Quasi periodic Hamiltonian Motions, Scale Invariance, harmonic Oscillators

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Abstract

The work of Kolmogorov and Moser appeared just before the renormalization group approach to statistical mechanics was proposed by [1]: it can be classified as a multiscale approach which also appeared in works on the convergence of Fourier's series, [2, 3], or construction of Euclidean quantum fields, [4], or the scaling analysis of the short scale behaviour of Navier-Stokes fluids, [5], to name a few which originated a great variety of further problems. Here the proof of the KAM theorem will be presented as a classical renormalization problem with the harmonic oscillator as a "trivial" fixed point.

1 Introduction

The KAM theorem can be regarded as a multiscale analysis of the stability of the harmonic oscillator viewed as a fixed point of a transformation which enlarges a region of phase space focused around a nonresonant quasi periodic motion. The problem considers a Hamiltonian

$$H_0(\mathbf{A},\boldsymbol{\alpha})=\frac12(\mathbf{A},J_0\mathbf{A})+\boldsymbol{\omega}_0\cdot\mathbf{A}+f_0(\mathbf{A},\boldsymbol{\alpha})\equiv h_0+f_0 \tag{1.1}$$

real analytic for $(\mathbf{A},\boldsymbol{\alpha})\in(\mathcal{D}_\varrho\times\mathcal{T}^\ell)$ with: $\mathcal{D}_\varrho=\{\mathbf{A}\in R^\ell,|A_j|<\varrho\}$, $\mathcal{T}^\ell$ the ℓ-dimensional torus $[0,2\pi]^\ell$ identified with unit circle $\{\mathbf{z}|\,z_j=e^{i\alpha_j},j=1,\ldots,\ell\}$, $\boldsymbol{\omega}_0\in R^\ell$ and J_0 could be a $\ell\times\ell$ *non degenerate* symmetric matrix ($\det J_0\ne0$) but here it will be taken just the identity matrix time a constant, to simplify notations.

The Hamiltonian is supposed holomorphic in the region

$$\begin{aligned}\mathcal{C}_{\varrho_0,\kappa_0}&\stackrel{def}{=}\{(\mathbf{A},\mathbf{z})||A_j|\le\varrho_0,\ e^{-\kappa_0}\le|e^{i\alpha_j}|\le e^{\kappa_0},j=1,\ldots,\ell\}\subset\mathcal{C}^{2\ell}\\ \varepsilon_0&=||f_0||_{\varrho_0,\kappa_0}\stackrel{def}{=}\max_{\mathcal{C}_{\varrho_0,\kappa_0}}|f_0(\mathbf{A},\mathbf{z})|\end{aligned}\tag{1.2}$$

with $\varrho_0>0,\kappa_0>0$, $z_j\equiv e^{i\alpha_j}$ and f_0 defined on $\mathcal{D}_{\varrho_0}\times T^\ell$; generally $\mathcal{C}_{\varrho,\kappa}(\overline{\mathbf{A}})$ will denote a polydisk centered at $\overline{\mathbf{A}}$, *i.e.* defined as in Eq.(1.2) with $|A_j-\overline{A}_j|\le\varrho_0$

replacing $|A_j| \le \varrho$ and $e^{-\kappa} \le |z_j| \le e^{\kappa}$; polydisks centered at the "origin" will be simply denoted $\mathcal{C}_{\varrho,\kappa}$ and called "centered polydisks".

It is supposed, no loss of generality, that the $\boldsymbol{\alpha}$-average of $f_0(\mathbf{0}, \boldsymbol{\alpha})$ vanishes.

Set $|\mathbf{A}| = \max |A_j|, |\mathbf{z}| = \max |z_j|,\ \forall \mathbf{A}, \mathbf{z} \in \mathcal{C}^\ell$.

The idea is to focus attention on the center of $\mathcal{C}_{\varrho_0,\kappa_0}$ where, if $\varepsilon_0 = 0$, a motion ("free motion") takes place which is quasi periodic "with spectrum" $\boldsymbol{\omega}_0$. This is done by changing variables in a small centered polydisk and transforming it into a region that is then recentered and essentially enlarged back to the original size so that it contains $\mathcal{C}_{\varrho_0,\frac{1}{2}\kappa_0}$.

The motions developing in the initial polydisk can be studied as "through a microscope": in the good cases (*i.e.* under suitable assumption on the initial Hamiltonian, *i.e.* on $J_0, \boldsymbol{\omega}_0$ and f_0) the Hamiltonian will turn out to be substantially closer to that of a harmonic oscillator (described by its "normal" Hamiltonian $\boldsymbol{\omega}_0 \cdot \mathbf{A}$ in the variables $\mathbf{A}, \boldsymbol{\alpha}$). Iterating the process the Hamiltonian evolves and, *remaining analytic in the same polydisk* $C_{\varrho_0,\frac{1}{2}\kappa_0}$, converges to a harmonic oscillator: the interpretation will be that, looking very carefully in the vicinity of the torus $\mathcal{T}_{\boldsymbol{\omega}_0} = \{\mathbf{A} = \mathbf{0}, \boldsymbol{\alpha} \in [0, 2\pi]^\ell\}$, also the perturbed Hamiltonian exhibits a harmonic motion with spectrum $\boldsymbol{\omega}_0$.

This is not only reminiscent of the methods called "renormalization group", RG, in quantum field theory but in this review it will be shown to be just a realization of them, following [6] with adaptation to more recent views on the RG.

2 Formal coordinate change

The Hamiltonian Eq.(1.1), considered as a holomorphic function on a domain $\mathcal{C}_{\varrho_0,\kappa_0}$ (Eq.(1.2)), will be denoted $H_0 = h_0 + f_0$. The label 0 is attached since the beginning because $H_n, f_n, \varrho_n, \kappa_n$ will arise later with $n = 1, 2, \ldots$.

The frequency spectrum $\boldsymbol{\omega}_0$ will be supposed "Diophantine", *i.e.* for all $\mathbf{0} \ne \boldsymbol{\nu} \in \mathcal{Z}^\ell$,

$$|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}|^{-1} < C_0 \, |\boldsymbol{\nu}|^\ell, \quad \forall \boldsymbol{\nu} \ne \mathbf{0} \tag{2.1}$$

and the latter inequality will be repeatedly used to define canonical transformations with generating functions of the form $\Phi(\mathbf{A}', \boldsymbol{\alpha}) + \boldsymbol{\alpha} \cdot \mathbf{a}$:

$$\mathbf{A} = \mathbf{A}' + \boldsymbol{\partial}_{\boldsymbol{\alpha}} \Phi(\mathbf{A}', \boldsymbol{\alpha}) + \mathbf{a}, \quad \boldsymbol{\alpha}' = \boldsymbol{\alpha} + \boldsymbol{\partial}_{\mathbf{A}'} \Phi(\mathbf{A}', \boldsymbol{\alpha}) \tag{2.2}$$

with the function Φ and the shift $\mathbf{a}$ chosen so that in the new coordinates $(\mathbf{A}', \boldsymbol{\alpha}')$ the perturbation is *weaker*, at the price that the new coordinates will cover a (much) smaller domain, inside the $\mathcal{D}_\varrho \times \mathcal{T}^\ell$.

To simplify the notations the functions of $\boldsymbol{\alpha}$ *will always be implicitly regarded as functions of* $z_j = e^{i\alpha_j}$ *whenever referring to their holomorphy properties, and without further comments their arguments will be written as* $\mathbf{z}$ *or* $\boldsymbol{\alpha}$*, as convenient.*

At first the natural choice for Φ, *temporarily forgetting* the determination of the domain of definition of the transformation, would be

$$\Phi(\mathbf{A}',\boldsymbol{\alpha}) = -\sum_{\mathbf{0}\neq\boldsymbol{\nu}\in\mathcal{Z}^\ell} \frac{f_{0,\boldsymbol{\nu}}(\mathbf{A}')}{i(\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}+\mathbf{A}'\cdot J_0\boldsymbol{\nu})} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\alpha}}, \quad \mathbf{a} = -J_0^{-1}\boldsymbol{\partial}\overline{f}_0(\mathbf{a}) \tag{2.3}$$

where $\mathcal{Z}^\ell$ is the lattice of the integers, $f_{0,\boldsymbol{\nu}}(\mathbf{A}) \stackrel{def}{=} \frac{1}{2\pi i}\oint_{|\zeta_j|=1} f_0(\mathbf{A},\boldsymbol{\zeta})\frac{\prod_j \zeta_j^{\nu_j}\,d\zeta_j}{\zeta_j}$ is the Fourier's transform of $f_0(\mathbf{A},\boldsymbol{\alpha})$, and $\overline{f}_0(\mathbf{A}')$ denotes the average of $f_0(\mathbf{A}',\boldsymbol{\alpha})$ over $\boldsymbol{\alpha}$. The symbol $\boldsymbol{\partial}$ with no further labels means gradient with respect to the argument of the function to which it is applied. The shift $\mathbf{a}$ is formally implicitly defined in Eq.(2.3).

Then inserting Eq.(2.2) into H_0 the Hamiltonian is transformed into

$$\begin{aligned} H'(\mathbf{A}',\boldsymbol{\alpha}') &= \frac{1}{2}(\mathbf{A}'\cdot J_0\mathbf{A}') + \boldsymbol{\omega}_0\cdot\mathbf{A}' \\ &+ \Big\langle \frac{1}{2}\mathbf{a}\cdot J_0\mathbf{a} + \boldsymbol{\omega}_0\cdot\mathbf{a} + \overline{f}_0(\mathbf{a})\Big\rangle \\ &+ \Big[f_0(\mathbf{A}'+\mathbf{a},\boldsymbol{\alpha}) - f_0(\mathbf{A}',\boldsymbol{\alpha}) - \overline{f}_{\mathbf{0}}(\mathbf{A}'+\mathbf{a}) + \overline{f}_{\mathbf{0}}(\mathbf{A}')\Big] \\ &+ \Big[(J_0\mathbf{A}'+\boldsymbol{\omega}_0)\cdot\partial_{\boldsymbol{\alpha}}\Phi + f_0(\mathbf{A}',\boldsymbol{\alpha}) - \overline{f}_{\mathbf{0}}(\mathbf{A}')\Big] \\ &+ \Big[\mathbf{A}'\cdot J_0\mathbf{a} + \mathbf{A}'\cdot\boldsymbol{\partial}\overline{f}_0(\mathbf{0})\Big] + \Big[\mathbf{A}'\cdot(\boldsymbol{\partial}\overline{f}_0(\mathbf{a}) - \boldsymbol{\partial}\overline{f}_0(\mathbf{0}))\Big] \\ &+ \Big\{\overline{f}_0(\mathbf{A}'+\mathbf{a}) - \overline{f}_0(\mathbf{a}) - \boldsymbol{\partial}\overline{f}_0(\mathbf{a})\cdot\mathbf{A}'\Big\} \\ &+ \Big(\frac{1}{2}\boldsymbol{\partial}_{\boldsymbol{\alpha}}\Phi\cdot J_0\boldsymbol{\partial}_{\boldsymbol{\alpha}}\Phi + \mathbf{a}\cdot J_0\boldsymbol{\partial}_{\boldsymbol{\alpha}}\Phi + f_0(\mathbf{A}'+\mathbf{a}+\boldsymbol{\partial}_{\boldsymbol{\alpha}}\Phi,\boldsymbol{\alpha}) - f_0(\mathbf{A}'+\mathbf{a},\boldsymbol{\alpha})\Big) \end{aligned} \tag{2.4}$$

where a few terms have been added and subtracted so that:

- (1) the term in angle-shape bracket is a constant,
- (2) the first term in square brackets is inserted for convenience, to simplify the following line
- (3) the second and third terms in square brackets vanish if $\Phi,\mathbf{a}$ are conveniently defined, *i.e.* if defined via Eq.(2.3),
- (4) the fourth term in square brackets and the terms in curly brackets or in parenthesis as well as the term in item (2), are *formally* of higher order in the size ε_0 of f_0 (*i.e.* $O(\varepsilon_0^{\frac{3}{2}})$) provided $|\mathbf{A}'|,|\mathbf{a}| < O(\sqrt{\varepsilon_0})$.

In a domain in which the transformation Eq.(2.2) could be defined, the motions would be described by a simpler Hamiltonian which is still an integrable Hamiltonian plus a $O(\varepsilon_0^{\frac{3}{2}})$ perturbation.

However to make sense of the transformation in Eq.(2.2) it is not only necessary to restrict the variables $(\mathbf{A}',\boldsymbol{\alpha})$ to a smaller domain, to be able to solve the implicit functions problem in Eq.(2.2) (namely expressing $(\mathbf{A},\boldsymbol{\alpha})$ in terms of $(\mathbf{A}',\boldsymbol{\alpha}')$ and viceversa), but also the denominator in Eq.(2.3) will have to be modified to avoid dividing by 0: which will happen, for generic f_0 and for some $\boldsymbol{\nu}$ on a dense set of $\mathbf{A}'\in\mathcal{D}_\varrho$, at least if J_0 is not singular (as it is being supposed). Therefore the map in Eq.(2.2) will now be modified and defined properly after recalling the notion of dimensional estimate.

3 Dimensional estimates

The very nature of the stability of quasi periodic motions is that it is a multiscale problem: like many other problems in analysis, from the almost everywhere convergence of Fourier series of $L_2([0,2\pi])$-functions ([3]), to the study of the possible singularities of the Navier-Stokes problem ([5]), to the convergence of the functional integrals arising in quantum field theory ([7]), to name a few. The *renormalization group* method, [8, 9], unifies the approaches developed to study such problems.

The main feature of the renormalization group applications is their being based on what will be called here "*dimensional estimates*".

Dimensional estimates deal with elementary bounds on holomorphic functions. Let $g(z)$ be any holomorphic function in a closed domain $C\subset\mathcal{C}$ (domain $\Rightarrow$ closure of an open set in the complex plain $\mathcal{C}$). The function g can be bounded, toghether with its Taylor coefficients, in terms of $||g||_C=\max_{z\in C}|g(z)|$, inside the region C_δ consisting of the points in C at distance $\ge\delta$ from the boundary of C:

$$|\partial_z^n g(z)|\le\, n!\,||g||_C\,\delta^{-n},\qquad \forall z\in\mathcal{C}_\delta,\ n\ge0 \tag{3.1}$$

A consequence is that if g is holomorphic in a disk $C_\varrho=\{z|\,|z|\le\varrho\}$ or in an annulus $\Gamma_\kappa=\{z|e^{-\kappa}\le|z|\le e^\kappa\}$ then the following elementary bounds on the derivatives of g or, respectively, the Fourier coefficients g_ν of the function $g(e^{i\alpha})$ hold

$$\begin{aligned}&||\partial_z^n g||_{C_{\varrho'}}\le\, n!\,||g||_{C_\varrho}\,(\varrho-\varrho')^{-n},\qquad\forall n\ge0\\ &|g_{\boldsymbol{\nu}}|\le\,||g||_{\Gamma_\kappa}\,e^{-\kappa|\boldsymbol{\nu}|},\qquad\forall\boldsymbol{\nu}\in\mathcal{Z},\ |\boldsymbol{\nu}|=\sum_{i=1}^{\ell}|\nu_i|\end{aligned} \tag{3.2}$$

Holomorphic functions of ℓ or 2ℓ arguments will be considered, in the following, in domains

$$\begin{aligned}&\mathcal{C}_\varrho=\{\mathbf{A}||A_j|\le\varrho,j=1,..,\ell\},\quad\Gamma_\kappa=\{\mathbf{z}|e^{-\kappa}\le|z_j|\le e^\kappa,j=1,..,\ell\}\\ &\mathcal{C}_{\varrho,\kappa}=\mathcal{C}_\varrho\times\Gamma_\kappa\end{aligned} \tag{3.3}$$

and their maxima will be denoted by appending labels ϱ or κ or ϱ,κ, as appropriate, to the symbol $||g||$.

Hence if $||g||_{\varrho,\kappa}=\varepsilon$ the bounds

$$\begin{aligned} ||g_{\boldsymbol{\nu}}||_{\varrho} &\le \varepsilon\, e^{-\kappa|\boldsymbol{\nu}|}, && \forall \boldsymbol{\nu} \in \mathcal{Z}^{\ell}, \mathbf{A} \in \mathcal{C}_{\varrho'} \\ ||\boldsymbol{\partial}^n_{\mathbf{A}} g_{\boldsymbol{\nu}}||_{\varrho',\kappa} &\le n!\, \varepsilon\, e^{-\kappa|\boldsymbol{\nu}|}(\varrho-\varrho')^{-n}, && \forall \boldsymbol{\nu} \in \mathcal{Z}^{\ell}, \mathbf{A} \in \mathcal{C}_{\varrho'} \end{aligned} \tag{3.4}$$

hold and will be called *dimensional bounds.*

Summarizing: the dimensional bounds say that the n-th derivatives of a function holomorphic in a domain C are bounded at a point z at distance δ from the boundary of C by the maximum of the function in C divided by the n-th power of the distance of z to the boundary ∂C of C times $n!$ ("Cauchy's theorem").

4 A canonical map

The renormalization group generates a map $\mathcal{R}$ whose iterations can be interpreted as successive magnifications zooming on ever smaller regions of phase space in which motions develop closer and closer to the searched quasi periodic motion of spectrum $\boldsymbol{\omega}_0$.

At step $n = 0, 1, \dots$ the motions will be described by a Hamiltonian $H_n + f_n$ which will be the sum of three terms

$$\frac{1}{2}\mathbf{A}\cdot J_n\mathbf{A} + \boldsymbol{\omega}_0\cdot\mathbf{A} + f_n(\mathbf{A},\mathbf{z}), \tag{4.1}$$

see Eq.(1.1). In the renormalization group nomenclature and *under the conditions Eq.(2.1) and* $\det J_0 \neq 0$ the first and third terms would be called "*irrelevant*" and the intermediate (*i.e.* the normal form for the ℓ-dimensional harmonic oscillators Hamiltonian) would be called "*trivial fixed point*": the reason behind the latter names will become clear.

Introducing the parameters $\varepsilon_n, J_n, \varrho_n, C_n, \kappa_n$, characterizing H_n in the same sense in which $\varepsilon_0, J_0, \varrho_0, C_0, \kappa_0$ characterize H_0, it is convenient, for the purpose of a rapid evalutation of several estimates, to keep in mind that the following "dimensionless" quantities,

$$\eta_n = \varepsilon_n C_n \varrho_n^{-1},\ \theta_n = \varepsilon_n J_n^{-1}\varrho_n^{-2},\ e^{\kappa_n} \tag{4.2}$$

will naturally occurr in the dimensional estimates: the latter will, therefore, be expressed as products of selected dimensionless quantities times a suitable factor chosen among the dimensional parameters $\varepsilon_n, \varrho_n, C_n, J_n$.

All bounds will be carefully written so that they will involve only dimensionless constants and, when needed, a factor to fix the dimensions. Furthermore the construction of the sequence H_n will be so designed that

$$C_n \equiv C_0, \varrho_n \equiv \varrho_0, \kappa_n = \kappa_{n-1} - 4\delta_n \tag{4.3}$$

with δ_n defined so that $\kappa_0 \ge \kappa_n \ge \frac{1}{2}\kappa_0$; to fix the ideas δ_n will be fixed as $\delta_n = \kappa_0(n+9)^{-2}$. J_n and f_n will tend to 0, with $J_n \le J_0, \varepsilon_n \le \varepsilon_0$.

The difficulty will be to control that J_n does not tend to zero so fast that θ_n diverges.

It will not be restrictive to suppose

$$\eta_0, \theta_0 < 1, 2^{-1} < e^{\frac{\kappa_0}{2}} < e^{\kappa_n} < e^{\kappa_0} < 2, C_0 \varrho_0 J_0 < 1 \tag{4.4}$$

because the theorem will apply for ε_0 small enough and ϱ_0, κ_0 can be *initially* restricted as needed (if J_n will turn out to be $\le J_0$). Furthermore it is important to keep in mind that the bounds that follow are derived *without any optimization attempt*, yet they will suffice for a complete proof.

To define properly a tranformation inspired by Eq.(2.2) and to eliminate the mentioned possible divisions by 0, *while still keeping H' in Eq.(2.4) formally close to* H_0 as in Sec.2, define $\mathbf{a}$ as $-J_0^{-1}\boldsymbol{\partial}\overline{f}_0(\mathbf{0})$ and replace Φ with its formal first order expansion in J_0:

$$\begin{aligned}\mathbf{a} &= -\, J_0^{-1}\boldsymbol{\partial}\overline{f}_0(\mathbf{0})\\ \Phi_0(\mathbf{A}', \boldsymbol{\alpha}) &= -\sum_{\mathbf{0}\ne\boldsymbol{\nu}\in\mathcal{Z}^\ell} \frac{1}{i\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}}(1 - \frac{J_0\mathbf{A}'\cdot\boldsymbol{\nu}}{\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}}) f_{0,\boldsymbol{\nu}}(\mathbf{A}') e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}}\end{aligned} \tag{4.5}$$

The function Φ_0 is well defined in a polydisk $\mathcal{C}_{\varrho',\kappa'}$, $\varrho' \le \varrho_0, \kappa' < \kappa_0$: this is seen via the general dimensional bounds given in Eq.(3.4) on functions bounded by ε_0 and holomorphic in a domain $\mathcal{C}_{\varrho_0,\kappa_0}$.

Taking into account the Diophantine inequality Eq.(2.1), for $0 \le \delta_0 < \kappa_0$, the definitions Eq.(4.2),(4.3) and the dimensional inequality Eq.(3.4) leads to:

$$\begin{aligned}&||\Phi_0||_{\varrho_0,\kappa_0-\delta_0} \le \varepsilon_0 \sum_{\boldsymbol{\nu}\ne\mathbf{0}} \frac{e^{-\delta_0|\boldsymbol{\nu}|}}{|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}|}(1 + \frac{|J_0|\varrho_0|\boldsymbol{\nu}|}{|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}|}) \le \gamma_1\eta_0\,\varrho_0\delta_0^{-(3\ell+1)}\\ &|\Phi_{0,\boldsymbol{\nu}}(\mathbf{A}')| < \gamma_2\eta_0\,\varrho_0|\boldsymbol{\nu}|^{(2\ell+1)}e^{-\kappa_0|\boldsymbol{\nu}|}, \qquad \forall|\mathbf{A}'| < \varrho_0\end{aligned} \tag{4.6}$$

with γ_1, γ_2 are dimensionless constants.

Then the functions in the *r.h.s* of Eq.(2.2) admit the bounds:

$$\begin{aligned}&||\boldsymbol{\partial}_{\boldsymbol{\alpha}}\Phi_0||_{\frac12\varrho_0,\kappa_0-\delta_0} \le \gamma_3\eta_0\,\varrho_0\delta_0^{-(3\ell+2)},\ ||\boldsymbol{\partial}_{\mathbf{A}'}\Phi_0||_{\frac12\varrho_0,\kappa_0-\delta_0} \le \gamma_4\eta_0\,\delta_0^{-(3\ell+1)}\\ &||\boldsymbol{\partial}^2_{\boldsymbol{\alpha}\mathbf{A}'}\Phi_0||_{\frac12\varrho_0,\kappa_0-\delta_0} \le \gamma_5\eta_0\,\delta_0^{-3(\ell+2)},\ ||\boldsymbol{\partial}^2_{\mathbf{A}'\mathbf{A}'}\Phi_0||_{\frac12\varrho_0,\kappa_0-\delta_0} \le \gamma_6\eta_0\varrho_0^{-1}\delta_0^{-(3\ell+1)}\end{aligned} \tag{4.7}$$

where the derivatives with respect to α_j should be interpreted as $iz_j\partial_{z_j}$ for $z_j = e^{i\alpha_j}$ in the domain $(\mathbf{A}', \boldsymbol{\alpha}) \in C_{\frac12\varrho_0,\kappa_0-\delta_0}$, and a possible choice for the constants γ_i can be found in the appendix.

To define the canonical transformation $(\mathbf{A}', \boldsymbol{\alpha}') \to (\mathbf{A}, \boldsymbol{\alpha})$ the implicit functions in Eq.(2.2) have to be solved. This can be done quite easily if one is willing to define the map only for $(\mathbf{A}', \boldsymbol{\alpha}')$ contained in a small enough domain.

The condition of solubility for $(\mathbf{A}', \boldsymbol{\alpha}') \in \mathcal{C}_{\varrho',\kappa'}$ for $\varrho' = \frac12\varrho_0, \kappa' = \kappa_0 - 2\delta_0$ is prescribed (usual implicit function theorem ("Dini's theorem"), for analytic functions, see for instance Appendix N in [10]) on dimensional grounds simply by

$$||\boldsymbol{\partial}^2_{\mathbf{A}'\boldsymbol{\alpha}}\Phi||_{\frac12\varrho_0,\kappa_0-\delta_0} < \gamma_7\eta_0\,\delta_0^{-(3\ell+2)} < 1 \tag{4.8}$$

where the first inequality is just the bound Eq.(4.7) on the *l.h.s.* with γ_5 modified, by a suitable factor, into a smaller γ_7 as discussed below (see appendix A for more details).

The solution can be obtained by first fixing $\mathbf{A}'\in C_{\frac12\varrho_0}$ so that the second inequality in Eq.(4.8) simply *implies injectivity* of the map $\boldsymbol{\alpha}'=\boldsymbol{\alpha}+\partial_{\mathbf{A}'}\Phi_0(\mathbf{A}',\boldsymbol{\alpha})$ for $\boldsymbol{\alpha}\in C_{\kappa_0-\delta_0}$, for all $\mathbf{A}'$ fixed in $C_{\frac12\varrho_0}$; it implies also $\boldsymbol{\alpha}'\in C_{\kappa_0-\delta_0}$ for $\boldsymbol{\alpha}\in C_{\kappa_0-2\delta_0}$. Therefore, given $\mathbf{A}'\in C_{\frac12\varrho_0}$ and using the injectivity, $\boldsymbol{\alpha}$ can be computed from $\boldsymbol{\alpha}'$ in the form

$$\begin{aligned}
&\boldsymbol{\alpha}=\boldsymbol{\alpha}'+\boldsymbol{\Delta}(\mathbf{A}',\boldsymbol{\alpha}'),\qquad \boldsymbol{\alpha}'\in C_{\kappa_0-2\delta_0},\forall\mathbf{A}'\in C_{\frac12\varrho_0}\\
&\boldsymbol{\Delta}(\mathbf{A}',\boldsymbol{\alpha}')\equiv-\partial_{\mathbf{A}'}\Phi_0(\mathbf{A}',\boldsymbol{\alpha})\\
&||\boldsymbol{\Delta}||_{\frac12\varrho_0,\kappa_0-2\delta_0}<\gamma_4\eta_0\delta_0^{-(3\ell+1)}<\delta_0,\qquad \text{if}\quad \gamma_7\ \text{small enough}
\end{aligned} \tag{4.9}$$

where the second line in Eq.(4.9) is an identity which implies, via Eqs.(4.7),(4.8), the inequalities in the third line.

The second inequality in Eq.(4.8) also insures the injectivity of $\mathbf{A}=\mathbf{A}'+\partial_{\boldsymbol{\alpha}}\Phi_0(\mathbf{A}',\boldsymbol{\alpha})+\mathbf{a}$ for $\mathbf{A}'$ in $C_{\frac12\varrho_0}$, for all $\boldsymbol{\alpha}$ fixed in $C_{\kappa_0-\delta_0}$. Furthermore, for $\theta_0\equiv\varepsilon_0J_0^{-1}\varrho_0^{-2}$ (see Eq.(4.2)), $\mathbf{a}=-J_0^{-1}\boldsymbol{\partial}\overline{f}_0(\mathbf{0})$ is dimensionally bounded by

$$|\mathbf{a}|=||J_0^{-1}\partial_{\mathbf{A}}\overline{f}_0||_{\varrho_0}<\varrho_0\theta_0 \tag{4.10}$$

Having defined $\boldsymbol{\Delta}(\mathbf{A}',\boldsymbol{\alpha}')$ the angles $\boldsymbol{\alpha}$ can be expressed in terms of $\boldsymbol{\alpha}',\mathbf{A}'$; hence it is possible to express, for each $\boldsymbol{\alpha}'\in C_{\kappa_0-2\delta_0}$, $\mathbf{A}$ in terms of $\mathbf{A}'$, $\forall\mathbf{A}'\in C_{\varrho_0\sqrt{\theta_0}}$: simply by substituting the $\boldsymbol{\alpha}$ by $\boldsymbol{\alpha}'+\boldsymbol{\Delta}(\mathbf{A},\boldsymbol{\alpha}')$ and γ_7 is small enough:

$$\begin{aligned}
&\mathbf{A}=\mathbf{a}+\mathbf{A}'+\boldsymbol{\Xi}(\mathbf{A}',\boldsymbol{\alpha}')\\
&\boldsymbol{\Xi}(\mathbf{A}',\boldsymbol{\alpha}')\equiv\partial_{\boldsymbol{\alpha}}\Phi_0\Big(\mathbf{A}',\boldsymbol{\alpha}'+\boldsymbol{\Delta}(\mathbf{A}',\boldsymbol{\alpha}')\Big),\\
&|\mathbf{A}-\mathbf{a}|<\varrho_0\sqrt{\theta_0}+||\boldsymbol{\Xi}||_{\frac12\varrho_0,\kappa_0-2\delta_0}\le\varrho_0\sqrt{\theta_0}+\gamma_3\eta_0\varrho_0\delta_0^{-(3\ell+2)}<\frac18\varrho_0
\end{aligned} \tag{4.11}$$

with the inequalities again obtained via Eqs.(4.7),(4.8), if $\gamma_3\eta_0\delta_0^{-(3\ell+2)}<\frac1{16}$ and $\sqrt{\theta_0}<\frac1{16}$.

For $(\mathbf{A}',\boldsymbol{\alpha}')\in C_{\frac12\varrho_0,\kappa_0-2\delta_0}$ the $(\mathbf{A},\boldsymbol{\alpha})$ will vary inside the original domain and actually in $C_{\frac14\varrho_0,\kappa_0-\delta_0}$.

Collecting all conditions the Eq.(2.2) will be solved in the form

$$\begin{aligned}
\mathbf{A}=&\mathbf{A}'+\boldsymbol{\Xi}(\mathbf{A}',\boldsymbol{\alpha}')+\mathbf{a},\qquad \boldsymbol{\alpha}=\boldsymbol{\alpha}'+\boldsymbol{\Delta}(\mathbf{A}',\boldsymbol{\alpha}')\\
&||\boldsymbol{\Xi}||_{\frac14\varrho_0,\kappa_0-2\delta_0}<\gamma_8\eta_0\,\varrho_0\delta_0^{-(3\ell+2)}\\
&||\boldsymbol{\Delta}||_{\frac14\varrho_0,\kappa_0-2\delta_0}<\gamma_8\eta_0\,\delta_0^{-(3\ell+1)}\\
&|\mathbf{a}|<||J_0^{-1}\partial_{\mathbf{A}}\overline{f}_0||_{\frac14\varrho_0}<\theta_0\varrho_0
\end{aligned} \tag{4.12}$$

defining a map from a polydisk $C_{\frac{1}{8}\varrho_0,\kappa_0-2\delta_0}$ into $C_{\frac{1}{2}\varrho_0,\kappa_0-\delta_0}$. And the conditions imposed in the construction can be all implied by a single condition

$$\gamma_9(\eta_0+\theta_0)\delta_0^{-c_9}<1 \tag{4.13}$$

for γ_9, c_9 large enough, *together with* the initial conditions $\varepsilon_0 C_0\varrho_0^{-1}, \varepsilon_0 J_0^{-1}\varrho_0^{-2}$, $C_0\varrho_0 J_0<1, e^{\kappa_0}<2$ which are not restrictive as mentioned after Eq.(4.2).

The map in Eq.(4.12), *will now be restricted to a smaller polydisk* $(\mathbf{A}',\boldsymbol{\alpha}')\in\mathcal{C}_{\widetilde\varrho,\widetilde\kappa}\subset C_{\frac{1}{4}\varrho_0,\kappa_0-4\delta_0}$ with

$$\widetilde\varrho=\varrho_0\sqrt{\theta_0},\qquad \widetilde\kappa=\kappa_0-\overline\delta_0 \tag{4.14}$$

where $\overline\delta_n=4\delta_n=\kappa_0(n+9)^{-2}$, fixed here so that (as mentioned after Eq.(4.3)) $\sum_{n\ge0}\overline\delta_n<\frac{1}{4}\kappa_0$, and this is possible if ε_0 is small enough to fulfill the conditions imposed so far, *i.e.* Eq.(4.13). The $\widetilde\varrho$ is chosen so small to define properly, in the next section, the renormalization map as a map with an essentially scale invariant domain.

The domain of variability in the initial variables $(\mathbf{A},\boldsymbol{\alpha})$, where the canonical map is defined, will now contain (at least) a small domain of shape close to a polydisk (*eccentric* because of the translation by $\mathbf{a}$ in Eq.(4.12)) inside the initial domain C_{ϱ_0,κ_0} of the Hamoltonian H_0. The small eccentric polydisk is the image of a *centered* polydisk $\mathcal{C}_{\widetilde\varrho,\widetilde\kappa}$ in the new variables $(\mathbf{A}',\boldsymbol{\alpha}')$.

5 Renormalization

The implicit equations for $\boldsymbol{\alpha}$ in terms of $\mathbf{A}',\boldsymbol{\alpha}'$ in Eq.(3.2) are solved in a polydisk $C_{\widetilde\varrho,\widetilde\kappa}$ in Eq.(4.9). The Hamiltonian H_0+f_0 in the new coordinates $\mathbf{A}',\boldsymbol{\alpha}'$ becomes:

$$H'(\mathbf{A}',\boldsymbol{\alpha}')=\frac{1}{2}\mathbf{A}'\cdot J_0\mathbf{A}'+\boldsymbol{\omega}_0\cdot\mathbf{A}'+f',\qquad(\mathbf{A}',\boldsymbol{\alpha}')\in\mathcal{C}_{\widetilde\varrho,\widetilde\kappa} \tag{5.1}$$

in the domain $(\mathbf{A}',\boldsymbol{\alpha}')\in\mathcal{C}_{\widetilde\varrho,\widetilde\kappa}$, Eq.(4.14). The function f' is defined, in the variables $(\mathbf{A}',\boldsymbol{\alpha})$, by Eq.(2.4) deprived of the trivial constant term, which is the first of the terms in brackets (which in this case are of angular type).

1) The third line contribution, in square brackects in Eq.(2.4), *does not vanish: but it carries the key cancellation* that shows that the sum of three terms individually formally $O(\varepsilon_0)$ is in fact of higher order in ε_0 as can be seen via the Fourier's transform of $f_0-\overline f_0=\sum_{\mathbf{0}\ne\boldsymbol{\nu}}f_{0,\boldsymbol{\nu}}e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}}$ which, after a few simplifications, is:

$$(\boldsymbol{\omega}_0+J_0\mathbf{A}')\cdot\boldsymbol{\partial}_{\boldsymbol{\alpha}}\Phi_0+f_0(\mathbf{A}',\boldsymbol{\alpha})-\overline f_0(\mathbf{A}')=\sum_{\mathbf{0}\ne\boldsymbol{\nu}}f_{0,\boldsymbol{\nu}}\frac{(J_0\mathbf{A}'\cdot\boldsymbol{\nu})^2}{(\boldsymbol{\omega}_0\cdot\boldsymbol{\nu})^2}e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}} \tag{5.2}$$

which admits the dimensional bound

$$\gamma_{11}\varepsilon_0(J_0\widetilde{\varrho}C_0)^2\delta_0^{-(3\ell+2)} = \gamma_{11}\varepsilon_0\,(\varepsilon_0 C_0^2 J_0)\,\delta_0^{-(3\ell+2)} \tag{5.3}$$

in the polydisk $\mathcal{C}_{\widetilde{\varrho},\widetilde{\kappa}}$ (where $|\mathbf{A}'| < \varrho_0\sqrt{J_0^{-1}\varrho_0^{-2}\varepsilon_0}$): hence it is formally of second order. This is the first non trivial contribution to the higher order terms

2) The fourth term in square backets is exactly $\mathbf{0}$ by the definition of $\mathbf{a}$, Eq.(2.3).

3) The fifth term, among the terms in brackets in Eq.(2.4), is in square brackets and is bounded, see the last of Eq.(4.12), by

$$\gamma_{11}\varepsilon_0\theta_0 \tag{5.4}$$

yielding a second non trivial contribution to the higher order terms.

4) The second line contribution in Eq.(2.4), as well as the term in curly brackets and the terms in parenthesis (naturally split into a sum of three addends), are also formally of second order and add (altogheter five) further contributions to the higher order terms which are orderly bounded, see also Eq.(4.14), by:

$$\begin{aligned}&\gamma_{12}\varepsilon_0\big(\theta_0\delta_0^{-1} + \varepsilon_0 C_0^2 J_0)\delta_0^{-2(3\ell+2)}\\ &+\ (\varepsilon_0 C_0\varrho_0^{-1})\delta_0^{-(3\ell+1)} + (\varepsilon_0 C_0\varrho_0^{-1})\delta_0^{-(3\ell+2)}\big)\\ &< \gamma_{13}\varepsilon_0^2(C_0\varrho_0^{-1} + C_0^2 J_0)\delta_0^{-2(3\ell+4)} < 2\gamma_{13}\varepsilon_0(\varepsilon_0 C_0\varrho_0^{-1})\delta_0^{-(3\ell+4)}\end{aligned} \tag{5.5}$$

as $C_0\varrho_0 J_0 < 1$.

Adding the bound Eq.(5.3) and the bounds (5.4),(5.5) it is found

$$|f'|_{\frac{1}{8}\varrho,\kappa_0-4\delta_0} < \gamma\varepsilon_0(\varepsilon_0 C_0\varrho_0^{-1})\delta_0^{-c} \tag{5.6}$$

for $\gamma, c > 0$ suitably fixed if $C_0\varrho_0 J_0 < 1$.

At this point the polydisk $C(\widetilde{\varrho},\widetilde{\delta})$, see Eq.(4.10), is scaled back to the original size ϱ_0 by scaling the variables $\mathbf{A}'$ by $\sqrt{\theta_0} = \widetilde{\varrho}/\varrho_0 = (J_0^{-1}\varrho_0^{-2}\varepsilon_0)^{\frac{1}{2}}$, calling $\mathbf{A}_1 = \frac{1}{\sqrt{\theta_0}}\mathbf{A}'$, $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}'$, multiplying the Hamiltonian by $\frac{1}{\sqrt{\theta_0}}$ (to keep the form of the equations of motion) and defining $f_1 = \frac{1}{\sqrt{\theta_0}}f'(\sqrt{\theta_0}\mathbf{A}_1,\boldsymbol{\alpha}_1)$. The result is that in the coordinates $\mathbf{A}_1,\boldsymbol{\alpha}_1$ the motion is Hamiltonian with Hamiltonian H_1, recalling the definitions of the dimensionless quantities in Eq.(4.2):

$$\begin{aligned}&H_1 = \mathbf{A}_1\cdot J_1\mathbf{A}_1 + \boldsymbol{\omega}_0\cdot\mathbf{A}_1 + f_1(\mathbf{A}_1,\boldsymbol{\alpha}_1)\\ &J_1 = \sqrt{\theta_0}J_0, \quad \varrho_1 = \varrho_0, \quad \kappa_1 = \kappa_0 - \overline{\delta}_0\\ &\varepsilon_1 = \gamma\varepsilon_0\big(\varepsilon_0 C_0\varrho_0^{-1}\big)^{\frac{1}{3}}, \qquad \theta_1 = \theta_0^{\frac{1}{2}}\big(\varepsilon_0 C_0\varrho_0^{-1}\big)^{\frac{1}{3}}\end{aligned} \tag{5.7}$$

where γ is a constant and $\overline{\delta}_n = 4\delta_n = \kappa_0(n+9)^{-2}$. The constant $\frac{1}{3}$ is introduced to simplify the bound $\varepsilon_1 = \gamma\varepsilon_0\big(\varepsilon_0 C_0\varrho_0^{-1}\big)^{\frac{1}{2}}\,\log\big(\varepsilon_0 C_0\varrho_0^{-1}\big)\,\overline{\delta}_0^{-c}$ that would naturally arise.

The above transformation of coordinates, which will be denoted $\mathcal{K}_0$, is well defined and holomorphic in the domain $\mathcal{C}_{\varrho_1,\kappa_1}$ *provided* ε_0 *is small enough* so that the conditions imposed during the construction, namely Eq.(4.13), and the ones following it, are satisfied.

Furthermore $\mathcal{K}_0$ is seen from Eq.(4.9) to be close to the identity, after taking the rescaling into account, within $\varepsilon_0^{\frac{1}{2}} C_0 \varrho_0^{-1} \delta_0^{-3\ell-3}$. This means that if η_0 (*i.e.* ε_0) is small enough the map in Eq.(5.7) generates a sequence with ε_n tending to 0 superexponentially while J_n, θ_n, δ_n tend to 0 still very fast but far slower than ε_n so that the condition Eq.(4.13) remains true for all n and therefore the renormalization procedure can be iterated defining a sequence of transformations $\mathcal{K}_n$ under the only initial condition Eq.(4.13) with γ, c large enough.

In the polydisk $\mathcal{C}_{\varrho_n,\kappa_n}$, *which is essentially n-independent as it contains* $\mathcal{C}_{\varrho_0,\frac{1}{2}\kappa_0}$ *for all* $n \geq 0$, the motion becomes closer and closer to the motion of a harmonic oscillator with frequency spectrum $\boldsymbol{\omega}_0$ and in the limit $n \to \infty$ all motions in the polydisk are harmonic with spectrum $\boldsymbol{\omega}_0$. However all of them represent only an infinitesimal vicinity of a single invariant torus.

The transformation of $H_0 \to H_1 \to H_2 \to \ldots$ is a rescaling transformation that renormalizes the Hamiltonian which remains analytic in a *fixed polydisk* $\mathcal{C}_{\varrho,\kappa}$ because in the iteration κ_n remains strictly positive since $\varepsilon_n \xrightarrow[n\to\infty]{} 0$ superexponentially fast: this shows that the J_0 and the perturbation f are, after renormalization, "irrelevant operators" while the harmonic oscillator is a "fixed point": the transformation has the harmonic oscillator as an attractive fixed point.

The composition of the maps $\mathcal{K}_n \mathcal{K}_{n-1} \cdots \mathcal{K}_0$ converges to a limit $\mathcal{K}$ which is analytic in the polydisk $\mathcal{C}_{\varrho,\kappa}$, $\varrho = \varrho_0, \kappa = \frac{1}{2}\kappa_0$: the strong contraction implies that the $\mathcal{K}$ image of $\{\mathbf{A}\} \times \mathcal{T}^\ell$ *does not depend on* $\mathbf{A} \in \mathcal{C}_{\varrho_0}$ and is a torus on which motion is quasi periodic with spectrum $\boldsymbol{\omega}_0$. This is the KAM theorem, [11, 12, 13, 14, 15]

Remarks: (1) *a simpler analysis* (and an instructive warm-up exercise) *can be carried also if* $J_0 = 0$ provided the perturbation depends only on the angles $\boldsymbol{\alpha}$. *The independence of* f_0 *from* $\mathbf{A}$ *has the consequence, in the proof development, that all terms appearing to involve* J_0^{-1} *actually do not arise at all: since the condition* $\det J_0 \neq 0$ *is called "anisochrony condition" or "twist condition" the invariant tori that exist when* f_0 *depends only on* $\boldsymbol{\alpha}$ *are called "twistless", [16].* (2) The estimates in the above analysis are far from otpimal and optimization is desirable

The analysis here is a reformulation of the original proof by Kolmogorov, [11], reproduced in full detail in [15] and ported to a rigorous computation algorithm in [17]. The feature of the approach, common also to Moser's work, [14], is to use canonical maps with fixed small denominators: this avoids dealing with $\mathbf{A}$ dependent divisors appearing in [13, p.105], reproduced in [10]. The renormalization group interpretation has been proposed in [6] still dealing with $\mathbf{A}$-dependent divisors: the approach developed in Sections 4,5 is inspired by the latter development but avoids $\mathbf{A}$-dependent divisors, hence it is close to [11, 14, 15, 17]. The relation between the KAM theorem and the renormalization group has been pointed out, and used in various forms for its proof, in several papers, for instance [18, 19, 6, 20, 21].

The analysis of the singularity at $\varepsilon_0 = 0$, in the case of resonant quasi periodic motions (*i.e.* motions which dwell on lower dimensional tori), can also be pursued via multiscale methods conveniently interpreted as methods of performing the resummations of the perturbative series, which unlike the KAM case, are divergent power series, [22, 23, 24].

A Appendix 10

Remark that the distance of the boundary of the polyannulus Γ_{κ_0} to that of $\Gamma_{\kappa_0-\delta_0}$ is bounded, if $\frac12 \le e^{\pm\kappa_0} \le 2$, below by $\frac12\delta_0$ and above by $2\delta_0$:

$$\begin{aligned}
\gamma_1 =& \max_{\delta_0>0} 2(2\delta_0)^{3\ell+1} \sum_{\mathbf{0}\ne\boldsymbol{\nu}} |\boldsymbol{\nu}|^{2\ell+1} e^{-\delta_0|\boldsymbol{\nu}|}\\
\gamma_3 =& \max_{\delta_0>0} 2(2\delta_0)^{3\ell+1} \sum_{\mathbf{0}\ne\boldsymbol{\nu}} |\boldsymbol{\nu}|^{2\ell+4} e^{-\delta_0|\boldsymbol{\nu}|}\\
\gamma_2 =& 2, \qquad \gamma_4 = 2\gamma_1, \qquad \gamma_5 = 2\gamma_3, \quad \gamma_6 = 4\gamma_1
\end{aligned} \tag{A.1}$$

The injectivity follows by writing $z_j = e^{i\alpha_j}$ and, integrating along the shortest path enclosed in $C_{\kappa_0-\delta_0}$ connecting $\mathbf{z}_1$ and $\mathbf{z}_2$:

$$\begin{aligned}
&|\mathbf{z}'_{1,j} - \mathbf{z}'_{2,j}| = \int_{\mathbf{z}_1}^{\mathbf{z}_2} \sum_j dz_j\, \partial_{z_j} \Big(z_j \exp(i\partial_{A'_j} \Phi_0(\mathbf{A}', \boldsymbol{\alpha}))\Big)\\
&\ge |\mathbf{z}_1 - \mathbf{z}_2| \Big(1 - \ell e^{\gamma_4\eta_0\delta_0^{-3\ell-3}} \gamma_4\eta_0\delta_0^{-3\ell-3} - \frac{\pi\ell}{2}\gamma_5\eta_0\delta_0^{-3\ell-4}\Big)\\
&\ge \frac12 |\mathbf{z}_1 - \mathbf{z}_2|
\end{aligned} \tag{A.2}$$

deduced from the choice $\gamma_7 = 4\ell(\gamma_4 + \frac{\pi}{2}\gamma_5)$ (after taking into account the inequalities Eq.(4.7),(4.8).

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