SOBOLEV SPACES $W^{1,p}((a,b)) := \{ u \in L^p((a,b)) ; \exists u' \in L^p((a,b)) \}$

THEOREM CHARACTERIZATION OF SOBOLEV SPACES

LET $u \in L^p((a,b))$ FOR $p \in (1,+\infty]$. THEN, THE FOLLOWING ARE EQUIVALENT:

a) $u \in W^{1,p}((a,b))$

b) $\exists C>0$ SUCH THAT $\left|\int_a^b u\varphi'\right| \le C\|\varphi\|_{L^{p'}((a,b))}$ $\forall \varphi \in C_0^1((a,b))$ $\left(\frac{1}{p}+\frac{1}{p'}=1\right)$

c) $\exists C>0$ SUCH THAT $\|u(x+h)-u(x)\|_{L^p((c,d))} \le C|h|$ $\forall \varphi \in C_0^1((a,b))$
c, d, h SUCH THAT $[c,d] \subset (a+|h|, b-|h|)$

C CAN BE CHOSEN AS $\|u'\|_{L^p}$

IF $p=1$ WE HAVE a $\Rightarrow$ b $\Leftrightarrow$ c

REMARKS $p=1$ b $\not\Rightarrow$ a $u(x) = \mathrm{sign}(x)$ IS NOT WEAKLY DERIVABLE
$u \notin W^{1,1}((a,b))$ BUT u SATISFIES b AND c : $\left|\int_a^b u\varphi'\right| = |2\varphi(0)| \le 2\|\varphi\|_{L^\infty}$
AND $\int_a^b |u(x+h)-u(x)| \le 2|h|$ "$u' = 2\delta_0$" $\rightarrow$ A MEASURE, NOT A FUNCTION

$p=\infty$ c IS EQUIVALENT TO $u \in Lip((a,b))$ LIPSCHITZ CONTINUOUS
THE THEOREM SAYS $W^{1,\infty}((a,b)) = Lip((a,b))$

COROLLARY IF $u \in W^{1,p}((a,b))$ AND $H \in C^1(\mathbb{R})$ THEN $H(u) \in W^{1,p}((a,b))$ $p>1$

PROOF $\underset{a}{u \in W^{1,p}} \Rightarrow \underset{c}{\|u(x+h)-u\|_{L^p} \le C|h|} \Rightarrow \|H(u(x+h)) - H(u(x))\|$ c

$\le (\sup|H'|)\,\|u(x+h)-u(x)\| \le C|h|$

$\Downarrow$

$H(u) \in W^{1,p}$ a

PROOF OF THEOREM a $\Rightarrow$ b ASSUME $u \in W^{1,p}((a,b))$, TAKE $\varphi \in C_0^1((a,b))$

$\Rightarrow \left|\int_a^b u\varphi'\right| = \left|-\int u'\varphi\right| \le \|u'\|_{L^p}\|\varphi\|_{L^{p'}} \le C\|\varphi\|_{L^{p'}}$

b $\Rightarrow$ a BY HYPOTHESIS, $\varphi \xrightarrow{L} -\int_a^b u\varphi'$ IS A CONTINUOUS LINEAR FUNCT. ON $\left(C_0^1((a,b)), \|\cdot\|_{L^{p'}}\right)$, WE EXTEND L TO $\tilde{L} \in (L^{p'})^*$ USING HAHN-BANACH

SINCE $p>1$, THEN BY RIESZ' THEOREM $\exists g \in L^p((a,b))$, SUCH THAT $\tilde{L}: f \mapsto \int^b fg$ NOW $\forall \varphi \in C_0^1((a,b))$ $L\varphi = \tilde{L}\varphi = \int^b \varphi g \Rightarrow g = u'$

[illegible]

$\tilde{L}: f \to \int_a^b fg$. NOW, $\forall \varphi \in C_0^1((a,b))$ $\quad L\varphi = \tilde{L}\varphi = \int_a^b \varphi g \Rightarrow g = u' \Rightarrow u \in W^{1,p}$

$L\varphi = -\int_a^b u\varphi'$

(b) $\Rightarrow$ (c) TAKE $\varphi \in C_0^1((a,b))$, TAKE h SUCH THAT $\operatorname{supp}\varphi \subset (a+|h|, b-|h|)$

$$\int_a^b \frac{u(x+h)-u(x)}{h}\varphi(x)\,dx = \int_a^b \frac{u(x+h)}{h}\varphi(x)\,dx - \int_a^b \frac{u(x)}{h}\varphi(x)\,dx \underset{y=x+h}{=} \int_a^b \frac{u(y)}{h}\varphi(y-h)\,dy$$

$$-\int_a^b \frac{u(y)}{h}\varphi(y)\,dy = -\int_a^b u(y)\frac{\varphi(y)-\varphi(y-h)}{h}\,dy$$

$\frac{\varphi(y)-\varphi(y-h)}{h} \longrightarrow \varphi'(y)$ POINTWISE, $\left|\frac{\varphi(y)-\varphi(y-h)}{h}\right| \leq C$ BECAUSE $\varphi \in C_0^1$

$u \in L^1_{loc} \Rightarrow \left|u(y)\frac{\varphi(y)-\varphi(y-h)}{h}\right| \leq C|u|\chi_{\operatorname{supp}(\varphi)} \in L^1$

BY DOMINATED CONVERGENCE THEOREM, $\int_a^b u(y)\frac{\varphi(y)-\varphi(y-h)}{h} \to \int_a^b u\varphi' \overset{(b)}{\leq} C\|\varphi\|_{L^{p'}(a,b)}$

$$\left|\int_a^b \frac{u(x+h)-u(x)}{h}\varphi(x)\,dx\right| \leq C\|\varphi\|_{L^{p'}}$$

$\|\varphi\|_{L^{p'}} = \left(\int |\varphi|^{\frac{p}{p-1}}\right)^{1-\frac{1}{p}}$

TAKE $\varphi_n \in C_0^1((a,b))$ APPROXIMATING $|u(x+h)-u(x)|^{p-2}(u(x+h)-u(x))$

$$\int_c^b \frac{|u(x+h)-u(x)|^p}{|h|} \leq C\|u(x+h)-u(x)\|_{L^p}^{p-1}$$

$$\| \;\; = \frac{\|u(x+h)-u(x)\|_{L^p}^p}{|h|} \Rightarrow \|u(x+h)-u(x)\|_{L^p} \leq C|h|$$

(c) $\Rightarrow$ (b) $\left|\int_a^b \frac{u(x+h)-u(x)}{h}\varphi(x)\,dx\right| \leq \left\|\frac{u(x+h)-u(x)}{h}\right\|_{L^p}\|\varphi\|_{L^{p'}} \overset{(c)}{\leq} C\|\varphi\|_{L^{p'}}$

$h \to 0$, SAME PROOF AS BEFORE

$\left|-\int_a^b u\varphi'\right| \Rightarrow$ (b) HOLDS TRUE.

DEF LET X, Y BE NORMED SPACES AND $A \in \mathcal{L}(X,Y)$. A IS SAID TO

DEF LET X, Y BE NORMED SPACES AND $A \in \mathcal{L}(X,Y)$. A IS SAID TO BE A <u>COMPACT</u> OPERATOR IF $\forall C \subset X$ BOUNDED $A(C) \subset Y$ IS REL. COMPACT. IN OTHER WORDS, IF $\{x_n\}$ IS BOUNDED IN X THEN $\{Ax_n\}$ HAS A CONVERGING SUBSEQUENCE IN Y.

WE DENOTE THE SET OF COMPACT OPERATORS AS $\mathcal{K}(X,Y) \subset \mathcal{L}(X,Y)$ $\quad (\mathcal{K}(X) := \mathcal{K}(X,X))$

REMARK ① IN GENERAL, $A(C)$ IS BOUNDED IF C IS BOUNDED, $A(C)$ IS COMPACT IF C IS COMPACT BUT $A(C)$ MAY NOT BE COMPACT IF C IS BOUNDED

② IF $A \in \mathcal{L}(X,Y)$, $B \in \mathcal{L}(Y,Z)$ AND <u>EITHER</u> A OR B IS COMPACT, THEN $B \circ A \in \mathcal{K}(X,Z)$

PROP $\mathcal{K}(X,Y) \triangleleft \mathcal{L}(X,Y)$

<u>PROOF</u> IT IS IMMEDIATE TO VERIFY THAT $A, B \in \mathcal{K}(X,Y) \Rightarrow \alpha A + \beta B \in \mathcal{K}(X,Y)$.
LET US SHOW THAT $\mathcal{K}(X,Y)$ IS CLOSED: TAKE $\{A_n\}$ SEQUENCE IN $\mathcal{K}(X,Y)$ SUCH THAT $\underline{A_n \to A}$ IN $\mathcal{L}(X,Y)$, WE WANT TO SHOW $A \in \mathcal{K}(X,Y)$.
TAKE $\{x_m\}_{m \in \mathbb{N}}$ BOUNDED SEQUENCE IN X. SINCE A_1 IS COMPACT, $A_1 x_m$ CONVERGES UP TO A SUB-SEQ.; $A_2 x_m$ AGAIN CONVERGES, ... $A_n x_m \xrightarrow[m \to \infty]{} y_n \in Y$ $\forall n \in \mathbb{N}$. IN PARTICULAR IT IS CAUCHY.

$\forall \varepsilon > 0$ $\exists N$ SUCH THAT $\|A - A_N\| < \varepsilon$ AND $\|A_N x_m - A_N x_n\| < \varepsilon$ $n, m \geq N_\varepsilon$

$$\Rightarrow \|Ax_n - Ax_m\| \leq \underbrace{\|Ax_n - A_N x_n\|}_{\leq \underbrace{\|x_n\|}_{\leq C} \underbrace{\|A - A_N\|}_{< \varepsilon}} + \underbrace{\|A_N x_n - A_N x_m\|}_{< \varepsilon} + \underbrace{\|A_N x_m - A x_m\|}_{\underbrace{\|x_m\|}_{\leq C} \underbrace{\|A - A_N\|}_{\leq \varepsilon}}$$

$$\|x_n\|, \|x_m\| \leq C \quad \leq (2C+1)\varepsilon$$

$\Rightarrow \{Ax_n\}$ IS CAUCHY, SO IT CONVERGES, SO A IS COMPACT

EXAMPLES ① $\dim Y < +\infty \Rightarrow \mathcal{L}(X,Y) = \mathcal{K}(X,Y)$ BECAUSE ANY BOUNDED SET IN Y IS REL. COMPACT. SINCE $\mathcal{K}(X,Y) \triangleleft \mathcal{L}(X,Y)$, IF A IS THE LIMIT OF A SEQ. OF FINITE RANGE OPERATORS $(\dim(\operatorname{ran}(A_n)) < +\infty)$, THEN A IS COMPACT

② $I : X \to X$ IDENTITY IS <u>NOT</u> COMPACT IF $\dim X = +\infty$: IN FACT, TAKE $C = B_1(0)$ UNIT BALL IS BOUNDED BUT $I(C) = C$ IS NOT COMPACT

– TAKE $C = B_1(0)$ UNIT BALL IS BOUNDED BUT $I(C) = C$ IS NOT COMPACT

③ $A \in \mathcal{L}(\ell_2)$ $\quad (x(1), x(2), \dots x(k), \dots) \xrightarrow{A} \left(x(1), \frac{x(2)}{2}, \dots \frac{x(k)}{k}, \dots\right)$ IS COMPACT.

IN FACT, A IS THE LIMIT OF $A_n : (x(1), \dots, x(k), \dots) \longrightarrow \left(x(1), \dots, \frac{x(n)}{n}, 0, \dots, 0\right)$

$\dim(\mathrm{ran}(A_n)) = n$, LET US SHOW THAT $\|A - A_n\| \longrightarrow 0$

$A - A_n : (x(1) \dots x(k), \dots) \to \left(0, \dots, 0, \frac{x(n+1)}{n+1}, \dots\right)$

$$\Rightarrow \|A - A_n\| = \sup_{\|x\| \leq 1} \sqrt{\sum_{k \geq n+1} \frac{x(k)^2}{k^2}} \leq \frac{1}{n+1} \underbrace{\sqrt{\sum_{k \geq n+1} x(k)^2}}_{\leq \|x\|} \leq \frac{1}{n+1} \longrightarrow 0$$

DEF LET $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ BE NORMED SPACES, WITH $X \subset Y$

WE SAY <u>X IS CONTINUOUSLY EMBEDDED IN Y</u> IF $\begin{aligned} i : X &\to Y \\ x &\to x \end{aligned}$

IS A CONTINUOUS MAP, THAT IS $\|x\|_Y \leq C \|x\|_X \quad \forall x \in X$

WE SAY <u>X IS COMPACTLY EMBEDDED IN Y</u> IF $\begin{aligned} i : X &\to Y \\ x &\to x \end{aligned}$ IS

A COMPACT OPERATOR, THAT IS IF $\|x_n\|_X \leq C$ THEN x_n CONVERGES W.R.TO $\|\cdot\|_Y$

EXAMPLES ① IF $E \subset X$ IS A LINEAR SUBSPACE, THE EMBEDDING $(E, \|\cdot\|_X)$ IN $(X, \|\cdot\|_X)$ IS ALWAYS CONTINUOUS, COMPACT IFF $\dim E < +\infty$

② $L^p((a,b))$ IS CONTINUOUSLY EMBEDDED IN $L^q((a,b))$ IF $a, b \in \mathbb{R}$, $p \geq q$

③ <u>ℓ_p</u> IS CONTINUOUSLY EMBEDDED IN <u>ℓ_q</u> IF $p \leq q$