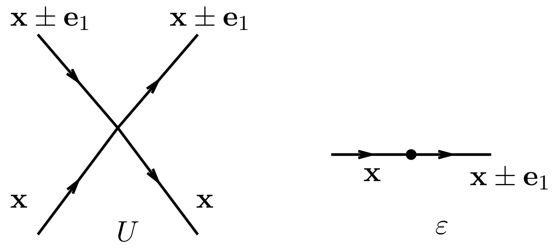

FIG. 1: Graphical representation of the three terms in $\mathcal{V}(\psi)$ eq.(9)

Even in the non-interacting case $U = 0$ the perturbation theory fails to converge everywhere, due to a small divisor problem caused by the irrationality of ω. The peculiarity of the quasi-periodic potential, with respect to the periodic case, is that the propagator can be *arbitrarily large* when $x \neq \widehat{x}$. If we set $x = x' + \bar{x}_\rho$, $\rho = \pm$,

$$\bar{x}_+ = \widehat{x} \qquad \bar{x}_- = -\widehat{x} - 2\theta/\omega \tag{19}$$

then $\cos 2\pi(\omega(x' + \bar{x}_\rho) + \theta) - \cos(2\pi(\omega\widehat{x} + \theta)) = \rho v_0 (\omega x')_{\text{mod.1}} + r_{\rho,x'}$ with $r_{\rho,x'} = O(((\omega x')_{\text{mod.1}})^2)$, $v_0 = \sin 2\pi(\omega\widehat{x} + \theta)$. Therefore the propagators (and then the Feynman graphs) are unbounded as $(\omega x')_{\text{mod.1}}$ can be arbitrarily small and

$$\widehat{g}(x' + \bar{x}_\rho, k_0) \sim \frac{1}{-ik_0 \pm v_0(\omega x')_{\text{mod.1}}} \tag{20}$$

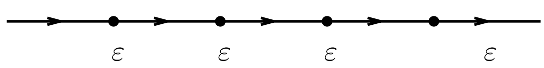

FIG. 2: A graph with $n_\varepsilon = 4$, $n_U = n_\nu = 0$

If we consider not all possible irrational ω, but only the ones veryfing a diophantine condition (which are a full mesure set) the propagators are bounded for $x \neq \rho\widehat{x}$; using that $||\omega x'|| = ||\omega(x - \rho\widehat{x}) + 2\delta_{\rho,-1}\theta|| \geq C|x - \rho\widehat{x}|^{-\tau}$ one finds

$$|\widehat{g}(x' + \bar{x}_\rho, k_0)| \leq C|x - \rho\widehat{x}|^\tau \tag{21}$$

A naive bound using the above estimate is however still not sufficient to achieve convergence, as it is easy to identify graphs bounded by $O(n!^\tau)$ (assume for instance $\alpha_i = +$ in (19) for any i). There is indeed a striking similarity between the expansion when $U = 0$ and the *Lindstedt series* for KAM invariant tori in quasi-integrable Hamiltonian systems [29],[30]; in both cases the expansion can be represented in terms of graphs with no loops and plagued by a *small divisor problem.* A direct proof of convergence of such series, which were known to converge as consequence of KAM theorem, was a non trivial problem which was finally solved in in [29],[30] by Renormalization Group methods. A similar approach was also used in [27] to prove Theorem 1.1 in the absence of many-body interaction $U = 0$.

In the expansion for the 2-point function in presence of many body interaction much more complex graphs can appear, namely graphs with *loops*; an example is Fig. 3 whose

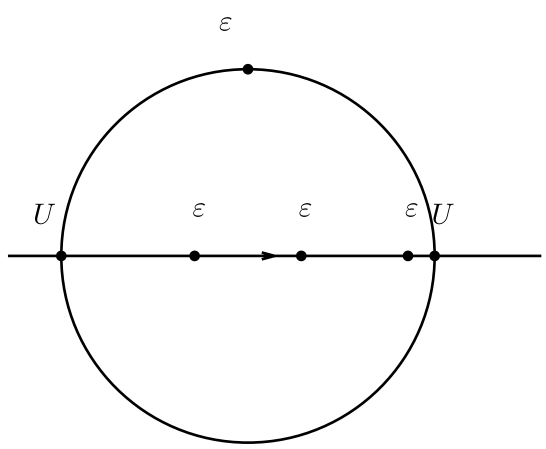

FIG. 3: A graph with $n_U = 2, n_\varepsilon = 4$

value is the following

$$\varepsilon^4 U^2 \sum_x \int dx_{0,1}...dx_{0,6}\eta_x \bar g(x; x_{0,1}-x_{0,2})\bar g(x+1, x_{0,2}-x_{0,3})\bar g(x; x_{0,3}-x_{0,4}) \qquad (22)$$
$$\bar g(x+1; x_{0,4}-x_{0,5})\bar g(x+1; x_{0,1}-x_{0,5})\bar g(x+1; x_{0,1}-x_{0,6})\bar g(x+2; x_{0,6}-x_{0,5})\eta_{x+2,x_{5,0}}$$

The appearance of graphs with loops plagued by small divisors like (22) produces a number of new problems. First of all, a $Cost^n$ bound on each Feynman graph is not sufficient to achieve convergence; the number of graphs with loops is $O(n!^2)$ and one has to take into account *cancellations* between graphs. In addition, the presence of loops has the effects that the structure of small divisors is much more complex and the dangerous subgraphs can have any number of external lines (not only two as in the $U = 0$ case). The presence of loops is the signature of an interacting many-body system, and their presence makes the problem genuinely different with respect to KAM theory.

2. PROOF OF THEOREM 1.1

A. Multiscale decomposition

We start by describing the integration of the generating function in the case $\eta = 0$ (the partition function); we will describe how to adapt the expansion to the study of the two point function in §2.I below.

We introduce a function $\chi_h(t, k_0) \in C^\infty(\mathbb{T} \times \mathbb{R})$, such that $\chi_h(t, k_0) = \chi_h(-t, -k_0)$ and $\chi_h(t, k_0) = 1$, if $\sqrt{k_0^2 + v_0^2 ||t||^2} \leq a\gamma^{h-1}$ and $\chi_h(t, k_0) = 0$ if $\sqrt{k_0^2 + v_0^2 ||t||^2} \geq a\gamma^h$ with a and $\gamma > 1$ suitable constants. We define $\bar{x}_+ = \widehat{x}$ $\bar{x}_- = -\widehat{x} - 2\theta/\omega$ and we choose a so that the supports of $\chi_0(\omega(x - \widehat{x}_+), k_0)$ and $\chi_0(\omega(x - \widehat{x}_-), k_0)$ are disjoint; we also define $\chi^{(1)}(\omega x, k_0) = 1 - \chi_0(\omega(x - \bar{x}_+), k_0) - \chi_0(\omega(x - \bar{x}_-), k_0)$. For reasons which will appear clear below, see Lemma 2.4, we choose $\gamma > 2^{\frac{1}{\tau}}$. We can write then

$$g^{(\leq M)}(\mathbf{x}, \mathbf{y}) = g^{(1)}(\mathbf{x}, \mathbf{y}) + g^{(\leq 0)}(\mathbf{x}, \mathbf{y}) \tag{23}$$

and

$$g^{(\leq 0)}(\mathbf{x}, \mathbf{y}) = \sum_{\rho = \pm} g_\rho^{(\leq 0)}(\mathbf{x}, \mathbf{y}) \tag{24}$$

where, for M large enough

$$\begin{aligned} g^{(1)}(\mathbf{x}, \mathbf{y}) &= \frac{\delta_{x,y}}{\beta} \sum_{k_0 \in D_\beta} \chi^{(1)}(\omega x, k_0) \bar{\chi}(\gamma^{-M}|k_0|) \frac{e^{-ik_0(x_0 - y_0)}}{-ik_0 + \cos 2\pi(\omega x + \theta) - \cos 2\pi(\omega\widehat{x} + \theta)} \\ g_\rho^{(\leq 0)}(\mathbf{x}, \mathbf{y}) &= \frac{\delta_{x,y}}{\beta} \sum_{k_0 \in D_\beta} \chi_0(\omega(x - \bar{x}_\rho), k_0) \frac{e^{-ik_0(x_0 - y_0)}}{-ik_0 + \cos 2\pi(\omega x + \theta) - \cos 2\pi(\omega\widehat{x} + \theta)} \end{aligned} \tag{25}$$

We use the following property; if $P_g(d\psi)$ is a Gaussian Grassmann integration with propagator g and $g = g_1 + g_2$, then $P_g(d\psi) = P_{g_1}(d\psi_1) P_{g_2}(d\psi_2)$, in the sense that for every polynomial f

$$\int P_g(d\psi) f(\psi) = \int P_{g_1}(d\psi_1) \int P_{g_2}(d\psi_2) f(\psi_1 + \psi_2)\ . \tag{26}$$

By using such property

$$e^{W(0)} = \int P(d\psi) e^{-\mathcal{V}(\psi)} = \int P(d\psi^{(\le 0)}) \int P(d\psi^{(1)}) e^{-\mathcal{V}(\psi^{(\le 0)}+\psi^{(1)})} \tag{27}$$

where $P(d\psi^{(1)})$ and $P(d\psi^{(\le 0})$ are gaussian Grassmann integrations with propagators respectively $g^{(1)}(\mathbf{x},\mathbf{y})$ and $g^{(\le 0)}(\mathbf{x},\mathbf{y})$ and $\psi^{(1)}$ and $\psi^{(\le 0)}$ are independent Grassmann variables. We can write

$$\int P(d\psi^{(1)}) e^{-\mathcal{V}(\psi^{(\le 0)}+\psi^{(1)})} = e^{\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \mathcal{E}_1^T(\mathcal{V};n)} \equiv e^{-\beta L E_0 - \mathcal{V}^{(0)}(\psi^{(\le 0)})} \tag{28}$$

where $\mathcal{E}_1^T$ is the fermionic truncated expectation with respect to $P(d\psi^{(1)})$. By the above definition

$$\mathcal{V}^{(0)} = \sum_{m=1}^{\infty} \sum_{x_1} \int dx_{0,1} .... \sum_{x_m} \int dx_{0,m} W_m^{(0)}(\mathbf{x}_1, ..., \mathbf{x}_n) [\prod_{i=1}^m \psi^{(\varepsilon_i)(\le 0)}_{\mathbf{x}'_i,\rho_i}] \tag{29}$$

with $\mathbf{x} = \mathbf{x}' + \bar{\mathbf{x}}_\rho$, $\bar{\mathbf{x}}_\rho = (\bar{x}_\rho, 0)$ and E_0 is a constant; moreover

$$e^{W(0)} = e^{-\beta L E_0} \int P(d\psi^{(\le 0)}) e^{-\mathcal{V}^{(0)}(\psi^{(\le 0})} \tag{30}$$

It was proved in Lemma 2.1 [17] that the constant E_0 and the kernels $W_m^{(0)}$ are given by power series in U, ε, ν convergent for $|U|, |\varepsilon|, |\nu| \le \varepsilon_0$, for ε_0 small enough and independent of β, L. They satisfy the following bounds:

$$|W_m^{(0)}|_{L_1} \le L\beta C^m \varepsilon_0^{k_m} \ , \tag{31}$$

for some constant $C > 0$ and $k_m = \max\{1, m-1\}$. Moreover the limit $M \to \infty$ exists and is reached uniformly.

We will show in the following section that we can integrate the fields $\psi^{(0)} ... \psi^{(h+1)}$ obtaining

$$e^{-\beta L E_0} \int P(d\psi^{(\le 0)}) e^{-\mathcal{V}^{(0)}(\psi^{(\le 0)})} = e^{-\beta L E_h} \int P(d\psi^{(\le h)}) e^{-\mathcal{V}^{(h)}(\psi^{(\le h)})} \tag{32}$$

where $P(d\psi^{(\le h)})$ is the gaussian Grassman integration with propagator, $\rho = \pm$

$$g_\rho^{(\le h)}(\mathbf{x}', \mathbf{y}') = \delta_{x',y'} \bar{g}_\rho^{(\le h)}(x', x_0 - y_0) \tag{33}$$

with, if $x = x' + \bar{x}_\rho$

$$\bar{g}_\rho^{(\le h)}(x', x_0 - y_0) = \int dk_0 e^{-ik_0(x_0-y_0)} \chi_h(\omega x', k_0) \frac{1}{-ik_0 + v_0 \rho (\omega x')_{\text{mod}.1} + r_{\rho,x'}} \tag{34}$$

and the corresponding fields are denoted by $\psi^{(\varepsilon,\leq h)}_{\mathbf{x}',\rho}$. The effective potential $\mathcal{V}^{(h)}$ can be written as sum of terms of the form

$$\sum_{x'_1}\int dx_{0,1}....\int dx_{0,n}H^{(h)}_{n;\rho_1,..,\rho_n}(x'_1;x_{0,1},.,x_{0,n})[\prod_{i=1}^{n}\psi^{\varepsilon_i(\leq h)}_{\mathbf{x}'_i,\rho_i}] \tag{35}$$

and x'_i are functions of x_1.

Definition 2.1 (Resonances):The contribution to the effective potential $\mathcal{V}^{(h)}$ of the form (35) such that $x'_i = x'_1$ for any $i = 1,..,n$ are called *resonant terms*; the other are called *non-resonant terms*.

Lemma 2.1 *In a resonant term $\rho_i = \rho_1$ for any $i = 1,..,n$.*

Proof. The second of (12) implies $\frac{2\theta}{\omega} \notin \mathbb{Z}/\{0\}$; as $x_i - x_j \in \mathbb{Z}$ and $x'_i = x'_j$ then $(\bar{x}_{\rho_i} - \bar{x}_{\rho_j}) + N = 0$, $N \in \mathbb{Z}$ so that $\rho_i = \rho_j$ as $\bar{x}_+ = \widehat{x}$ and $\bar{x}_- = -\widehat{x} - 2\theta/\omega$ and $\widehat{x} \in \mathbb{Z}$. ∎

Remark. There are several ways in which the multiscale integration (32) can be performed. The most naive one would be simply to proceed as in the integration of $\psi^{(1)}$ (27); that is, writing, by using (26), $P(d\psi^{(\leq 0)}) = P(d\psi^{(\leq -1)})P(d\psi^{(0)})$ and integrating $\psi^{(0)}$ so obtaining $\mathcal{V}^{(-1)}$ and proceeding in this way. This procedure would lead to a sequence of $\mathcal{V}^{(h)}$ with kernels $H^{(h)}_{n;\rho_1,..,\rho_n}$ admitting bounds increasing as $h \to -\infty$, producing a lack of convergence. Such problem is due to the fact that, according to the usual terminology of Renormalization Group, the theory is dimensionally non-renormalizable; the scaling dimension D is

$$D = 1 \tag{36}$$

for any term in the effective potential. One has to device a more clever integration procedure, which will be described in the following section. The idea behind it is that the resonant and the non resonant terms behave in a quite different way; one needs to *renormalize* the resonant terms extracting as usual the local part; as we will see, the local part is vanishing except for the kernels with two external fields, an essential fact which avoid the presence of an infinite number of running coupling constants. On the other hand, the dimensional bound can be dramatically improved in the case of the non resonant terms using the Diophantine condition, as we will show in §2.F.

B. Renormalized expansion

The sequence of effective potentials $\mathcal{V}^{(h)}$, $h = 0, -1, -2, ..$ is constructed iteratively in the following way; assume that we have already integrated the fields $\psi^{(0)}, \psi^{(-1)}, .., \psi^{(h+1)}$ obtaining the r.h.s. of (32) which we rewrite as

$$e^{-\beta L E_h} \int P(d\psi^{(\leq h)}) e^{-\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\leq h)}) - \mathcal{R}\mathcal{V}^{(h)}(\psi^{(\leq h)})} \tag{37}$$

with $\mathcal{L} = 1 - \mathcal{R}$ and $\mathcal{R}$ acts on the terms (35) in $\mathcal{V}^{(h)}$ in the following way:

1. If $n = 2$ then $\mathcal{R} = 1$ if (35) is non resonant, while if (35) is resonant

$$\begin{aligned} &\mathcal{R} \sum_{x'} \int dx_{0,1} dx_{0,2} H^{(h)}_{2;\rho,\rho}(x'; x_{0,1}, x_{0,2}) \psi^{+(\leq h)}_{x',x_{0,1},\rho} \psi^{-(\leq h)}_{x',x_{0,2},\rho} \\ &= \sum_{x'} \int dx_{0,1} dx_{0,2} H^{(h)}_{2;\rho,\rho}(x'; x_{0,1}, x_{0,2}) \psi^{+(\leq h)}_{x',x_{0,1},\rho} T^{-(\leq h)}_{x',x_{0,1},x_{0,2}\rho} \end{aligned} \tag{38}$$

with

$$T^{-(\leq h)}_{x',x_{0,1},x_{0,2}\rho} = \psi^{-(\leq h)}_{x',x_{0,2},\rho} - \psi^{-(\leq h)}_{x',x_{0,1},\rho} - (x_{0,1} - x_{0,2}) \partial \psi^{-(\leq h)}_{x',x_{0,1},\rho} \tag{39}$$

2. If $n \geq 4$ the $\mathcal{R}$ operation consists in replacing any monomial of fields with the same x', ε, ρ in (35), that is $\psi^{\varepsilon(\leq h)}_{x',x_{0,1},\rho} \prod_i \psi^{\varepsilon(\leq h)}_{x',x_{0,i},\rho}$, with

$$\psi^{\varepsilon(\leq h)}_{x',x_{0,1},\rho} \prod_i D^{\varepsilon(\leq h)}_{x',x_{0,1},x_{0,i},\rho} \tag{40}$$

with

$$D^{\pm(\leq h)}_{x',x_{0,1},x_{0,2},\rho} = \psi^{\pm(\leq h)}_{x',x_{0,1},\rho} - \psi^{\pm(\leq h)}_{x',x_{0,2},\rho} \tag{41}$$

Remark When $n \geq 4$ the $\mathcal{R}$ operation is simply the identity, as $\psi^{\varepsilon(\leq h)}_{x',x_{0,1},\rho} \prod_i \psi^{\varepsilon(\leq h)}_{x',x_{0,i},\rho} = \psi^{\varepsilon(\leq h)}_{x',x_{0,1},\rho} \prod_i D^{\varepsilon(\leq h)}_{x',x_{0,1},x_{0,i},\rho}$; however, as we will see in the following sections, the equivalent representation of the monomials given by (40) has the effect that certain dimensional gains in the bounds can be extracted more easily. Note that in all resonances with $n \geq 4$ there are at least two D-fields, by Lemma 2.1; as we will see below, this will change the scaling dimension from 1 to -1. Finally note that, in presence of the spin, the $\mathcal{L}$ action would be non vanishing and a quartic running coupling constant is generated.

By definition $\mathcal{L}\mathcal{V}^{(h)}$ is given by the following expression

$$\mathcal{L}\mathcal{V}^{(h)} = \gamma^h \nu_h F^{(h)}_\nu + F^{(h)}_\zeta + F^{(h)}_\alpha \tag{42}$$

where

$$
\begin{aligned}
F_\nu^{(h)} &= \sum_\rho \sum_{x'} \int dx_0 \psi^{+(\le h)}_{\mathbf{x}',\rho} \psi^{-(\le h)}_{\mathbf{x}',\rho} \\
F_\zeta^{(h)} &= \sum_\rho \sum_{x'} \int dx_0 (\omega x')_{\mathrm{mod.1}} \zeta_{h,\rho}(x') \psi^{+(\le h)}_{\mathbf{x}',\rho} \psi^{-(\le h)}_{\mathbf{x}',\rho} \\
F_\alpha^{(h)} &= \sum_\rho \sum_{x'} \int dx_0 \alpha_{h,\rho}(x') \psi^{+(\le h)}_{\mathbf{x}',\rho} \partial \widehat{\psi}^{-(\le h)}_{\mathbf{x}',\rho}
\end{aligned}
\tag{43}
$$

The ν_h coefficients are *independent* from ρ, as (8) is invariant under parity $x \to -x$, $\theta \to -\theta$; and this implies invariance under the transformation $\psi^{\pm(h)}_{x_0,x',\rho} \to \psi^{\pm(h)}_{x_0,-x',-\rho}$; therefore, if $\varepsilon = \pm$

$$
H^{(h)}_{2,\rho}(x', x_0, y_0) = H^{(h)}_{2,-\rho,}(-x', x_0, y_0) \tag{44}
$$

so that the fact that ν_h is independent of ρ follows. Note also that $(\widehat{g}^{(k)})^*(x, k_0) = \widehat{g}^{(k)}(x, -k_0)$ so that $(\widehat{H}^{(h)}_{2,\rho}(x', k_0))^* = \widehat{H}^{(h)}_{2,\rho}(x', -k_0)$, and this implies that ν_h is real.

With the above definitions we finally write (32) as

$$
\int P(d\psi^{(\le h-1)}) \int P(d\psi^{(h)}) e^{-\mathcal{L}\mathcal{V}^{(h)} - \mathcal{R}\mathcal{V}^{(h)}} = e^{-\beta L E_h} \int P(d\psi^{(\le h-1)}) e^{-\mathcal{V}^{(h-1)}(\psi^{(\le h-1)})} \tag{45}
$$

where $P(d\psi^{(\le h-1)})$ has propagator $g^{(\le h-1)}$ defined by a formula analogous to (34) with $h-1$ replacing h, and $P(d\psi^{(h)})$ has propagator $g^{(h)}$ defined by a formula analogous to (34) with χ_h replaced by $f_h = \chi_h - \chi_{h-1}$, with f_h a smooth compact support function vanishing for $c_1 \gamma^{h-1} \le \sqrt{k_0^2 + v_0^2 ||\omega x'||_1^2} \le c_2 \gamma^{h+1}$, for suitable constants c_1, c_2. From the r.h.s. of (45), the procedure can be iterated.

The single scale propagator $g^{(h)}$ verifies the following bound, for any integer N and a suitable constant C_N

$$
|\bar{g}^{(h)}_\rho(x', x_0 - y_0)| \le \frac{C_N}{1 + (\gamma^h |x_0 - y_0|)^N} \tag{46}
$$

which can be easily obtained by integrating by parts.

The above procedure allows to write the $W(0)$ (27) in terms of an expansion in the *running coupling constants* $\vec{v}_h = (\nu_h, \zeta_{h,\rho}, \alpha_{h,\rho})$ with $h \le 0$; as it is clear from the above construction, they verify a recursive equation of the form

$$
\vec{v}_{h-1} = \vec{v}_h + \vec{\beta}_h(\vec{v}_h, ..\vec{v}_0) \tag{47}
$$

We will describe more explicitly such expansion in the following section.

C. Trees

The effective potential $\mathcal{V}^{(h)}$ can be written as a sum over *trees* [32], defined below.

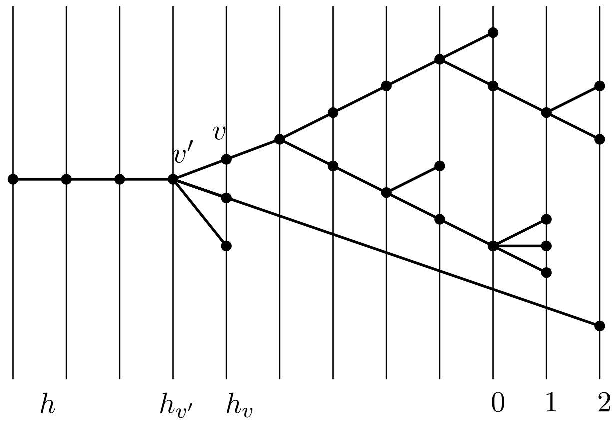

FIG. 4: A tree $\tau \in \mathcal{T}_{h,n}$ with its scale labels.

Definition 2.2 (τ-Trees):

1. The *labeled trees* $\tau \in \mathcal{T}_{h,n}$ with n endpoints (to be called simply trees in the following) are defined by associating some labels with the unlabeled trees, which are constructed by joining a point r, the *root*, with an ordered set of $n \geq 1$ points, the *endpoints* of the *unlabeled tree*, so that r is not a branching point. Starting from the unlabeled trees, the labeled trees are defined associating a label $h \leq 0$ with the root; moreover, we introduce a family of vertical lines, labeled by an integer taking values in $[h, 2]$, and we represent any tree $\tau \in \mathcal{T}_{h,n}$ so that, if v is an endpoint or a non trivial vertex (the branching points), it is contained in a vertical line with index $h_v > h$, to be called the *scale* of v, while the root r is on the line with index h. In general, the tree will intersect the vertical lines in set of points different from the root, the endpoints and the branching points; these points will be called *trivial vertices*. Every vertex v of a tree will be associated to its scale label h_v, defined, as above, as the label of the vertical line whom v belongs to.

2. There is only one vertex immediately following the root, whose scale is $h + 1$. Given a vertex v of $\tau \in \mathcal{T}_{h,n}$ that is not an endpoint, we can consider the subtrees of τ with

root v, which correspond to the connected components of the restriction of τ to the vertices $w \geq v$; the number of endpoint of these subtrees will be called N_v. If a subtree with root v contains only v and one endpoint on scale h_v+1, it will be called a *trivial subtree*. With each endpoint v of scale $h_v \leq 1$ we associate $\mathcal{L}\mathcal{V}^{(h_v-1)}$, and there is the constrain that $h_v = h_{\bar{v}}+1$, if $\bar{v}$ is the non trivial vertex immediately preceding it or the first vertex after the root; to the end-points of scale $h_v = 2$ are associated one of the terms contributing to $\mathcal{V}$ and there is not such a constrain.

3. The set of field labels associated with the endpoint v will be called I_v; if v is not an endpoint, we shall call I_v the set of field labels associated with the endpoints following the vertex v. Finally with each trivial or non trivial vertex v with $h < h_v \leq 0$, which is not an endpoint, we associate the $\mathcal{R} = 1-\mathcal{L}$ operator, acting on the corresponding kernel.

The effective potential appearing in (37) can be written as sum over trees in the following way, if $h \leq -1$

$$\mathcal{V}^{(h)}(\psi^{(\leq h)}) + L\beta E_{h+1} = \sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{T}_{h,n}} V^{(h)}(\tau,\psi^{(\leq h)}) \tag{48}$$

where, if $\bar{v}_0$ is the first vertex of τ and $\tau_1,..,\tau_s$ ($s = s_{\bar{v}_0}$) are the subtrees of τ with root $\bar{v}_0$, $V^{(h)}(\tau,\psi^{(\leq h)})$ is defined inductively by the relation

$$V^{(h)}(\tau,\psi^{(\leq h)}) = \frac{(-1)^{s+1}}{s!}\mathcal{E}^T_{h+1}[\bar{V}^{(h+1)}(\tau_1,\psi^{(\leq h+1)});..;\bar{V}^{(h+1)}(\tau_s,\psi^{(\leq h+1)})] \tag{49}$$

where $\bar{V}^{(h+1)}(\tau_i,\psi^{(\leq h+1)})$:

1. it is equal to $\mathcal{R}\mathcal{V}^{(h+1)}(\tau_i,\psi^{(\leq h+1)})$, with $\mathcal{R}$ given by (38),(40) if the subtree τ_i is non trivial;

2. if τ_i is trivial, it is equal to $\mathcal{L}\mathcal{V}^{(h+1)}$.

Starting from the above inductive definition, the effective potential can be written in a more explicit way.

Definition 2.3 (Q,P-Subsets):

1. We associate with any vertex v of the tree a subset P_v of I_v, the *external fields* of v, and the set $\mathbf{x}_v$ of all space-time points associated with one of the end-points following

v. The subsets P_v must satisfy various constraints. First of all, $|P_v| \geq 2$, if $v > v_0$; moreover, if v is not an endpoint and $v_1, \ldots, v_{S_v}$ are the $S_v \geq 1$ vertices immediately following it, then $P_v \subseteq \cup_i P_{v_i}$; if v is an endpoint, $P_v = I_v$. If v is not an endpoint, we shall denote by Q_{v_i} the intersection of P_v and P_{v_i}; this definition implies that $P_v = \cup_i Q_{v_i}$. The union $\mathcal{I}_v$ of the subsets $P_{v_i} \setminus Q_{v_i}$ is, by definition, the set of the *internal fields* of v, and is non empty if $S_v > 1$.

2. Given $\tau \in \mathcal{T}_{h,n}$, there are many possible choices of the subsets P_v, $v \in \tau$, compatible with all the constraints. We shall denote $\mathcal{P}_\tau$ the family of all these choices and $\mathbf{P}$ the elements of $\mathcal{P}_\tau$.

3. Given a tree τ and $\mathbf{P} \in \mathcal{P}_\tau$, we shall define the χ-vertices as the vertices v of τ , such that $\mathcal{I}_v$ (the union of the subsets $P_{v_i} \setminus Q_{v_i}$ defined before (50), that is the set of lines contracted in v) is non empty; note that $|V_\chi|$ is smaller than $4n$.

4. We call v' is the first vertex $\in V_\chi$ preceding v, and v_0 the first vertex $v \in V_\chi$ in τ.

With these definitions, we can rewrite $\mathcal{V}^{(h)}(\tau, \psi^{(\leq h)})$ as

$$\mathcal{V}^{(h)}(\tau, \psi^{(\leq h)}) = \sum_{\mathbf{P} \in \mathcal{P}_\tau} \mathcal{V}^{(h)}(\tau, \mathbf{P}) \quad \bar{\mathcal{V}}^{(h)}(\tau, \mathbf{P}) = \int d\mathbf{x}_{v_0} \widetilde{\psi}^{(\leq h)}(P_{v_0}) K_{\tau,\mathbf{P}}^{(h+1)}(\mathbf{x}_{v_0}) \,, \tag{50}$$

where $K_{\tau,\mathbf{P}}^{(h+1)}(\mathbf{x}_{v_0})$ is defined inductively and $\widetilde{\psi}^{(h_v)}(P_v) = \prod_{f \in P_v} \psi^{\varepsilon(f)(h_v)}_{\mathbf{x}'(f),\rho(f)}$.

The tree structure provides an arrangement of endpoints into a hierarchy of *clusters*, see Fig.5. Given a cluster with scale h_v, one can imagine that the fields $\widetilde{\psi}^{(h_v)}(P_{v_1} \setminus Q_{v_1})$,..,$\widetilde{\psi}^{(h_v)}(P_{v_{S_v}} \setminus Q_{v_{S_v}})$ are external to the S_v inner clusters, and the $\mathcal{E}^T_{h_v}$ operation contracts them in pairs.

In order to get the final form of our expansion, we need a convenient representation for the truncated expectation. Let us put $P_i := P_{v_i} \setminus Q_{v_i}$; moreover we order in an arbitrary way the sets $P^{\pm}_{v_i} := \{f \in P_{v_i}, \varepsilon(f) = \pm\}$, we call $f^{\pm}_{ij}$ their elements and we define $\mathbf{x}^{(i)} = \cup_{f \in P_i^-} \mathbf{x}(f)$, $\mathbf{y}^{(i)} = \cup_{f \in P_i^+} \mathbf{y}(f)$, $\mathbf{x}_{ij} = \mathbf{x}(f^-_{ij})$, $\mathbf{y}_{ij} = \mathbf{x}(f^+_{ij})$. A couple $l := (f^-_{ij}, f^+_{i'j'}) := (f^-_l, f^+_l)$ will be called a line joining the fields with labels $f^-_{ij}, f^+_{i'j'}$. Then, we use the *Brydges-Battle-Federbush* formula [33],[34] saying that , if $S_v > 1$,

$$\mathcal{E}^T_{h_v}(\widetilde{\psi}^{(h_v)}(P_i), \cdots, \widetilde{\psi}^{(h_v)}(P_{S_v}))) = \sum_{T_v} \prod_{l \in T_v} \left[\delta_{x_l, y_l} \bar{g}^{(h_v)}_{\rho_l}(x'_l, x_{0,l} - y_{0,l})\right] \int dP_T(\mathbf{t}) \det G^{h_v,T}(\mathbf{t}) \,, \tag{51}$$

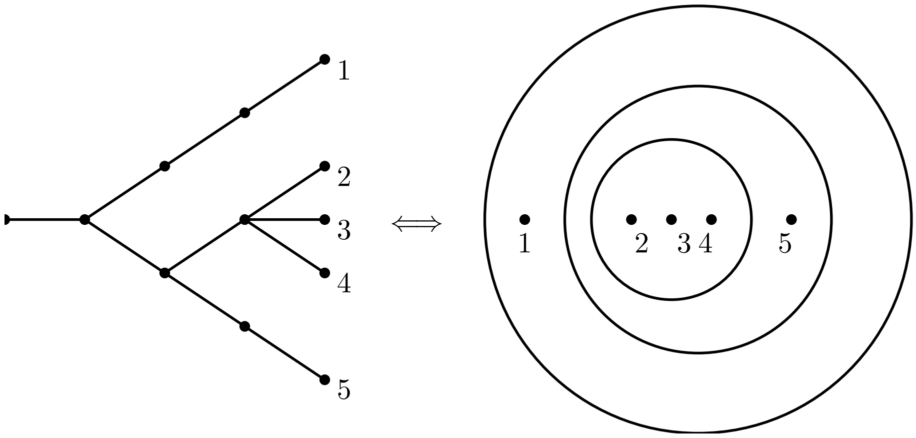

FIG. 5: A tree of order 5 and the corresponding clusters. Only the vertices $v \in V_\chi$ are represented.

where T_v is a set of lines forming an *anchored tree graph* between the clusters of points $\mathbf{x}^{(i)} \cup \mathbf{y}^{(i)}$, see Fig.6, that is T_v is a set of lines, which becomes a tree graph if one identifies all the points in the same cluster. Moreover $\mathbf{t} = \{t_{ii'} \in [0,1], 1 \le i, i' \le S_v\}$, $dP_{T_v}(\mathbf{t})$ is a probability measure with support on a set of $\mathbf{t}$ such that $t_{ii'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i \in \mathbb{R}^{S_v}$ of unit norm.

$$G^{h_v,T}_{ij,i'j'} = t_{ii'} \delta_{x_{ij},y_{i'j'}} \bar{g}^{(h_v)}_{\rho_{ij}}(x_{ij}, x_{0,ij} - y_{0,i'j'}) \; , \tag{52}$$

with $(f^-_{ij}, f^+_{i'j'})$ not belonging to T_v.

Definition 2.4 (*T*-trees):

1. We define $\bar{T}_v = \bigcup_{w \ge v} T_w$ starting from T_v and attaching to it the trees $T_{v_1}, .., T_{v_{S_v}}$ associated to the vertices $v_1, .., v_{S_v}$ following v, and repeating this operation until the end-points are reached. The tree $\bar{T}_v$ is composed by a set of lines, representing propagators with scale $\ge h_v$, connecting the end-points w of the tree τ.

2. To each line i_w attached to w in $\bar{T}_v$ is associated a factor $\delta^{i_w}_w$, and a) $\delta^i_w = 0$ if w corresponds to a ν_h, α_h, ζ_h end-point; b) $\delta^i_w = \pm 1$ if it corresponds to an ε end-point; c) $\delta^i_w = (0, \pm 1)$ if it corresponds to a U end-point.

3. Given w_1, w_2 in $\bar{T}_v$ such that x'_{w_1} and x'_{w_2} are coordinates of the external fields $\widetilde{\psi}(P_v)$, and let be c_{w_1,w_2} the set of end-points in the path in $\bar{T}_v$ connecting w_1 with w_2, including w_1, w_2 (in the example in Fig. 7 the path is composed by w_1, w_a, w_b, w_c, w_2). We call

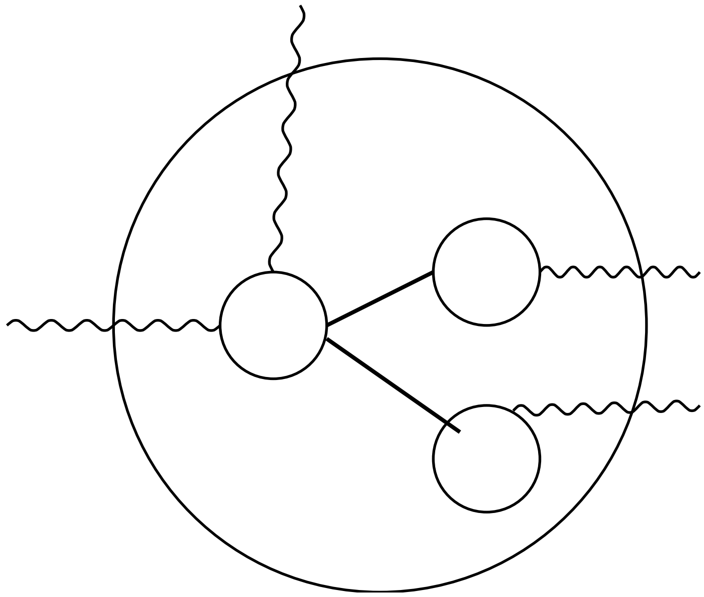

FIG. 6: A symbolic representation of a contribution to (51); the solid lines represent the propagators $g^{(h_v)}$ in the tree T_v connecting the $S_v = 3$ clusters, represented as circles, the wiggly lines are the external fields $\widetilde{\psi}(P_v)$; the fields in the determinant are not represented. Inside the 3 clusters other trees connecting inner clusters or points must be imagined, and so on.

i^*_w the line following w in c_{w_1,w_2} starting from w_1. We call $|c_{w_1,w_2}|$ the number of vertices in c_{w_1,w_2}.

By using the above definitions

$$x'_{w_1} - x'_{w_2} = (\bar{x}_{\rho_{\ell_{w_2}}} - \bar{x}_{\rho_{\ell_{w_1}}}) + \sum_{w \in c_{w_1,w_2}} \delta_w^{i^*_w} \tag{53}$$

The above relation implies, in particular, that the coordinates of the external fields $\widetilde{\psi}(P_{v_0})$ are determined once that the choice of a single one of them and of $\tau, \bar{T}_{v_0}$ and $\mathbf{P}$ is done.

Definition 2.5 (L and H vertices)

1. If the coordinates x' of the fields $\widetilde{\psi}(P_v)$ are the *same* we say that v is a *resonant vertex*, while if the coordinates are different is called *non resonant vertex*; the set of resonant vertices in V_χ is denoted by H_χ and the set of non-resonant vertices is denoted by L_χ.

2. We define $\bar{H}_\chi$ the union of H_χ and the non-resonant end-points (that is the ε, U end-points) $\bar{L}_\chi$ the union of L_χ and the resonant end-points (that is the ν_h, ζ_h, α_h end-points).

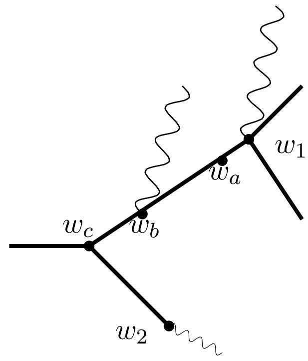

FIG. 7: A tree $\bar{T}_v$ with attached wiggly lines representing the external lines P_v; the lines represent propagators with scale $\geq h_v$ connecting w_1, w_a, w_b, w_c, w_2, representing the end-points following v in τ.

3. If v_i $i = 1, \ldots, v_{S_v}$ are the vertices (including end-points) such that $v'_i = v_i$; among such vertices there are S_v^L vertices belonging to $\bar{L}_\chi$ and S_v^H vertices belonging to $\bar{H}_\chi$ so that
$$S_v = S_v^L + S_v^H \tag{54}$$

D. Graphs

Let us first set $\mathcal{R} = 1$ and we can write
$$\mathcal{V}^{(h)}(\psi^{(\leq h)}) = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{h,n}} \sum_{G \in \mathcal{G}(\tau)} \text{Val}(G) \tag{55}$$
where $\mathcal{G}(\tau)$ is the set of Feynman graphs of order n obtained associating to each end-point a graph element as in Fig.1, and joining (contracting) the lines with consistent orientation so that all the n vertices are connected. With respect to the Feynman graph seen in the previous section, each propagator carries an index h_v, if v is the minimal cluster containing the propagator.

An immediate bound for each Feynman graph is, if $|U|, |\varepsilon| \leq \varepsilon_0$, and remembering that

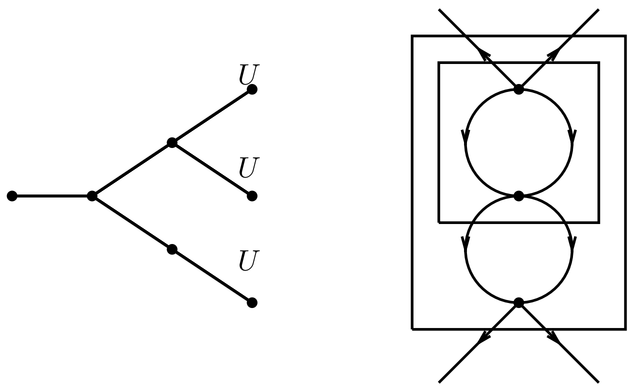

FIG. 8: A tree τ (only the vertices $v \in V_\chi$ are represented), the corresponding clusters, represented as boxes, and a Feynman graph; the propagators have scale h_{v_1} and h_{v_2} respectively.

S_v is the number of clusters contained in the cluster v

$$|\mathrm{Val}(G)| \leq \varepsilon_0^n C^n \prod_{v \in V_\chi} \gamma^{-(S_v-1)h_v} \tag{56}$$

The above estimate is immediately obtained considering a tree of propagators connecting all vertices, bounding by a constant the propagators not belonging to such tree and by γ^{-h_v} the integrals of each one of the $S_v - 1$ propagators in the tree connecting the vertices in the cluster v. The above bound can be rewritten as

$$|\mathrm{Val}(G)| \leq \varepsilon_0^n C^n \gamma^{h_{v_0}} [\prod_{v \in H_\chi} \gamma^{-D(h_{v'}-h_v)}][\prod_{v \in e.p.}^{*} \gamma^{-h_{v'}}] \tag{57}$$

where $D = 1$ is the *scaling dimension*. The bound (57) do not provide a finite result when summed over the scales h_v. As we will see in §2.F the $\mathcal{R}$ operation produces an extra factor $\prod_{v \in H_\chi} \gamma^{2(h_{v'}-h_v)}$ in the bound, making the dimension of the resonant vertices negative $D = -1$. Moreover, as we will see in the following section, the diophantine condition implies that the dimension of the non resonant vertices can be improved.

E. The non resonant terms

Consider a non resonant vertex v and x'_{w_1} and x'_{w_2} are coordinates of two external fields, with $x'_{w_1} - x'_{w_2}$ given by (53). The Diophantine conditions imply a relation between the scale

h_v and the number of vertices between w_2 and w_1 in $\bar{T}_v$.

Lemma 2.2 *Given $\tau, \mathbf{P}, \mathbf{T}$, let us consider $v \in \bar{L}_\chi$ and w_1, w_2 two vertices (possibly coinciding) in $\bar{T}_v$, see (53), with $x'_{w_1} \neq x'_{w_2}$; then*

$$|c_{w_1,w_2}| \geq A\gamma^{\frac{-h_{\bar{v}'}}{\tau}} \tag{58}$$

with a suitable constant A.

Proof. Note that $||\omega x'_{w_i}||_1 \leq c v_0^{-1}\gamma^{h_{v'}-1}$, $i = 1, 2$ by the compact support properties of the propagator; therefore by using (53) and the Diophantine condition, if

$$2cv_0^{-1}\gamma^{h_{v'}} \geq ||(\omega x'_{w_1})|| + ||(\omega x'_{w_2})|| \geq ||\omega(x'_{w_1} - x'_{w_2})|| = \tag{59}$$

$$||(\bar{x}_{\rho_{\ell_{w_2}}} - \bar{x}_{\rho_{\ell_{w_1}}})\omega + \omega \sum_{w \in c_{w_1,w_2}} \delta_w^{i_w}|| \tag{60}$$

If $\rho_{\ell_{w_2}} = \rho_{\ell_{w_1}}$ by the first of (12) we get

$$2cv_0^{-1}\gamma^{h_{v'}} \geq \frac{C_0}{|\sum_{w \in c_{w_1,w_2}} \delta_w^{i_w}|^\tau} \tag{61}$$

.

If $\rho_{\ell_{w_2}} = \varepsilon$, $\rho_{\ell_{w_1}} = -\varepsilon$, $\varepsilon = \pm$ then

$$||(\bar{x}_{\rho_{\ell_{w_2}}} - \bar{x}_{\rho_{\ell_{w_1}}})\omega + \omega \sum_{w \in c_{w_1,w_2}} \delta_w^{i_w}|| = ||2\varepsilon\omega\widehat{x} + 2\varepsilon\theta + \omega \sum_{w \in c_{w_1,w_2}} \delta_w^{i_w}|| \tag{62}$$

and if $\sum_{w \in c_{w_1,w_2}} \delta_w^{i_w} + 2\varepsilon\widehat{x} \neq 0$ by the second of (12)

$$2cv_0^{-1}\gamma^{h_{v'}} \geq \frac{C_0}{|2\varepsilon\widehat{x} + \sum_{w \in c_{w_1,w_2}} \delta_w^{i_w}|^\tau} \geq \frac{C_0}{(2|\widehat{x}| + |\sum_{w \in c_{w_1,w_2}} \delta_w^{i_w}|)^\tau} \geq \frac{C_0}{|\sum_{w \in c_{w_1,w_2}} \delta_w^{i_w}|^\tau} \tag{63}$$

Finally if $\sum_{w \in c_{w_1,w_2}} \delta_w^{i_w} + 2\varepsilon\widehat{x} = 0$ then $cv_0^{-1}\gamma^{h_{v'}} \geq ||2\theta|| \geq ||2\theta|| \frac{|2\widehat{x}|^\tau}{|\sum_{w \in c_{w_1,w_2}} \delta_w^{i_w}|^\tau}$. The fact that $|\sum_{w \in c_{w_1,w_2}} \delta_w^{i_w}| \leq |c_{w_1,w_2}|$ ends the proof. ∎

Lemma 2.2 says that there is a relation between the number of end-points following $v \in L_\chi$ and the scales of the external lines coming out from v. In particular the U, ε-endpoints with scale $h_v = 2$ have $|c_{w_1,w_2}| = 1$, hence the scale of the first vertex $v \in V_\chi$ preceding the end-point is bounded by a constant.

Lemma 2.3 *Given $\tau, \mathbf{P}, \mathbf{T}$ the following inequality holds, for any $0 < c < 1$*

$$c^n \leq \prod_{v \in \bar{L}_\chi} c^{A\gamma^{\frac{-h_{v'}}{\tau}} 2^{h_{v'}-1}} \tag{64}$$

Proof. If $v \in V_\chi$ and $N_v = \sum_{i, v_i^* > v} 1$ is the number of end-points following v in τ then

$$c^n \leq \prod_{v \in V_\chi} c^{N_v 2^{h_{v'}-1}} \tag{65}$$

Indeed we can write $c = \prod_{h=-\infty}^{0} c^{2^{h-1}}$. Given a tree $\tau \in \mathcal{T}_{h,n}$, we consider an end-point v^* and the path in τ from v^* to the root v_0; to each vertex $v \in V_\chi$ in such path with scale h_v we associate a factor $c^{2^{h_v-2}}$; repeating such operation for any end-point, the vertices v followed by N_v end-points are in N_v paths, therefore we can associate to them a factor $c^{N_v 2^{h_v-2}}$; finally we use that $c^{2^{h_v-2}} < c^{2^{h_{\bar{v}'}-2}}$.

Note that if v is non resonant, there exists surely two external fields with coordinates x_1', x_2' such that $x_1' \neq x_2'$; note that

$$N_v \geq |c_{w_1,w_2}| \geq A \gamma^{\frac{-h_{v'}}{\tau}} \tag{66}$$

therefore, by (65), (64) follows, ■.

By combing the above results we get the following final lemma which will play a crucial role in the following. We choose $\gamma^{\frac{1}{\tau}}/2 \equiv \gamma^\eta > 1$; for instance $\gamma = 2^{2\tau}$, $\eta = \frac{1}{2\tau}$.

Lemma 2.4 *Given $\tau, \mathbf{P}, \mathbf{T}$ the following inequality holds*

$$[\prod_{v \in \bar{L}_\chi} c^{A\gamma^{\frac{-h_{v'}}{\tau}} 2^{h_{v'}}}] \leq \bar{C}^n [\prod_{v \in V_\chi} \gamma^{h_v S_v^L}][\prod_{v \in \bar{L}_\chi} \gamma^{h_{v'}}] \tag{67}$$

with $\bar{C} = [\frac{2}{|\log |c||A}]^2 e^{-2}]$.

Proof As we assumed $\gamma^{\frac{1}{\tau}}/2 \equiv \gamma^\eta > 1$ than, for any N

$$c^{A\gamma^{\frac{-h}{\tau}} 2^h} = e^{-|\log c| A \gamma^{-\eta h}} \leq \gamma^{N\eta h} \frac{N}{|\log |c||A]^N e^N} \tag{68}$$

as $e^{-\alpha x} x^N \leq [\frac{N}{\alpha}]^N e^{-N}$, and (67) follows choosing $N = 2/\eta$. ■

F. The resonant terms

In the previous section, and in particular in Lemma 2.4, we have seen that the Diophantine condition implies an extra factor $\gamma^{2h_{v'}}$ for any non resonant vertex $v \in \bar{L}_\chi$, at the cost of an harmless constant c^{-n}, where n is the perturbative order. There is no such a gain for the resonant vertices, and one has to exploit the $\mathcal{R}$ operation in order to gain factors allowing at the end to sum over all the scales h_v of the tree τ. In addition, the $\mathcal{R}$ operation, when applied over vertices with a large number of external fields, gives also, combined with lemma 2.3, a factor allowing the sum over P_v.

Let us start considering the resonant vertices. The effect of the $\mathcal{R}$ operation on the vertices $v \in H_\chi$ consists in replacing a ψ fields with a T field (39) when $|P_v| = 2$, or to replace at least two fields with D-fields (41) if $|P_v| \geq 4$; if such fields are contracted at a scale $h_{v'}$, the replacement of a ψ with a D fields implies the replacement of a propagator $\bar{g}^{(h_{v'})}(x', x_{0,1} - z_0)$ with

$$\bar{g}^{(h_{v'})}(x', x_{0,1} - z_0) - \bar{g}^{(h_{v'})}(x', x_{0,2} - z_0) \tag{69}$$

In the bounds, it can be convenient to write such difference as

$$(x_{0,1} - x_{0,2}) \int_0^1 dt \partial \bar{g}^{(h_{v'})}(x', \widehat{x}_{0,1,2}(t) - z_0) \tag{70}$$

where $\widehat{x}_{0,1,2}(t) = x_{0,1} + t(x_{0,2} - x_{0,1})$ is an interpolated point between $x_{0,1}$ and $x_{0,2}$; note that replacing $g^{(h_{v'})}(x', x_{0,1} - z_0)$ with (70) produces at least an extra factor $\gamma^{h_{v'} - h_v}$ in the bounds. Similarly replacing a ψ with a T field can produce an improvement $\gamma^{2(h_{v'} - h_v)}$ so that, in conclusion, for each $v \in H_\chi$ the $\mathcal{R}$ operation produces an extra factor $\gamma^{2(h_{v'} - h_v)}$.

In order to get a finite bound on the kernels of the effective potential, in addition to the sum over the trees and the scale labels h_v, there is also the sum over the sets P_v. Let us consider the vertices v with a large $|P_v|$. Note that the external lines have labels $(\varepsilon_i, \rho_i) = (\pm, \pm)$; therefore P_v can be decomposed in 4 groups, and we denote by $\bar{\rho}, \bar{\varepsilon}$ the labels of the external fields whose number is maximal; we call m_v this subset of P_v and $|m_v| \geq |P_v|/4$; we replace the D fields in P_v not belonging to m_v with ψ fields. We consider a tree $\bar{T}_v$ and we define a pruning operation associating to it another tree $\widehat{T}_v$ eliminating from $\bar{T}_v$ all the non branching vertices w in $\bar{T}_v$ not associated to any external line with label $\bar{\rho}, \bar{\varepsilon}$, and all the subtrees not containing any external line with label $\bar{\rho}, \bar{\varepsilon}$ (see Fig. 9 for an example), so that there is an external line associated to all end-points. The vertices w of $\widehat{T}_v$ are then only branching vertices or non branching vertices with external lines $\bar{\rho}, \bar{\varepsilon}$; all the end-points have associated an external line. We define a procedure to group in couples the fields in m_v, such that every field belongs to a couple and at most to two couples, and the paths in $\widehat{T}_v$ connecting the coordinates of the points in the couple are non overlapping.

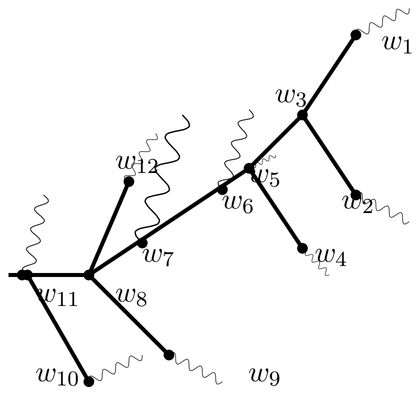

FIG. 9: In the picture the lines represent the propagators with scale $\le h_v$ in $\widehat{T}_v$ and the wiggly lines represent the external lines P_v with label $\bar{\rho}$; note that, by definition of the pruning operation, all the end-points have associated wiggly lines, contrary to what happens in $\bar{T}_v$, see Fig. 7.

The procedure starts by a first pruning operation considering the end-points w_a immediately followed by vertices w_b with external lines (in the tree in Fig. 9 the vertices are w_{10}, w_{11} or w_4, w_5); we say that the couple of fields in w_a, w_b is of type 1 if $x'_{w_a} = x'_{w_b}$, while it is of type 2 if $x'_{w_a} \neq x'_{w_b}$. We now cancel the end-points w_a already considered and the resulting subtrees with no external lines; in the resulting tree we select an end-point w_a immediately followed by vertices w_b with wiggly lines, and again such a couple can be of type 1 or 2; we continue unless there are no end-points w followed by vertices with wiggly line (the result of this pruning operation on the tree in Fig 9 is Fig. 10).

In the second pruning operation we consider (if they are present, otherwise the tree is trivial and the procedure ends) a couple of endpoints followed by a branching vertex (in the picture w_1, w_2 or w_9, w_{12}); we call them w_a, w_b and we proceed exactly as above distinguishing the two kind of couples. We then cancel such end-points w_a, w_b and the subtrees not containing external lines, (the result of this operation on the tree in Fig. 10 is in Fig 11). If the resulting tree has again end-points with external lines followed by vertices with external lines, we prune such vertices as in the first step and we continue in this way so that at the end all except at most one vertex with external lines are considered.

Note that by construction the paths c_{w_a,w_b} in $\widehat{T}_v$ do not overlap; for instance in Fig.8 the paths are $c_{w_{10},w_{11}}$, c_{w_4,w_5}, c_{w_1,w_2}, $c_{w_9,w_{12}}$, c_{w_5,w_6}, $c_{w_6,w_7}, c_{w_7,w_{11}}$. Therefore, given a vertex

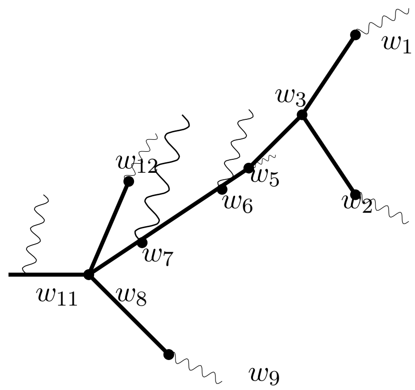

FIG. 10: The tree in Fig. 9 after the first pruning operation.

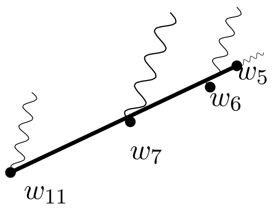

FIG. 11: The tree in Fig. 9 after the second pruning operation.

v in the tree τ, we have that every external field belongs to a couple and at most to two couples, and the paths in $\widehat{T}_v$ connecting the coordinates of the points in the couple are non overlapping. The fields in the couples can have the same x' or different x'. In a couple of fields with the same x' one is surely a D-fields; we then write it as (70) which will produce in the bounds a factor $\gamma^{(h_{v'}-h_v)} \leq \gamma^{-1}$, as $h_{v'} - h_v \leq -1$. On the other hand given w, w' with $x'_w \neq x'_{w'}$, we have $|c_{w,w''}| \geq B\gamma^{-h_{v'}/\tau}$ by lemma 2.2; moreover by Lemma 2.3 we can associate to each $v \in V_\chi$ a factor $c^{N_v 2^{h_{v'}-1}}$ with N_v the vertices in $\bar{T}_v$; as the paths $c_{w,w'}$ are non overlapping, we get one factor $c^{|c_{w,w'}|2^{h_{v'}}} \leq c^{B\gamma^{-h_{v'}/\tau}2^{h_{v'}}} \leq \gamma^{-1}$ for c small enough, for each of the couples. As we can associate a factor $\gamma^{-\frac{1}{2}}$ to each field in a couple, we get at the end a factor $\gamma^{-|m_v|/2} \leq \gamma^{-|P_v|/8}$.

The result of the above operations is the following representation for the effective potential (for more details on how derive such representation in a similar case see for instance §3.3 of [35])

$$V^{(h)} = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{h,n}} \sum_{T \in \mathbf{T}} \sum_{\mathbf{P} \in \mathcal{P}_\tau} \sum_{\alpha \in A_T} \sum_{x} \int dx_{0,v_0} H_{\tau,\mathbf{P},T,\alpha}(x, x_{0,v_0}) \prod_{f \in P_{v_0}} \partial^{q_\alpha(f)}_{\gamma^h x_0(f)} \psi^{(\leq h)\varepsilon(f)}_{\widehat{\mathbf{x}}'(f),\rho(f)} \tag{71}$$

and

$$H_{\tau,\mathbf{P},T,\alpha}(x, x_{0,v_0}) = K_{\tau,\mathbf{P},T,\alpha} \prod_{v \text{ not e.p.}} \frac{1}{S_v!} \int dP_T(\mathbf{t}) \det \widetilde{G}^{h_v,T_v}_\alpha(\mathbf{t}_v) \tag{72}$$

$$\prod_{l \in T_v} \partial^{q_\alpha(f_l^+)}_{\gamma^{h_v} x_{0,l}} \partial^{q_\alpha(f_l^-)}_{\gamma^{h_v} x_{0,l}} (\gamma^{h_l}(x_{0,l} - y_{0,l}))^{b_\alpha(l)} \bar{g}^{(h_v)}_{\rho_l}(x'_l; x_{0,l} - y_{0,l}))\Big|\Bigg]$$

where $\mathbf{T}$ is the set of the tree graphs on $\mathbf{x}_{v_0}$, obtained by putting together an anchored tree graph T_v for each non trivial vertex v, A_T is a set of indices which allows to distinguish the different terms produced by the non trivial $\mathcal{R}$ operations and the iterative decomposition of the zeros $G^{h_v,T_v}_\alpha(\mathbf{t}_v)$ has elements

$$G^{h_v,T_v}_{\alpha,ij,i'j'} = t_{v,i,i'} \delta_{x_{ij},y_{i'j'}} \partial^{q_\alpha(f^+_{ij})}_{\gamma^{h_v} x_{0ij}} \partial^{q_\alpha(f^-_{ij})}_{\gamma^{h_v} x_{0ij}} g^{(h_v)}(x_{ij}, x_{0,ij} - y_{0,i'j'}) \tag{73}$$

The indices $q_\alpha, b_\alpha \in (0, 2)$ are such that, by construction and for $c < 1$

$$|K_{\tau,\mathbf{P},T,\alpha}| \leq c^{-n} \prod_{v \in H_\chi} \gamma^{2(h_{v'} - h_v)} \gamma^{-\frac{1}{8}|P_v|} \tag{74}$$

The factor $\prod_{v \in H_\chi} \gamma^{2(h_{v'} - h_v)}$ is obtained by the action of $\mathcal{R}$ on the resonant term; the factor $\gamma^{-\frac{1}{2}|P_v|}$ is obtained by the by the action of $\mathcal{R}$ and by Lemma 2.3.

Regarding the flow equation for ν_h we get

$$\nu_{h-1} = \gamma \nu_h + \gamma^{-h} \sum_{n \geq 2} \sum_{\tau \in \mathcal{T}_{h,n}} \sum_{T \in \mathbf{T}} \sum_{\mathbf{P} \in \mathcal{P}_\tau} \sum_{\alpha \in A_T} \int dx_{0,v_0} H_{\tau,\mathbf{P},T}(0, x_{0,v_0}) \tag{75}$$

where by construction on the first vertex of the trees v_0 the $\mathcal{L}$ operation acts and $v_0 \in V_\chi$; a similar expression holds for the $\zeta_{h,\rho}, \alpha_{h,\rho}$.

G. Bounds for the effective potential

In this section we get a bound for the kernels of the effective potential defined in (71).

Lemma 2.5 *If* $n = n_\nu + n_U + n_\varepsilon + n_\alpha + n_\zeta$ *the following bound holds*

$$\frac{1}{\beta L}\sum_{\tau\in\mathcal{T}_{h,n}}\sum_{T\in\mathbf{T}}\sum_{\mathbf{P}\in\mathbf{P}_\tau}\sum_{x}\int dx_{0,v_0}|H_{\tau,\mathbf{P},T,\alpha}(x,x_{0,v_0})| \leq$$

$$C^n\gamma^{h_{v_0}}(\sup_{k\geq h}|\nu_k|)^{n_\nu}(\sup_{x',\rho,k\geq h}|\zeta_{k,\rho}|)^{n_\zeta}(\sup_{x',\rho,k\geq h}|\alpha_{k,\rho}|)^{n_\alpha}|U|^{n_U}|\varepsilon|^{n_\varepsilon} \tag{76}$$

where C *is a suitable constant.*

Proof We start from (72) and, in order to bound the matrix $\widetilde{G}^{h,T}_{ij,i'j'}$, we introduce an Hilbert space $\mathcal{H} = \ell^2\otimes\mathbb{R}^s\otimes L^2(\mathbb{R}^1)$ so that

$$\widetilde{G}^{h,T}_{ij,i'j'} = \Big(\mathbf{v}_{x_{ij}}\otimes\mathbf{u}_i\otimes A(x_{0,ij}-,x_{ij})\ ,\ \mathbf{v}_{y_{i',j'}}\otimes\mathbf{u}_{i'}\otimes B(y_{0,i'j'}-,x_{ij})\Big)\ , \tag{77}$$

where $\mathbf{v}\in\mathbb{R}^L$ are unit vectors such that $(\mathbf{v}_i,\mathbf{v}_j)=\delta_{ij}$, $\mathbf{u}\in\mathbb{R}^s$ are unit vectors $(u_i,u_i)=t_{ii'}$, and A,B are vectors in the Hilbert space with scalar product

$$(A,B) = \int dz_0 A(x',x_0-z_0)B^*(x',z_0-y_0) \tag{78}$$

given by

$$A(x',x_0-z_0) = \frac{1}{\beta}\sum_{k_0}e^{-ik_0(x_0-z_0)}\sqrt{f_h(\omega x',k_0)}$$

$$B(x',y_0-z_0) = \frac{1}{\beta}\sum_{k_0}\frac{e^{-ik_0(y_0-z_0)}\sqrt{f_h(\omega x',k_0)}}{-ik_0+\cos 2\pi(\omega x'+\bar{x}_\rho+\theta)-\cos 2\pi(\bar{x}_\rho+\theta)}$$

Moreover

$$||A_h||^2 = \int dz_0|A_h(x',z_0)|^2 \leq C\gamma^h\ , \qquad ||B_h||^2 \leq C\gamma^{-h}\ , \tag{79}$$

for a suitable constant C. Therefore by Gram-Hadamard indequality we get:

$$|\det\widetilde{G}^{h_v,T_v}(\mathbf{t}_v)| \leq C^{\sum_{i=1}^{S_v}|P_{v_i}|-|P_v|-2(S_v-1)}\ . \tag{80}$$

Assume first that v_0 is non resonant; by using (64),(67) we get

$$\frac{1}{L\beta}\sum_x\int dx_{0,v_0}|H_{\tau,\mathbf{P},T,\alpha}(x,x_{0,v_0})| \leq \tag{81}$$

$$c^{-n}[\prod_v\frac{1}{S_v!}][\prod_{v\in V_\chi}\gamma^{h_vS_v^L}][\prod_{v\in\bar{L}_\chi}\gamma^{h_{v'}}][\prod_{v\in H_\chi,v\neq v_0}\gamma^{2(h_{v'}-h_v)}]$$

$$[\prod_{v\in V_\chi}\gamma^{-\frac{1}{8}|P_v|}][\prod_{v\in V_\chi}\gamma^{-h_v(S_v^H+S_v^L-1)}][\prod^{*}_{v\in e.p.}\gamma^{h_{v'}}](\sup_{k\geq h}|\nu_k|)^{n_\nu}(\sup_{x',\rho,k\geq h}|\zeta_{k,\rho}|)^{n_\zeta}(\sup_{x',\rho,k\geq h}|\alpha_{k,\rho}|)^{n_\alpha}|U|^{n_U}|\varepsilon|^{n_\varepsilon}$$

where $\prod^{*}_{v\in e.p.}$ is over the ν_h, α_h, ζ_h end-points and by construction in $\prod^{*}_{v\in e.p.}\gamma^{h_{v'}}$ one has $h_{v'} = h_v - 1$. We use that $\prod_{v\in V_\chi}\gamma^{-h_v S^H_v} = \prod_{v\in H_\chi, v\neq v_0}\gamma^{-h_{v'}}\prod^{*}_{v\in e.p.}\gamma^{-h_{v'}}$ and $\prod_{v\in V_\chi}\gamma^{h_v} = \gamma^{h_{v_0}}\prod_{v\in H_\chi, v\neq v_0}\gamma^{h_v}$; therefore

$$[\prod_{v\in V_\chi}\gamma^{-h_v(S^H_v-1)}][\prod_{v\in H_\chi}\gamma^{h_{v'}-h_v}][\prod^{*}_{v\in e.p.}\gamma^{h_{v'}}] \leq \gamma^{h_{v_0}} \tag{82}$$

so that

$$\frac{1}{L\beta}\sum_x\int dx_{v_0}|H_{\tau,\mathbf{P},T,\alpha}(x,\mathbf{x}_{v_0})| \leq \gamma^{h_{v_0}}[\prod_v\frac{1}{S_v!}][\prod_{v\in\bar{L}_\chi}\gamma^{h_{v'}}] \tag{83}$$
$$[\prod_{v\in H_\chi}\gamma^{(h_{v'}-h_v)}][\prod_{v\in V_\chi}\gamma^{-\alpha|P_v|}](\sup_{k\geq h}|\nu_k|)^{n_\nu}(\sup_{x',\rho,k\geq h}|\zeta_{k,\rho}|)^{n_\zeta}(\sup_{x',\rho,k\geq h}|\alpha_{k,\rho}|)^{n_\alpha}|U|^{n_U}|\varepsilon|^{n_\varepsilon}$$

Note that $\sum_{\mathbf{P}}[\prod_{v\in V_\chi}\gamma^{-\frac{1}{8}|P_v|}] \leq C^n$, see for instance §3.7 of [28] for a proof; moreover $\sum_{\mathbf{T}}[\prod_v\frac{1}{S_v!}] \leq C^n$, see Lemma 2.4 of [28]. The sum over the trees τ is done performing the sum of unlabeled trees and the sum over scales. The unlabeled trees can be bounded by 4^n by Caley formula, and the sum over the scales reduces to the sum over h_v, with $v \in V_\chi$, as given a tree with such scales assigned, the others are of course determined. We use that $\prod_{v\in\bar{L}_\chi}\gamma^{h_{v'}} = \prod_{v\in L_\chi}\gamma^{h_{v'}}\prod^{**}_{v\in e.p.}\gamma^{h_{v'}}$ where $\prod^{**}_{v\in e.p.}$ is over the v corresponding to the ε, U end-points; moreover trivially $\prod_{v\in L_\chi}\gamma^{h_{v'}} \leq \prod_{v\in L_\chi}\gamma^{(h_{v'}-h_v)}$. Therefore

$$\sum_{\{h_v\}}][\prod_{v\in\bar{L}_\chi}\gamma^{h_{v'}}][\prod_{v\in H_\chi}\gamma^{(h_{v'}-h_v)}] \leq \sum_{\{h_v\}}][\prod_{v\in V_\chi}\gamma^{(h_{v'}-h_v)}][\prod^{**}_{v\in e.p.}\gamma^{h_{v'}}] \leq C^n \tag{84}$$

where we have summed over the all possible difference of scales (the scale of the root is fixed) and we have bounded by 1 the factor $[\prod^{**}_{v\in e.p.}\gamma^{h_{v'}}]$. A similar boud is obtained if v_0 is resonant, using that $h_{v'_0} \equiv h$ and an extra factor $\gamma^{2(h-h_{v_0})}$ appears in (82). ■

H. The flow of the running coupling constants

The above lemma ensures convergence provided that the running coupling constant v_k remain small for any k; this is obtained by choosing properly the counterterm ν. We can write

$$\nu_{h-1} = \gamma\nu_h + \sum_{n=2}^{\infty}\beta^{\nu}_{h,n} \quad \alpha_{h-1,\rho} = a_{h,\rho} + \sum_{n=2}^{\infty}\beta^{\alpha}_{h,n} \quad \zeta_{h-1,\rho} = \zeta_{h-1} + \sum_{n=2}^{\infty}\beta^{\zeta}_{h,n} \tag{85}$$

Lemma 2.6 *If $v = \nu, \alpha, \zeta$ and $(\omega x')$ in the support of χ_h*

$$|\beta^v_{h,n}| \leq C^n \gamma^{\frac{h}{2}} (\sup_{k\geq h} |\nu_k|)^{n_\nu} (\sup_{x',\rho,k\geq h} |\zeta_{k,\rho}|)^{n_\zeta} (\sup_{x',\rho,k\geq h} |\alpha_{k,\rho}|)^{n_\alpha} |U|^{n_U} |\varepsilon|^{n_\varepsilon} \tag{86}$$

Proof By (75)

$$\gamma^h \beta^\nu_{h,n} = \sum_{\tau\in\mathcal{T}_{h,n}} \sum_{T\in\mathbf{T}} \sum_{\mathbf{P}\in\mathcal{P}_\tau} \sum_{\alpha\in A_T} \int dx_{0,v_0} H_{\tau,\mathbf{P},T}(0, x_{0,v_0}) \tag{87}$$

and $v_0 \in V_\chi$. The r.h.s. of (87) verifies the same bound as the r.h.s. of (76); indeed in v_0 no $\mathcal{R}$ is applied and by definition $h_{v_0} = h$; the same is true for $\beta^\zeta_h, \beta^\alpha_h$. Moreover no contributions from trees with all the endpoints associated to ν_h, α_h, ζ_h are possible; the corresponding graphs are chains, whose value is vanishing by the compact support properties of the propagator. Therefore in the trees giving a nonvanishing contribution there is at least a vertex with scale 0 corresponding to an ε or U end-point so that (84) is replaced by

$$\sum_{\{h_v\}}][\prod_{v\in\bar{L}_\chi} \gamma^{h_{v'}}][\prod_{v\in H_\chi} \gamma^{(h_{v'}-h_v)}] \leq \gamma^{\frac{h}{2}} \sum_{\{h_v\}}][\prod_{v\in V_\chi} \gamma^{(h_{v'}-h_v)/2}][\prod_{v\in e.p.}^{**} \gamma^{h_{v'}/2}] \leq C^n \gamma^{\frac{h}{2}} \tag{88}$$

■

It remains to prove that we can choose ν so that the running coupling constants are bounded uniformly in h. First we write, for $h \leq -1$, if $\beta^\nu_k = \sum_{n=2}^\infty \beta^\nu_{k,n}$

$$\nu_h = \gamma^{-h}(\nu_0 + \sum_{k=h+1}^{0} \gamma^{k-1} \beta^\nu_k) \tag{89}$$

Lemma 2.7 *There exists ν_0 such that*

$$\sup_k |\nu_k| + \sup_{x',\rho,k} |\zeta_{k,\rho}| + \sup_{x',\rho,} |\alpha_{k,\rho}| \leq C \max(|\varepsilon|, |U|) \tag{90}$$

for a suitable constant C.

Proof. In order to fix $\nu_{-\infty} = 0$ we choose

$$\nu_0 = -\sum_{k=-\infty}^{0} \gamma^{k-1} \beta^\nu_k \tag{91}$$

so that

$$\nu_h = -\sum_{k=-\infty}^{h} \gamma^{k-h-1} \beta^\nu_k \tag{92}$$

We consider the space $\mathcal{M}$ of sequences $\underline{\nu}$ such that $|\nu_h| \leq C\max(|\varepsilon|,|U|)$; we shall think to $\mathcal{M}$ as a Banach space with norm $||\underline{\nu}|| = \sup_{k\leq 0}|\nu_k|$. We look for a fixed point of the operator $T:\mathcal{M}\to\mathcal{M}$ defined as

$$(T\nu)_h = -\sum_{k=-\infty}^{h}\gamma^{k-h-1}\beta^{\nu}_k(\underline{\nu}) \tag{93}$$

By using (86) we see that T leaves $\mathcal{M}$ invariant; moreover

$$|\beta^{\nu}_k(\underline{\nu})-\beta^{\nu}_k(\underline{\nu}')| \leq C(\max(|\varepsilon|,|U|))\gamma^{\frac{h}{2}}||\underline{\nu}-\underline{\nu}'|| \tag{94}$$

as $\beta^{(h)}_n$ is vanishing if $n_\varepsilon = n_U = 0$. Therefore a unique fixed point for T exists. Finally with the above choice for ν ones has, from (86)

$$\begin{aligned}|\alpha_{h,\rho}| &\leq \sum_{k=h}^{0}|\beta^{\alpha}_k| \leq \sum_{k=h}^{0}C(\max(|\varepsilon|,|U|))\gamma^{\frac{h}{2}} \leq C_1\max(|\varepsilon|,|U|)\\ |\zeta_{h,\rho}| &\leq \sum_{k=h}^{0}|\beta^{\zeta}_k| \leq \sum_{k=h}^{0}C(\max(|\varepsilon|,|U|))\gamma^{\frac{h}{2}} \leq C_1\max(|\varepsilon|,|U|)\end{aligned} \tag{95}$$

■

By using lemma 2.5 and 2.7 the convergence of the expansion for the kernel of the effective potential follows.

I. The 2-point function

We have finally to get a bound for the two-point function, which can be written as

$$S(\mathbf{x},\mathbf{y}) = \sum_{n=2}^{\infty}H_n(\mathbf{x},\mathbf{y}) \tag{96}$$

where $H_n(\mathbf{x},\mathbf{y})$ is sum over trees with n end-points and any value of h_{v_0}, among which there are 2 special end-points associated to the external lines and $n-2$ are associated normal end-points of type $\varepsilon, U, \nu_h, \alpha_h, \zeta_h$. Note that there is necessarily a path c_{w_1,w_2} in $\widehat{T}_v$ connecting the points w_1, with $\mathbf{x}_{w_1} = \mathbf{x}$ and w_2 with $\mathbf{x}_{w_2} = \mathbf{y}$ such that by (53) $|x-y| \leq |c_{w_1,w_2}|$; moreover $|c_{w_1,w_2}| \leq n$ so that $H_n = 0$ for $n < |x-y|$. No $\mathcal{R}$ operation is applied in v_0 and with respect to the bound to the effective potential (76) there is an extra $\gamma^{-h_{v_0}}$ for the presence of the external lines and of one integral missing due to the fact that the coordinates of the external lines are fixed. The sum over the scales is bounded by $|\bar{h}|$ with

$$\gamma^{-\bar{h}} \leq \max_{k\in 0,n}\max_{\rho=\pm 1}\frac{1}{||\omega(x+k)-\omega\rho\widehat{x}-2\delta_{\rho,-1}\theta||}\leq$$
$$C(1+\min\{|x|,|y|\}+n)^{\tau}\leq C(1+\min\{|x|,|y|\})^{\tau}(1+\frac{n}{1+\min\{|x|,|y|\}})^{\tau} \tag{97}$$

so that in conclusion, using Lemma 2.6 and 2.7

$$|S(\mathbf{x},\mathbf{y})|\leq\sum_{n\geq|x-y|}(\max(|\varepsilon|,|U|))^{n}C^{n}\log[(1+\min\{|x|,|y|\})^{\tau}(1+\frac{n}{1+\min\{|x|,|y|\}})^{\tau}]$$
$$\leq e^{-\frac{\alpha}{2}|\log max(|\varepsilon|,|U|)||x-y|}\log[(1+\min\{|x|,|y|\})^{\tau}] \tag{98}$$

We can get another bound, which is better for large $|x_0-y_0|$; by integrating by parts and using that each derivative carry an extra $\gamma^{-h_{v_0}}$ one gets

$$|S(\mathbf{x},\mathbf{y})|\leq e^{-\frac{\alpha}{2}|\log\max(|\varepsilon|,|U|)||x-y|}\frac{C_N}{1+(\min\{|x|,|y|\}^{-\tau}|x_0-y_0|)^{N}} \tag{99}$$

and combining the above two bounds, Theorem 1.1 follows.

---

[1] P. W. Anderson:*Absence of Diffusion in Certain Random Lattices.* Phys. Rev. 109, 1492 (1958)

[2] J. Froehlich and T. Spencer:*Absence of diffusion in the Anderson tight binding model for large disorder or low energy.* Comm. Math. Phys. 88, 151 (1983)

[3] M. Aizenman and S. Molchanov: *Localization at large disorder and at extreme energies: an elementary derivation.* Comm. Math. Phys. 157, 245 (1993)

[4] S. Aubry and G. André: *Analyticity breaking and Anderson localization in incommensurate lattices* . Ann. Israel Phys. Soc 3, 133 (1980).

[5] Ya. Sinai: *Anderson Localization for one dimensional difference Schroedinger operator with quasiperiodic potential.* J. Stat. Phys. 46, 861 (1987)

[6] J. Froehlich, T. Spencer, T. Wittwer: *Localization for a class of one-dimensional quasi-periodic Schrdinger operators.* Comm. Math. Phys.132,1, 5 (1990)

[7] S. Ya. Jitomirskaya: *Metal-insulator transition for the almost Mathieu operator.* Ann. of Math. (2) 150 (1999), no. 3, 11591175.

[8] L.Fleishmann, P.W. Anderson: *Interactions and the Anderson transition.* Phys. Rev B 21, 2366 (1980)

[9] T. Giamarchi, H.J. Schulz:*Localization and interaction in onedimensional quantum fluids.* Europhys. Lett. 3, 1287 (1987)

[10] I. Gornyi, A. Mirlin, A., D. Polyakov: *Interacting electrons in disordered wires.* Phys. Rev. Lett. 9, 206603 (2005)

[11] D.M. Basko, I. Alteiner , B. L. Altshuler: *Metal-insulator transition in a weakly interacting many-electron system with localized single-particle states.* Ann. Phys. 321, 1126 (2006)

[12] V. Oganesyan, D. A. Huse:*Localization of interacting fermions at high temperature.* Phys. Rev. B 75, 155111 (2007)

[13] A. Pal, D.A. Huse: *Many-body localization phase transition.* Phys. Rev. B 82, 174411 (2010)

[14] S. Iyer, V. Oganesyan, G. Refael, D. A. Huse: *Many-body localization in a quasiperiodic system.* Phys. Rev. B 87, 134202 (2013)

[15] S. Goldstein, D. A. Huse, J. L. Lebowitz, R. Tumulka: *Thermal equilibrium of a macroscopic quantum system in a pure state* Phys. Rev. Lett. 115, 100402 (2015)

[16] J. Imbrie: *On Many-Body Localization for Quantum Spin Chains.* arxiv1403.7837

[17] V. Mastropietro: *Localization in the ground state of an interacting quasi-periodic fermionic chain* Comm. Math. Phys. 342, 1, 217-250 (2016)

[18] M Schreiber S. Hodgman P Bordia, H P. Lschen, M. H. Fischer, R. Vosk, E. Altman, U. Schneider, I. Bloch *Observation of many-body localization of interacting fermions in a quasirandom optical lattice* Science Vol. 349, Issue 6250, pp. 842-845 2015

[19] A. Avila, S. Jitomirskaya: *The ten Martin problem.* Ann. of. Math. 170 303 (2009)

[20] E. Dinaburg, E, Y. Sinai:*The one-dimensional Schrdinger equation with a quasiperiodic potential.* Funct. Analysis and its App. 9, 279 (1975)

[21] L. Pastur: *Spectra of random self-adjoint operators.* Russian Mathematical Surveys, 28, 1 (1973)

[22] L.H. Eliasson: *Floquet solutions for the 1 -dimensional quasi-periodic Schroedinger equation.* Comm. Math. Phys 146, 447 (1992)

[23] J. Bellissard, R. Lima, D. Testard: *A metal-insulator transition for the almost Mathieu model.* Comm. Math. Phys. 88, 207 (1983)

[24] J.,Moser, J.,Poschel: *An extension of a result by Dinaburg and Sinai on quasi-periodic potentials.*:" Comment. Math. Helv. 59, 3985 (1984)

[25] G.Benfatto, G. Gentile, V.Mastropietro: *Electrons in a lattice with an incommensurate po-*

tential. J. Stat. Phys. 89, 655 (1997)

[26] V.Mastropietro: *Small denominators and anomalous behaviour in the incommensurate Hubbard-Holstein model.* Comm. Math. Phys. 201, 81 (1999)

[27] G. Gentile, V.Mastropietro. *Anderson Localization for the Holstein Model.* Comm. Math.Phys. 215, 69 (2000)

[28] V.Mastropietro *Constructive Renormalization* World Scientific 2006

[29] G. Gallavotti: *Twistless KAM tori.* Comm.Math. Phys 164, 1, 145 (1994)

[30] G.Gentile, V.Mastropietro: *Methods for the analysis of the Lindstedt series for KAM tori and renormalizability in classical mechanics. A review with some applications.* Rev. Math. Phys. 8, 3, 393 (1996).

[31] V. Mastropietro *Localization of interacting fermions in the Aubry-André model* Phys. Rev. Lett. 115, 180401 (2015)

[32] G. Gallavotti: *Renormalization theory and ultraviolet stability for scalar fields via renormalization group methods* Rev. Mod. Phys. 57, 471 (1985)

[33] D. C. Brydges. *A short course on cluster expansions.* In Phenomenes critiques, systemes aleatoires, theories de jauge, (Les Houches, 1984), pages 129183. North-Holland, Amsterdam, 1986.

[34] A. Lesniewski.*Effective action for the Yukawa2 quantum field theory.* Comm. Math. Phys., 108:437467, 1987.

[35] G. Benfatto, V. Mastropietro:*Renormalization group, hidden symmetries and approximate Ward identities in the XYZ model.* Rev. Math. Phys. 13, 1323 (2001)