PROPOSITION LET X BE A UNIFORMLY CONVEX SPACE, $\{x_n\}$ SEQUENCE IN X AND $x \in X$. THEN,
$$x_n \to x \iff \begin{cases} x_n \rightharpoonup x \\ \|x_n\| \to \|x\| \end{cases}$$

REMARKS ① IF X IS A HILBERT SPACE,
$$\|x_n - x\|^2 = \|x\|^2 + \underbrace{\|x_n\|^2}_{\|x\|^2} - 2\underbrace{(x_n, x)}_{\|x\|^2} \longrightarrow \|x\|^2 + \|x\|^2 - 2\|x\|^2 = 0$$

② $X = \ell_1$ IS NON UNIFORMLY CONVEX BUT THE RESULT IS STILL TRUE BECAUSE $x_n \to x \iff x_n \rightharpoonup x$

③ IN GENERAL, THE RESULT IS FALSE FOR NON-UNIFORMLY CONVEX SPACES
$$X = c_0 \qquad x_n = (1, 0, \dots, 0, \underset{\substack{\uparrow \\ n}}{1}, 0, \dots) = e_1 + e_n \rightharpoonup e_1 = (1, 0, \dots) = x$$
$\|x_n\|_\infty = 1 = \|x\|$ BUT $x_n \not\to x$ BECAUSE $\|x_n - x\| = \|e_n\| = 1$.

PROOF ($\Rightarrow$) WE KNOW THAT $x_n \to x \Rightarrow x_n \rightharpoonup x$ AND $x_n \to x \Rightarrow \|x_n\| \to \|x\|$ (NORM IS CONTINUOUS)

($\Leftarrow$) IT IS OBVIOUS IF $x = 0$; OTHERWISE, CONSIDER $y_n := \frac{x_n}{\|x_n\|}$, $y := \frac{x}{\|x\|}$. ($\|x_n\| \neq 0$ FOR n LARGE)
SINCE $\|x_n\| \to \|x\|$, $y_n \rightharpoonup y$. $\frac{y_n + y}{2} \rightharpoonup y$ BY LINEARITY. BY THE WEAK SEMI-CONTINUITY OF $\|\cdot\|$, $\liminf_{n \to \infty} \left\|\frac{y_n + y}{2}\right\| \geq \|y\| = 1$. IF $y_n \not\to y$, THEN
$\|y_n - y\| \geq \varepsilon > 0 \Rightarrow \left\|\frac{y_n + y}{2}\right\| \leq 1 - \delta$, IMPOSSIBLE, SO $y_n \to y$.
$$\|x_n - x\| \leq \left\|x_n - \frac{\|x\|}{\|x_n\|} x_n\right\| + \left\|\frac{\|x\|}{\|x_n\|} x_n - x\right\| = \underbrace{\big|\|x_n\| - \|x\|\big|}_{0} + \|x\| \underbrace{|y_n - y|}_{0}$$

PROP. (HANNER'S INEQUALITIES)
LET $f, g \in L^p(\mu)$ WITH $1 \leq p \leq 2$, THEN
$$\boxed{\|f+g\|^p + \|f-g\|^p \geq (\|f\| + \|g\|)^p + \big|\|f\| - \|g\|\big|^p}$$
IF $p \geq 2$, THEN $\|f+g\|^p + \|f-g\|^p \leq (\|f\| + \|g\|)^p + \big|\|f\| - \|g\|\big|^p$

REMARK ① $p = 2$, WE GET AN EQUALITY WHICH IS THE PARALLELOGRAM'S RULE
$$\|f+g\|^2 + \|f-g\|^2 = (\|f\| + \|g\|)^2 + (\|f\| - \|g\|)^2 = 2\|f\|^2 + 2\|g\|^2$$

... WHICH IS THE PARALLELOGRAM'S RULE

$$\|f+g\|^2+\|f-g\|^2 = (\|f\|+\|g\|)^2+(\|f\|-\|g\|)^2 = 2\|f\|^2+2\|g\|^2$$

(2) $p=1$, $\|f+g\|+\|f-g\| \geq \max\{2\|f\|, 2\|g\|\} \underset{\uparrow}{\geq} \|2f\| = \|f+g+f-g\|$

UP TO EXCHANGE

$\Rightarrow$ HANNER'S INEQUALITY IS THE TRIANGLE INEQUALITY

$\boxed{\text{COROLLARY}}$ L^p SPACES ARE UNIFORMLY CONVEX FOR $1<p<+\infty$.

$\boxed{\text{PROOF}}$ $\underline{p\geq 2}$. TAKE f,g WITH $\|f\|_p=\|g\|_p=1$ AND $\|f-g\|_p\geq\varepsilon$, I WANT $\left\|\frac{f+g}{2}\right\|\leq 1-\delta$

HANNER'S INEQUALITY SAYS $\|f+g\|^p+\|f-g\|^p\leq 2^p \Rightarrow \left\|\frac{f+g}{2}\right\|\leq\left(1-\frac{\|f-g\|^p}{2^p}\right)^{\frac{1}{p}} \leq \left(1-\frac{\varepsilon^p}{2^p}\right)^{\frac{1}{p}} = 1-\delta$

$$\delta := 1-\left(1-\frac{\varepsilon^p}{2^p}\right)^{\frac{1}{p}}.$$

$\underline{p\leq 2}$ APPLY HANNER'S INEQUALITY TO $\frac{f+g}{2}$, $\frac{f-g}{2}$

$$\Rightarrow \|f\|^p+\|g\|^p \geq \left(\left\|\frac{f+g}{2}\right\|+\left\|\frac{f-g}{2}\right\|\right)^p+\left|\left\|\frac{f+g}{2}\right\|-\left\|\frac{f-g}{2}\right\|\right|^p$$

BY CONTRADICTION, I ASSUME $\exists f_n, g_n$ SUCH THAT $\|f_n\|=\|g_n\|=1$, $\|f_n-g_n\|\geq\varepsilon>0$ AND $\left\|\frac{f_n+g_n}{2}\right\|\to 1$ $\Rightarrow$

$$\underbrace{\|f_n\|^p+\|g_n\|^p}_{=2} \geq \underbrace{\left(\left\|\frac{f_n+g_n}{2}\right\|+\left\|\frac{f_n-g_n}{2}\right\|\right)^p+\left|\left\|\frac{f_n+g_n}{2}\right\|-\left\|\frac{f_n-g_n}{2}\right\|\right|^p}_{\downarrow}$$

$$\left(1+\frac{\varepsilon}{2}\right)^p+\left(1-\frac{\varepsilon}{2}\right)^p>2$$

A CONTRADICTION.

PROOF OF HANNER'S INEQUALITY

($p\leq 2$) I CAN ASSUME $\|f\|\geq\|g\|>0$. I NEED TO SHOW:

$$\begin{aligned}
\|f+g\|^p+\|f-g\|^p &\geq (\|f\|+\|g\|)^p+(\|f\|-\|g\|)^p \\
&= (\|f\|+\|g\|)^{p-1}(\|f\|+\|g\|)+(\|f\|-\|g\|)^{p-1}(\|f\|-\|g\|) \\
&= \left((\|f\|+\|g\|)^{p-1}+(\|f\|-\|g\|)^{p-1}\right)\|f\| \\
&\quad+\left((\|f\|+\|g\|)^{p-1}-(\|f\|-\|g\|)^{p-1}\right)\|g\| \\
&= \|f\|^p\left(\left(1+\frac{\|g\|}{\|f\|}\right)^{p-1}+\left(1-\frac{\|g\|}{\|f\|}\right)^{p-1}\right) \\
&\quad+\|f\|^{p-1}\left(\left(1+\|g\|\right)^{p-1}-\left(1-\|g\|\right)^{p-1}\right)\|g\|^p
\end{aligned}$$

$$+\frac{\|f\|^{p-1}}{\|g\|^{p-1}}\left(\left(1+\frac{\|g\|}{\|f\|}\right)^{p-1}-\left(1-\frac{\|g\|}{\|f\|}\right)^{p-1}\right)\|g\|^p$$

$$=\|f\|^p\, a\left(\frac{\|g\|}{\|f\|}\right)+\|g\|^p\, b\left(\frac{\|g\|}{\|f\|}\right)$$

$$a(r)=(1+r)^{p-1}+(1-r)^{p-1} \qquad b(r)=\frac{(1+r)^{p-1}-(1-r)^{p-1}}{r^{p-1}} \qquad r=\frac{\|g\|}{\|f\|}\in(0,1]$$

WE SUFFICE TO PROVE:

$$|f(x)+g(x)|^p+|f(x)-g(x)|^p\geq |f(x)|^p\, a\left(\frac{\|g\|}{\|f\|}\right)+|g(x)|^p\, b\left(\frac{\|g\|}{\|f\|}\right) \quad \text{FOR a.e. } x$$

THAT IS,

$$|A+B|^p+|A-B|^p\geq |A|^p a(r)+|B|^p b(r) \quad \forall r\in(0,1]\ \forall A,B\in\mathbb{R}.$$

STEP I: I CAN ASSUME $|A|\geq|B|$. IN FACT, CONSIDER $F(r)=a(r)-b(r)$

$$F'(r)=a'(r)-b'(r)=(p-1)\left(1+\frac{1}{r^p}\right)\left(\frac{1}{(1+r)^{2-p}}-\frac{1}{(1-r)^{2-p}}\right)\leq 0 \Rightarrow F \text{ DECREASES}$$

$$F(r)\geq F(1)=0$$

$$\Rightarrow 0\leq F(r)=\frac{a(r)|B|^p-b(r)|A|^p-\left(a(r)|A|^p-b(r)|B|^p\right)}{|B|^p-|A|^p}$$

SO, IF $|B|>|A|$, THEN $a(r)|A|^p-b(r)|B|^p\leq a(r)|B|^p-b(r)|A|^p$

THEREFORE I CAN ASSUME $|A|\geq|B|$

STEP II: PROVE IN THE CASE $|A|\geq|B|$. CONSIDER $G(r)=a(r)|A|^p+b(r)|B|^p$

$$G'(r)=\underbrace{(p-1)\left(\frac{1}{(1+r)^{2-p}}-\frac{1}{(1-r)^{2-p}}\right)}_{\leq 0}\left(|A|^p-\frac{|B|^p}{r^p}\right) \quad \text{HAS A MAX IF } r=\frac{|B|}{|A|}$$

$$\Rightarrow G(r)\leq G\left(\frac{|B|}{|A|}\right)=\big||A|+|B|\big|^p+\big||A|-|B|\big|^p=|A+B|^p+|A-B|^p,$$

SO THE INEQ. IS PROVED

$p\geq 2$ THE SAME BUT WITH OPPOSITE INEQUALITIES (G HAS A MIN. IN $\frac{|B|}{|A|}$).

II ESONERO 2019 - EXERCISE 1

LET X,Y BE NORMED SPACES, $Z\subset X$ DENSE AND $\{A_n\}$ A BOUNDED SEQ. IN $\mathcal{L}(X,Y)$

1. SHOW THAT IF $A_n z\to 0\ \forall z\in Z$, THEN $A_n x\to 0\ \forall x\in X$.

↳ FIX $x\in X$, $\forall\varepsilon>0\ \exists z\in Z$ WITH $\|z-x\|<\varepsilon$

$\hookrightarrow$ FIX $x \in X$, $\forall \varepsilon > 0$ $\exists z \in Z$ WITH $\|z - x\| < \varepsilon$,

$$\|A_n x\| \leq \|A_n z\| + \|A_n x - A_n z\| \underset{\substack{\uparrow \\ n \geq N_\varepsilon}}{\leq} \varepsilon + \|A_n\| \left(\|x - z\|\right) \leq \varepsilon \left(1 + \sup_n \|A_n\|\right)$$

$\Rightarrow A_n x \longrightarrow 0$

(2) LET $\{x_n\}$ BE A SEQUENCE IN ℓ_2. SHOW THAT $x_n \longrightarrow 0 \Leftrightarrow$ $\{x_n\}$ IS BOUNDED AND $x_n(k) \longrightarrow 0$ $\forall k \in \mathbb{N}$.

$\hookrightarrow$ ($\Rightarrow$) $x_n \rightharpoonup 0 \Rightarrow \{x_n\}$ BOUNDED BECAUSE OF BANACH-STEINHAUS THEOREM

$\Downarrow$ $x_n(k) = L x_n$ FOR $L: x \to x(k) \in X^*$ $\Rightarrow L x_n \to 0$

($\Leftarrow$) $L_n e_k \longrightarrow 0$ $\forall k \in \mathbb{N}$ FOR $L_n : y \to \sum x_n(k) y(k)$, BY LINEARITY

$L_n y \longrightarrow 0$ FOR $y = c_1 e_1 + \dots + c_k e_k \in \operatorname{span}\{e_n\}$

BUT $\operatorname{span}\{e_n\} =: Z$ IS DENSE IN $X = \ell_2$, $\{L_n\}$ IS BOUNDED BY HYPOTHESES

$\Rightarrow L_n y \longrightarrow 0$ $\forall y \in \ell_2$, THAT IS $\cancel{x_n} \rightharpoonup 0$.

(3) $x(k) : n e_n = \begin{cases} k & k = n \\ 0 & k \neq n \end{cases}$. SAY WHETHER:

(a) $\{x_n\}$ IS BOUNDED? (b) $x_n(k) \longrightarrow 0$ $\forall k$? (c) $x_n \rightharpoonup 0$?