LAST TIME: FREDHOLM'S ALTERNATIVE $I - \underset{\hookrightarrow \text{COMPACT}}{A}$

SPECTRAL THEOREM FOR COMPACT OPERATORS

LET A BE A COMPACT OPERATOR ON A COMPLEX BANACH SPACE X

THEN:

① $0 \in \sigma(A)$ IF $\dim X = +\infty$

② $\lambda \in \sigma(A) \setminus \{0\} \Rightarrow \lambda$ IS AN EIGENVALUE

③ $\forall \lambda \in \sigma(A) \setminus \{0\}$ $\exists !$ A COUPLE $K(\lambda), R(\lambda) \triangleleft X$ SUCH THAT $X = K(\lambda) \oplus R(\lambda)$

AND $(A - \lambda I)\big|_{R(\lambda)} : R(\lambda) \to R(\lambda)$ IS INVERTIBLE $\quad (A - \lambda I)\big|_{K(\lambda)} : K(\lambda) \to K(\lambda)$ IS NILPOTENT,

MOREOVER $\dim K(\lambda) < +\infty \quad \dim \dfrac{X}{R(\lambda)} < +\infty$

④ $K(\lambda) \subset \ker(A - \lambda I)$, $R(\lambda) \supset \operatorname{ran}(A - \lambda I)$

⑤ $\sigma(A) \setminus \{0\}$ IS DISCRETE $\Rightarrow \sigma(A)$ IS COUNTABLE (EITHER FINITE OR SEQUENCE TENDING TO 0)

⑥ IF $\lambda, \mu \in \sigma(A) \setminus \{0\}$, $\lambda \neq \mu$ THEN $K(\mu) \subset R(\lambda)$

PROOF ① BY CONTRADICTION, ASSUME A IS INVERTIBLE $\Rightarrow I = \underset{\text{COMPACT}}{A} \circ \underset{\text{CONTINUOUS}}{A^{-1}}$

$\Rightarrow I$ IS A COMPACT OPERATOR, IMPOSSIBLE IF $\dim X = +\infty$

② ASSUME $\lambda \neq 0$ IS NOT AN EIGENVALUE, THAT IS $A - \lambda I$ IS INJECTIVE.

$A - \lambda I = -\lambda \big(I - \underbrace{\tfrac{A}{\lambda}}_{\text{COMPACT}} \big)$ $\quad I - \frac{A}{\lambda}$ IS INJECTIVE. BY FREDHOLM'S ALTERNATIVE, $I - \frac{A}{\lambda}$ IS ALSO SURJECTIVE $\Rightarrow A - \lambda I$ IS SURJECTIVE $\Rightarrow \lambda \notin \sigma(A)$

③ WE APPLY FREDHOLM'S ALTERNATIVE TO $\frac{A}{\lambda}$ AND DEFINE $K(\lambda) = K_N$, $R(\lambda) = R_N$

WE JUST NEED TO SEE THE UNIQUENESS OF DECOMPOSITION.

$X = K' \oplus R'$

λ $K(\lambda)=K_N$

$R(\lambda)=R_N$

WE JUST NEED TO SEE THE UNIQUENESS OF DECOMPOSITION, THAT IS:

$X=K'\oplus R'$

$A-\lambda I: K'\to K'$ NILPOTENT

$A-\lambda I: R'\to R'$ INVERTIBLE

$\Rightarrow K'=K(\lambda)$, $R'=R(\lambda)$

TAKE $\underset{y+z}{\overset{\|}{x}}\in K'$ $\quad y\in K(\lambda)$ $z\in R(\lambda)$

$$\Rightarrow \underset{\substack{\| \\ x\in K' \Rightarrow 0}}{(A-\lambda I)^N x} = \underset{\substack{\| \\ y\in K(\lambda)\Rightarrow 0}}{(A-\lambda I)^N y} + (A-\lambda I)^N z \Rightarrow (A-\lambda I)^N z = 0 \xRightarrow{(A-\lambda I)\text{ INVERTIBLE ON } R(\lambda)} z=0 \Rightarrow x=y\in K(\lambda)$$

$\Rightarrow \underline{K'\subset K(\lambda)}$

SIMILARLY, $R'\ni x = \underset{\in K(\lambda)}{y} + \underset{\in R(\lambda)}{z}$ $\Rightarrow \underset{\substack{\in \\ (A-\lambda I)^N R' = R'}}{(A-\lambda I)^N x} = \underset{\substack{\| \\ 0}}{(A-\lambda I)^N y} + (A-\lambda I)^N z \in R(\lambda)$

$\Rightarrow \underline{R'\subset R(\lambda)}$

EXCHANGING K', R' WITH $K(\lambda), R(\lambda)$ WE GET $K(\lambda)\subset K'$, $R(\lambda)\subset R'$ $\Rightarrow K'=K(\lambda)$, $R'=R(\lambda)$ $\Downarrow$ UNIQUENESS

④ FOLLOWS AFTER FREDHOLM'S ALTERNATIVE, BECAUSE $K(\lambda)=\mathrm{Ker}((A-\lambda I)^N)$ $\quad R(\lambda)=\mathrm{RAN}((A-\lambda I)^N)$

⑤ TAKE $\lambda\in\sigma(A)\setminus\{0\}$, WE WANT TO SHOW $\mu\notin\sigma(A)$ IF μ CLOSE TO λ.

WE SHOWED $(A-\lambda I)K(\lambda)\subset K(\lambda)$, $(A-\lambda I)R(\lambda)\subset R(\lambda)$, SINCE $A-\mu I = A-\lambda I+(\lambda-\mu)I$ WE HAVE $(A-\mu I): K(\lambda)\to K(\lambda)$

$(A-\mu I): R(\lambda)\to R(\lambda)$ ✓

$\lambda\notin\sigma(A|_{R(\lambda)})$ BY ③, SO IF μ IS CLOSE TO λ THEN $\mu\notin\sigma(A|_{R(\lambda)})$.

WE SUFFICE TO SHOW $\mu\notin\sigma(A|_{K(\lambda)})$, THIS IS TRUE $\forall\,\mu\neq\lambda$

$$(a+b)(a^{N-1}-a^{N-2}b+\dots \pm b^{N-1}) = a^N+b^N$$

$$(A-\lambda I+(\lambda-\mu)I)\left((\lambda-\mu)^{N-1}-(A-\lambda I)(\lambda-\mu)^{N-2}-\dots+(-1)^{N-1}(A-\lambda I)^{N-1}\right) = (\lambda-\mu)^N I + \underset{\substack{\| \\ 0 \text{ ON } K(\lambda)}}{(A-\lambda I)^N}$$

$$\Rightarrow (A-\mu I)\,\frac{\left((\lambda-\mu)^{N-1}-\dots\right)}{(\lambda-\mu)^N} = I$$ ON $K(\lambda)$ $\Rightarrow$ $A-\mu I$ INVERTIBLE ALSO ON $K(\lambda)$.

⑥ TAKE $x\in K(\mu)$, $x=\underset{\substack{\cap \\ K(\lambda)}}{y}+\underset{\substack{\cap \\ R(\lambda)}}{z}$. AS BEFORE, $(A-\mu I)^N: K(\lambda)\to K(\lambda)$

$(A-\mu I)^N: R(\lambda)\to R(\lambda)$

$(A-\mu I)^N x = (A-\mu I)^N y + (A-\mu I)^N z$ $\quad$ SINCE THE DEC

$$\underset{\substack{\| \\ 0}}{(A-\mu I)^N x} = \underset{\in K(\mu)}{(A-\mu I)^N y} + \underset{\in R(\mu)}{(A-\mu I)^N z}$$ $x \in K(\mu)$, SINCE THE DECOMPOSITION IS UNIQUE, $(A-\mu I)^N y = 0$, $(A-\mu I)^N z = 0$

$(A-\mu I)^N$ IS INVERTIBLE, AS IN ⑤ $\Rightarrow y = 0$ $x = z \in R(\mu) \Rightarrow K(\mu) \subset R(\mu)$.

EXAMPLES

① $A: \ell_2 \to \ell_2$ $a_n \xrightarrow[n\to\infty]{} 0$ A IS COMPACT

$(x(1), x(2), x(3) ..) \to (a_1 x(1), a_2 x(2), a_3 x(3), ...)$ $a_n = \frac{1}{n}$

$\lambda = 0 \in \sigma(A)$

- $0 \in \underline{\sigma_c(A)}$ BECAUSE ran(A) IS DENSE ($c_{00} \subset$ ran(A)) IF $a_n \neq 0$
- $0 \in \underline{\sigma_p(A)}$ IF $a_n = 0$ FOR SOME k. ($Ae_n = 0$)

$\left.\begin{matrix}\lambda \neq 0 \\ \lambda \in \sigma(A)\end{matrix}\right\} \Rightarrow \lambda$ IS EIGENVALUE. LET US FIND λ: $Ax = \lambda x \Leftrightarrow$ $\lambda x(1) = a_1 x(1)$, $\lambda x(2) = a_2 x(2)$, - - - . NONTRIVIAL SOLUTIONS $\Leftrightarrow \lambda = a_n$ FOR SOME k $Ae_k = \lambda k$

$\Rightarrow \sigma(A) = \{a_1, a_2, a_3 ...., 0\}$ IF $a_n \equiv 0$ FOR $k \geq K_0$ THEN $\sigma(A)$ IS FINITE.

② $B: \ell_2 \to \ell_2$ $B =$ RIGHT SHIFT $\circ A$ $\xrightarrow{}$ COMPACT, RIGHT SHIFT $\to$ CONTINUOUS $\Rightarrow$ B IS COMPACT.

$(x(1) x(2), x(3) ..) \to (0, a_1 x(1) a_2 x(2) ...)$

LET US FIND EIGENVALUES: $Bx = \lambda x \Leftrightarrow$ $\lambda x(1) = 0 \to x(1) = 0$; $\lambda x(2) = a_1 x(1) \to x(2) = \frac{a_1}{\lambda} x_1 = 0$; $\lambda x(3) = a_2 x(2) \to x(3) = 0$ ..

NO EIGENVALUES! $\Rightarrow \sigma(B) = \{0\}$

$0 \in \underline{\sigma_r(A)}$ BECAUSE ran(A) $\subset e_1^\perp$ CANNOT BE DENSE.

③ $A: C([0,1]) \to C([0,1])$ IS COMPACT BECAUSE OF ASCOLI-ARZELÀ THEOREM

$f(x) \to \int_0^x f$

TAKE f_n BOUNDED, WE WANT TO SHOW Af_n IS <u>BOUNDED</u> AND EQUICONTINUOUS.

$\|f_n\|_{L^\infty} \leq C \Rightarrow \|Af_n\|_\infty \leq C$ BOUNDED.

$|x-y| < \delta \Rightarrow |Af_n(x) - Af_n(y)| = |\int_x^y f| \leq C|x-y| \leq C\delta =: \varepsilon \Rightarrow$ EQUICONTINUOUS

LET US FIND EIGENVALUES: $Af = \lambda f \Leftrightarrow \int_0^x f = \lambda f(x) \quad \forall x \in [0,1]$

$\Downarrow$ DERIVE BOTH SIDES

SINCE ~~THERE~~ ARE NO EIGENVALUES, $\sigma(A) = \{0\}$

ran $A = \{f \in C^1([0,1]);\ \underline{f(0) = 0}\}$ IS NOT DENSE

$\left.\begin{matrix} f = \lambda f' \\ \lambda f(0) = 0 \end{matrix}\right.$ DIFF. EQUATION $\Downarrow$

$\mathrm{ran}\, A = \{ f \in C^1([0,1]) ; \underline{f(0)=0} \}$ IS NOT DENSE $\begin{cases} f = \lambda f' \\ \lambda f(0) = 0 \end{cases}$ DIFF. EQUATION $\Downarrow$ $f \equiv 0 \Rightarrow$ NO EIGENVALUES

$\Rightarrow 0 \in \sigma_r(A)$

④ $A : C([0,1]) \to C([0,1])$

$$f(x) \to \int_0^x f - \underbrace{\int_0^1 dx \int_0^x f}$$

AS BEFORE, $A \in \underline{K(X)}$

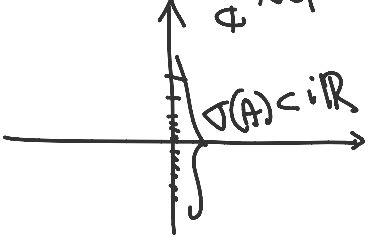
LET US FIND EIGENVALUES! $Af(x) = \lambda f(x) \iff \int_0^x f - \int_0^1 dx \int_0^x f = \lambda f(x)$

$\Updownarrow$

SOLUTIONS TO $f(x) = \lambda f'(x)$ ARE $f(x) = C e^{\frac{x}{\lambda}}$

$\begin{cases} f(x) = \lambda f'(x) \\ 0 = \int_0^1 \lambda f(x) \end{cases}$

WE ALSO WANT $\underline{\int_0^1 f = 0}$ IS POSSIBLE ONLY IF $\lambda = \frac{i}{2n\pi}$ $n \in \mathbb{Z}$



$$\sigma(A) = \{0\} \cup \left\{ \frac{i}{2n\pi} \right\}_{n \in \mathbb{Z}}$$

$0 \in \sigma_r(A)$ BECAUSE $\mathrm{ran}(A) = \{ f \in C^1([0,1]) : \int_0^1 f = 0 \}$ $\underline{\text{NOT}}$ DENSE.

ADJOINT OPERATOR

$\boxed{\text{DEF}}$ LET $A \in \mathcal{L}(X,Y)$ BE AN OPERATOR BETWEEN COMPLEX BANACH SP.

ITS $\underline{\text{ADJOINT OPERATOR}}$ IS $A^* \in \mathcal{L}(Y^*, X^*)$ IS DEFINED AS

$$Y^* \to X^*$$

$$L \longrightarrow A^* L : x \to L(Ax)$$

$\boxed{\text{REMARK}}$ ① A^* IS WELL-DEFINED, UNIQUE, CONTINUOUS AND $\|A^*\|_{\mathcal{L}(Y^*,X^*)} = \|A\|_{\mathcal{L}(X,Y)}$

② $A, B \in \mathcal{L}(X,Y)$, $\alpha, \beta \in \mathbb{C}$ $\Rightarrow (\alpha A + \beta B)^* = \bar{\alpha} A^* + \bar{\beta} B^*$

$C \in \mathcal{L}(Y,Z)$ $(A \circ C)^* = C^* \circ A^* \in \mathcal{L}(Z^*, X^*)$

③ X, Y ARE REFLEXIVE, THEN $A^{**} \in \mathcal{L}(X^{**}, Y^{**})$ $A^{**} = A$

UP TO THE CANONICAL IDENTI

EXIVE, THEN $A^{**} \in \mathcal{L}(X^{**}; Y^{**})$ $A^{**} = A$
UP TO THE CANONICAL IDENTIFICATION

④ IF $X = Y = H$ IS A COMPLEX HILBERT SPACE AND $A \in \mathcal{L}(H)$ THEN $A^* \in \mathcal{L}(H)$ UP TO IDENTIFICATON GIVEN BY RIESZ THEOREM
$\Rightarrow$ A^* IS THE ADJOINT OF A IF AND ONLY IF $\underline{(Ax, y) = (x, A^* y)}$