Reversible viscosity and Navier–Stokes fluids

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Abstract: *Exploring the possibility of describing a fluid flow via a time-reversible equation and its relevance for the fluctuations statistics in stationary turbulent (or laminar) incompressible Navier-Stokes flows.*

I. INTRODUCTION

Studies on non equilibrium statistical mechanics progressed after the introduction of thermostats, [1]. Finite thermostats have not only permitted a new series of simulations of many particle systems, but have been essential to clarify that *irreversibility* and dissipation *should not* be identified.

Adopting the terminology of [2] it is convenient to distinguish the finite system of interest, *i.e.* particles forming the *test system* in a container $\mathcal{C}_0$, from the thermostats. The thermostats $\mathcal{T}_1, \mathcal{T}_2, \ldots$ are also particle systems, forming the *interaction systems*, acting on the test systems: they are in infinite containers and, *asymptotically at infinity*, are always supposed in equilibrium states with given densities $\rho_1, \rho_2, \ldots$ and temperatures $T_1, T_2, \ldots$.

The thermostats particles in each thermostat may interact with each other and with the particles of the test system *but not directly* with the particles of the other thermostats. The test system and the interaction systems, together, form a Hamiltonian system (classical or quantum) that can be symbolically illustrated as in Fig.0:

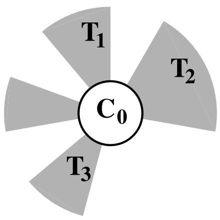

Fig.0: The "*test*" system are particles enclosed in $\mathcal{C}_0$ while the external $\mathcal{C}_j$ systems are thermostats or, following the terminology of Feynman–Vernon, [2], *"interaction"* systems.

Finite thermostats have been introduced recently and fulfill the main function of replacing, [1], the above test systems and "perfect thermostats", consisting of infinite systems of particles in a state in a well defined equilibrium state at infinity, with finite systems suitable for simulations.

The perfect thermostats, being infinite, are not suited in simulations, while the finite ones have the drawback that their equations of motion contain "unphysical forces".

The basic idea is that, asymptotically *e.g.* for large number of particles ("thermodynamic limit"), most statistical properties of the "test" system do not depend on the particular thermostat model but only on its equilibrum parameters defined at infinity.

Several finite thermostats employed in simulations are governed by reversible equations of motion: denoting $u \to S_t u, t \in R$ the time evolution of a point u in phase space $\mathcal{F}$, this means that the map $u \to Iu$ in which all velocities in u are reversed is such that $S_t I = IS_{-t}$, so that if $u(t), t \in R$ is a possible solution of the equations of motion also $I\,u(-t), t \in R$ is a possible one.

If u describes the state of a system in which dissipation occurs, *i.e.* in which external forces perform work on the test subsystem, it might be thought that, unless the interaction systems are infinite, the motion is not reversible: this has been clearly shown to be not true by the many simulations performed since the early '80s, reviewed in [1]. And the simulations have added evidence that the same physical phenomenon occurring in the test system is largely independent of several (appropriate) realizations of thermostat models (reversible or not).

A remarkable instance is an example of a system of particles interacting with a single thermostat at temperature $\beta^{-1} = T$ which has a stationary state described by a probability distribution $\mu(du)$ which is different from the canonical distribution (say) but which is nevertheless equivalent to it in the sense, [3], of the theory of ensembles, *i.e.* in the thermodynamic limit, see [1].

In the different context of turbulence theory a similar example can be found in the simulation in [4]: where viscosity is set $= 0$ but "unphysical forces" are introduced to constrain the energy value on each "energy shell" to fulfill the OK "$\frac{5}{3}$ law". The stationary distribution of the velocity field for many observables, *e.g.* the large scale velocity components, remains the same as in the viscous unconstrained system and in the reversible new one, at very large Reynolds number.

Then one is led to think that the root of the equivalence between very different equations of motion for the same physical system lies in the fundamental microscopic reversibility of the equations of motion, [5, 6], and to a precise formulation of the "conjecture" that "*in microscopically reversible (chaotic) systems time reversal symmetry cannot be spontaneously broken, but only phenomenologically so*" and a program to test it, was proposed, [7]. The program has been followed so far in a few works, [8, 9], with results apparently not always satisfactory [10].

Here, after a general discussion of the conjecture and its precise formulation, several tests will be proposed, on the statistical properties of the stationary states of the $2D$ incompressible Navier-Stokes equation, and performed with results described in some detail.

II. IRREVERSIBLE AND REVERSIBLE ODE'S

More generally an ODE $\dot{x} = h(x)$ on the "phase space" R^N has a *time reversal symmetry* I if the solution operator $x \to S_t x, x \in R^N$, and the map I are such that $I^2 = 1$, $S_t I = IS_{-t}$.

Non trivial examples are provided, as mentioned, by many Hamiltonian equations, but there are also interesting examples not immediately related to Hamiltonian systems, as the equations of the form $\dot{x}_j = f_j(x)$, $\nu > 0$, $j = 1, \ldots, N$ with $f_j(x) = f_j(-x)$, like the Lorenz96 model at $\nu = 0$:

$$\dot{x}_j = x_{j-1}(x_{j+1} - x_{j-2}) + F - \nu x_j, \qquad (2.1)$$

with $F = const$ and periodic b.c. $x_0 = x_N$.

Another example is provided by the GOY*shell model*, [11, 12], given by:

$$\begin{aligned} \dot{u}_n =& -\nu k_n^2 u_n + g\delta_{n,4} \\ &+ik_n\big(-\frac{1}{4}\overline{u}_{n-1}\overline{u}_{n+1} + \overline{u}_{n+1}\overline{u}_{n+2} - \frac{1}{8}\overline{u}_{n-1}\overline{u}_{n-2}\big) \end{aligned} \qquad (2.2)$$

where $k_n = 2^n$, $u_n = u_{n,1} + iu_{n,2}$, with $u_n = 0$ for $n = -1, 0$ or $n > N$, if $\nu = 0$.

A reversible equation often evolves initial data x into functions $x(t)$ which are unbounded as $t \to \infty$. The case of Hamiltonian systems with bounded energy surfaces are an important exception. Therefore, particularly in problems dealing with stationary states in chaotic systems, the equations contain additional terms which arise by taking into account that the systems under study are also subject to stabilizing mechanisms forcing motions to be confined to some sphere in phase space.

A typical additional term is $-\nu x_j$ or $-\nu(Lx)_j$ with $\nu > 0$ and L a positive defined matrix: such extra terms are often introduced empirically. This is the case in the above two examples. And they can be thought as empirical realizations of the action of thermostats acting on the systems.

At this point it is necessary to distinguish the models in which
(1) the equations $\dot{x} = h(x) - \nu Lx$ arise, possibly in some limit case, from a system of particles, as the one of the Feynman-Vernon system in Sec.1, Fig.0, or
(2) the equations are not directly related to a fundamental microscopic description of the system.

The above Lorenz96 and GOY models are examples of the second case, while the Navier-Stokes equations, since the beginning, were considered macroscopic manifestations of particles interacting via Newtonian forces, [13, Eq.(128)].

The success of the simulations using artificial thermostat forces with finite thermostats and the independence of the results from the particular choice of the thermostats used to contain energy growth in nonequilibrium, [1], induces to think that there might be alternative ways to describe the same systems via equations that maintain the time reversal symmetry shown by the non thermostatted equations. A first proposal that seems natural is the following.

Consider an equation

$$\dot{x} = h(x) - \nu Lx, \text{ with } h(x) = h(-x) \qquad (2.3)$$

time reversible if $\nu = 0$, for the time reversal $Ix = -x$; suppose that $|x \cdot h(x)| \le \Gamma(x \cdot Lx)$. Then the motions will be asymptotically confined, if $\nu > 0$, to the ellipsoid $(x \cdot Lx) \le \frac{G}{\nu}$ and the system will be able to reach a stationary state, *i.e.* an invariant probability distribution $\mu^C_{\frac{1}{\nu}}$ of the phase space points. Frequently, if ν is small enough, the motions will be chaotic and there will be a unique stationary distribution, the "SRB distribution", [18].

The family of stationary distributions forms what will be called the "*viscosity ensemble*" $\mathcal{F}^C$ whose elements are parameterized by ν (and possibly by an index distingushing the extremal distributions which can be reached as stationary states, for the same ν, from different initial data); then consider the new equation

$$\dot{x} = h(x) - \alpha(x)Lx, \qquad \alpha(x) = \frac{(Lx \cdot h(x))}{(Lx \cdot Lx)} \qquad (2.4)$$

where α has been determined so that the observable $\mathcal{D}(x) \stackrel{def}{=} (x \cdot Lx)$ is an exact constant of motion. For each choice of the parameter $\mathcal{E}$ the evolution will determine a family $\mu^M_{\mathcal{E}}$ of stationary probability distributions parameterized by the value $\mathcal{E}$ that $\mathcal{D}$ takes on the initial x generating the distribution. The collection $\mathcal{F}^M$ of such distributions will be called "*reversible viscosity ensemble*" because the distributions are stationary states for Eq.(2.4) which is reversible (for $Ix = -x$).

Also in this case if $\mathcal{E}$ is large the evolution Eq.(2.4) is likely to be chaotic and for each such $\mathcal{E}$ the distribution $\mu^M_{\mathcal{E}}$ is unique: *if not* extra parameter needs to be introduced the identify each of the extremal ones.

Suppose for simplicity that $\frac{1}{\nu}, \mathcal{E}$ are large enough and the stationary states $\mu^C_{\frac{1}{\nu}}, \mu^M_{\mathcal{E}}$ are unique. Then say that $\mu^C_{\frac{1}{\nu}}$ and $\mu^M_{\mathcal{E}}$ are *correspondent* if

$$\mu^M_{\mathcal{E}}(\alpha) = \frac{1}{\nu}, \qquad \text{or if} \qquad \mu^C_{\frac{1}{\nu}}(\mathcal{D}) = \mathcal{E} \qquad (2.5)$$

Then the following proposal appears in [5, 6] about the properties of the fluctuations of "K-local observables", *i.e.* of observables $F(x)$ depending only on the coordinates x_i with $i < K$

If $\frac{1}{\nu}$ and $\mathcal{E}$ are large enough so that the motions generated by the equations Eq.(2.3),(2.4) are chaotic, e.g. satisfy the "Chaotic hypothesis", [14, 15], then corresponding distributions $\mu^C_{\frac{1}{\nu}}, \mu^M_{\mathcal{E}}$ give the same distribution to the fluctuations of a given K-local observable F in the sense that

$$\mu^M_{\mathcal{E}}(F) = \mu^C_{\frac{1}{\nu}}(F)(1 + o(F, \nu)) \qquad (2.6)$$

with $o(F, \nu) \xrightarrow[\frac{1}{\nu} \to \infty]{} 0$.

There have been a few attempts to check this idea, [8, 9] and more recently in [16].

III. REVERSIBLE VISCOSITY

The ideas of the preceding section will next be studied in the case of the Navier-Stokes equation. This is particularly interesting becasuse the equation can be formally derived as an equation describing the macroscopic evolution of microscopic Newtonian particles (*i.e.* point massses interacting via a short range force), [13]. Hence the equation belongs to the rather special case (1) in Sec.2.

The incompressible Navier-stokes equations with viscosity ν for a velocity field $\mathbf{v}(\boldsymbol{\xi},t)$ in a periodic container of size L and with a forcing $\mathbf{F}=F\mathbf{g}$ acting on large scale, *i.e.* with Fourier components $\mathbf{F_k}\neq 0$ only for a few $|\mathbf{k}|$. *To fix the ideas* in 2 dimensions choose $F_{\mathbf{k}}\neq 0$ only for the single mode $\mathbf{k}=\pm\frac{2\pi}{L}(2,-1)$ with $||\mathbf{F}||_2=F$ (*i.e.* $\mathbf{g}_{\pm\mathbf{k}}=\frac{e^{\pm i\theta}}{\sqrt{2}}$ for some phase θ).

The equations can be written in dimensionless form: introduce rescaling parameters V,T for velocity and time, and write $\underset{\sim}{\mathbf{v}}(\boldsymbol{\xi},\tau)=V\underset{\sim}{\mathbf{u}}(\boldsymbol{\xi}/L,\tau/T)$. Define $V=(FL)^{\frac12}$, $T=(\frac{L}{F})^{\frac12}$ and fix $\frac{TV}{L}=1$ and $\frac{FT}{V}=1$; then the equation for $\mathbf{u}(\mathbf{x},t)$ can be written as, "I-NS":

$$\dot{\underset{\sim}{\mathbf{u}}}+(\mathbf{u}\cdot\boldsymbol{\partial})\mathbf{u}=\frac{1}{R}\Delta\underset{\sim}{\mathbf{u}}+\mathbf{g}-\boldsymbol{\partial}p\ ,\quad \boldsymbol{\partial}\cdot\mathbf{u}=0 \tag{3.1}$$

where $R\equiv\frac{LV}{\nu}\equiv(\frac{FL^3}{\nu^2})^{\frac12}$ and p is the pressure. In this way the inverse of the viscosity can be identified with the dimensionless parameter R, "Reynolds number".

The units for L,F will be fixed so that $F=1$ and $L=2\pi$: hence the modes $\mathbf{k}$ will be pairs of integers $\mathbf{k}=(k_1,k_2)$. The reality conditions $\mathbf{u_k}=\overline{\mathbf{u}}_{-\mathbf{k}},F_{\mathbf{k}}=\overline{F}_{-\mathbf{k}}$ implies that only the components with

$$\mathbf{k}=(k_1,k_2)\in I^+\stackrel{def}{=}\{k_1>0 \text{ or } k_1=0,k_2\geq 0\} \tag{3.2}$$

are independent components (and it is assumed that $\mathbf{u_0}=0$).

We shall consider the case of 2 dimensional incompressible fluids to avoid the problem that the 3 dimensional equations have not yet been proved to admit a (classical or even just constructive) solution. In spite of this, below, the 3 dimensional case will also be commented and essentially everything that will be presented in the 2 dimensional case *turns out also relevant in* 3 *dimensions*.

Proceding as in sec.2, define the family $\mathcal{F}^C$ of stationary probability distribution $\mu^{\mathbf{C}}_R(d\mathbf{u})$ on the fields $\mathbf{u}$ corresponding to the *stationary state* for the Eq.(3.1).

Consider, *alternatively*, the equation (reversible for the symmetry $I\mathbf{u}=-\mathbf{u}$), "R-NS":

$$\dot{\underset{\sim}{\mathbf{u}}}+(\mathbf{u}\cdot\boldsymbol{\partial})\underset{\sim}{\mathbf{u}}=\alpha(\underset{\sim}{\mathbf{u}})\Delta\underset{\sim}{\mathbf{u}}+\underset{\sim}{\mathbf{F}}-\underset{\sim}{\boldsymbol{\partial}}p\ ,\quad \underset{\sim}{\boldsymbol{\partial}}\cdot\underset{\sim}{\mathbf{u}}=0 \tag{3.3}$$

in which the viscosity $\nu=\frac{1}{R}$, *c.f.r.* Eq.(3.1), is replaced by the multiplier $\alpha(\mathbf{u})$ which is fixed so that

$$\mathcal{D}(\mathbf{u})=\int|\underset{\sim}{\boldsymbol{\partial}}\mathbf{u}(\mathbf{x})|^2d\mathbf{x}=\text{exact const. of motion} \tag{3.4}$$

Therefore, if the space dimension is 2, the multiplier $\alpha(\mathbf{u})$ will be expressed, in terms of the Fourier transform $\mathbf{u_k}$ (defined via $\mathbf{u}(\mathbf{x})=\sum_{\mathbf{k}}e^{2\pi i\mathbf{k}\cdot\mathbf{x}}\mathbf{u_k}$) as:

$$\alpha(\mathbf{u})=\frac{\sum_{\mathbf{k}}\mathbf{k}^2\overline{\mathbf{g}}_{\mathbf{k}}\cdot\mathbf{u_k}}{\sum_{\mathbf{k}}\mathbf{k}^4|\mathbf{u_k}|^2}\equiv\frac{\sum_{\mathbf{k}\in I^+}\mathbf{k}^2(g^r_{\mathbf{k}}\mathbf{u}^r_{\mathbf{k}}+g^i_{\mathbf{k}}\mathbf{u}^i_{\mathbf{k}})}{2\sum_{\mathbf{k}\in I^+}\mathbf{k}^4|\mathbf{u_k}|^2} \tag{3.5}$$

and the stationary distribution for Eq.(3.5) with the value of $\mathcal{D}(\mathbf{u})$ fixed to $\mathcal{E}$, will be denoted $\mu^{\mathbf{M}}_{\mathcal{E}}(d\mathbf{u})$.

The collection of all stationary distributions $\mu^{\mathbf{C}}_R$ as R varies and of all stationary distributions $\mu^{\mathbf{M}}_{\mathcal{E}}$ as $\mathcal{E}$ varies will be denoted $\mathcal{F}^C$ and $\mathcal{F}^M$ and called *viscosity ensemble*, as in sec.2, and, respectively, *enstrophy ensemble.*

Call K-*local* an observable $f(\mathbf{u})$ which depends on the finite number of components $\mathbf{u_k}$ with $|\mathbf{k}|<K$, of the velocity field; then in the above cases the *conjecture* proposed in [5, 6] becomes

In the limit of large Reynolds number the distribution μ^C_R attributes to any given K-local observable $f(\mathbf{u})$ the same average, in the sense of Eq.(2.6) with $R\equiv\frac{1}{\nu}$, as the distribution $\mu^{\mathbf{M}}_{\mathcal{E}}$ if

$$\mathcal{E}=\int\mu^{\mathbf{C}}_R(d\mathbf{u})\mathcal{D}(\mathbf{u}) \tag{3.6}$$

Remarks: (1) The size of R might (of course ?, see however Sec.4) depend on the observable f, *i.e.* on how many Fourier modes are needed to define f.
(2) Therefore locality in Fourier space is here analogous to locality in space in the equivalence between equilibrium ensembles.
(3) The notations $\mu^{\mathbf{C}}_R,\mu^{\mathbf{M}}_{\mathcal{E}}$ have been used to evoke the analogy of the equivalence between canonical and microcanonical ensembles in equilibrium statistical mechanics: the viscosity ensemble can be likened to the canonical ensemble, with the viscosity $\nu=\frac{1}{R}$ corresponding to β, and the enstrophy ensemble to the microcanonical one, with the enstrophy corresponding to the total energy.
(4) The equivalence has roots in the *chaotic hypothesis*, [15]: if the motion is sufficiently chaotic, as expected if R or $\mathcal{E}$ are large, [17, 18], the multiplier $\alpha(\mathbf{u})$ fluctuates in time and the conjecture is based on a possible "*self-averaging*" of α implying homogeneization of $\alpha(\mathbf{u})$ in Eq.(3.3) to a constant value, namely $\nu=\frac{1}{R}$.
(5) the latter remark, if μ^C_R is equivalent to $\mu^M_{\mathcal{E}}$ (*e.g.* if $\mu^C_R(\mathcal{D})=\mathcal{E}$, see Eq.(3.6),(2.5)), leads to expect a relation like:

$$\mu^{\mathbf{M}}_R(\alpha)=\frac{1}{R}(1+o(\frac{1}{R})), \tag{3.7}$$

(6) The property $\alpha(\mathbf{u})=-\alpha(-\mathbf{u})$ implies that the evolution defined by Eq.(3.3) is *time reversible*, so that $\alpha(\mathbf{u})$ can be called "*reversible viscosity*".

IV. REGULARIZATION

In Eq.(2.6),(3.7) the question on how large should R be for equivalence is implicitly raised. An answer, which may become relevant in simulations, that it would be interesting to investigate, is that the equivalence might hold much more generally, at least in the cases (1) in Sec.2 above: therefore for the Navier Stokes equations in dimension 2 (and 3, see below).

The Navier-Stokes equation in $2D$ is known to admit unique evolution of smooth initial data, [19]. The same question has not yet been studied for the reversible viscosity case. In *both cases*, however, simulations impose that the field $\mathbf{u}$ must be represented by a finite number of data, *i.e.* it must be "regularized", to use the language of field theory, [20].

Here the regularization will simply be enforced by considering Eq.(3.1),(3.3) with fields with $\mathbf{u_k} \neq 0$ only if $\mathbf{k} \in I_N \stackrel{def}{=} \{|k_j| \le N\}$. Consequently all statements will depend on the cut-off value N. In particular the conjecture of equivalence will have to be studied also as a function of N and for a fixed local observable.

Pursuing the analogy wih equilibrium statistical mechanics, SM, of a system with energy E, temperature β^{-1} and observables localized in a volume V_0, mentioned above, consider

(a) the cut-off N as analogous to the total volume in SM,
(b) K–local observables (defined before Eq.(2.5)) as analogous to the observables localized in a volume $V_0 = K$ in SM
(c) the enstrophy $\mathcal{D}(\mathbf{u})$ as analogous to the energy in SM

Furthermore the incompressible Navier Stokes equations (as well as the Euler equations or the more general transport equations) can be regarded, if $N = \infty$, as macroscopic versions of the atomic motion: the latter is certainly reversible (if appropriately described together with the external interactions) and essentially always strongly chaotic.

Therefore, for $N = \infty$ and at least for 2 dimensions, no matter whether R is small or large, the equivalence should not only remain valid but could hold in stronger form. Let $\mu^M_{\mathcal{E},N}, \mu^C_{R,N}$ be the stationary distributions for the regularized Navier-Stokes equations, then

Fixed K let F be a K-local observable; suppose that the equivalence condition $\mu^C_{R,N}(\mathcal{D}) = \mathcal{E}$ (or $\mu^M_{\mathcal{E},N}(\alpha) = \frac{1}{R}$) holds, then:

$$\begin{aligned} &(a)\mu^C_R = \lim_{N\to\infty} \mu^C_{R,N},\ \mu^M_{\mathcal{E}} = \lim_{N\to\infty} \mu^M_{\mathcal{E},N} \ \text{exist}\\ &(b)\ \mu^C_R(F) = \mu^M_{\mathcal{E}}(F), \qquad \text{for all} \quad R, \mathcal{E} \end{aligned} \tag{4.1}$$

Remarks: (1) The statement is much closer in spirit to the familiar thermodynamic limit equivalence between canonical and microcanonical ensembles.
(2) Since the basis is that the microscopic motions that generate the Navier-Stokes equations are chaotic and reversible the limit $N \to \infty$ is essential.
(3) The full Navier stokes equations at *low Reynolds number* admit, for the same R, fixed point solutions, periodic solutions or even coexisting chaotic solutions, [21, 22], the condition of equivalence must be interpreted as meaning that when there are several coexisting stationary *ergodic* distributions then there is a one-to-one correspondence between the ones that in the two ensembles $\mathcal{F}^C, \mathcal{F}^M$ obey the equivalence condition and the averages of local observables obey Eq.(4.1).
(4) The possibility of coexisting stationary distributions is analogous to the phase coexistence in equilibrium statistical mechanics (and in that case too the equivalence can hold only in the thermodynamic limit).
(5) It is remarkable that above conjecture *really deals only with the regularized equations*: therefore it makes sense irrespective of whether the non regularized equations dimensionality is 2 or 3.
Of course in the 3–dimensional equation the $\alpha(\mathbf{u})$ has a somewhat different form, [19]; furthermore in the developed turbulence regimes, *in dimension* 3, the picture may become simpler: this is so because of the *natural cut-off due to the OK41 $\frac{5}{3}$-law*: namely $|k_j| \le N = R^{\frac{3}{4}\varepsilon}$, $\varepsilon > 0$, [19].
(6) The equivalence also suggests that there might be even some relation between the "*T-local Lyapunov exponents*" of pairs of equivalent distributions. Here $T-$local exponents are defined via the Jacobian matrix $M_T(\mathbf{u}) = \partial S_T(\mathbf{u})$ and its *RU*-decomposition: they are the averages of the diagonal elements $\lambda_j(\mathbf{u})$ of the R-matrix over T time steps of integration, [23]. Although the "local exponents" cannot be considered to be among the K-local observables it is certainly worth to compare the two spectra.
(7) A suggestion emerges that it would be interesting to study the R-NS equations with $\alpha(\mathbf{u})$ *replaced by a stochastic process* like a white noise centered at $\frac{1}{R}$ with the reversibility taken into account by imposing the width of the fluctuations to be also $\frac{1}{R}$, as required by the fluctuation relation, see below. As R varies stationary states describe a new ensemble which could be equivalent to $\mathcal{E}^C$ in the sense of the conjecture.
(8) A heuristic comment: if the *Chaotic hypothesis*, [15], is assumed for the evolution in the regularized equations the *fluctuation relation*, see below, should also hold, thus yielding a prediction on the large fluctuations of the observable "divergence of the equations of motion" $\sigma_N(\mathbf{u})$ in the distributions $\mu^{\mathbf{M}}_{\mathcal{E},N}$ which, in the 2-dimensional case, is:

$$\sigma_N(\mathbf{u}) = -\frac{\sum_{\mathbf{h}\in I_N} \mathbf{h}^4 \mathrm{Re}(\overline{\mathbf{g}}_\mathbf{h} \cdot \mathbf{u_h}) - 2\alpha E_6}{E_4} - \alpha \sum_{\mathbf{h}\in I_N} \mathbf{h}^2 \tag{4.2}$$

where $I_N \stackrel{def}{=} \{|k_j|\} \le N$, $E_{2m} = \sum_{\mathbf{h}\in I_N} \mathbf{h}^{2m} |\mathbf{u_h}|^2$ which follows, if $g_\mathbf{h} \stackrel{def}{=} g^r_\mathbf{h} + i\, g^i_\mathbf{h}$, from

$$\frac{\partial \alpha}{\partial \mathbf{u}^b_\mathbf{h}} = \frac{\mathbf{h}^2 g^b_\mathbf{h}}{E_4} - 2\alpha \frac{\mathbf{h}^4 \mathbf{u}^b_\mathbf{h}}{E_4}, \quad b = r, i \tag{4.3}$$

Notice that the cut-off N is essential to define $\sigma_N(\mathbf{u})$ as the last (and main) term in Eq.(4.2) would be, otherwise, infinite.

If $\sigma_{N,+}$ is the infinite time average of $\sigma_N(S_t\mathbf{u})$, *i.e.* $\sigma_{+,N} \equiv \int \mu_R^M(d\mathbf{u})\,\sigma_N(\mathbf{u})$ and if $p_\tau(\mathbf{u}) = \frac{1}{\tau}\int_0^\tau \frac{\sigma_N(S_t\mathbf{u})}{\sigma_{N,+}}dt$ then the variable $p_\tau(\mathbf{u})$ satisfies the fluctuation relation in $\mu_{En,N}^M$ if, asymptotically as $\tau \to \infty$,

$$\frac{\mu_{En,N}^{\mathbf{M}}(p_\tau(\mathbf{u}) \sim p)}{\mu_{En,N}^{\mathbf{M}}(p_\tau(\mathbf{u}) \sim -p)} = e^{p\sigma_{N,+}\tau + o(\tau)} \qquad (4.4)$$

The average $\sigma_{N,+}$ becomes infinite in the limit $N \to \infty$: which implies that the probability of $|p-1| > \varepsilon$ tends to 0 (exponentially in N^4, *i.e.* proportionally to $\varepsilon^2 \sum_{|\mathbf{k}|<N} \mathbf{k}^2$) so that the reversible viscosity (proportional to $\alpha \sim \frac{\sigma}{\sigma_+}$) will have probability tending to 0 as $N \to \infty$ (if the large deviation function has a quadratic maximum at $p = 1$ or faster if the maximum is steeper). Large fluctuations of the reversible viscosity away from $\frac{1}{R}$ are still possible if $N < \infty$ but not observable, [24, Eq.(5.6.3)].

Some of the questions raised in the remarks in the above sections will now be analyzed in a series of simulations in the next Appendix. They are very preliminary tests and are meant just to propose tests to realize in the future to test validity, dependence/stability of the results as N, R vary. Source-codes (in progress) available on request.

V. APPENDIX: REVERSIBLE VISCOSITY AND REYNOLDS NUMBER

We first analyze the evolution and distribution of the reversible viscosity $\alpha(\mathbf{u})$ defined in Eq.(3.5) considered as an observable for the evolution Eq.(3.1), *i.e.* for the *irreversible NS2D evolution.*

Consider the NS2D with regularization $(2N+1) \times (2N+1)$. For $N = 3$ a simulation gives the running

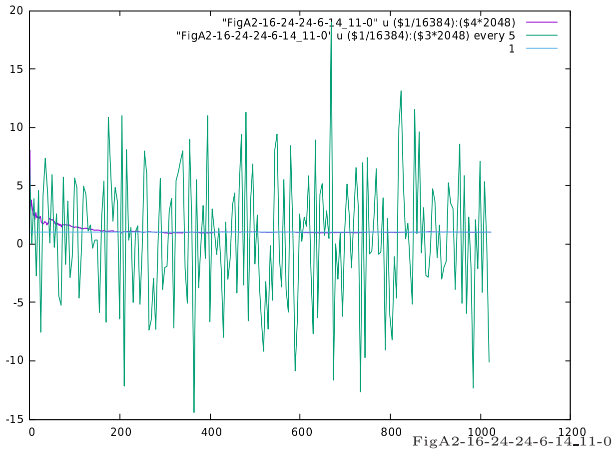

Fig.1: the modes are in the 7×7 box centered at the origin, corresponding to a cut-off $N = 3$; the Reynolds number is $R = 2^{11}$; the time step is 2^{-14} and the time axis is in units of 2^{14} (*i.e.* the evolution history is obtained via 2^{24} time steps).

average of the value of $R\alpha(\mathbf{u})$ (drawn every 5 data to avoid a too dense a figure), the actual fluctuating values of $R\alpha(\mathbf{u})$ and the straight line at quota 1. It shows that $R\alpha(\mathbf{u})$ fluctuates strongly, yet $R\alpha(\mathbf{u})$ averages to a value close ($\sim 2\%$) to 1, *i.e.* $\alpha(\mathbf{u})$ averages to the viscosity value: the analogy, mentioned earlier in Eq.(3.7), with equilibrium thermodynamics would suggest checking that at large R, $\mu_{R,N}^C(\alpha) = \frac{1}{R}(1 + o_{R,N})$ with $o_{R,N}$ small. A check is also necessary because $\alpha(\mathbf{u})$ is not a K-local observable.

The same data considered in Fig.1 for $R = 2014$ and 2^{26} integration steps of size 2^{-15} drawn every $10 \cdot 2^{15}$ yield:

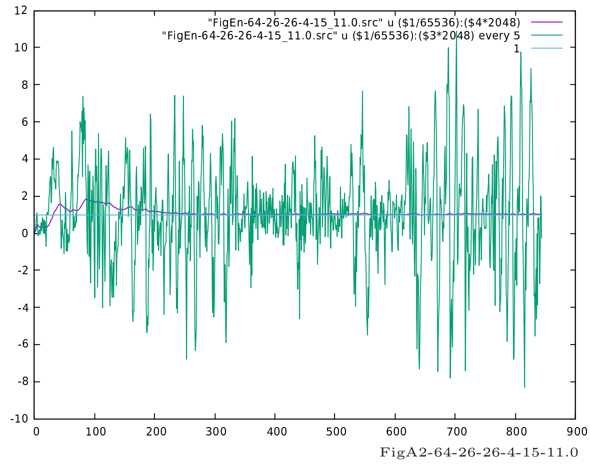

Fig.2: At 960 modes and $R = 2048$: the evolution of the observable "*reversible viscosity*", *i.e.* $\alpha(\mathbf{u})$ in Eq.(3.5) in the I-NS: the time average of α should be $\frac{1}{R}(1+o(\frac{1}{R}))$. Represents the fluctuating values of α every $5 \cdot 2^{16}$ integration steps; the middle line is the running average of α and it *is close* to $\frac{1}{R}$ (horiz. line).

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