WEAK DERIVATIVES AND SOBOLEV SPACES

$\boxed{\text{DEF}}$ LET $a,b \in \mathbb{R}\cup\{\pm\infty\}$ WITH $a<b$ AND $u \in L^1_{loc}((a,b))$

WE SAY THAT u HAS A <u>WEAK DERIVATIVE</u> IF $\forall \varphi \in C^1_0((a,b))$ ONE HAS $\int_a^b u\varphi' = -\int_a^b g\varphi$ FOR SOME $g \in L^1_{loc}((a,b))$.

WE SAY g IS THE <u>WEAK DERIVATIVE</u> OF u AND WE WRITE $u' = g$

$\boxed{\text{EXAMPLES}}$ (1) IF u IS DERIVABLE THEN IT IS WEAKLY DERIVABLE AND THE WEAK DERIVATIVE IS THE "CLASSICAL" ONE.

(2) $u(x)=|x|$ IS WEAKLY DERIVABLE (ALTHOUGH NOT DERIVABLE ON $\mathbb{R}$)

IN FACT, TAKE $\varphi \in C^1_0(\mathbb{R}) \Rightarrow \int_{-\infty}^{+\infty} u\varphi' = -\int_{-\infty}^{0} x\varphi'(x)dx + \int_0^{+\infty} x\varphi(x)dx =$

$= \left[\cancel{-x\varphi(x)}\right]_{-\infty}^{0} \int_{-\infty}^{0}\varphi(x)dx + \left[\cancel{x\varphi(x)}\right]_0^{+\infty} - \int_0^{+\infty}\varphi(x)dx = -\int_{-\infty}^{+\infty} g\varphi$ WITH

$$g(x) = \begin{cases} 1 & x>0 \\ -1 & x<0 \end{cases} = \text{sign}(x)$$

$g(0)=?$ NOT IMPORTANT, WE SUFFICE g DEFINED A.E.

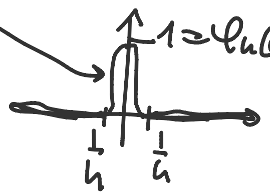
(3) $u(x)=\text{sign}(x)$ DOES <u>NOT</u> HAVE A WEAK DERIVATIVE ON (a,b) IF $a<0<b$: IF $\exists u'$, TAKE $\varphi \in C^1_0((a,b))$

$$\underset{\sim\sim\sim}{-\int_a^b u'\varphi} = \int_a^b u\varphi' = -\int_a^0 \varphi' + \int_0^b \varphi' = -\varphi(0) \cancel{+\varphi(a)} + \cancel{\varphi(b)} - \varphi(0) = \underset{\sim\sim\sim}{-2\varphi(0)}$$

CONTRADICTION.

$$\underbrace{\int_a^b g\varphi_n}_{\downarrow\; 0} = \underbrace{2\varphi_n(0)}_{=\;2}$$

IMPOSSIBLE TAKE φ_n



$\boxed{\text{REMARK}}$ (1) IF $u \equiv v$ A.E, THEN u HAS A WEAK DERIVATIVE $\Leftrightarrow$ v HAS A WEAK DERIVATIVE, IF SO $u' \equiv v'$ A.E.

(2) THE WEAK DERIVATIVE IS UNIQUE: IF $u'=g_1=g_2$, THEN $\forall \varphi\in C_0^1((a,b))$

$\int_a^b u\varphi' = -\int_a^b g_1\varphi = -\int_a^b g_2\varphi \Rightarrow \int_a^b \varphi(g_1-g_2)=0$, TAKE φ APPROXIMATING

$\mathrm{sign}(g_1-g_2) \Rightarrow \int_a^b |g_1-g_2| = 0 \Rightarrow g_1\equiv g_2$ A.E.

(3) BY DENSITY, IN THE DEFINITION I CAN TAKE EQUIVALENTLY $\varphi\in C_0^\infty((a,b))$

(4) THE RULES OF DERIVATION HOLD TRUE: $(f+g)'=f'+g'$

$(fg)'=fg'+f'g$ IF f (OR g) IS $C^1((a,b))$

$\left(H(f)\right)' = H'(f)f'$ IF $H\in C^1(\mathbb{R})$

DEF A FUNCTION $u\in L^1_{loc}((a,b))$ IS <u>ABSOLUTELY CONTINUOUS</u>

IF u IS DERIVABLE A.E. AND $u'\in L^1_{loc}((a,b))$ $u(y)=u(x)+\int_x^y u'$ FOR A.E. $x,y\in(a,b)$

WE DENOTE $u\in AC([a,b])$

EXAMPLE THE CANTOR FUNCTION f IS CONTINUOUS ON $[0,1]$, DERIVABLE A.E. WITH $f'\equiv 0$ BUT IT IS NOT ABSOLUTELY CONTINUOUS BECAUSE $f(1)-f(0)=1$ $\int_0^1 f'=0$

PROP LET $u\in L^1_{loc}(a,b)$. THEN, u HAS A WEAK DERIVATIVE IF AND ONLY IF $\exists v\in AC([a,b])$ SUCH THAT $v\equiv u$ A.E.. IF SO, THEN $u'\equiv v'$ A.E.

LEMMA ASSUME $h\in L^1_{loc}((a,b))$ SATISFIES $\int_a^b h\varphi'=0$ $\forall\varphi\in C_0^1((a,b))$

THEN, h IS CONSTANT A.E.

<u>PROOF</u> FIX $\psi_0\in C_0^1((a,b))$ SUCH THAT $\int_a^b \psi_0=1$. GIVEN $\eta\in C_0^1((a,b))$

DEFINE $\varphi(x)=\int_a^x \eta - \left(\int_a^b \eta\right)\int_a^x \psi_0 \Rightarrow \varphi\in C_0^1((a,b))$

$$\Rightarrow 0=\int_a^b h\varphi' = \int_a^b h\left(\eta-\left(\int_a^b\eta\right)\psi_0\right) = \int_a^b \eta\left(h-\int_a^b h\psi_0\right) \quad \forall\eta$$

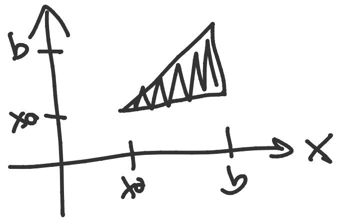
$\Rightarrow h-\int^b h\psi_0\equiv 0 \Rightarrow h\equiv\int^b h\psi_0$ IS CONSTANT

$\Rightarrow h - \int_a^b h\varphi_0 \equiv 0 \Rightarrow h \equiv \int_a^b h\varphi_0$ IS CONSTANT.

PROOF OF THE PROPOSITION

($\Leftarrow$) ASSUME $u \in AC([a,b])$. TAKE $\varphi \in C_c^1((a,b))$ AND $x_0 \in (a,b)$ SUCH THAT $\varphi', \varphi|_{[a,x_0]} \equiv 0$

$$\int_a^b u\varphi' = \int_a^b \varphi'(x)\left(u(x_0) + \int_{x_0}^x u'(t)\,dt\right)dx \overset{\varphi'|_{[a,x_0]}\equiv 0}{=} \cancel{u(x_0)(\varphi(b)-\varphi(a))} + \int_{x_0}^b \varphi'(x)\left(\int_{x_0}^x u'(t)\,dt\right)dx$$



$$\overset{\text{FUBINI}}{=} \int_{x_0}^b u'(t)\left(\int_t^b \varphi'(x)\,dx\right)dt = \int_{x_0}^b u'(t)\left(-\varphi(t)\right)dt \overset{\varphi|_{[a,x_0]}\equiv 0}{=} -\int_a^b u'\varphi$$

($\Rightarrow$) ASSUME u IS WEAKLY DERIVABLE, FIX $x_0 \in (a,b)$ AND DEFINE $v(x) = u(x_0) + \int_{x_0}^x u'$. BY CONSTRUCTION $v \in AC([a,b])$, WE JUST SAW THAT v HAS A WEAK DERIVATIVE u', THAT IS $\int_a^b v\varphi' = -\int_a^b u'\varphi$ $\forall \varphi \in C_c^1((a,b))$. WE ALSO HAVE $\int_a^b u\varphi' = -\int_a^b u'\varphi$, THEREFORE $\int_a^b (u-v)\varphi' = 0 \Rightarrow u-v$ IS CONSTANT A.E., BY CONSTRUCTION $v(x_0) = u(x_0) \Rightarrow u - v \equiv 0 \Rightarrow u \equiv v$ IS ABS. CONTINUOUS.

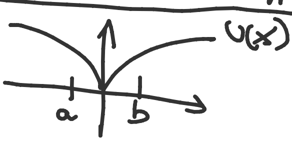
DEF: FOR $p \in [1,+\infty]$ WE DEFINE THE SOBOLEV SPACE $W^{1,p}((a,b))$ AS

$$W^{1,p}((a,b)) = \{u \in L^p((a,b)) : u' \text{ HAS A WEAK DERIVATIVE IN } L^p((a,b))\} \subset L^p((a,b))$$

$$W^{1,p}_{loc}((a,b)) = \{u \in L^p_{loc}((a,b)) : u' \;\;\text{"}\;\;\text{"}\;\;\text{"} \text{ IN } L^p_{loc}((a,b))\}$$

FOR $u \in W^{1,p}((a,b))$ WE DEFINE THE NORM $\|u\|_{W^{1,p}((a,b))} := \|u\|_{L^p((a,b))} + \|u'\|_{L^p((a,b))}$

EXAMPLE: $u(x) = |x|^\alpha$, $0 < \alpha < 1$



u IS WEAKLY DERIVABLE

$u'(x) = \frac{\text{sign}(x)}{|x|^{1-\alpha}} \in L^p((a,b))$ IFF $p < \frac{1}{\;}$ $\Rightarrow u \in W^{1,p}((a,b))$ ...

$u'(x) = \frac{\text{sign}(x)}{|x|^{1-\alpha}} \in L^p((a,b))$ IFF $p < \frac{1}{1-\alpha} \Rightarrow u \in W^{1,p}((a,b))\ \forall p < \frac{1}{1-\alpha}$ IF $a,b \in \mathbb{R}$

$a<0<b$ ... NOT INTEGRABLE

$u(x) = |x| \Rightarrow u \in W^{1,\infty}_{loc}(\mathbb{R})$ $\quad u \in W^{1,p}_{loc}(\mathbb{R})\ \forall p < \frac{1}{1-\alpha}$

REMARKS ① $W^{1,p}((a,b)) \subset W^{1,q}((a,b))$ IF $p \geq q$, $a,b \in \mathbb{R}$, $W^{1,p}_{loc} \subset W^{1,q}_{loc}\ \forall p \geq q$

② AN EQUIVALENT NORM ON $W^{1,p}((a,b))$ IS $\left(\|u\|^p_{L^p((a,b))} + \|u'\|^p_{L^p((a,b))}\right)^{\frac{1}{p}}$

③ $p=2 \Rightarrow \left(\|u\|^2_{L^2} + \|u'\|^2_{L^2}\right)^{\frac{1}{2}}$ IS GIVEN BY A SCALAR PRODUCT

$(u,v)_{W^{1,2}((a,b))} := \int_a^b (uv + u'v')$

PROP $W^{1,p}((a,b))$ IS A BANACH SPACE W.R.T. $\|u\|_{W^{1,p}((a,b))}$

$p=2 \Rightarrow W^{1,2}((a,b))$ IS A HILBERT SPACE

PROOF TAKE $\{u_n\}$ CAUCHY SEQUENCE IN $W^{1,p}((a,b)) \Rightarrow \{u_n\}, \{u_n'\}$ ARE CAUCHY IN $L^p((a,b))$. SINCE L^p IS COMPLETE, $u_n \to u$, $u_n' \to g$ IN $L^p((a,b))$

WE NEED TO SHOW $g = u'$.

$\varphi \in C^1_0((a,b)) \Rightarrow -\int_a^b u_n' \varphi = \int_a^b u_n \varphi' \longrightarrow \int_a^b u \varphi' \Rightarrow u' = g$

$\downarrow$

$-\int_a^b g\varphi$

$\Rightarrow \|u_n - u\| = \|u_n - u\|_{L^p} + \|u_n' - u'\|_{L^p} \to 0$
